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Some equilibrium problems under uncertainty and random variational inequalities

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Abstract In this paper we describe some nonlinear equilibrium problems under uncertainty arising from economics and operations research. In particular we treat Wardrop equilibria in traffic networks. We show how the theory of monotone random variational inequalities, where random variables occur both in the operator and the constraint set, can be applied to model these problems.

Therefore in this contribution we introduce the topic of random variational inequalities and present some of our recent results in this field. In particular, we treat the more structured case where a finite Karhunen-Loève expansion leads to a separation of the random and the deterministic variables. Here we describe a norm convergent approximation procedure based on averaging and truncation. We illustrate this procedure by means of some small sized numerical examples.

Keywords Random variational inequality · Monotone operator · Cassel-Wald equilibrium · Distributed market equilibrium · Traffic network · Wardrop equilibrium

1 Introduction

Although relatively recent, the variational inequality (V.I.) approach to a variety of equilibrium problems that encompass (convex) optimization and minimax problems and that arise in various fields of applied sciences, such as economics, game theory and transportation science, has developed very rapidly (see e.g. Facchinei and Pang 2003; Giannessi and

Dedicated to Professor A. Prékopa and to the memory of Professor W. Oettli.

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Maugeri 1995; Konnov 2007; Nagurney 1993). Since the data of most of the above mentioned problems are often affected by uncertainty, the question arises of how to introduce uncertainty or randomness in their V.I. formulation. In fact, while the topic of stochastic programming is already a well established field of optimization theory (see e.g. Dempster 1980; Prekopa 1995), the theory of random (or stochastic) variational inequalities is much less developed.

Here we first quote (Chen and Fukushima 2005; Gürkan et al. 1999). Chen and Fukushima (2005) present an expected residual method that finds a surrogate solution to stochastic (random) linear complementarity problems (LCP) by transforming the random LCP to a random minimization problem using a NCP (or gap) function and then minimizing its expectation. Gürkan et al. (1999) extend sample-path optimization, the simulation-based method of sample average approximation, to the approximate solution of a class of stochastic (random) variational inequalities on a polyhedral subset in finite dimension. There is further recent work on sample average approximation methods (see Shapiro 2003) for generalized equations (Shapiro et al. 2009, Sect. 5.2), for stochastic mathematical programs with equilibrium constraints (Shapiro and Xu 2008), and for various equilibrium problems, including Stackelberg games (DeMiguel and Xu 2009) and Nash games (Ravat and Shanbhag 2010, 2011). In particular, we draw the reader's attention to the recent paper (Xu 2010) (see also Xu and Zhang 2009) on sample average approximation methods for a class of stochastic variational inequality problems. Roughly said, sample average approximation treats a mean value model as a deterministic surrogate model and uses random samples of realizations of the random vectors such that almost sure, even exponential convergence to the solution of the surrogate deterministic model can be established.

Before presenting our analysis and some specific nonlinear equilibrium problems in the subsequent sections, some words about our approach to modelling uncertainty in equilibrium problems by random variational inequalities are in order. In the present paper, randomness may affect all data of the equilibrium problems: firstly extending random (nonlinear) operator equations, randomness may occur in the right hand side and in the operator that may come from an optimization model via the gradient of the cost functional. More importantly, we admit randomness in the constraints and require these constraints to be satisfied with probability 1. Thus we consider a special instance of probability constraints, here with probability level $p = 1$; see (Shapiro et al. 2009, Chap. 4) for stochastic optimization models with probabilistic constraints. We follow Shapiro et al. (2009, p. 87) and "emphasize that imposing constraints on the probability of events is particularly appropriate whenever high uncertainty is involved and reliability is a central issue." In such cases a mere mean value model may not be appropriate.

Our treatment of random variational inequalities started with Gwinner (2000) where the author studied a class of V.I. in infinite dimension with a linear random operator, presented an existence and full discretization theory and applied this theory to a unilateral boundary value problem stemming from continuum mechanics, where the coefficients of the elliptic differential operator are admitted to be random to model uncertainty in material parameters. The functional setting introduced therein, and extended in Gwinner and Raciti (2006) in order to include randomness also in the constraints, can also be utilized to model many finite dimensional random equilibrium problems, which only in special cases admit an optimization formulation. Furthermore, recently in Gwinner and Raciti (2009), the authors have extended the theory consisting of the existence results and the deterministic approximation procedure in Gwinner and Raciti (2006) to the monotone nonlinear case. This extension is motivated by the need to cope with the nonlinearity in many equilibrium problems arising in operations research, such as the random traffic equilibrium problem which is studied in detail in this article. In contrast to Gwinner and Raciti (2006, 2009), we also discuss here the nonuniqueness case, when still monotonicity of the operator is present, but uniform monotonicity is abandoned. Moreover, under stronger assumptions, we can show norm convergence of our deterministic approximation procedure, even when the substitute problems are only solved inaccurately.

The relevant information that we can extract from our random solution concept is the mean of the random solution. We can understand this mean as a deterministic surrogate solution. By our deterministic approximation procedure we provide an approximation of this unknown surrogate solution. We show in this paper how the well-known Chebyshev inequality can be applied in virtue of the L^2 convergence of our approximation procedure to derive a probabilistic error estimate for the approximation obtained.

This paper consists of 6 sections. In the following Sect. 2 we specialize the abstract formulation of Gwinner and Raciti (2009) to the case in which the deterministic variables belong to a finite dimensional space, so as to make our theory readily applicable to economics and operations research problems; in Sect. 3 we consider the special separable case where a finite Karhunen-Loève expansion separates deterministic and random variables; in Sect. 4 we recall the approximation procedure given in Gwinner and Raciti (2009) and refine its convergence analysis. Then in Sect. 5, we show how the theory of monotone random V.I., where random variables occur both in the operator and the constraint set, can be applied to model various nonlinear equilibrium problems under uncertainty arising from economics and operations research. In the last Sect. 6 we focus on the modelling of the nonlinear random traffic equilibrium problem and, in order to explain the role of monotonicity, we also discuss the fact that this problem (as every network equilibrium problem) can be formulated by using two different sets of variables, connected by a linear transformation. Here we illustrate our theory of monotone random V.I by two small sized numerical examples of traffic equilibrium problems. In the final concluding remarks we delineate some open research avenues for future research.

2 The random variational inequality problem—the pointwise and the integral formulation

Let $(\Omega, \mathcal{A}, P)$ be a probability space. For all $\omega \in \Omega$, let $\mathcal{K}(\omega)$ be a closed, convex and nonempty subset of $\mathbb{R}^k$. Consider a random vector λ and a Carathéodory function $F : \Omega \times \mathbb{R}^k \mapsto \mathbb{R}^k$, i.e. for each fixed $x \in \mathbb{R}^k$, $F(\cdot, x)$ is measurable with respect to $\mathcal{A}$, and for every $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous. Moreover, for each $\omega \in \Omega$, let $F(\omega, \cdot)$ be a monotone map on $\mathbb{R}^k$, i.e. $\langle F(\omega, x) - F(\omega, y), x - y\rangle \geq 0, \forall x, y \in \mathbb{R}^k$.

With these data we consider the following

Problem 1 *For almost all $\omega \in \Omega$, find $x = x(\omega) \in \mathcal{K}(\omega)$ such that*

$$\big\langle F(\omega, x), y - x\big\rangle \geq \big\langle \lambda(\omega), y - x\big\rangle, \quad \forall y \in \mathcal{K}(\omega). \tag{2.1}$$

This means that the preceding inequality holds pointwise on Ω except a fixed null set, depending on the solution.

Obviously if a solution of Problem 1 that depends on ω, exists for all $\omega \in \Omega$, we arrive at a set-valued map $\Sigma : \Omega \rightrightarrows \mathbb{R}^k$ which, to each $\omega \in \Omega$, associates the solution set of (2.1). The measurability of Σ (with respect to the algebra $\mathcal{B}(\mathbb{R}^k)$ of the Borel sets on $\mathbb{R}^k$ and to the σ-algebra $\mathcal{A}$ on Ω) has been proved in Gwinner and Raciti (2006) for the case of a bilinear form on a general separable Hilbert space. The proof given therein can be adapted in a straightforward fashion to nonlinear maps.

We are not interested in investigating a merely parameter dependent variational inequality. Instead we shall formulate the problem in an appropriate functional analytic setting. We introduce for fixed $p \geq 2$, the reflexive Banach space $L^p(\Omega, P, \mathbb{R}^k)$ of random vectors V from Ω to $\mathbb{R}^k$ such that the expectation

$$E^P \|V\|^p = \int_\Omega \big\|V(\omega)\big\|^p dP(\omega) < \infty. \tag{2.2}$$

Then it makes sense to introduce the nonvoid closed convex subset

$$K := \big\{V \in L^p\big(\Omega, P, \mathbb{R}^k\big) : V(\omega) \in \mathcal{K}(\omega), \text{P-almost sure}\big\}.$$

In addition, we assume that the map F satisfies the growth condition

$$\big\|F(\omega, z)\big\| \leq \alpha(\omega) + \beta(\omega)\|z\|^{p-1} \quad \forall z \in \mathbb{R}^k \text{ P-almost sure}, \tag{2.3}$$

for some $\alpha \in L^p(\Omega, P)$, $\beta \in L^\infty(\Omega, P)$.

It is noteworthy that in many applications, such as the traffic equilibrium problem, the modelling is often done with polynomial cost functions. Then the growth of those polynomial cost functions determines the parameter p of the L^p space and the growth condition (2.3) is naturally satisfied.

Under the growth condition (2.3) a map $\hat{F}$ that acts from $L^p(\Omega, P, \mathbb{R}^k)$ to the dual $L^{p'}(\Omega, P, \mathbb{R}^k)$, $\frac{1}{p} + \frac{1}{p'} = 1$ can be derived from F by

$$\hat{F}(V)(\omega) := F\big(\omega, V(\omega)\big), \quad \omega \in \Omega.$$

Moreover, let $\lambda \in L^{p'}(\Omega, P, \mathbb{R}^k)$. Then the integrals and the associated duality forms

$$\int_\Omega \big\langle F\big(\omega, U(\omega)\big), V(\omega) - U(\omega)\big\rangle dP(\omega) =: \big[\hat{F}(U), V - U\big]$$

$$\int_\Omega \big\langle \lambda(\omega), V(\omega) - U(\omega)\big\rangle dP(\omega) =: [\lambda, V - U]$$

are well defined for all $U, V \in L^p(\Omega, P, \mathbb{R}^k)$. Therefore, we can consider the following

Problem 2 *Find $U \in K$ such that*, $\forall V \in K$,

$$\big[\hat{F}(U), V - U\big] \geq [\lambda, V - U]. \tag{2.4}$$

Both problem formulations are related in the following way.

Proposition 1 *Let $U \in L^p(\Omega, P, \mathbb{R}^k)$. Suppose U solves Problem 1 such that $x(\omega) := U(\omega)$ satisfies (2.1) on Ω except a null set, then U solves Problem 2 satisfying (2.4). Vice versa, let us assume $\mathcal{K}$ does not depend on ω. Then any solution of Problem 2 solves Problem 1.*

Proof With $U \in L^p(\Omega, P, \mathbb{R}^k)$ a solution of Problem 1, we have $U \in K$. Let $V \in K$ arbitrarily chosen, then plug in $y = V(\omega) \in \mathcal{K}(\omega)$ and (2.1) holds pointwise in Ω except a null set depending on U and V. Then integrate a nonnegative $L^1(\Omega, P)$ function to obtain that U solves Problem 2.

The converse statement follows by contradiction. Let $U \in K$ be a solution of Problem 2. Then, $U(\omega) \in \mathcal{K}$ for almost all ω. Assume $\exists A \subset \Omega$, $P(A) > 0$, $\exists z \in \mathcal{K}$:

$$\langle F(\omega, U(\omega)),\ z - U(\omega)\rangle < \langle \lambda(\omega), z - U(\omega)\rangle, \quad \forall \omega \in A.$$

Then by construction,

$$\int_A \langle F(\omega, U(\omega)), z - U(\omega)\rangle\, dP(\omega) < \int_A \langle \lambda(\omega), z - U(\omega)\rangle\, dP(\omega).$$

Take $V \in L^p(\Omega, P, \mathbb{R}^k)$ by

$$V(\omega) = \begin{cases} U(\omega) & \omega \notin A \\ z & \omega \in A \end{cases}$$

Then $V \in K$ and

$$\int_\Omega \langle F(\omega, U(\omega)),\ V(\omega) - U(\omega)\rangle\, dP(\omega) < \int_\Omega \langle \lambda(\omega), V(\omega) - U(\omega)\rangle dP(\omega)$$

contradicting that U solves Problem 2. Using the separability of $\mathcal{K}$ as a closed subset of $\mathbb{R}^k$, we can get rid of the dependence on the null set of $z \in \mathcal{K}$. Therefore we conclude that U solves Problem 1. □

Remark 2.1 Clearly, when unique solutions exist to Problems 1 and 2 (as considered in Gwinner and Raciti 2006, 2009), then both problem formulations are equivalent for a general ω dependent constraint set K.

Since the continuity of $F(\omega, \cdot)$ implies the continuity of the monotone superposition operator $\hat{F}$ (see Appel and Zabrejko 1990), monotone operator theory (see Kinderlehrer and Stampacchia 1980; Zeidler 1990) applies. Thus we have unique solvability of Problem 2, if F is uniformly strongly monotone, i.e. if there is some constant $c_0 > 0$ such that

$$\langle F(\omega, x) - F(\omega, y), x - y\rangle \geq c_0 \|x - y\|^2 \quad \forall x, y \in \mathbb{R}^k, \forall \omega \in \Omega.$$

For existence results without the assumption of uniform strong monotonicity, we can refer to the existence theory of monotone variational inequalities (Kinderlehrer and Stampacchia 1980), see also Maugeri and Raciti (2009) for a recent exposition. Here we use the notation $K_R := \{V \in K : \|V\| \leq R\}$ and quote from Kinderlehrer and Stampacchia (1980, Theorem III 1.7) the following characterization result.

Theorem 2.1 *A necessary and sufficient condition in order that Problem 2 is solvable is the existence of $R > 0$ such that a solution U_R of the variational inequality*

$$U_R \in K_R, \left[\hat{F}(U_R), V - U_R\right] \geq [\lambda, V - U_R], \quad \forall V \in K_R$$

satisfies $\|U_R\| < R$.

Remark 2.2 For purposes of completeness let us point out that time dependent V.I. (Gwinner 2003) with their application to time dependent equilibrium problems (Daniele et al. 1999), when considered without any delay effects or time integration/differentiation and treated in an L^p framework, can be seen as a subclass of random V.I. Indeed, without any loss of generality, consider the time interval $(0, 1)$. Then take $\Omega := (0, 1)$; $P := dt$, the Lebesgue measure leading to the special uniform probability distribution on $(0, 1)$; see also Example 6.1 in the paper.

3 The separable case

In the following we want to present applications not only with implicit constraints described by a constraint set, but also with explicit inequality constraints. Moreover we have to simplify somewhat our random V.I. problems to arrive at numerically treatable problems. Therefore here and in the sequel we shall posit further assumptions on the structure of the constraint set and on the operator. More precisely, with a matrix $A \in \mathbb{R}^{m\times k}$ and a random m-vector D being given, we consider the random set

$$M(\omega) := \left\{x \in \mathbb{R}^k : Ax \le D(\omega)\right\}, \quad \omega \in \Omega.$$

Moreover, let $G, H : \mathbb{R}^k \to \mathbb{R}^k$ be two (nonlinear) maps; $b, c \in \mathbb{R}^k$ fixed vectors and R and S two real valued random variables on Ω. We assume that $S \in L^\infty(\Omega)$ and $R \in L^p(\Omega)$. We simplify Problem 1 to that of finding $\hat{X} : \Omega \to \mathbb{R}^k$, such that $\hat{X}(\omega) \in M(\omega)$ (P-a.s.) and the following inequality holds for P-almost every elementary event $\omega \in \Omega$ and $\forall x \in M(\omega)$

$$\left\langle S(\omega)G\left(\hat{X}(\omega)\right) + H\left(\hat{X}(\omega)\right), x - \hat{X}(\omega)\right\rangle \ge \left\langle b + R(\omega)c, x - \hat{X}(\omega)\right\rangle. \tag{3.1}$$

Thus the operator F defined by

$$F(\omega, x) := S(\omega)G(x) + H(x)$$

inherits (uniform strong) monotonicity from the (uniform strong) monotonicity of $s_0\ G$ and H, provided s_0 is a positive constant such that there holds $S \ge s_0$ P-a.s. (almost sure). We also require that F satisfies the growth condition (2.3). Moreover, we assume that $D \in L^p_m(\Omega) := L^p(\Omega, P, \mathbb{R}^m)$. Hence we can introduce the following closed convex nonvoid subset of $L^p_k(\Omega)$:

$$M^P := \left\{V \in L^p_k(\Omega) : AV(\omega) \le D(\omega), P - a.s.\right\}$$

and arrive at the following problem: Find $\hat{U} \in M^P$ such that, $\forall V \in M^P$,

$$\begin{aligned}\int_\Omega &\left\langle S(\omega)G\left(\hat{U}(\omega)\right) + H\left(\hat{U}(\omega)\right), V(\omega) - \hat{U}(\omega)\right\rangle dP(\omega)\\ &\ge \int_\Omega \left\langle b + R(\omega)c, V(\omega) - \hat{U}(\omega)\right\rangle dP(\omega).\end{aligned} \tag{3.2}$$

The r.h.s. of (3.2) defines a continuous linear form on $L^p_k(\Omega)$, while the l.h.s. defines a continuous monotone form on $L^p_k(\Omega)$. Similar relations as in Sect. 2 hold for the pointwise formulation (3.1) and integral formulation (3.2).

In order to get rid of the abstract sample space Ω, we consider the joint distribution $\mathbb{P}$ of the random vector (R, S, D) and work with the special probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P})$, where the dimension $d := 2 + m$. To simplify our analysis we shall suppose that R, S and D are independent random vectors. Let $r = R(\omega)$, $s = S(\omega)$, $t = D(\omega)$, $y = (r, s, t)$. For each $y \in \mathbb{R}^d$, consider the set

$$M(y) := \left\{x \in \mathbb{R}^k : Ax \le t\right\}$$

Then the pointwise version of our problem now reads: Find $\hat{x}$ such that $\hat{x}(y) \in M(y)$, $\mathbb{P}$-a.s., and the following inequality holds for $\mathbb{P}$-almost every $y \in \mathbb{R}^d$ and $\forall x \in M(y)$,

$$\left\langle sG\big(\hat{x}(y)\big) + H\big(\hat{x}(y)\big), x - \hat{x}(y)\right\rangle \ge \left\langle b + rc, x - \hat{x}(y)\right\rangle. \tag{3.3}$$

In order to obtain the integral formulation of (3.3), consider the space $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ and introduce the closed convex nonvoid set

$$M_{\mathbb{P}} := \left\{v \in L^p\big(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k\big) : Av(r, s, t) \le t, \mathbb{P} - a.s.\right\}. \tag{3.4}$$

This leads to the problem: Find $\hat{u} \in M_{\mathbb{P}}$ such that, $\forall v \in M_{\mathbb{P}}$,

$$\int_{\mathbb{R}^d} \left\langle sG\big(\hat{u}(y)\big) + H\big(\hat{u}(y)\big), v(y) - \hat{u}(y)\right\rangle d\mathbb{P}(y) \ge \int_{\mathbb{R}^d} \left\langle b + rc, v(y) - \hat{u}(y)\right\rangle d\mathbb{P}(y). \tag{3.5}$$

Again similar relations as in Sect. 2 hold for the pointwise formulation (3.3) and the integral formulation (3.5).

Remark 3.1 Our approach and analysis here and in the next section readily applies also to more general finite Karhunen-Loève expansions

$$\lambda(\omega) = b + \sum_{l=1}^{L} R_l(\omega)c_l, \quad F(\omega, x) = H(x) + \sum_{l=1}^{L} S_l(\omega)G_l(x).$$

However, such an extension does not only need a more lengthy notation, but—more importantly—leads to more computational work; see Gwinner and Raciti (2006) for a more thorough discussion of those computational aspects.

4 An approximation procedure by discretization of distributions

Without loss of generality, we can suppose that $R \in L^p(\Omega, P)$ and $D \in L^p_m(\Omega, P)$ are nonnegative (otherwise we can use the standard decomposition in the positive part and the negative part). Moreover, we assume that the support (the set of possible outcomes) of $S \in L^\infty(\Omega, P)$ is the interval $[s_0, s_1) \subset (0, \infty)$. Furthermore we assume that the distributions P_R, P_S, P_D are continuous with respect to the Lebesgue measure, so that according to the theorem of Radon-Nikodym, they have the probability densities $\varphi_R, \varphi_S, \varphi_{D_i} i = 1, \ldots, m$, respectively. Hence, $\mathbb{P} = P_R \otimes P_S \otimes P_D$, $dP_R(r) = \varphi_R(r)dr$, $dP_S(s) = \varphi_S(s)ds$ and $dP_{D_i}(t_i) = \varphi_{D_i}(t_i)dt_i$ for $i = 1, \ldots, m$. Let us note that $v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ means that $(r, s, t) \mapsto \varphi_R(r)\varphi_S(s)\varphi_D(t)v(r, s, t)$ belongs to the standard Lebesgue space $L^p(\mathbb{R}^d, \mathbb{R}^k)$

with respect to the Lebesgue measure, where shortly $\varphi_D(t) := \prod_i \varphi_{D_i}(t_i)$. Thus our problem (3.5) reads as follows: Find $\hat{u} \in M_{\mathbb{P}}$ such that, $\forall v \in M_{\mathbb{P}}$,

$$\int_0^\infty \int_{s_0}^{s_1} \int_{\mathbb{R}_+^m} \langle sG(\hat{u}) + H(\hat{u}), v - \hat{u} \rangle \varphi_R(r)\varphi_S(s)\varphi_D(t)dy$$
$$\geq \int_0^\infty \int_{s_0}^{s_1} \int_{\mathbb{R}_+^m} \langle b + rc, v - \hat{u} \rangle \varphi_R(r)\varphi_S(s)\varphi_D(t)dy. \tag{4.1}$$

In order to give an approximation procedure for the solution $\hat{u}$, let us start with a discretization of the space $X := L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ and introduce a sequence $\{\pi_n\}_n$ of partitions of the support $\Upsilon := [0, \infty) \times [s_0, s_1) \times \mathbb{R}_+^m$ of the probability measure $\mathbb{P}$ induced by the random elements R, S, D. To be precise, let $\pi_n = (\pi_n^R, \pi_n^S, \pi_n^D)$, where

$$\pi_n^R := \left(r_n^0, \dots, r_n^{N_n^R}\right), \quad \pi_n^S := \left(s_n^0, \dots, s_n^{N_n^S}\right), \quad \pi_n^{D_i} := \left(t_{n,i}^0, \dots, t_{n,i}^{N_n^{D_i}}\right)$$
$$0 = r_n^0 < r_n^1 < \cdots r_n^{N_n^R} = n$$
$$s_0 = s_n^0 < s_n^1 < \cdots s_n^{N_n^S} = s_1$$
$$0 = t_{n,i}^0 < t_{n,i}^1 < \cdots t_{n,i}^{N_n^{D_i}} = n (i = 1, \dots, m)$$
$$\left|\pi_n^R\right| := \max\left\{r_n^j - r_n^{j-1} : j = 1, \dots, N_n^R\right\} \to 0 \quad (n \to \infty)$$
$$\left|\pi_n^S\right| := \max\left\{s_n^k - s_n^{k-1} : k = 1, \dots, N_n^S\right\} \to 0 \quad (n \to \infty)$$
$$\left|\pi_n^{D_i}\right| := \max\left\{t_{n,i}^{h_i} - t_{n,i}^{h_i-1} : h_i = 1, \dots, N_n^{D_i}\right\} \to 0 \quad (i = 1, \dots, m; n \to \infty).$$

These partitions give rise to the exhausting sequence $\{\Upsilon_n\}$ of subsets of Υ, where each Υ_n is given by the finite disjoint union of the intervals:

$$I_{jkh}^n := \left[r_n^{j-1}, r_n^j\right) \times \left[s_n^{k-1}, s_n^k\right) \times I_h^n,$$

where we use the multiindex $h = (h_1, \dots, h_m)$ and

$$I_h^n := \prod_{i=1}^m \left[t_{n,i}^{h_i-1}, t_{n,i}^{h_i}\right).$$

For each $n \in \mathbb{N}$ let us consider the space of the $\mathbb{R}^l$-valued simple functions ($l \in \mathbb{N}$) on Υ_n, extended by 0 outside of Υ_n:

$$X_n^l := \left\{v_n : v_n(r, s, t) = \sum_j \sum_k \sum_h v_{jkh}^n 1_{I_{jkh}^n}(r, s, t), v_{jkh}^n \in \mathbb{R}^l\right\},$$

where 1_I denotes the $\{0, 1\}$-valued characteristic function of a subset I.

To approximate an arbitrary function $w \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R})$ we employ the mean value truncation operator μ_0^n associated to the partition π_n given by

$$\mu_0^n w := \sum_{j=1}^{N_n^R} \sum_{k=1}^{N_n^S} \sum_h \left(\mu_{jkh}^n w\right) 1_{I_{jkh}^n}, \tag{4.2}$$

where

$$\mu^n_{jkh} w := \begin{cases} \frac{1}{\mathbb{P}(I_{jkh})} \int_{I^n_{jkh}} w(y) d\mathbb{P}(y) & \text{if } \mathbb{P}(I^n_{jkh}) > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Likewise for a L^p vector function $v = (v_1, \dots, v_l)$, we define $\mu^n_0 v := (\mu^n_0 v_1, \dots, \mu^n_0 v_l)$.

From Lemma 2.5 in Gwinner (2000) (and the remarks therein) we obtain the following result.

Lemma 4.1 *The linear operator* $\mu^n_0 : L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l) \to L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$ (*for fixed* $d, l \in \mathbb{N}$) *is bounded with* $\|\mu^n_0\| = 1$ *and for* $n \to \infty$, μ^n_0 *converges pointwise in* $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$ *to the identity.*

This lemma reflects the well-known density of the class of the simple functions in a L^p space. It shows that the mean value truncation operator μ^n_0, which acts as a projector on $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$, can be understood as a conditional expectation operator introduced by Kolmogorov in 1933, see also (Doob 1953), and thus our approximation method is a projection method according to the terminology of (Lepp 1994).

In order to construct approximations for $M_{\mathbb{P}}$ given by (3.4) we introduce the orthogonal projector $q : (r, s, t) \in \mathbb{R}^d \mapsto t \in \mathbb{R}^m$ and let, for each elementary quadrangle I^n_{jkh},

$$\overline{q}^n_{jkh} = \left(\mu^n_{jkh} q\right) \in \mathbb{R}^m, \qquad \left(\mu^n_0 q\right) = \sum_{jkh} \overline{q}^n_{jkh} 1_{I^n_{jkh}} \quad \in X^m_n.$$

Thus we arrive at the following sequence of convex, closed sets

$$M^n_{\mathbb{P}} := \left\{ v \in X^k_n : A v^n_{jkh} \le \overline{q}^n_{jkh}, \forall j, k, h \right\}.$$

Note that the sets $M^n_{\mathbb{P}}$ are of polyhedral type. In Gwinner and Raciti (2009) it has been proved that the sequence $\{M^n_{\mathbb{P}}\}$ approximate the set $M_{\mathbb{P}}$ in the sense of Mosco (Attouch 1984; Mosco 1969), i.e.

$$\text{weak-limsup}_{n\to\infty} M^n_{\mathbb{P}} \subset M_{\mathbb{P}} \subset \text{strong-liminf}_{n\to\infty} M^n_{\mathbb{P}}. \tag{4.3}$$

For the reader not familiar with Mosco set convergence we add that the above relations state that for each point u of $M_{\mathbb{P}}$ there exists a sequence of points $u_n \in M^n_{\mathbb{P}}$ which converges strongly to u, and that given any sequence $u_n \in M^n_{\mathbb{P}}$, any weak limit of any subsequence of u_n belongs to $M_{\mathbb{P}}$. Here a sequence w_n converges *strongly* to w in L^p iff $\|w_n - w\|_{L^p} \to 0$ for $n \to \infty$; whereas w_n converges *weakly* to w, iff for all $f \in L^{p'}$ we have $\int_{\mathbb{R}^d} f \, w_n \, d\mathbb{P} \to \int_{\mathbb{R}^d} f \, w \, d\mathbb{P}$ for $n \to \infty$.

Moreover we want to approximate the random variables R and S and introduce

$$\rho_n = \sum_{j=1}^{N^R_n} r^{j-1}_n 1_{[r^{j-1}_n, r^j_n)} \in X_n, \qquad \sigma_n = \sum_{k=1}^{N^S_n} s^{k-1}_n 1_{[s^{k-1}_n, s^k_n)} \in X_n.$$

We observe that $\sigma_n(r, s, t) \to \sigma(r, s, t) = s$ in $L^\infty(\mathbb{R}^d, \mathbb{P})$ while, as a consequence of the Chebyshev inequality (see e.g. Billingsley 1995), $\rho_n(r, s, t) \to \rho(r, s, t) = r$ in $L^p(\mathbb{R}^d, \mathbb{P})$.

Thus we are led to consider, $\forall n \in \mathbb{N}$, the following substitute problem: Find $\hat{u}_n \in M^n_{\mathbb{P}}$ such that, $\forall v_n \in M^n_{\mathbb{P}}$,

$$\int_{\mathbb{R}^d} \langle \tilde{F}^n(\hat{u}_n), v_n - \hat{u}_n \rangle d\mathbb{P}(y) \geq \int_{\mathbb{R}^d} \langle \tilde{c}^n, v_n - \hat{u}_n \rangle d\mathbb{P}(y), \tag{4.4}$$

where

$$\tilde{F}^n := \sigma_n G + H, \quad \tilde{c}^n := b + \rho_n c.$$

We observe that (4.4) splits in a finite number of finite dimensional monotone variational inequalities: For $\forall n \in \mathbb{N}, \forall j, k, h$ find $\hat{u}^n_{jkh} \in M^n_{jkh}$ such that, $\forall v^n_{jkh} \in M^n_{jkh}$,

$$\langle \tilde{F}^n_k(\hat{u}^n_{jkh}), v^n_{jkh} - \hat{u}^n_{jkh} \rangle \geq \langle \tilde{c}^n_j, v^n_{jkh} - \hat{u}^n_{jkh} \rangle, \tag{4.5}$$

where

$$M^n_{jkh} := \{ v^n_{jkh} \in \mathbb{R}^k : A v^n_{jkh} \leq \overline{q}^n_{jkh} \},$$

$$\tilde{F}^n_k := s_n^{k-1} G + H, \quad \tilde{c}^n_j := b + r_n^{j-1} c.$$

Clearly, this gives

$$\hat{u}_n = \sum_j \sum_k \sum_h \hat{u}^n_{jkh} 1_{I^n_{jkh}} \quad \in X^k_n.$$

Now, we can state the following convergence results. The first result holds for general monotone maps.

Theorem 4.1 *Any weak limit point of the sequence $\hat{u}_n$ generated by the substitute problems in* (4.4) *is a solution of* (3.5).

Proof For the proof see part (2) of the proof of Theorem 4.1 in Gwinner and Raciti (2009), which uses only monotonicity arguments, but not the assumption of uniform monotonicity. □

Remark 4.1 Clearly, if the solution $\hat{u}$ of (3.5) is unique, then the entire sequence $\hat{u}_n$ converges weakly to $\hat{u}$. Hence if the probability measure has compact support (as e.g. the uniform distribution), then the means of $\hat{u}_n$ converge to the mean of $\hat{u}$.

The next result (see Gwinner and Raciti 2009, Theorem 4.1) deals with uniformly monotone maps and gives norm convergence.

Theorem 4.2 *Suppose uniform monotonicity. Then the sequence $\hat{u}_n$ generated by the substitute problems in* (4.4) *converges strongly in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ for $n \to \infty$ to the unique solution $\hat{u}$ of* (3.5).

The preceding convergence theorem can be refined under the additional assumption of Lipschitz continuity, because in this case (and in virtue of uniform strong monotonicity), it is enough to solve the finite dimensional substitute problem (4.4) only inaccurately.

Theorem 4.3 *Suppose, both maps G and H are uniformly strongly monotone and Lipschitz continuous. Let $\varepsilon_n > 0$. Introduce the monotone operator T_n by*

$$T_n(u)(y) := \sigma_n(y)G(u)(y) + H(u)(y) - b - \rho_n(y)c$$

and the associated natural map

$$F_n^{\text{nat}}(u) = u - Proj_{M_{\mathbb{P}}^n}\big[u - T_n(u)\big],$$

both acting in $X_n^l(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ (where $Proj$ is the minimum norm projection). Let $\tilde{u}_n \in M_{\mathbb{P}}^n$ satisfy

$$\big\| F_n^{\text{nat}}(\tilde{u}_n)\big\| \le \varepsilon_n. \tag{4.6}$$

Suppose that in (4.6), $\varepsilon_n \to 0$ for $n \to \infty$. Then the sequence $\tilde{u}_n$ converges strongly in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ to the unique solution $\hat{u}$ of (3.5).

Proof It will be enough to show that $\lim_n \|\tilde{u}_n - \hat{u}_n\| = 0$.

Let us observe that obviously a zero $\hat{u}_n$ of F_n^{nat} is an exact solution of (4.4). Instead we solve (4.4) only inaccurately. In fact, we can estimate (see Facchinei and Pang 2003, Vol. I, Theorem 2.3.3)

$$\|\tilde{u}_n - \hat{u}_n\| \le \frac{L_n + 1}{c_n}\big\| F_n^{\text{nat}}(\tilde{u}_n)\big\|,$$

where L_n, respectively c_n is the Lipschitz constant, respectively the uniform monotonicity constant of T_n. Since the support of the random variable $S \in L^\infty(\Omega, P)$ is the interval $[s_0, s_1) \subset (0, \infty)$ and $s_0\ G + H$ is uniformly strongly monotone with some constant $c_0 > 0$, respectively $s_1\ G + H$ is Lipschitz continuous with some constant L_0, we have $0 < c_0 \le c_n, L_n \le L_0 < \infty$. Therefore by construction, $\lim_n \|\tilde{u}_n - \hat{u}_n\| = 0$ follows. □

We emphasize that with $p = 2$ we have proved the convergence of the means and the variances of $u^n := \hat{u}_n$ (respectively of $u^n := \tilde{u}_n$) towards $\langle \hat{u} \rangle$, the mean, respectively towards $\sigma^2(\hat{u})$, the variance of the unique solution $\hat{u}$. Therefore the well-known Chebyshev inequality becomes applicable to estimate errors in probability, as we shortly discuss here.

Let $0 < \delta < 1$; $0 < \eta \ll 1$. Then for any component j for $j = 1, \dots l$, we have

$$\mathbb{P}\big(\big|\hat{u}_j - \big\langle u_j^n\big\rangle\big| \ge \delta\big) \le \mathbb{P}\left(\big|\hat{u}_j - \langle \hat{u}_j \rangle\big| \ge \frac{\delta}{2}\right) \le \frac{4}{\delta^2}\sigma^2(\hat{u}_j) \le \frac{4}{\delta^2}\sigma^2\big(u_j^n\big) + \frac{4\eta}{\delta^2}, \tag{4.7}$$

as long as n is chosen is large enough such that

$$\big|\langle \hat{u}_j \rangle - \big\langle u_j^n\big\rangle\big| \le \frac{\delta}{2}, \quad \big|\sigma^2(\hat{u}_j) - \sigma^2\big(u_j^n\big)\big| \le \eta.$$

5 Some random nonlinear equilibrium problems

In this section we describe some simple equilibrium problems from economics, while equilibrium problems using a more involved network structure are deferred to the next section. Here we discuss where uncertainty can enter in the data of the problems and show how our theory of random V.I., where we can admit that random variables occur both in the operator and the constraint set, can be applied to model those nonlinear equilibrium problems under uncertainty.

5.1 A random Cassel-Wald economic equilibrium model

We follow Konnov (2007) and describe a Cassel-Wald type economic equilibrium model. This model deals with n commodities and m pure factors of production. Let c_k be the price of the k-th commodity, let b_i be the total inventory of the i-th factor, and let a_{ij} be the consumption rate of the i-th factor which is required for producing one unit of the j-th commodity, so that we set $c = (c_1, \dots, c_n)^T$, $b = (b_1, \dots, b_m)^T$, $A = (a_{ij})_{m\times n}$. Next let x_j denote the output of the j-th commodity and p_i denote the shadow price of the i-th factor, so that $x = (x_1, \dots, x_n)^T$ and $p = (p_1, \dots, p_m)^T$. In this model it is assumed that the prices are dependent on the outputs, so that $c : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a given mapping. Now in contrast to Konnov (2007) we do not consider b as a fixed vector, but we admit that the total inventory vector may be uncertain and model it as a random vector $b = B(\omega)$. Thus we arrive at the following

Problem CW-1 *For each $\omega \in \Omega$, find $\hat{X}(\omega) \in \mathbb{R}^n_+$, $\hat{P}(\omega) \in \mathbb{R}^m_+$ such that*

$$\big\langle c\big(\hat{X}(\omega)\big), \hat{X}(\omega) - x\big\rangle + \big\langle \hat{P}(\omega), Ax - A\hat{X}(\omega)\big\rangle \geq 0, \quad \forall x \in \mathbb{R}^n_+;$$

$$\big\langle p - \hat{P}(\omega), B(\omega) - A\hat{X}(\omega)\big\rangle \geq 0, \quad \forall p \in \mathbb{R}^n_+.$$

This is nothing but the optimality condition for the variational inequality problem:

Problem CW-2 *For each $\omega \in \Omega$, find $\hat{X}(\omega) \in \mathcal{K}(\omega)$ such that*

$$\big\langle c\big(\hat{X}(\omega)\big), \hat{X}(\omega) - x\big\rangle \geq 0, \quad \forall x \in \mathcal{K}(\omega),$$

where here

$$\mathcal{K}(\omega) = \big\{x \in \mathbb{R}^n \mid x \geq 0, Ax \leq B(\omega)\big\}.$$

Both Problems CW-1 and CW-2 are special instances of the general Problem 1, Problem CW-1 is a random variational inequality with a generally nonlinear map c over a fixed (nonrandom) constraint set, while randomness in CW-2 also affects the constraint set.

5.2 A random distributed market equilibrium model

We follow Gwinner (1995) and consider a single commodity that is produced at n supply markets and consumed at m demand markets. There is a total supply g_i in each supply market i, where $i = 1, \dots, n$. Likewise there is a total demand f_j in each demand market j, where $j = 1, \dots, m$. Since the markets are spatially separated, x_{ij} units of the commodity are transported from i to j. Introducing the excess supply s_i and the excess demand t_j we must have

$$g_i = \sum_{j=1}^{m} x_{ij} + s_i, \quad i = 1, \dots, n; \tag{5.1}$$

$$f_j = \sum_{i=1}^{n} x_{ij} + t_j, \quad j = 1, \dots, m; \tag{5.2}$$

Moreover the transportation from i to j gives rise to unit costs π_{ij}. Further we associate with each supply market i a supply price p_i and with each demand market j a demand price q_j. We assume there is given a fixed minimum supply price $\underline{p}_i \geq 0$ ('price floor') for each supply market i and also a fixed maximum demand price $\bar{q}_j > 0$ ('price ceiling') for each demand market j. These bounds can be absent and the standard spatial price equilibrium model due to Dafermos (Dafermos 1990, see also Konnov 2007) results, where the markets are required to be cleared, i.e.

$$s_i = 0 \quad \text{for } i = 1, \ldots, n; \qquad t_j = 0 \quad \text{for } \ j = 1, \ldots, m$$

are required to hold. Since $s_i \geq 0$ and $t_j \geq 0$ are admitted, the model is also termed a disequilibrium model. As is common in operations research models, we also include upper bounds $\bar{x}_{ij} > 0$ for the transportation fluxes x_{ij} on our bipartite graph of distributed markets.

Let us group the introduced quantities in vectors omitting the indices i and j. This gives the total supply vector $g \in \mathbb{R}^n$, the supply price vector $p \in \mathbb{R}^n$, the total demand vector $f \in \mathbb{R}^m$, the demand price vector $q \in \mathbb{R}^m$, the flux vector $x \in \mathbb{R}^{nm}$, and the unit cost vector $\pi \in \mathbb{R}^{nm}$.

Thus in our constrained distributed market model the feasible set for the unknown vector $u = [p, q, x]$ is given by the product set

$$M := \prod_{i=1}^{n} [\underline{p}_i, \infty) \times \prod_{j=1}^{m} [0, \bar{q}_j] \times \prod_{i=1}^{n} \prod_{j=1}^{m} [0, \bar{x}_{ij}].$$

Assuming perfect equilibrium the economic market conditions take the following form

$$s_i > 0 \quad \Rightarrow \quad p_i = \underline{p}_i, \quad p_i > \underline{p}_i \quad \Rightarrow \quad s_i = 0 \quad i = 1, \ldots, n; \tag{5.3}$$

$$t_j > 0 \quad \Rightarrow \quad q_j = \bar{q}_j, \quad q_j < \bar{q}_j \quad \Rightarrow \quad t_j = 0 \quad j = 1, \ldots, m; \tag{5.4}$$

$$p_i + \pi_{ij} \begin{cases} \geq q_j & \text{if } x_{ij} = 0 \\ = q_j & \text{if } 0 < x_{ij} < \bar{x}_{ij} \\ \leq q_j & \text{if } x_{ij} = \bar{x}_{ij} \end{cases} \quad i = 1, \ldots, n; \ j = 1, \ldots, m. \tag{5.5}$$

The last condition (5.5) extends the classic equilibrium conditions in that $p_i + \pi_{ij} < q_j$ can occur because of the flux constraint $x_{ij} \leq \bar{x}_{ij}$. As in unconstrained market equilibria (Dafermos 1990) we assume that we are given the functions

$$g = \breve{g}(p), \qquad f = \breve{f}(q), \qquad \pi = \breve{\pi}(x).$$

Then under the natural assumptions that for each $i = 1, \ldots, n$; $j = 1, \ldots, m$ there holds

$$q_j = 0 \quad \Rightarrow \quad \breve{f}_j(q) \geq 0; \qquad x_{ij} > 0 \quad \Rightarrow \quad \breve{\pi}_{ij}(x) > 0.$$

it can be shown (see Gwinner 1995) that a market equilibrium $u = (p, q, x)$ introduced above by the conditions (5.1)–(5.5) can be characterized as a solution to the following variational inequality: Find $u = (p, q, x) \in M$ such that

$$\sum_{i=1}^{n} \left(\breve{g}_i(p) - \sum_{j=1}^{m} x_{ij} \right) (\tilde{p}_i - p_i) - \sum_{j=1}^{m} \left(\breve{f}_j(q) - \sum_{i=1}^{n} x_{ij} \right) (\tilde{q}_j - q_j)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{m}\big(p_i+\breve{\pi}_{ij}(x)-q_j\big)(\tilde{x}_{ij}-x_{ij})\geq 0,\quad \forall \tilde{u}=(\tilde{p},\tilde{q},\tilde{x})\in M.$$

As soon as the given bounds are uncertain and we model these bounds as random variables, we obtain the random constraint set

$$\mathcal{M}(\omega):=\prod_{i=1}^{n}[\underline{p}_i(\omega),\infty)\times\prod_{j=1}^{m}\big[0,\bar{q}_j(\omega)\big]\times\prod_{i=1}^{n}\prod_{j=1}^{m}\big[0,\bar{x}_{ij}(\omega)\big],$$

and the above variational inequality becomes a random variational inequality.

6 A random traffic equilibrium problem

The purpose of this Sect. 6 is to discuss the applicability of our results to network equilibrium problems. A common characteristic of these problems is that they admit two different formulations based either on link variables or on path variables. These are actually related to each other through a linear transformation; we stress that in general, in the path approach, the strong monotonicity assumption is not reasonable. However, since a Mosco convergence result holds for the transformed sequence of sets, see the subsequent Remark 6.1, we can work in the "right"group of variables. To be more precise we need first some preliminary notations commonly used to state the standard traffic equilibrium problem from the user's point of view in the stationary case (see for instance Smith 1979; Dafermos 1980; Patriksson 1994).

A traffic network consists of a triple (N,A,W) where $N=\{N_1,\dots,N_p\}$, $p\in\mathbb{N}$, is the set of nodes, $A=(A_1,\dots,A_n)$, $n\in\mathbb{N}$, represents the set of the directed arcs connecting couples of nodes and $W=\{W_1,\dots,W_m\}\subset N\times N$, $m\in\mathbb{N}$ is the set of the origin-destination (O,D) pairs. The flow on the arc A_i is denoted by f_i, $f=(f_1,\dots,f_n)$; for the sake of simplicity we shall consider arcs with infinite capacity. We call a set of consecutive arcs a path, and assume that each (O,D) pair W_j is connected by r_j, $r_j\in\mathbb{N}$, paths whose set is denoted by P_j, $j=1,\dots,m$. All the paths in the network are grouped in a vector $(R_1,\dots,R_k)$, $k\in\mathbb{N}$, We can describe the arc structure of the paths by using the arc-path incidence matrix $\Delta=(\delta_{ir})_{\substack{i=1,\dots,n\\ r=1,\dots,k}}$ with the entries

$$\delta_{ir}=\begin{cases}1 & \text{if } A_i\in R_r\\ 0 & \text{if } A_i\notin R_r\end{cases}. \tag{6.1}$$

To each path R_r there corresponds a path flow F_r. The path flows are grouped in a vector $(F_1,\dots,F_k)$ which is called the network flow. The flow f_i on the arc A_i is equal to the sum of the path flows on the paths which contain A_i, so that $f=\Delta F$. Let us now introduce the unit cost of going through A_i as a real function $c_i(f)\geq 0$ of the flows on the network, so that $c(f)=(c_1(f),\dots,c_n(f))$ denotes the arc cost vector on the network. The meaning of the cost is usually that of travel time. Analogously, one can define a cost on the paths as $C(F)=(C_1(F),\dots,C_k(F))$. Usually $C_r(F)$ is just the sum of the costs on the arcs which build that path: $C_r(F)=\sum_{i=1}^{n}\delta_{ir}c_i(f)$ or in compact form,

$$C(F)=\Delta^T c(\Delta F). \tag{6.2}$$

For each pair W_j there is a given traffic demand $D_j \geq 0$, so that $(D_1, \dots, D_m)$ is the demand vector. Feasible path flows are nonnegative and satisfy the demands, i.e. belong to the set

$$K = \{F \in \mathbb{R}^k : F_r \geq 0 \text{ for any } r = 1, \dots, k \text{ and } \Phi F = D\},$$

where Φ is the pair-path incidence matrix whose entries for $j = 1, \dots, m$, $r = 1, \dots, k$, are

$$\varphi_{jr} = \begin{cases} 1 & \text{if the path } R_r \text{ connects the pair } W_j \\ 0 & \text{elsewhere} \end{cases}.$$

A path flow H is called an equilibrium flow or *Wardrop Equilibrium*, if and only if $H \in K$ and for any (O, D) pair $W_j \in W$ and any $R_q, R_s \in P_j$ there holds

$$C_q(H) < C_s(H) \quad \Longrightarrow \quad H_s = 0. \tag{6.3}$$

This means that all the used paths connecting a given (O, D) pair share the same cost and any path with a higher cost is not used. It can be shown that condition (6.3) is equivalent (see, for instance, Dafermos 1980 and Smith 1979) to the variational inequality

$$H \in K \quad \text{and} \quad \langle C(H), F - H \rangle \geq 0, \quad \forall F \in K. \tag{6.4}$$

Although the Wardrop equilibrium principle is expressed in the path variables, it is clear that the "physical" (and measured) quantities are expressed in the link variables; moreover, the strong monotonicity hypothesis on $c(f)$ is quite common, but as noticed for instance in (Bertsekas and Gafni 1982) this does not imply the strong monotonicity of $C(F)$ in (6.2), unless the matrix $\Delta^T \Delta$ is nonsingular. Although one can give a procedure for buildings networks preserving the strong monotonicity property (see, for instance, Falsaperla and Raciti 2007), the condition fails for a generic network, even for a very simple one as we shall illustrate in the sequel. Thus, it is useful to consider the following variational inequality problem:

$$h \in \Delta K \text{ and } \langle c(h), f - h \rangle \geq 0 \quad \forall f \in \Delta K. \tag{6.5}$$

If c is strongly monotone, one can prove that for each solution H of (6.4), $C(H) = \text{const.}$, i.e. all possibly nonunique solutions of (6.4) share the same cost. From an algorithmic point of view it is worth noting that one advantage in working in the path variables is the simplicity of the corresponding convex set but the price to be paid is that the number of paths grows exponentially with the size of the network.

Let us now consider the random version of (6.4) and (6.5):

$$H(\omega) \in K(\omega) \quad \text{and} \quad \langle C(\omega, H(\omega)), F - H(\omega) \rangle \geq 0, \quad \forall F \in K(\omega), \tag{6.6}$$

where, for any $\omega \in \Omega$,

$$K(\omega) = \{F \in \mathbb{R}^k : F_r \geq 0 \text{ for any } r = 1, \dots, k \text{ and } \Phi F = D(\omega)\},$$

The random variational inequality in the link-flow variables is:

$$h(\omega) \in \Delta K(\omega) \quad \text{and} \quad \langle c(\omega, h(\omega)), f - h(\omega) \rangle \geq 0, \quad \forall f \in \Delta K(\omega). \tag{6.7}$$

Moreover (6.6) is equivalent to the random Wardrop principle: For any $\omega \in \Omega$, for any $H(\omega) \in K(\omega)$, and for any $W_j \in W$, $R_q, R_s \in P_j$, there holds

$$C_q\big(\omega, H(\omega)\big) < C_s\big(\omega, H(\omega)\big) \quad \Longrightarrow \quad H_s(\omega) = 0.$$

In order to use our approximation scheme we require the assumption that the deterministic and random variables are separated. However this assumption is very natural in many applications where the random perturbation is treated as a *modulation* of a deterministic model. Under the above mentioned assumptions, (6.6) assumes the particular form:

$$S(\omega)\big\langle A\big(H(\omega)\big), F - H(\omega)\big\rangle \geq R(\omega)\big\langle b, F - H(\omega)\big\rangle, \quad \forall F \in K(\omega) \tag{6.8}$$

In (6.8), both the l.h.s. and the r.h.s. can, be replaced with any (finite) linear combination of monotone and separable terms, where each term satisfies the hypothesis of the previous sections:

$$\sum_i S_i(\omega)\big\langle A_i^T\big(H(\omega)\big), F - H(\omega)\big\rangle \geq \sum_j R_j(\omega)\langle b_j, F - H(\omega)\rangle, \quad \forall F \in K(\omega) \tag{6.9}$$

In this way, in (6.8) $R(\omega)$, $S(\omega)$ can be replaced by random vectors. Hence, in the traffic network, we could consider the case where the random perturbation has a different weight for each path.

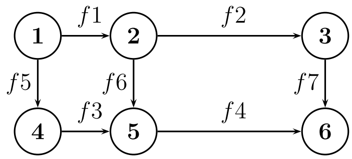
Remark 6.1 When applying our theory to the random traffic equilibrium problem we consider the integral form of (6.8), which, after the transformation to the image space, is defined on the set:

$$K_{\mathbb{P}} = \big\{F \in L^p\big(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k\big) : \Phi F(r,s,t) = t,\ F(r,s,t) \geq 0 \quad \mathbb{P}\text{-}a.s.\big\}$$

Let $K_{\mathbb{P}}^n$ be the approximate sets constructed as described in Sect. 4. It can be easily verified that the sets $K_{\mathbb{P}}^n$ are uniformly bounded. Moreover, the arc-path incidence matrix Δ induces a linear operator mapping $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ to $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^n)$. This operator, which by abuse of notation is still denoted by Δ, is continuous. Thus, from Mosco convergence $K_{\mathbb{P}}^n \to K_{\mathbb{P}}$ it follows easily that also $\Delta K_{\mathbb{P}}^n \to \Delta K_{\mathbb{P}}$ in Mosco's sense.

In what follows we present two small scale examples. In the first example we build a small network and we study the random variational inequality in the path-flow variables. The network is built in such a way that if the cost operator is strongly monotone in the link-flow variables, the transformed operator, is still strongly monotone in the path-flow variables. Moreover, this small network can be considered as an elementary block of an arbitrarily large network with the same property of preserving strong monotonicity. On the other hand, the second example, which we solve exactly, shows that even very simple networks can fail to preserve the strong monotonicity of the operator when passing from the link to the path-flow variables. In this last case, two possible strategies can be followed. The first possibility is to work from the beginning in the link-variables and use the previous remark to apply our approximation procedure. The other option is to regularize (in the sense of Tichonov) the problem in the path-variables. We stress the fact that if one is interested in the cost shared by the network users, it does not matter which solution is obtained from the regularized problem, because, thanks to the particular structure of the operator, the cost is constant on the whole solution set.

Fig. 1 Network which preserves strong monotonicity



Example 6.1 In the network under consideration, (see Fig. 1), there are 7 links and one origin-destination pair, $1-6$, which can be connected by 3 paths, namely:

$$R_1 = A_1A_2A_7,\ R_2 = A_1A_6A_4,\ R_3 = A_5A_3A_4.$$

The traffic demand is represented by the nonnegative random variable D with uniform probability distribution in DeMiguel and Xu (2009), Dentcheva and Ruszczyński (2003) so that $F_1 + F_2 + F_3 = D$, while link-cost functions are given by:

$$c_1 = \rho f_1^2 + f_1, \quad c_2 = \rho f_2^2 + 2f_2, \quad c_3 = \rho f_3^2 + f_3, \quad c_4 = \rho f_4^2 + 2f_4 + f_6,$$

$$c_5 = \rho f_5^2 + f_5, \quad c_6 = \rho f_6^2 + 2f_6, \quad c_7 = \rho f_7^2 + f_7 + 0.5f_5.$$

The linear part of the operator above is represented by a nonsymmetric positive definite matrix, while the nonnegative parameter ρ represents the weight of the nonlinear terms. Such a functional form is quite common in many network equilibrium problems (Nagurney 1993). Since we want to solve the variational inequality associated to the Wardrop Equilibrium we have to perform the transformation to the path-flow variables, which yields for the cost functions the following expressions:

$$\begin{aligned}
C_1 &= 3\rho F_1^2 + \rho F_2^2 + 2\rho F_1F_2 + 4F_1 + F_2 + 0.5F_3,\\
C_2 &= \rho F_1^2 + 3\rho F_2^2 + \rho F_3^2 + 2\rho F_1F_2 + 2\rho F_2F_3 + F_1 + 6F_2 + 2F_3,\\
C_3 &= \rho F_2^2 + 3\rho F_3^2 + 2\rho F_2F_3 + 4F_3 + 3F_2.
\end{aligned}$$

For the resolution of the discretized, finite dimensional variational inequalities, many algorithms are available. Due to the simple structure of our example we employ the extragradient algorithm.

Table 1 describes the case where the parameter ρ is equal to 0.1, while in Table 2 we have $\rho = 1$. This parameter controls the "degree" of nonlinearity of the problem. For $\rho = 0.1$ the problem can be considered as a linear problem with a small perturbation, and the solutions are quite stable with respect to the random demand. On the other hand, when $\rho = 1$ the problem is fairly nonlinear and the solutions change qualitatively according to the random demand (in particular, near $D = 10$ there is a zero component, H_2, in the solution vector, while in most of the interval the equilibrium solution has nonzero components). As a result, the variances are larger and the convergence slower.

Example 6.2 We consider the simple network of Fig. 2 below which consists of four arcs and one origin–destination pair, which can be connected by four different paths. Let us assume that the traffic demand between O and D is given by a real random variable $T > 0$ and that the link cost functions are given by $c_1 = 2f_1, c_2 = 3f_2, c_3 = f_3, c_4 = f_4$. The link flows belong to the set $\{f \in \mathbb{R}^4 : \exists F \in K(T), f = \Delta F\}$, where $K(T)$ is the feasible set in

Table 1 Mean values and variances corresponding to various approximations for $D \in [10, 11]$ and $\rho = 0.1$

N	$\langle F_1 \rangle$	$\langle F_2 \rangle$	$\langle F_3 \rangle$	σ_1^2	σ_2^2	σ_3^2
10	4.5396	1.4756	4.4346	0.0153	0.0017	0.0148
100	4.5590	1.4821	4.4537	0.0154	0.0017	0.0149
1000	4.5610	1.4821	4.4556	0.0154	0.0017	0.0149
10000	4.5612	1.4828	4.4558	0.0154	0.0017	0.0149

Table 2 Mean values and variances corresponding to various approximations for $D \in [10, 11]$ and $\rho = 1$

N	$\langle F_1 \rangle$	$\langle F_2 \rangle$	$\langle F_3 \rangle$	σ_1^2	σ_2^2	σ_3^2
10	3.1602	2.6005	4.6891	4.2853	2.3442	0.1010
100	3.6964	2.1968	4.6017	3.0077	1.6456	0.0759
1000	3.6460	2.2390	4.6143	3.1668	1.7326	0.0791
10000	3.6505	2.2436	4.6157	3.1837	1.7418	0.0794

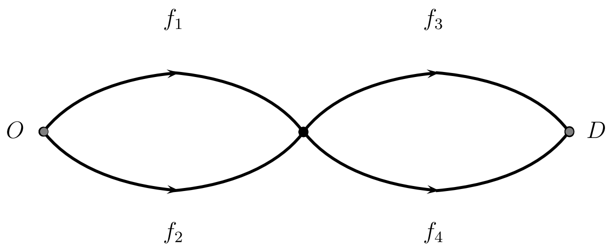


Fig. 2 Loss of strong monotonicity

the path flow variables given by

$$K(T) = \{F_1, F_2, F_3, F_4 \geq 0 \text{ such that } F_1 + F_2 + F_3 + F_4 = T\},$$

and Δ is the link-path matrix. Hence, if F is known, one can derive f from the equations

$$f_1 = F_1 + F_2,\ f_2 = F_3 + F_4,\ f_3 = F_1 + F_3,\ f_4 = F_2 + F_4.$$

The path–cost functions are given by the relations

$$\begin{aligned} C_1 &= c_1 + c_3 = 3F_1 + 2F_2 + F_3, \\ C_2 &= c_1 + c_4 = 2F_1 + 3F_2 + F_4, \\ C_3 &= c_2 + c_3 = F_1 + 4F_3 + 3F_4, \\ C_4 &= c_2 + c_4 = F_2 + 3F_3 + 4F_4 \end{aligned}$$

The associated variational inequality can be solved exactly (see e.g. Falsaperla and Raciti 2007 for a non-iterative algorithm) and the solution expressed in term of the second path variable is

$$\left(\frac{3t}{5} - G(t), G(t), G(t) - \frac{t}{10}, -G(t) + \frac{t}{2} \right),$$

where $G : (0, \infty) \to \mathbb{R}$ is any function of the realization t of the random variable T that satisfies the constraint $G(t) \in [\frac{t}{10}, \frac{t}{2}]$. Let us observe that for each feasible choice of $G(t)$

the cost at the corresponding solution is always equal to $\frac{17}{10}t \cdot (1, 1, 1, 1)$. One can also solve the variational inequality in the link variables by using the relations

$$f_1 + f_2 = T, \quad f_3 + f_4 = T.$$

We are then left with the problem:

$$(c_2 - c_1)\big(f_2 - h_2(T)\big) + (c_4 - c_3)\big(f_4 - h_4(T)\big) \geq 0,$$

which yields for any realization t,

$$h(t) = t \cdot \left(\frac{3}{5}, \frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right).$$

As an example we assume that our random parameter follows the *lognormal distribution*. This statistical distribution is used for numerous applications to model random phenomena described by nonnegative quantities. It is also known as the Galton Mc Alister distribution and, in economics, is sometimes called the Cobb-Douglas distribution, and has been used to model production data. Thus, let the *normal* distribution given by the density

$$g_{\mu,\sigma^2}(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\dfrac{(t-\mu)^2}{2\sigma^2}}$$

then the *lognormal distribution* is defined by the density

$$\begin{cases} (1/t) g_{\mu,\sigma^2}(\log t), \text{if} t > 0; \\ 0, \text{if} t \leq 0. \end{cases}$$

The numerical evaluation of the mean values and variances, corresponding to $\mu = 0$ and $\sigma = 1$ yields:

$$\big(\langle h_1 \rangle, \langle h_2 \rangle, \langle h_3 \rangle, \langle h_4 \rangle\big) = 1.64 \cdot (3/5, 2/5, 1/2, 1/2)$$

$$\big(\sigma^2(h_1), \sigma^2(h_2), \sigma^2(h_3), \sigma^2(h_4)\big) = 4.68 \cdot (3/5, 2/5, 1/2, 1/2).$$

7 Some concluding remarks

Let us first discuss our main results. We are aware that our definition of uniform strong monotonicity is very strong. It guarantees uniqueness of the solution, uniqueness and boundedness of the solutions of the substitute problems, and norm convergence in our approximation procedure. When we relax to mere monotonicity, we still have Theorem 4.1. At this level, our convergence result compares to Kryazhimskii and Ruszczyński (2001), where weak convergence of a constraint aggregation procedure is established in infinite-dimensional spaces.

The postulated separation of the deterministic and random variables in Sects. 3 and 4 is itself the outcome of an approximation process, namely Karhunen-Loève expansion, whose convergence behaviour remains unexplored in this paper. Moreover, we have not presented any analysis of the convergence speed of our approximation procedure. This would open the possibility to make the estimates (4.7) given at the end of Sect. 4 more rigorous to obtain confidence regions for the computed results.

Here we have studied static models of equilibrium of random systems. We did not consider multistage problems of stochastic programming or dynamic stochastic processes, as e.g. the problem of adaptive routing in a network with failures (Ferris and Ruszczyński 2000). Also we have only studied almost sure constraints, we did not consider the variety of constraints in stochastic programming, like chance contraints or the more recent dominance constraints (Dentcheva and Ruszczyński 2003).

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References

Appel, J., & Zabrejko, P. P. (1990). *Nonlinear superposition operators*. Cambridge: Cambridge University Press.

Attouch, H. (1984). *Variational convergence for functions and operators*. Boston: Pitman.

Bertsekas, D., & Gafni, E. (1982). Projection methods for variational inequalities with application to the traffic assignment problem. *Mathematical Programming Studies*, *17*, 139–159.

Billingsley, P. (1995). *Probability and measure*. New York: Wiley.

Chen, X., & Fukushima, M. (2005). Expected residual minimization method for stochastic linear complementarity problems. *Mathematics of Operations Research*, *30*, 1022–1038.

Dafermos, S. (1980). Traffic equilibrium and variational inequalities. *Transportation Science*, *14*, 42–54.

Dafermos, S. (1990). Exchange price equilibria and variational inequalities. *Mathematical Programming*, *46*, 391–402.

Daniele, P., Maugeri, A., & Oettli, W. (1999). Time-dependent traffic equilibria. *Journal of Optimization Theory and Applications*, *103*, 543–555.

Dempster, M. A. H. (1980). Introduction to stochastic programming. In *Stochastic programming. Proc. Int. Conf.*, Oxford (pp. 3–59).

DeMiguel, V., & Xu, H. (2009). A stochastic multiple-leader Stackelberg model: analysis, computation, and application. *Operations Research*, *57*, 1220–1235.

Dentcheva, D., & Ruszczyński, A. (2003). Optimization with stochastic dominance constraints. *SIAM Journal on Optimization*, *14*, 548–566.

Doob, J. L. (1953). *Stochastic processes*. New York: Wiley.

Facchinei, F., & Pang, J.-S. (2003). *Finite-dimensional variational inequalities and complementarity problems* (Vol. 2). New York: Springer.

Falsaperla, P., & Raciti, F. (2007). An improved, non-iterative algorithm for the calculation of the equilibrium in the traffic network problem. *Journal of Optimization Theory and Applications*, *133*, 401–411.

Ferris, M. C., & Ruszczyński, A. (2000). Robust path choice in networks with failures. *Networks*, *35*, 181–194.

Giannessi, F., & Maugeri, A. (Eds.) (1995). *Variational inequalities and network equilibrium problems*. New York: Plenum Press. Erice (1994).

Gürkan, G., Özge, A. Y., & Robinson, S. M. (1999). Sample-path solution of stochastic variational inequalities. *Mathematical Programming*, *84*, 313–333.

Gwinner, J. (1995). Stability of monotone variational inequalities with various applications. In F. Giannessi & A. Maugeri (Eds.), *Variational inequalities and network equilibrium problems* (pp. 123–142). New York: Plenum Press.

Gwinner, J. (2000). A class of random variational inequalities and simple random unilateral boundary value problems: existence, discretization, finite element approximation. *Stochastic Analysis and Applications*, *18*, 967–993.

Gwinner, J. (2003). Time dependent variational inequalities—some recent trends. In P. Daniele et al. (Eds.), *Equilibrium problems and variational models* (pp. 225–264). Boston: Kluwer Academic.

Gwinner, J., & Raciti, F. (2006). On a class of random variational inequalities on random sets. *Numerical Functional Analysis and Optimization*, *27*, 619–636.

Gwinner, J., & Raciti, F. (2009). On monotone variational inequalities with random data. *Journal of Mathematical Inequalities*, *3*, 443–453.

Kinderlehrer, D., & Stampacchia, G. (1980). *An introduction to variational inequalities and their applications*. New York: Academic Press.

Konnov, I. V. (2007). *Equilibrium models and variational inequalities*. Amsterdam: Elsevier.
Kryazhimskii, A. V., & Ruszczyński, A. (2001). Constraint aggregation in infinite-dimensional spaces and applications. *Mathematics of Operations Research*, *26*, 769–795.
Lepp, R. (1994). Projection and discretization methods in stochastic programming. *Journal of Computational and Applied Mathematics*, *56*, 55–64.
Maugeri, A., & Raciti, F. (2009). On existence theorems for monotone and nonmonotone variational inequalities. *Journal of Convex Analysis*, *16*, 899–911.
Mosco, U. (1969). Convergence of convex sets and of solutions of variational inequalities. *Advances in Mathematics*, *3*, 510–585.
Nagurney, A. (1993). *Network economics: a variational inequality approach*. Dordrecht: Kluwer Academic.
Patriksson, M. (1994). *The traffic assignment problem*. Utrecht: VSP.
Prékopa, A. (1995). *Stochastic programming*. Budapest, Dordrecht: Akadémiai Kiadó, Kluwer Academic.
Ravat, U., & Shanbhag, U. V. (2010). On the characterization of solution sets of smooth and nonsmooth stochastic Nash games. In *Proceedings of the American control conference (ACC)*, Baltimore.
Ravat, U., & Shanbhag, U. V. (2011). On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games. *SIAM Journal on Optimization*, *21*, 1168–1199.
Shapiro, A. (2003). Monte Carlo sampling methods. In A. Shapiro & A. Ruszczyński (Eds.), *Handbooks in operations research and management science* (pp. 353–426). Amsterdam: Elsevier.
Shapiro, A., Dentcheva, D., & Ruszczyński, A. (2009). *Lectures on stochastic programming—modeling and theory*. Philadelphia: SIAM.
Shapiro, A., & Xu, H. (2008). Stochastic mathematical programs with equilibrium constraints, modelling and sample average approximation. *Optimization*, *57*, 395–418.
Smith, M. J. (1979). The existence, uniqueness and stability of traffic equilibrium. *Transportation Research*, *138*, 295–304.
Xu, H. (2010). Sample average approximation methods for a class of stochastic variational inequality problems. *Asia-Pacific Journal of Operational Research*, *27*, 103–119.
Xu, H., & Zhang, D. (2009). *Stochastic nash equilibrium problems: sample average approximation and applications*. Available online at http://www.optimization-online.org/DB_HTML/2009/05/2299html.
Zeidler, E. (1990). *Nonlinear monotone operators: Vol. II/B. Nonlinear functional analysis and its applications*. New York: Springer.