

Cooperative oligopoly games with boundedly rational firms

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Abstract We analyze cooperative Cournot games with boundedly rational firms. Due to cognitive constraints, the members of a coalition cannot accurately predict the coalitional structure of the non-members. Thus, they compute their value using simple heuristics. In particular, they assign various non-equilibrium probability distributions over the outsiders' set of partitions. We construct the characteristic function of a coalition in such an environment and we analyze the core of the corresponding games. We show that the core is non-empty provided the number of firms in the market is sufficiently large. Moreover, we show that if two distributions over the set of partitions are related via first-order dominance, then the core of the game under the dominated distribution is a subset of the core under the dominant distribution.

Keywords Cooperative game · Externalities · Cournot market · Core · Bounded rationality

JEL Classification C71 · L2

1 Introduction

Collusion among firms in oligopolistic markets is a wide-spread phenomenon and constantly attracts the interest of economists. By colluding, firms can restrict output and raise market prices, thus extracting a higher surplus from consumers. From a methodological point of view, economists analyze collusion using either non-cooperative games (when agreements

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among firms are non-enforceable by an outside entity) or cooperative games (whenever the signing of enforceable agreements is possible). Under the latter approach, the focus usually is on the core of an appropriately defined cooperative game. The core consists of all allocations of total profits that cannot be blocked by any coalition of firms. Non-empty core means that cooperation among all firms in the market is a priori feasible.

When a coalition of firms contemplate the blocking of an agreement, they have to calculate their stand-alone payoff. In a market environment such a calculation is not a trivial task, as the coalition's worth depends on how the non-members act. In particular, it depends on the partition (coalition structure) that the outsiders will form. This calls for the formation of beliefs about the outsiders' coalitional actions.

Different conjectures about the reaction of the outsiders lead to different coalitional worths and thus to different notions of core. The α and β cores (Aumann 1959) are based on the assumption of min-max behavior on behalf of the non-members; the γ -core (Chander and Tulkens 1997) is based on the assumption that outsiders play individual best replies to the deviant coalition; the δ -core scenario (Hart and Kurz 1983) assumes that outsiders form a single coalition. Various authors applied these core notions to the study of Cournot markets. Rajan (1989) used the concept of γ -core and showed that it is non-empty for a market with 4 firms. A more general result for any number of firms is provided by Chander (2010). Currarini and Marini (2003) built a refinement of the γ -core by assuming that the deviant coalition acts as a Stackelberg leader in the product market. Zhao (1999) showed that the α and β cores of oligopolistic markets are non-empty.

The seminal work of Ray and Vohra (1999) goes one step further, as the worth of a coalition is deduced via arguments that satisfy a consistency criterion: a deviant coalition takes into account the fact that after its deviation, other deviations might follow, with the newly deviant coalitions thinking in a similar forward way. For games where binding agreements are feasible, Huang and Sjostrom (1998, 2003) and Koczy (2007) developed the recursive core. The recursive core is constructed under the assumption that the members of a coalition compute their value by looking recursively on the cores of the sub-games played among the outsiders.

Predicting the equilibrium coalitional formation in a game with many players is computationally cumbersome. Sandholm et al. (1999) showed that for an n -player game the number of different coalition structures is $O(n^n)$ and $\omega(n^{\frac{n}{2}})$. Hence, deducing the coalition structure that the outsiders form is a particularly difficult task (at least, for games with a large number of players). For example, finding the coalition structure that maximizes the sum of all players' payoffs is an *NP*-hard problem (Sandholm et al. 1999). Even finding sub-optimal solutions requires the search of an exponential number of cases.

The last considerations give the motivation of the current paper. We analyze an n -firm cooperative Cournot oligopoly assuming that no group of firms has the cognitive ability to accurately deduce the partition that the outsiders will form. As a result, the members of a coalition cannot compute their value with precision. Instead, they compute it by following simple procedures or heuristics.

Clearly, the number of different heuristics one can adopt is very large. Computer scientists, for example, model similar situations via search algorithms that give solutions within certain bounds from the optimal coalition structure (Sandholm et al. 1999; Dang and Jennings 2004). On the other hand, the economists' toolbox of heuristics includes models with players of various degrees of cognitive abilities (Stahl and Wilson 1994; Camerer 2003; Camerer et al. 2004; Haruvy and Stahl 2007), models with probabilistic choice rules (McKelvey and Palfrey 1995; Chen et al. 1997; Anderson et al. 2002), to name only a few.

In our paper, the heuristics are based on the assignment of non-equilibrium probability distributions over the set of the opponents' coalition structures. I.e., when contemplating a deviation from the grand coalition, the members of a coalition make the simplifying assumption that the reactions of the outsiders are governed by various plausible -but not necessarily optimal- probability distributions.

Our benchmark case assumes that the probability of a coalition structure is proportional to the profitability that the structure induces for the outsiders. Namely, a deviant coalition assumes it is more likely that its opponents will partition themselves according to the more efficient structures. This approach is motivated by the logit quantal response model of [McKelvey and Palfrey \(1995\)](#) in non-cooperative games, where the probability of a player choosing a certain strategy depends on its relative payoff, with the probability being positive even if the strategy is inferior. We derive the characteristic function of a coalition under such a distribution -i.e., a logit type of distribution- and we examine the core of the corresponding game. We show that if the number of firms in the market is sufficiently large then the core is non-empty. Hence, bounded rationality supports cooperation among all firms in the market.

In the second part of the paper, we extend our analysis by considering more general probability distributions. In particular we consider distributions which are related via first-order stochastic dominance. If a certain distribution dominates another one, it gives relatively higher weight to partitions consisting of many coalitions. Our analysis has two goals: first, given that a relatively less concentrated partition hurts a deviant coalition, we present a novel way of modeling pessimism in a cooperative game with externalities. Secondly, we utilize this machinery to analyze the core in a Cournot market under a large number of distributions (other than the logit).

We fix a pair of distributions satisfying the first-order dominance property. We show that the core of the game under the dominated distribution is contained in the core of the game under the dominant. In particular, this implies that the core under the logit distribution is contained in the core under any distribution that first-order dominates it. Thus we indirectly show that our Cournot game has a non-empty core for a large number of probability distributions.

In particular, the above inclusion holds for the case of γ -core. Namely, the core under the logit distribution is contained in the core constructed under the assumption that outsiders form singleton coalitions (in our terminology, the γ scenario corresponds to the degenerate distribution that assigns probability one to the singletons partition). Hence our core refines the γ -core.

In what follows, we present the basic model in Sect. 2. In Sects. 3 and 4 we present our results. Section 5 concludes.

2 The model

We consider a market with the set $N = \{1, 2, \dots, n\}$ of firms. Firms produce a homogeneous product. Firm l produces quantity q_l using the cost function $C(q_l) = cq_l$, $l \in N$. The market price p is determined via the inverse demand function $p = p(Q)$, where $Q = q_1 + q_2 + \dots + q_n$ is the total market quantity.

Assumptions

- A1. $\exists Q_0 > 0$ such that $p(Q) > 0$ for $Q < Q_0$ and $p(Q) = 0$ for $Q \geq Q_0$
- A2. $p'(Q) < 0$, whenever $p(Q) > 0$
- A3. $p'(Q) + q_i p''(Q) < 0$

where $p'(Q)$ and $p''(Q)$ denote the first and second derivatives of the inverse demand function. The above assumptions are standard and guarantee the existence and uniqueness of Cournot equilibrium (see, for example, [Vives 2001](#)).

Let $S \subset N$ denote a coalition of firms with $|S| = s$ members and let $N \setminus S$ denote the complementary set of S , where $|N \setminus S| = n - s$. The worth or value of S is the sum of its members' profits. These profits depend on how the members of $N \setminus S$ partition themselves into coalitions. The set $N \setminus S$ can be partitioned into disjoint subsets in B_{n-s} ways, where B_{n-s} is Bell's $(n - s)^{th}$ number ([Bell 1934](#)).

What matters for S is only the number of the opponent coalitions in the Cournot market. Consider for example the case $N = \{1, 2, 3, 4, 5\}$ and $S = \{1\}$. The set of outsiders is $N \setminus S = \{2, 3, 4, 5\}$. Consider the partitions $\{\{2, 3\}, \{4, 5\}\}$ and $\{\{2, 3, 4\}, \{5\}\}$ of outsiders. These partitions are equivalent for S (and so are all partitions with two coalitions) in the sense that both induce the same profit for S (in both cases, S would compete in a triopoly market). More generally, all partitions with j coalitions induce the same profit for S , irrespective of how the outsiders are grouped among the j coalitions. We will call these partitions j -similar, where $j = 1, 2, \dots, n - s$.

Denote the number of j -similar partitions by $K_{n-s,j}$, where $K_{n-s,j}$ gives the number of ways to partition a set of $n - s$ objects into j groups, or else the Stirling number of the second kind. Then

$$K_{n-s,j} = \frac{1}{j!} \sum_{i=0}^j (-1)^i \binom{j}{i} (j - i)^{n-s} \tag{1}$$

The basic assumption that underlies this paper is that the members of S use simple probabilistic models in order to predict the coalitional behavior of the non-members. As a benchmark case, we assume that the probability of a partition is proportional to the profitability that the partition induces for the outsiders. This approach is in line with the spirit of the logit quantal response model ([McKelvey and Palfrey 1995](#)) in non-cooperative games, where the probability of choosing a strategy depends on its relative payoff, the probability being positive even if the strategy is inferior.

Consider again a coalition S with s members and an outsiders' partition with j coalitions. Let Π_j denote the sum of the profits that the j coalitions earn under this partition (this sum is constant over all j -similar partitions). Define

$$f_{n,s}(j) = \frac{e^{\Pi_j} K_{n-s,j}}{\sum_{m=1}^{n-s} e^{\Pi_m} K_{n-s,m}} \tag{2}$$

Notice that $f_{n,s}(j) \in (0, 1)$ and $\sum_{j=1}^{n-s} f_{n,s}(j) = 1$. Then, $f_{n,s}(j)$ gives the total probability that S assigns to all j -similar structures. Note in (2) that the profitability of a j -similar partition is adjusted by the corresponding Stirling number. The results of the paper hold even if such an adjustment does not take place.

2.1 An example

Let us illustrate the above by considering an example with five firms, $N = \{1, 2, 3, 4, 5\}$. Assume that the inverse demand is $p = 1 - Q$ and that $c = 0$. Consider again a coalition S with s members. If the $n - s$ outsiders form j coalitions then there are $j + 1$ active players

in the market. By simple calculations, the total profits of the j outside coalitions then are

$$\Pi_j = \frac{j}{(j + 2)^2}, \quad j = 1, 2, \dots, n - s \tag{3}$$

Consider a singleton coalition, say $S = \{1\}$. Then $B_{n-s} = B_4 = 15$ and $K_{4,1} = K_{4,4} = 1, K_{4,2} = 7, K_{4,3} = 6$. Using (2) and (3) the probabilities that S assigns to outsiders' partitions are

$$f_{5,1}(1) = f_{5,1}(4) = \frac{e^{1/9}}{Z_1}, \quad f_{5,1}(2) = \frac{7e^{1/8}}{Z_1}, \quad f_{5,1}(3) = \frac{6e^{3/25}}{Z_1}$$

where $Z_1 = 2e^{1/9} + 7e^{1/8} + 6e^{3/25}$. Consider next a coalition with two members, say $S = \{1, 2\}$. Then $B_{n-s} = B_3 = 5$ and $K_{3,1} = K_{3,3} = 1, K_{3,2} = 3$. We then have

$$f_{5,2}(1) = \frac{e^{1/9}}{Z_2}, \quad f_{5,2}(2) = \frac{3e^{1/8}}{Z_2}, \quad f_{5,2}(3) = \frac{e^{3/25}}{Z_2}$$

where $Z_2 = e^{1/9} + 3e^{1/8} + e^{3/25}$. Consider next a coalition with three members $S = \{1, 2, 3\}$. In this case, $B_{n-s} = B_2 = 2$ and $K_{2,1} = K_{2,2} = 1$. Hence we have

$$f_{5,3}(1) = \frac{e^{1/9}}{Z_3}, \quad f_{5,3}(2) = \frac{e^{1/8}}{Z_3}$$

where $Z_3 = e^{1/9} + e^{1/8}$. Finally, if a deviant coalition has four members, it faces one outsider only and so there is no ambiguity.

2.2 The game (N, v^n)

In this section we compute the characteristic function of a deviant coalition. We use the j -similarity and focus for each j on one representative of the j -similar partitions. Let q_s denote the quantity of the deviant coalition S ; and let q_i^j denote the quantity of outside coalition $i, i = 1, 2, \dots, j$, under a partition with j members. The objective function that S faces is given by

$$\pi_f(S) = \sum_{j=1}^{n-s} f_{n,s}(j) \left(p \left(q_s + \sum_{i=1}^j q_i^j \right) - c \right) q_s \tag{4}$$

The objective function of coalition i is

$$\pi_i^j = \left(p \left(q_s + \sum_{r=1, r \neq i}^j q_r^j + q_i^j \right) - c \right) q_i^j, \quad i = 1, 2, \dots, j$$

Hence the maximization problems to solve for are

$$\max_{q_s} \pi_f(S) \tag{5}$$

and for $j = 1, 2, \dots, n - s$,

$$\max_{q_i^j} \pi_i^j, \quad i = 1, 2, \dots, j \tag{6}$$

Let $\tilde{q}_s(f) = \operatorname{argmax}_{q_s} \pi_f(S)$ denote the best reply function of S ; and let $\tilde{q}_i^j = \operatorname{argmax}_{q_i^j} \pi_i^j$ denote the best reply¹ of coalition $i, i = 1, 2, \dots, j$ where $j = 1, 2, \dots, n - s$. If we solve the system of equations defined by the best replies we end up with the reduced-form solution in quantities, which shall be denoted by $q_s(f)$ and $q_i^j(f)$. By straightforward calculations, the reduced-form solution is given implicitly by

$$q_s(f) = \frac{\sum_{j=1}^{n-s} f_{n,s}(j) p(q_s(f) + j q_i^j(f)) - c}{-\sum_{j=1}^{n-s} f_{n,s}(j) p'(q_s(f) + j q_i^j(f))} \tag{7}$$

and for $j = 1, 2, \dots, n - s$,

$$q_i^j(f) = \frac{p(q_s(f) + j q_i^j(f)) - c}{-p'(q_s(f) + j q_i^j(f))}, \quad i = 1, 2, \dots, j \tag{8}$$

where we used the fact that $q_1^j(f) = q_2^j(f) = \dots = q_j^j(f)$. Using (7) and (8) in (4), we obtain the characteristic function of S . We denote this function by $v^n(S)$ or $v^n(s)$. Letting $Q_j(f) = q_s(f) + j q_i^j(f)$ we have

$$v^n(S) = \frac{\left(\sum_{j=1}^{n-s} f_{n,s}(j) p(Q_j(f)) - c \right)^2}{-\sum_{j=1}^{n-s} f_{n,s}(j) p'(Q_j(f))} \tag{9}$$

Hence our game is the pair (N, v^n) where v^n is defined by (9). The value of the grand coalition is denoted by $v^n(N)$ or $v^n(n)$. This value is the monopoly profit (which is independent of n but for notational uniformity we keep the superscript). Denote by Q_M the monopoly output. Then it is easy to show that

$$v^n(N) = \frac{(p(Q_M) - c)^2}{-p'(Q_M)}$$

An allocation is a vector (x_1, x_2, \dots, x_n) such that $\sum_{i \in N} x_i = v^n(N)$. The core C_f of (N, v^n) is the set of all allocations that cannot be blocked by any coalition, given distribution $f_{n,s}$. I.e., the core is the set

$$C_f = \left\{ (x_1, \dots, x_n) : \exists S \text{ with } v^n(S) > \sum_{i \in S} x_i \right\}$$

Given the above, in the next sections we examine the core for various demand and probability functions.

¹ Clearly the best replies depend on the quantities of the opponents but for notational simplicity we drop writing them.

3 Results

The first result in this section states the following useful property.

Lemma 1 *For every positive integer k , the equality $v^n(s) = v^{n+k}(s + k)$ holds.*

Proof Consider two markets: in the first market there are n firms and the deviant coalition S has s members and in the second there are $n + k$ firms and S has $s + k$ members. Consider a number j of outside coalitions. Notice that in the first market, j runs from 1 up to $n - s$; in the second market, j runs from 1 up to $n + k - (s + k) = n - s$ again. The total profits of j outside coalitions under the first market are equal to their total profits under the second market. This is due to the constant returns to scale assumption. In both cases, these profits are Π_j . Hence

$$\begin{aligned}
 f_{n,s}(j) &= \frac{e^{\Pi_j} K_{n-s,j}}{\sum_{m=1}^{n-s} e^{\Pi_m} K_{n-s,m}} \\
 &= \frac{e^{\Pi_j} K_{n+k-(s+k),j}}{\sum_{m=1}^{n+k-(s+k)} e^{\Pi_m} K_{n+k-(s+k),m}} \\
 &= f_{n+k,s+k}(j)
 \end{aligned}
 \tag{10}$$

Moreover for each j , $q_s(f)$ and $q_i^j(f)$ are constant under the two markets (again due to constant returns to scale): namely, the quantity of S when it has s members and the market has n firms is equal to its quantity when S has $s + k$ members and the market has $n + k$ firms; the same holds for $q_i^j(f)$ and by consequence for $Q_j(f)$. Combining this fact with (10) and (9) proves the result. \square

The intuition behind Lemma 1 is clear. If coalition S has $s + k$ members in a market with $n + k$ firms, it faces $n + k - (s + k) = n - s$ outsiders. This equals the number of outsiders that S faces when it has s members in a market with n firms. Hence S faces the same set of potential coalition structures.

An almost immediate implication of Lemma 1 is the monotonicity of $v^n(s)$ in s .

Lemma 2 *For every n , $v^n(s)$ is strictly increasing in s .*

Proof The proof will be based on induction. For the base case, $n = 2$, we have to prove that $v^2(2) > v^2(1) > v^2(0)$. Recall that $v^2(2)$ is the monopoly profit and $v^2(1)$ is the profit of a firm when two firms are in the market. Under assumptions A1-A3 we have that the former profit is higher than the latter,² i.e., $v^2(2) > v^2(1)$. Moreover, $v^2(1) > 0 = v^2(0)$ and so we have the base case.

Assume for the induction hypothesis that in a game with n players and for an arbitrary s , we have that $v^n(s) > v^n(s - 1)$. We will prove that $v^{n+1}(s) > v^{n+1}(s - 1)$. By Lemma 1 and the induction hypothesis we have that $v^{n+1}(s) = v^n(s - 1) > v^n(s - 2) = v^{n+1}(s - 1)$. Note also that $v^{n+1}(s + 1) > v^{n+1}(s)$ (by Lemma 1) and that $v^{n+1}(1) > v^{n+1}(0) = 0$. This completes the proof. \square

² See Amir and Lambson (2000).

Lemmas 1 and 2 hold under any demand function (that satisfies assumptions A1-A3). In what follows, we use these Lemmas to derive conditions for core non-emptiness under certain demand functions. In particular we will focus on the family of demand functions

$$Q = 1 - p^b, \quad b > 0$$

which we borrow from Anderson and Engers (1992). Note that if $b > (<) 1$, demand is concave (convex); and if $b = 1$, demand is linear. In order to derive analytically the market solution for an arbitrary b we need to set $c = 0$. The relevant calculations appear in Lemma 5 in the Appendix. The solution, i.e, the quantities and the characteristic function, is given by

$$q_s(f) = \frac{\sum_{j=1}^{n-s} f_{n,s}(j)\psi_j^{\frac{1}{b}}}{\sum_{j=1}^{n-s} f_{n,s}(j)\psi_j^{\frac{1}{b}}(1+j+1/b)} \tag{11}$$

$$q_i^j(f) = b\psi_j \frac{\sum_{j=1}^{n-s} f_{n,s}(j)\psi_j^{\frac{1}{b}}(j+1/b)}{\sum_{j=1}^{n-s} f_{n,s}(j)\psi_j^{\frac{1}{b}}(1+j+1/b)}, \quad i = 1, 2, \dots, j \tag{12}$$

$$v^n(S) = \sum_{j=1}^{n-s} f_{n,s}(j) \left(\frac{\sum_{j=1}^{n-s} f_{n,s}(j)\psi_j^{\frac{1}{b}}(j+1/b)}{(bj+1)\sum_{j=1}^{n-s} f_{n,s}(j)\psi_j^{\frac{1}{b}}(1+j+1/b)} \right)^{\frac{1}{b}} q_s(f) \tag{13}$$

where $\psi_j = \frac{1}{bj+1}$. Finally it is easy to see that the value of the grand coalition is

$$v^n(N) = \frac{b}{(1+b)^{\frac{1}{b}+1}}$$

3.1 The case $b = 1$

As a benchmark case, we first present a result for the linear demand, i.e., $b = 1$. Afterwards, we discuss the non-linear case.

Proposition 1 *Assume the demand function is given by $Q = 1 - p$. If n is sufficiently large then $C_f \neq \emptyset$.*

Proof Since firms are identical, the core is non-empty if and only if for all $s \leq n$,

$$\frac{v^n(n)}{n} \geq \frac{v^n(s)}{s} \tag{14}$$

It is easy to verify that the above inequality does not hold for $3 \leq n \leq 11$. So for these values of n the core is empty.³ The inequality holds for $n = 12$ (Table 2 in the Appendix). We will prove the rest of the proposition using induction on n , where $n \geq 12$.

Base: Table 2 in the Appendix establishes the base case ($n = 12$).

Induction hypothesis: For all $S : |S| = s \leq n$, $\frac{v^n(n)}{n} \geq \frac{v^n(s)}{s}$.

³ For $3 \leq n \leq 11$ it holds that $v^n(1) > \frac{v^n(n)}{n}$ (see Table 3 in the Appendix). The relevant calculations were made using the Maple program and they are available by the authors upon request.

Induction step: We will show that for all $S : |S| = s \leq n + 1$,

$$\frac{v^{n+1}(n + 1)}{n + 1} \geq \frac{v^{n+1}(s)}{s}$$

By Lemma 1 we have that $v^{n+1}(s) = v^{n+1}((s - 1) + 1) = v^n(s - 1)$ and also $v^{n+1}(n + 1) = v^n(n)$. So we have to show that

$$\frac{v^n(n)}{n + 1} \geq \frac{v^n(s - 1)}{s} \tag{15}$$

From the Induction hypothesis we have

$$v^n(n) \geq \frac{n}{s - 1} v^n(s - 1)$$

and thus

$$(s - 1)v^n(n) \geq nv^n(s - 1) \tag{16}$$

Using Lemma 2,

$$v^n(n) > v^n(s - 1) \tag{17}$$

Adding (16) and (17) we have

$$sv^n(n) > (n + 1)v^n(s - 1)$$

which implies that (15) holds. So we have the proof for $n + 1$ and thus the proposition is proved. □

The monopoly profit is independent of the number of firms n . On the other hand, $v^n(s)$ decreases in n . As a result, for sufficiently large n the difference $v^n(n)/n - v^n(s)/s$ becomes positive for all s and the core is non-empty.

3.2 The case $b \neq 1$

In this section we discuss the non-emptiness of the core for the non-linear demand case. To this end, we will utilize the previous results, i.e., Lemmas 1 and 2, and Proposition 1. Recall that Lemmas 1 and 2 hold for any demand function. Furthermore, among the three steps of the induction proof of Proposition 1, i.e., base step, induction hypothesis and induction step, essentially only the base step depends on the demand function used. The other two steps work independently of the demand function (given of course the validity of the base step). Hence when extending Proposition 1 to cases where $b \neq 1$ we only need to ensure the validity of the base step. Namely, for a certain value of b we need to find a specific number of firms that provides the base step of the induction argument (we refer the reader to the proof of Proposition 1).

Due to the complexity of the model, we do not address the above task for all values of b . Nonetheless, Table 1 (next page) presents pairs of numbers $(b^*, n(b^*))$ that satisfy the property discussed above: given a certain value b^* (where $b^* \neq 1$), the table reports a value $n(b^*)$ which is the number of firms that establishes the base step of the induction process for the demand function $Q = 1 - p^{b^*}$. In other words, given a specific value b^* , the game has non-empty core for $n \geq n(b^*)$ and it has empty core for $n < n(b^*)$. We note that Table 1 provides values b^* for which the demand function can be either convex ($b^* < 1$) or concave ($b^* > 1$).

Table 1 b^* and $n(b^*)$ for base step of induction

b^*	$n(b^*)$
0.5	3
0.6	5
0.7	6
0.8	7
0.9	9
1.1	15
1.2	19
1.3	24
1.4	32
1.5	42
1.6	57
1.7	78
1.8	107
1.9	147
2.0	205

In the Table we note that as b^* increases, the core is non-empty less often (as b^* increases, the number $n(b^*)$ increases). Moreover, if b^* is sufficiently low, the core is non-empty for all $n \geq 3$.

4 First-order stochastic dominance

In this section we compare the cores of games that differ with respect to the probability schemes assigned to outsiders’ partitions. In particular, we consider distributions that are related via first-order stochastic dominance. We will show that if a certain distribution dominates at first-order another one, then the core under the dominated distribution is a subset of the core under the dominant distribution. An application of this result is that Proposition 1 holds under any distribution that dominates the distribution defined in (2).

Consider coalition S . Let $z_{n,s}$ and $w_{n,s}$ be two probability distributions over the set of the outsiders’ partitions. Assume that $z_{n,s}$ dominates $w_{n,s}$ at first-order, i.e., for all j^* ,

$$\sum_{j=1}^{j^*} z_{n,s}(j) \leq \sum_{j=1}^{j^*} w_{n,s}(j)$$

Denote by C_w and C_z the cores under the two distributions and assume that $C_z \neq \emptyset$. We have the following result.

Proposition 2 *Assume the inverse demand $p(Q)$ is weakly concave. If $z_{n,s}$ stochastically dominates $w_{n,s}$ at first-order then $C_w \subseteq C_z$.*

Proof For $j = 1, 2, \dots, n - s$, denote $\tilde{Q}_j^{-s} = \sum_{i=1}^j \tilde{q}_i^j$, where $\tilde{q}_i^j = \arg \max_{q_i^j} \pi_i^j$ (recall that π_i^j is the objective function of coalition i in a partition with j coalitions). Let $\pi_w(S)$ and $\pi_z(S)$

denote the objective functions of S under distributions $w_{n,s}$ and $z_{n,s}$ respectively. Let

$$\tilde{q}_s(w) = \arg \max_{q_s} \pi_w(S), \quad \tilde{q}_s(z) = \arg \max_{q_s} \pi_z(S)$$

Denote by $q_s(w)$, $q_i^j(w)$ and $Q_j^{-s}(w) = \sum_{i=1}^j q_i^j(w)$ the reduced-form solution of the system of equations defined by the above best replies when the probability distribution is $w_{n,s}$; and by $q_s(z)$, $q_i^j(z)$ and $Q_j^{-s}(z) = \sum_{i=1}^j q_i^j(z)$ the reduced-form solution when the probability distribution is $z_{n,s}$. Finally, denote by $v_w^n(S)$ and $v_z^n(S)$ the characteristic functions of coalition S under $w_{n,s}$ and $z_{n,s}$ respectively.

We first have that

$$\tilde{Q}_{j+1}^{-s} > \tilde{Q}_j^{-s}, \quad j = 1, 2, \dots, n - s - 1$$

This follows by Amir & Lambson (2000, Theorem 2.2 (b)). Next define

$$\tilde{\pi}_w(S) = \sum_{j=1}^{n-s} w_{n,s}(j) \left(p \left(q_s + \tilde{Q}_j^{-s} \right) - c \right) q_s \equiv \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s \left(q_s, \tilde{Q}_j^{-s} \right)$$

and

$$\tilde{\pi}_z(S) = \sum_{j=1}^{n-s} z_{n,s}(j) \left(p \left(q_s + \tilde{Q}_j^{-s} \right) - c \right) q_s \equiv \sum_{j=1}^{n-s} z_{n,s}(j) \pi_s \left(q_s, \tilde{Q}_j^{-s} \right)$$

where

$$\pi_s \left(q_s, \tilde{Q}_j^{-s} \right) \equiv \left(p \left(q_s + \tilde{Q}_j^{-s} \right) - c \right) q_s, \quad j = 1, 2, \dots, n - s$$

Notice that since $\tilde{Q}_{j+1}^{-s} > \tilde{Q}_j^{-s}$, we have

$$\pi_s \left(q_s, \tilde{Q}_j^{-s} \right) > \pi_s \left(q_s, \tilde{Q}_{j+1}^{-s} \right), \quad j = 1, 2, \dots, n - s - 1 \tag{18}$$

where the last inequality is due to the fact that the profit of a firm in a Cournot market is decreasing in the (individual or aggregate) quantities of the other firms. Notice next that

$$\begin{aligned} \tilde{\pi}_w(S) - \tilde{\pi}_z(S) &= \left(\underbrace{w_{n,s}(1) - z_{n,s}(1)}_{\geq 0} \right) \pi_s \left(q_s, \tilde{Q}_1^{-s} \right) + \sum_{j=2}^{n-s} \left(w_{n,s}(j) - z_{n,s}(j) \right) \pi_s \left(q_s, \tilde{Q}_j^{-s} \right) \\ &> \left(w_{n,s}(1) - z_{n,s}(1) \right) \pi_s \left(q_s, \tilde{Q}_2^{-s} \right) + \sum_{j=2}^{n-s} \left(w_{n,s}(j) - z_{n,s}(j) \right) \pi_s \left(q_s, \tilde{Q}_j^{-s} \right) \\ &= \left(\underbrace{w_{n,s}(1) + w_{n,s}(2) - z_{n,s}(1) - z_{n,s}(2)}_{\geq 0} \right) \pi_s \left(q_s, \tilde{Q}_2^{-s} \right) \\ &\quad + \sum_{j=3}^{n-s} \left(w_{n,s}(j) - z_{n,s}(j) \right) \pi_s \left(q_s, \tilde{Q}_j^{-s} \right) \end{aligned} \tag{19}$$

where the inequality is due to (18). Continuing the iterations on j , we eventually have that

$$\begin{aligned}
 \tilde{\pi}_w(S) - \tilde{\pi}_z(S) &> \left(\underbrace{\sum_{j=1}^{n-s-1} w_{n,s}(j) - \sum_{j=1}^{n-s-1} z_{n,s}(j)}_{\geq 0} \right) \pi_s(q_s, \tilde{Q}_{n-s-1}^{-s}) \\
 &+ (w_{n,s}(n-s) - z_{n,s}(n-s)) \pi_s(q_s, \tilde{Q}_{n-s}^{-s}) \\
 &> \left(\underbrace{\sum_{j=1}^{n-s} w_{n,s}(j) - \sum_{j=1}^{n-s} z_{n,s}(j)}_{=0} \right) \pi_s(q_s, \tilde{Q}_{n-s}^{-s}) = 0 \tag{20}
 \end{aligned}$$

where we again used (18). We conclude that for all q_s and the corresponding \tilde{Q}_j^{-s} , we have

$$\sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s, \tilde{Q}_j^{-s}) > \sum_{j=1}^{n-s} z_{n,s}(j) \pi_s(q_s, \tilde{Q}_j^{-s}) \tag{21}$$

In the Appendix we show that if the inverse demand function $p(Q)$ is weakly concave then $Q_j^{-s}(w) < Q_j^{-s}(z)$. Notice next that

$$\begin{aligned}
 v_w^n(S) &= \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s(w), Q_j^{-s}(w)) \\
 &\geq \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s(z), Q_j^{-s}(w)) \\
 &\geq \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s(z), Q_j^{-s}(z)) \\
 &> \sum_{j=1}^{n-s} z_{n,s}(j) \pi_s(q_s(z), Q_j^{-s}(z)) = v_z^n(S) \tag{22}
 \end{aligned}$$

where the first inequality is due to the fact that $q_s(w)$ is the optimal choice of coalition S against $Q_j^{-s}(w)$, $j = 1, 2, \dots, n-s$; the second inequality holds because $Q_j^{-s}(w) < Q_j^{-s}(z)$ and because the Cournot profit of a firm decreases in the quantities of the other firms; and the third inequality is due to (21). Since $v_w^n(S) > v_z^n(S)$ we conclude that $C_w \subseteq C_z$. \square

As an application of Proposition 2, we note that Proposition 1 holds not only under $f_{n,s}$ but also under any distribution that dominates $f_{n,s}$ at first order.

Corollary 1 Assume the demand function is given by $Q = 1 - p$. Consider any distribution $z_{n,s}$ that dominates distribution (2) at first-order. Then $C_z \neq \emptyset$ if n is sufficiently large.

Proof If demand is linear and n is large then $C_f \neq \emptyset$ (Proposition 1). Since $z_{n,s}$ dominates distribution (2) at first-order then $C_f \subseteq C_z$. Hence the latter core is non-empty. \square

Compared to the cumulative distribution of $f_{n,s}$, the cumulative distribution of $z_{n,s}$ assigns higher probabilities to events that include partitions with many coalitions. Clearly, these events are unfavorable for S , hence the use of $z_{n,s}$ indicates some sort of pessimism (relatively to $f_{n,s}$) on behalf of the members of S . This approach can be motivated by resorting to the theory of risk measurement, which often measures risk by assigning relatively high probabilities to unfavorable events (see e.g. [Acerbi 2002](#)).⁴ Our analysis provides a novel way of tracing the impact of pessimism or risk in a cooperative game with externalities.

Finally we note that a particular distribution that dominates $f_{n,s}$ is the distribution defined by $\widehat{z}_{n,s}(j) = 0$, for $j = 1, 2, \dots, n-s-1$ and $\widehat{z}_{n,s}(n-s) = 1$. This distribution corresponds to the γ -core scenario. It is known that the latter core is non-empty for general Cournot oligopolies ([Chander 2010](#)). Letting $C_{\widehat{z}}$ denote the γ -core, we have the following corollary.

Corollary 2 *The inclusion $C_f \subseteq C_{\widehat{z}}$ holds.*

The γ -core is based on the worst scenario for S : all $n-s$ firms remain separate entities. Under $f_{n,s}$, the singleton coalitions structure is just one of the partitions that S takes into account. Other, more favorable, partitions occur with positive probability. Hence, the value of S under $f_{n,s}$ is higher than its value under $\widehat{z}_{n,s}$.

Corollaries 1 and 2 are stated in terms of the linear demand function. We note that similar statements hold for the non-linear demand case (we refer the reader to Sect. 3.2 that presents cases of non-linear demand functions for which the core is non-empty).

5 Conclusions

This paper analyzed cooperative Cournot games. The analysis is based on the assertion that when a coalition contemplates a deviation from the grand coalition, it assigns various non-equilibrium distributions on the set of partitions that the outsiders can form. This assumption is justified by imposing cognitive constraints on behalf of the firms in the market. Provided that the number of firms is sufficiently high, the core is non-empty for a large number of probability distributions and demand functions.

Let us mention a few extensions of the current work. The analysis of oligopolistic markets with more general cost functions and/or other modes of competition (e.g., product differentiation, price competition) are natural future directions. Furthermore, the application of the current framework to other economic environments or to abstract cooperative games with externalities is of interest.

⁴ We thank an anonymous referee for pointing out this connection.

Appendix

Table 2 Values $v^n(s)$ and $v^n(s)/s$ with $n = 12$

s	$v^n(s)$	$v^n(s)/s$
1	0.02047	0.02047
2	0.02273	0.01136
3	0.02544	0.00848
4	0.02876	0.00719
5	0.03289	0.00657
6	0.03815	0.00635
7	0.04503	0.00643
8	0.05443	0.00680
9	0.06795	0.00755
10	0.08736	0.00873
11	0.11111	0.01010
12	0.25000	0.02083

Table 3 Values $v^n(1)$ and $v^n(n)/n$, $n \in \{3, 4, \dots, 11\}$

n	$v^n(1)$	$v^n(n)/n$
3	0.08736	0.08333
4	0.06795	0.06250
5	0.05444	0.05000
6	0.04503	0.04166
7	0.03815	0.03571
8	0.03289	0.03125
9	0.02876	0.02777
10	0.02544	0.02500
11	0.02273	0.02272

Lemma A1 Assume the demand function is $Q = 1 - p^b$. Then the characteristic function is given by (13).

Proof The profit function of coalition i under a partition with j members is

$$\pi_i^j = \left(1 - q_s - \sum_{r=1, r \neq i}^j q_r^j - q_i^j \right)^{\frac{1}{b}} q_i^j, \quad i = 1, 2, \dots, j \tag{23}$$

Note that

$$\frac{\partial \pi_i^j}{\partial q_i^j} = 0 \Leftrightarrow b \left(1 - q_s - q_i^j - \sum_{r=1, r \neq i}^j q_r^j \right) = q_i^j \tag{24}$$

By symmetry, all j outside coalitions produce the same. So let $q_r^j = q_i^j$, for all r . Therefore by (24), $b(1 - q_s - j q_i^j) = q_i^j$ and hence

$$\tilde{q}_i^j = \frac{b(1 - q_s)}{bj + 1} \tag{25}$$

The objective function of the deviant coalition S is

$$\pi_f(S) = \sum_{j=1}^{n-s} f_{n,s}(j) \left(1 - q_s - \sum_{i=1}^j q_i^j \right)^{\frac{1}{b}} q_s \tag{26}$$

Note that $\frac{\partial \pi_f(S)}{\partial q_s} = 0$ if

$$\sum_{j=1}^{n-s} f_{n,s}(j) \left(1 - q_s - \sum_{i=1}^j q_i^j \right)^{\frac{1}{b}} = \frac{1}{b} \sum_{j=1}^{n-s} f_{n,s}(j) \left(1 - q_s - \sum_{i=1}^j q_i^j \right)^{\frac{1}{b}-1} q_s \tag{27}$$

Using (25), (27) becomes

$$\sum_{j=1}^{n-s} f_{n,s}(j) \left(\frac{1 - q_s}{1 + bj} \right)^{\frac{1}{b}} = \frac{1}{b} \sum_{j=1}^{n-s} f_{n,s}(j) \left(\frac{1 - q_s}{1 + bj} \right)^{\frac{1}{b}-1} q_s$$

and hence

$$(1 - q_s)^{\frac{1}{b}} \sum_{j=1}^{n-s} f_{n,s}(j) \left(\frac{1}{1 + bj} \right)^{\frac{1}{b}} = \frac{1}{b} (1 - q_s)^{\frac{1}{b}-1} \sum_{j=1}^{n-s} f_{n,s}(j) \left(\frac{1}{1 + bj} \right)^{\frac{1}{b}-1} q_s$$

Define $\psi_j = \frac{1}{bj + 1}$. Then rearranging the above relation gives

$$q_s \left(\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} + \frac{1}{b} \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}-1} \right) = \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} \tag{28}$$

Notice that

$$\begin{aligned} \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} + \frac{1}{b} \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}-1} &= \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} \left(1 + \frac{1}{b \psi_j} \right) \\ &= \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} (1 + (bj + 1)/b) = \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} (1 + j + 1/b) \end{aligned} \tag{29}$$

Using (28) and (29) we get

$$q_s(f) = \frac{\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}}}{\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} (1 + j + 1/b)} \tag{30}$$

Using (30), (25) becomes

$$q_i^j(f) = b\psi_j \frac{\sum_{j=1}^{n-s} \psi_j^{\frac{1}{b}}(j + 1/b)}{\sum_{j=1}^{n-s} f_{n,s}(j)\psi_j^{\frac{1}{b}}(1 + j + 1/b)} \tag{31}$$

Plugging (30) and (31) in (26) gives us (13). □

Lemma A2 Assume the inverse demand $p(Q)$ is weakly concave. Then $Q_j^{-s}(w) < Q_j^{-s}(z)$, $j = 1, 2, \dots, n - s$.

Proof Fix Q_j^{-s} . For notational convenience, define $F(q_s) \equiv \sum_{j=1}^{n-s} w_{n,s}(j)\pi_s(q_s, Q_j^{-s})$ and

$H(q_s) \equiv \sum_{j=1}^{n-s} z_{n,s}(j)\pi_s(q_s, Q_j^{-s})$. Then $\tilde{q}_s(w)$ and $\tilde{q}_s(z)$ satisfy respectively the first-order

conditions $\frac{\partial F(q_s)}{\partial q_s} = 0$ and $\frac{\partial H(q_s)}{\partial q_s} = 0$ or equivalently

$$\sum_{j=1}^{n-s} w_{n,s}(j)p'(q_s + Q_j^{-s})q_s + \sum_{j=1}^{n-s} w_{n,s}(j)p(q_s + Q_j^{-s}) - c = 0 \tag{32}$$

and

$$\sum_{j=1}^{n-s} z_{n,s}(j)p'(q_s + Q_j^{-s})q_s + \sum_{j=1}^{n-s} z_{n,s}(j)p(q_s + Q_j^{-s}) - c = 0 \tag{33}$$

The function $F(q_s)$ is strictly concave in q_s (by assumptions A1-A3). Hence $\tilde{q}_s(w) > \tilde{q}_s(z)$ if and only if $\frac{\partial F(\tilde{q}_s(z))}{\partial q_s} > 0$. By (32) we have that

$$\frac{\partial F(\tilde{q}_s(z))}{\partial q_s} > 0 \Leftrightarrow \sum_{j=1}^{n-s} w_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s})\tilde{q}_s(z) + \sum_{j=1}^{n-s} w_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) - c > 0 \tag{34}$$

Solving for $\tilde{q}_s(z)$ by (33) and plugging in (34) we have that

$$\begin{aligned} \frac{\partial F(\tilde{q}_s(z))}{\partial q_s} > 0 \Leftrightarrow & \frac{\sum_{j=1}^{n-s} w_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s})}{-\sum_{j=1}^{n-s} z_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s})} \left(\sum_{j=1}^{n-s} z_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) - c \right) \\ & + \sum_{j=1}^{n-s} w_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) - c > 0 \end{aligned} \tag{35}$$

We now use the concavity assumption and claim that

$$\frac{\sum_{j=1}^{n-s} w_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s})}{-\sum_{j=1}^{n-s} z_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s})} \geq -1 \tag{36}$$

To show the above we can equivalently show

$$\sum_{j=1}^{n-s} w_{n,s}(j) p'(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j) p'(\tilde{q}_s(z) + Q_j^{-s}) \geq 0 \tag{37}$$

We have

$$\begin{aligned} & \sum_{j=1}^{n-s} w_{n,s}(j) p'(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j) p'(\tilde{q}_s(z) + Q_j^{-s}) \\ &= (w_{n,s}(1) - z_{n,s}(1)) p'(\tilde{q}_s(z) + Q_1^{-s}) + \sum_{j=2}^{n-s} (w_{n,s}(j) - z_{n,s}(j)) p'(\tilde{q}_s(z) + Q_j^{-s}) \\ &\geq (w_{n,s}(1) - z_{n,s}(1)) p'(\tilde{q}_s(z) + Q_2^{-s}) + \sum_{j=2}^{n-s} (w_{n,s}(j) - z_{n,s}(j)) p'(\tilde{q}_s(z) + Q_j^{-s}) \end{aligned}$$

where the inequality holds because $w_{n,s}(1) - z_{n,s}(1) \geq 0$ and because the weak concavity of price implies that $p'(\tilde{q}_s(z) + Q_1^{-s}) \geq p'(\tilde{q}_s(z) + Q_2^{-s})$ (recall that $Q_1^{-s} < Q_2^{-s}$). If we continue the process of iterating j , we end up with (37). Since the latter condition holds, we have that

$$\begin{aligned} & \frac{\sum_{j=1}^{n-s} w_{n,s}(j) p'(\tilde{q}_s(z) + Q_j^{-s})}{\sum_{j=1}^{n-s} z_{n,s}(j) p'(\tilde{q}_s(z) + Q_j^{-s})} \left(\sum_{j=1}^{n-s} z_{n,s}(j) p(\tilde{q}_s(z) + Q_j^{-s}) - c \right) \\ &+ \sum_{j=1}^{n-s} w_{n,s}(j) p(\tilde{q}_s(z) + Q_j^{-s}) - c \\ &\geq - \left(\sum_{j=1}^{n-s} z_{n,s}(j) p(\tilde{q}_s(z) + Q_j^{-s}) - c \right) + \sum_{j=1}^{n-s} w_{n,s}(j) p(\tilde{q}_s(z) + Q_j^{-s}) - c \\ &= \sum_{j=1}^{n-s} w_{n,s}(j) p(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j) p(\tilde{q}_s(z) + Q_j^{-s}) \end{aligned} \tag{38}$$

But expression (38) can be written as

$$\begin{aligned} & (w_{n,s}(1) - z_{n,s}(1)) p(\tilde{q}_s(z) + Q_1^{-s}) + \sum_{j=2}^{n-s} (w_{n,s}(j) - z_{n,s}(j)) p(\tilde{q}_s(z) + Q_j^{-s}) \\ &> (w_{n,s}(1) - z_{n,s}(1)) p(\tilde{q}_s(z) + Q_2^{-s}) + \sum_{j=2}^{n-s} (w_{n,s}(j) - z_{n,s}(j)) p(\tilde{q}_s(z) + Q_j^{-s}) \end{aligned}$$

where the last inequality holds because $Q_1^{-s} < Q_2^{-s}$. Continuing the iterations on j , we end up with

$$\sum_{j=1}^{n-s} w_{n,s}(j) p(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j) p(\tilde{q}_s(z) + Q_j^{-s}) > 0 \tag{39}$$

Combining (35), (38) and (39) we conclude that $\frac{\partial F(\tilde{q}_s(z))}{\partial q_s} > 0$ and hence $\tilde{q}_s(w) > \tilde{q}_s(z)$. But then $Q_j^{-s}(w) < Q_j^{-s}(z)$, since $Q_j^{-s}(w)$ and $Q_j^{-s}(z)$ emerge from \tilde{Q}_j^{-s} for $q_s = \tilde{q}_s(w)$ and $q_s = \tilde{q}_s(z)$ respectively and commodities in a Cournot market are substitutes. \square

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