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Application of Robust Optimization to the Sawmill Planning Problem

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Abstract Optimization models have been used to support decision making in the forest industry for a long time. However, several of those models are deterministic and do not address the variability that is present in some of the data. Robust Optimization is a methodology which can deal with the uncertainty or variability in optimization problems by computing a solution which is feasible for all possible scenarios of the data within a given uncertainty set. This paper presents the application of the Robust Optimization Methodology to a Sawmill Planning Problem. In the particular case of this problem, variability is assumed in the yield coefficients associated to the cutting patterns used. The main results show that the loss in the function objective value (the "Price of Robustness"), due to computing robust solutions, is not excessive. Moreover, the computed solutions remain feasible for a large proportion of randomly generated scenarios, and tend to preserve the structure of the nominal solution. We believe that these results provide an application area for Robust Optimization in which several source of uncertainty are present.

Keywords Uncertainty · Sawmill production planning · Modelling · Robust solutions · Linear programming

1 Introduction

The forest industry in Chile is one of the most important sectors in the national economy, with a participation of 3.1% in the national Gross Domestic Product (GDP). It is the second

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export sector, and the first based in a renewable natural resource. Annually, 40 millions of cubic meters of wood are harvested and the Central Bank indicates that forest exports represent 7.8% of the annual total, with US$53.000 millions in 2009.

Because of the importance of the forest industry, this industrial sector has been a pioneer in the use of operations research (OR) tools to support managements decisions. OR has produced a significant and positive change in the sector, generating annual savings in the order of US$13 millions (Epstein et al. 1999).

Optimization models have been the main type of OR tools used in this industry. These models have been successfully implemented to support decisions in several problems like transportation planning and truck assignment, harvesting planning, machinery location, road building, etc. (Epstein et al. 2007). The formulation of these models and their solution requires the use of large amounts of data. As in many other areas, this data is considered as being known exactly, but usually this assumption is wrong. In general, several elements in the models are uncertain although decision makers tend to use estimated average values, "most probable values" or the like. In some cases this could lead to wrong decisions.

Therefore, incorporating the consideration of uncertainty into the models is essential to guarantee that the decision, based on the solutions obtain by the model, be feasible and good despite the actual data. Some techniques have been applied to include explicitly the uncertainty in forestry. These techniques can be divided in two types of approach: one of them using probability-based models, and the other applying fuzzy models, but for the last one there are few reported applications. Some methods using probabilistic information are stochastic programming, dynamic programming, chance constrained linear programming, scenario analysis, Markov decision models and optimal control theory (Weintraub and Bare 1996; Martell et al. 1998). In Weintraub and Vera (1991) an algorithm is presented to solve chance constrained linear program, i.e., problems with random constraint coefficients. This algorithm is applied in forest planning models by Weintraub and Abramovich (1995). Kazemi Zanjani et al. (2010b) present a multi-period, multi-product production planning problem in sawmills with uncertainty in processes yield and in products demands. The results indicate a high quality model and provide some evidence of the advantages of a robust optimization approach over a 2-stage stochastic programming approach.

However, in general, the increase in complexity of the problems (both in the mathematical formulation as in the algorithms to solve it) and difficulties in developing reliable probability functions, due in part to the lack of reliable detailed data, have prevented their massive application. So, while theoretical methods are available, only few of them have been applied in practice (Weintraub and Romero 2006). This leads to the possibility of using other optimization techniques which do not require detailed probabilistic knowledge, like the ones that allow the computation of robust solutions, i.e. solutions which are less sensitive to uncertainty and, in some degree, independent of the data variability, at least within a certain range. Robust Optimization (RO) is a recently developed technique that follows those lines. The methodology attempts to compute solutions that remain feasible for all possible data scenarios defined in a certain set, and prevents non-completion of constraints (Ben Tal and Nemirovski 1999; El-Ghaoui et al. 1998; Bertsimas and Sim 2003).

In this work, we apply this methodology to a sawmill planning problem, which includes production decisions, inventory control and demand satisfaction. The problem belongs to the area of Forest Supply Chain Planning and presents demand for boards by final customers. Those constraints (demand satisfaction) can be considerate "hard" constraints, i.e., limitations that should be fulfilled due to the risk of customer dissatisfaction and the consequent market share loss. It has to be pointed out that Chilean forest companies are competing in several markets around the world with some of the world largest forest consortium. Maintaining a high level of service is very important but, at the same time, being efficient is very relevant for the financials of the business. It is in this context that adequate consideration of uncertainties becomes very relevant and we expect this work to make a contribution in Supply Chain Management in the forest industry. Particularly because the methodology we use in this paper preserves the problem structure (linearity) and has the potential of making easier the consideration of data variability in real applications. Sawmill planning is an interesting problem due the presence of uncertainty not only in demands but also in final products, production levels, and in particular, in the yields of the sawing process. Full consideration of these elements increases significantly the difficulty of the problem, so we will restrict to working with the uncertainty in the sawing process yield. We have also decided to consider the planning at a tactical level where the main decision is the quantity of logs to be requested to the forest in order to satisfy demand. There exists, of course, a problem in the short term, but the consideration of some aspects like setups or minimum production quantities will increase the difficulty of the problem at this point.

The consideration of methodologies to address uncertainty in this kind of problem was motivated by an actual case of a forest company in Chile. That company uses a sawmill planning model similar to the one presented in this paper, but with no consideration of uncertainty. Although the model is of help to have some estimates of production and log purchases, the actual yield was always different from the estimated one, generating problems in the correct use of logs supply. The need for a more robust tactical planning arises immediately from this situation. Moreover, companies like that one are not ready yet for the consideration of a full stochastic approach, mainly due to the lack of solid probabilistic information. In this sense, we believe that the results of this paper can be used to show that a robust optimization approach can be introduced in a more effective way into the decision process of this kind of companies, as it begins with an extension of the currently used optimization model and introduces elements that will allow managers to star considering uncertainty in a more formal way. Moreover, the possibility of easily analyzing the tradeoff between robustness and optimality, as we show in the results, should be an attractive way of assessing the impact of uncertainty in the decision making process. This is an important benefit of robust optimization, as it is possible to manage the degree of conservatism and not necessarily protect for the worst case, which is unlikely. Moreover, robust optimization gives a good solution to the tactical model where the data is aggregated and, because of the structure of the model, it is possible to obtain solutions quickly when required.

The paper is structured as follows. In the next section we describe sawmill planning problem and we show a deterministic linear programming model for it. In Sect. 3 we briefly explain the Robust Optimization methodology. Subsequently the document shows the way in which the methodology is applied to sawmill planning problem and we evaluate the behavior of robust optimization in a problem instance based on industrial data in a typical Chilean company. The paper ends with conclusions of the research and the future challenges in this area.

2 Sawmill Planning Problem

In a typical forest company in Chile, the Supply Chain is divided in areas like the Forest Division, Sawmill Division, Pulp Mill Division, etc. In this context, a complex set of interrelated decisions has to be addressed.

In particular, the Sawmill Division, with several facilities, seeks to maximize revenue originating in lumber sales. This lumber may consist of boards (“dry” or “green” according to the percentage of humidity). There are specific markets for both kinds of boards.

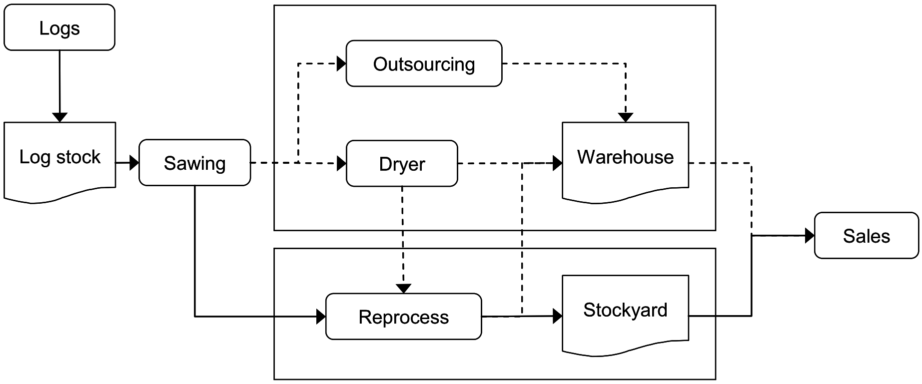

Fig. 1 Sawmill processes

The raw material of a sawmill consists of logs which come from different forests. Logs are classified by diameter, length and quality. Depending on the products demanded and the production process, each sawmill has to determine the quantity of different type of logs required to satisfy board demand. These requirements are to be satisfied by the Forest Division of the company, although, in general, the decision processes of both divisions are not coordinated (Epstein et al. 2007). In the case when the Forest Division cannot fulfill the demand from its own forests, the sawmill can purchase logs to small landowners. It is important to point out that in Chile, large forest firms own large extensions of forests, mainly radiata pine plantations.

Logs are transported from forests by trucks, and once they arrive to the sawmill logs are put in a temporary storage area where they are classified according to diameter and quality. Logs remain in this place until their utilization. When they are required in the process they enter the sawing line where they are cut in accordance with a cutting pattern, previously defined, adequate to the log diameter. This cutting process generates boards of different sizes. In some cases, the process is divided in two, some boards "in process" goes directly to the reprocessing phase and finally stored, resulting in "green" boards. Another quantity of boards "in process" goes to the drier unit where humidity is removed, producing "dry" boards. In some cases the drying process can be outsourced. Finally, a proportion of the "dry" boards can go to the reprocessing facility where they are further cut and dived.

The process concludes when the boards are ready for dispatch, either to the local market, or to ports to be sent to different destinations around the world. Each process has associated costs, and limited capacity. The same occurs in the inventory phases. Figure 1 represents the general scheme of operations in sawmills. This scheme is modeled in the next section.

2.1 Sawmill Planning Model

The sawmill production planning problem can be modeled as a linear program (Weintraub and Epstein 2002; Singer and Donoso 2007). This is the model we present now. We point out that uncertainty is not considered in this model although there are several sources of variability, decisions are risky, and demand might not be met. The consideration of uncertainty in this kind of problems has been carried out by combining optimization with real-time simulations (Kazemi Zanjani et al. 2010a) although this kind of analysis is time consuming. We later show how to apply robust optimization to this model.

The notation used and formulation of the model (based in Fig. 1) are shown below.

Variables

x_{jmt} Volume of boards type m ("green" or "dry") for sale in sawmill j at period t.
y_{jkt} Volume of logs type k demanding by sawmill j at period t.
z_{jkt} Volume of logs type k in inventory in sawmill j at period t.
w_{jmt} Volume of boards type m in inventory in sawmill j at period t.
v_{jmt} Volume of boards type m sent to outsourcing by sawmills j at period t.
r_{jmt} Volume of boards type m ("green" or "dry") to be produced by sawmill j at period t.
s_{jekt} Volume of logs type k processed with cutting pattern e in sawmill j at period t.

Parameters

δ_t Discount factor for period t.
A_m Price of boards type m(US$/m^3).
a_{jkt} Purchase cost of logs type k in sawmill j at period t (US$/m^3).
b_j Inventory cost of logs in sawmill j (US$/m^3).
c_j Inventory cost of boards in sawmill j (US$/m^3).
d Outsourcing cost (US$/m^3).
f_j Reprocessing cost in sawmill j (US$/m^3).
F Fraction of wood which goes to reprocessing (%).
g_j Drier cost in sawmill j (US$/m^3).
h_j Sawing cost in sawmill j (US$/m^3).
R_{jekm} Yield of cutting pattern e in sawmill j, applied to logs of type k, given as boards of type m produced (m^3 boards/m^3 logs).
ϕ_j Sawing capacity in the sawmill j (m^3).
ψ_j Inventory capacity of logs in sawmill j (m^3).
μ_j Inventory capacity of boards in sawmill j (m^3).
θ_j Drier capacity in sawmill j (US$/m^3).
η_j Reprocessing capacity in sawmill j (m^3).
B_{mt} Estimated demand for boards type m in period t (m^3).

Optimization Model

$$\text{Max } z = \sum_t \delta_t \left[\begin{array}{l} \sum_j \sum_m A_m \cdot x_{jmt} - \sum_j \sum_k a_{jk} \cdot y_{jkt} - \sum_j \sum_k b_j \cdot z_{jkt} - \\ \sum_j \sum_m c_j \cdot w_{jmt} - \sum_m \sum_j d \cdot v_{jmt} - \sum_j \sum_m F \cdot f_j \cdot r_{jmt} - \\ \sum_j \sum_m g_j \cdot r_{jmt} - \sum_j \sum_k \sum_e h_j \cdot s_{jekt} \end{array} \right] \tag{1}$$

subject to:

$$z_{jkt-1} + y_{jkt} = z_{jkt} + \sum_e s_{jekt} \quad \forall j, k, t \tag{2}$$

$$w_{jmt-1} + r_{jmt} + v_{jmt} = w_{jmt} + x_{jmt} \quad \forall j, m, t \tag{3}$$

$$\sum_e \sum_k R_{jekm} \cdot s_{jekt} = r_{jmt} \quad \forall j, m, t \tag{4}$$

$$\sum_e \sum_k s_{jekt} \le \varphi_j \quad \forall j, t \tag{5}$$

$$\sum_{k} z_{jkt} \leq \psi_j \quad \forall j, t \tag{6}$$

$$\sum_{m} w_{jmt} \leq \mu_j \quad \forall j, t \tag{7}$$

$$\sum_{m} r_{jmt} \leq \theta_j \quad \forall j, t \tag{8}$$

$$\sum_{m} r_{jmt} \leq \eta_j \quad \forall j, t \tag{9}$$

$$\sum_{j} x_{jmt} \geq B_{mt} \quad \forall m, t \tag{10}$$

$$x_{jmt}, y_{jkt}, z_{jkt}, w_{jmt}, v_{jmt}, r_{jmt}, s_{jekt} \geq 0 \quad \forall j, m, e, k, t \tag{11}$$

Expression (1) specifies the objective function of the problem, which maximizes the economic benefit obtained from board sales and considering the following costs: logs purchases, inventory, outsourcing, drying, reprocessing and sawing. Transportation costs are not considered. Constraints (2) and (3) correspond to the inventory constraints for logs and boards ("green" and "dry") respectively. Constraints (4) transform the logs into boards depending on the specific sawmill yield. This yield is function of the log type, cutting pattern, sawmill and the boards we need produce. Constraints (5), (6), (7), (8) and (9) correspond to capacity limits in each phase (sawing, log inventory, board inventory, drier and reprocess). Demand fulfillment is represented by constraints (10). Finally, constraints (11) represent variables specifications.

As we can see, the model includes several potentially uncertain parameters, for instance demands, prices, costs, yields or sawmills conversion efficiency. Carino and Willis (2001a, 2001b) present a linear model to solve the production-inventory problem, and the results show that the model is very sensitive to changes in sawmill conversion efficiency. In the present paper, we follow those results and we consider the uncertainty in yield parameter (R_{jekm}), and we will use the Robust Optimization methodology to find solutions which are immune, in a certain degree, to this variability.

We assume that the Forest Division is capable of delivering all the volume demanded by the sawmill. There is no lost of generality here as the sawmill can purchase from other producers. Moreover, we can also put bounds to the quantities of logs purchased, to represent other situations.

3 Uncertainty

As we mentioned before, the data used in real problems is not known exactly, and errors could have significant effect in the solutions of optimization problems. These solutions could be non optimal or even become infeasible for the actual instance. Due to this, it is desirable to consider explicitly the uncertainty in the optimization models.

The treatment of uncertainty in optimization was studied as early as the 50's by Dantzig (1955). The author deals with uncertainty using different scenarios, each one with a given probability of occurrence. This study can be considered as one of the first steps in stochastic programming.

The following approaches are the main ways to consider uncertainty in optimization models:

- Replace the random parameters by the most probable value or by the mean value (or average). The resultant mathematical model is deterministic. It does not consider data variability, so the solution could be suboptimal and even it could be not representative (Kouvelis 1997). Typically, the decision could be myopic and only valid for the data instance used.
- Use a finite number of scenarios and solve each of them independently. This technique is useful to compare solutions in different scenarios and select one solution according to the decision maker criterion. Maturana and Contesse (1998) show an application of this approach in a Mixed Integer Programming Problem in the Logistics of Sulfuric Acid in Chile.
- Sensitivity analysis. It measures the impact in the solutions produced by the disturbances in the input data. It's a "post mortem" approach, and doesn't provide a mechanism to control the sensitivity (Hillier and Liberman 2001).
- Stochastic Programming. This methodology arises as an extension to linear and nonlinear models, where uncertain coefficients are represented by random variables with a probability distribution. It attempts to optimize some expected behavior measurement and each situation is balanced by its occurrence probability (Birge and Louveaux 1997). However, the approach ignores higher order moments and preferences relative to risk by the decision maker (Vladimirou and Zenios 1997). One alternative to this methodology is Fuzzy Optimization. This approach makes a difference between randomness and imprecision, and assumes pertinence functions instead of probability distributions (Jensen and Maturana 2002). Robust approaches were considered, in a particular way, by Mulvey et al. (1995), in a context of Stochastic Programming. However, the cost of the Robust Programming solution is higher than the cost of the Stochastic Programming solution (Vladimirou and Zenios 1997). This robust approach incorporates explicitly the random parameters, and minimizes the expected value cost plus some penalization due to infeasibilities. The solution is robust if the optimal solution varies slightly when the data change. Mulvey et al. (1995), Bai et al. (1997), Leung et al. (2002), Takriti and Ahmed (2004), Yang and Zenios (1997) and Kazemi Zanjani et al. (2010a), among others, show applications and solution methods for this methodology. Specifically, Kazemi Zanjani et al. (2010a) apply Robust Optimization, assuming scenarios in a sawmill production planning, obtaining good results.

Another standard formulation in Stochastic Programming is to use multistage decision models with adaptive decisions. In a typical two stages Stochastic Programming approach the set of decisions can be divided in two groups: decisions that have to be taken "here and now", before the realization of the uncertain events, and decisions that are taken after the event and, hence, adapt to that and have a sequential characteristic. This modeling approach is powerful but, as Chen and Zhang (2009) point out, it could lead to large scale problems that are too difficult to solve to optimality. In addition to that, stochastic programming needs to know in detail the distribution of the uncertain data, which is typically unknown or hard to obtain.

In the next section, we explain in detail the Robust Optimization methodology we are going to apply in the sawmill planning problem.

4 Robust Optimization

The necessity to obtain robust solutions, i.e., solutions which are immune to data variability has been present for a long time in the formulation of mathematical programming models. Soyster (1973) was the first to work in this area. He proposes an approach to compute robust solutions to a linear model, assuming that parameters vary within certain intervals, and considering all possible scenarios within those intervals. The resulting model gives a high level of protection against uncertainty but it is a very conservative model and generates an optimal value much smaller than the deterministic one. In recent years some important progress has been done in Robust Optimization (Ben Tal and Nemirovski 1999; Bertsimas and Sim 2003), with different approaches and applications. The approach developed by Ben Tal and Nemirovski (1999) consider a linear programming problem of the form.

$$\text{Min} \quad \{x_0 : f_0(x, \zeta) - x_0 \leq 0, f_i(x, \zeta) \leq 0, i = 1, \dots, m\} \tag{12}$$

Here the input data is generally considered uncertain to some degrees in the real world, and then it is necessary to treat this uncertainty in some way.

To differentiate from the approach developed by Mulvey et al. (1995), Ben Tal and Nemirovski (1999) notice the existence of "hard" constraints, which means constraint that should be satisfied regardless of the specific data realization. In this way, the candidate solution (x_0, x) should satisfy a semi-infinite system of constraints:

$$f_0(x, \zeta) \leq x_0, f_i(x, \zeta) \leq 0, \quad i = 1, \dots, m \; \forall(\zeta \in U) \tag{13}$$

where U is the uncertain data set (Ben Tal and Nemirovski 2002). The original linear programming problem can be reformulated as:

$$\text{Min} \quad \{x_0 : f_0(x, \zeta) \leq x_0, f_i(x, \zeta) \leq 0, i = 1, \dots, m \; \forall(\zeta \in U)\} \tag{14}$$

The above problem is called the "robust counterpart" of the original. It is a semi-infinite linear problem that appears to be computationally intractable. However, depending on the specific set U, the robust counterpart could be a tractable convex mathematical problem. Typically the robust counterpart is a linear problem or a conic quadratic problem (Ben Tal et al. 2005), which can be solved using linear solvers or interior point methods (Ben Tal and Nemirovski 1998).

Some applications of this methodology can be found in Ben Tal and Nemirovski 1999, 2000, 2002), Ben Tal et al. (2000) and Ben Tal et al. (2004). Ben Tal and Nemirovski (1999) developed an example for portfolio optimization, they explain the methodology, and moreover they compare their results with those obtained with the methodology proposed by Mulvey et al. (1995). Also, Ben Tal and Nemirovski (2000) used the methodology to construct robust solutions for the NETLIB collection of problems; they demonstrate that for many of those problems robust solutions lost little in optimality. In Ben Tal et al. (2000), they present a portfolio problem example and the results were compared with those obtained by applying a linear programming model which replaces the uncertain data with their expected value, and with stochastic programming.

A different form of the approach was developed by Bertsimas and Sim (2003, 2004) based in the same idea. They propose a construction that preserves the linear structure of the problem, making it very attractive due to its applicability.

The authors consider a linear problem of the form:

$$\begin{aligned} &\text{Max } c^T x \\ &\text{s.t.} \quad Ax \leq b \\ &\qquad l \leq x \leq u \end{aligned} \tag{15}$$

The uncertainty affects the coefficients a_{ij} of the matrix A. Each entry a_{ij} can be modeled as a symmetric and bounded random variable $\tilde{a}_{ij}$ that takes values in $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$. Moreover, let J_i represent the set of coefficients a_{ij}, $j \in J_i$, which are subject to uncertainty. For every i we introduce a parameter Γ_i. This parameter indicates the protection level in the model. With this assumption, Bertsimas and Sim (2004) propose the following problem which is the robust counterpart of the original model:

$$
\begin{aligned}
&\text{Max } c^T x \\
&\qquad \sum_j a_{ij} x_j + z_i \Gamma_i + \sum_{j \in J_i} p_{ij} \le b_i \quad \forall i \\
&\qquad z_i + p_{ij} \ge \widehat{a}_{ij} y_j \quad \forall i, j \in J_i \\
&\qquad -y_j \le x_j \le y_j \quad \forall j \\
&\qquad l_j \le x_j \le u_j \quad \forall j \\
&\qquad p_{ij} \ge 0 \quad \forall i, j \in J_i \\
&\qquad y_j \ge 0 \quad \forall j \\
&\qquad z_i \ge 0 \quad \forall i
\end{aligned}
\tag{16}
$$

The original linear structure is preserved, and it is possible to control the degree of conservatism of the solution. Even more, this approach can be extended to discrete mathematical programming problems with no major modifications. Several applications have been developed, for instance, Bohle et al. (2010) use this approach in a wine grape harvesting scheduling problem and in Palma and Nelson (2009) a forest harvest scheduling model is solved, in both cases the results show that robust optimization is a useful methodology to deal with data variability.

5 Results

5.1 Data instance

We present now some computational testing based on a prototype problem instance corresponding to the typical situation of a Chilean forest company. The structure consist of a company which manages 3 sawmills. Each of them is capable of processing 6 different types of logs with 3 different cutting patterns. There are a total of 7 types of boards that are ready to be sold to the customers. The time horizon in which the planning takes pace is a year, divided in monthly periods.

The parameter in the model which is considered to be subject to uncertainty is the sawmill cutting pattern yield, (R_{jekm}). This parameter represents the yield of logs type k when using cutting pattern e in sawmills j, for producing boards of type m (m^3 boards/m^3 logs). This parameter affects sawmill production as it indicate the rate of conversion between logs and boards for each sawmill, and variations in yield will impact total production.

5.2 Deterministic problem

The sawmill planning model described in the previous sections was solved using AMPL/CPLEX. The size of the model considerer 1,296 decisions variables and 1,848 constraints. The results indicate that the optimal value for the deterministic model is 3,252,270 US$.

Table 1 A type of log demanding by sawmills

Sawmill	Log type					
	L1	L2	L3	L4	L5	L6
1				⊗	⊗	⊗
2	⊗		⊗			
3	⊗					

Table 2 Logs used to produce boards

Log	Board type						
	B1	B2	B3	B4	B5	B6	B7
L1	⊗		⊗		⊗		⊗
L2	⊗		⊗		⊗		⊗
L3	⊗	⊗	⊗	⊗	⊗	⊗	
L4	⊗	⊗	⊗	⊗	⊗	⊗	
L5	⊗	⊗	⊗		⊗		
L6	⊗	⊗	⊗		⊗		⊗

Table 3 Boards produce by each sawmill

Sawmill	Board type						
	B1	B2	B3	B4	B5	B6	B7
1	⊗	⊗	⊗	⊗	⊗	⊗	
2	⊗	⊗	⊗	⊗	⊗	⊗	⊗
3	⊗		⊗		⊗		⊗

The main decisions variables considered are the volume of logs type k processed with cutting pattern e in sawmill j at period t, (s_{jekt}). With this decision variable, using (3) in the model, it is possible to determine sawmill production. This also allows calculating the volume of logs type k demanded by sawmill j at period t (y_{jkt}). Table 1 indicates the type of logs demanded at each sawmill. The table indicates with a mark, for each sawmill, which types of logs are actually used in the optimal solution.

As we can see, not all sawmills use the same logs, there seems to be a "specialization" based on productivity. Moreover, type L2 is unnecessary for all sawmills. The reason for this is because L2 only participates in the production of 4 types of boards with lower yields, while the others logs can produce more types of boards with better yields, as indicated by Table 2, which shows which board types can be obtained from the different log types.

Table 3 shows the types of boards produce by each sawmill, where this is computed using (3) in an arbitrary period.

More details about variations in the optimal value and in the structure of the solution will be analyzed in the following sections.

5.3 Robust problem

To formulate the robust counterpart of the problem, we first replaced the variables r_{jmt} by its definition corresponding to (3). This substitution alters the objective function and several of the constraints but eliminates the equality constraint leaving the uncertain parameter in the demand constraint, which is an inequality. However, we did not considered the objective function to be affected by the yield uncertainty in the construction of the robust counterpart. This is justified on the basis that we are interested on determining robust solutions, so we focus on constraint satisfaction and subordinate the objective function to this.

The main constraint modified with the replacement is the demand constraint (relation (9)). After transformation and applying the robust specification described earlier, the demand constraint is replaced by the following inequalities:

$$\begin{aligned}
&\sum_j \left(w_{jmt-1} + \sum_e \sum_k R_{jekm} \cdot s_{jekt} + v_{jmt} - w_{jmt} \right) - \alpha_{mt} \Gamma_{mt} \\
&\quad - \sum_j \sum_e \sum_k \rho_{jekm} \geq B_{mt} \quad \forall m, t \\
&\alpha_{mt} + \rho_{jekm} \geq \widehat{a}_{jekm} \beta_{jekt} \quad \forall j, m, e, k, t \\
&-\beta_{jekt} \leq s_{jekt} \leq \beta_{jekt} \quad \forall j, e, k, t \\
&\beta_{jekt}, \alpha_{mt}, \rho_{jekm} \geq 0 \quad \forall j, m, e, k, t
\end{aligned} \tag{17}$$

Here, β_{jekt}, α_{mt}, ρ_{jekm} are new variables introduced by the robust reformulation, following Bertsimas and Sim approach. The Γ_{mt} parameter represents the grade of robustness to each constraint. In our study, we used Γ in all constraints.

Uncertainty was modeled assuming that the yield parameter could vary within a maximum of 5%, 10%, 15% and 20% of the center value. Besides, we also considered variation in the total "budget of uncertainty" which is represented by the parameter Γ, which takes values between 0 (nominal or deterministic case) and 104 (worst case) in the specific instance we consider. Under these assumptions, we analyze the behavior of the robust counterpart in comparison with the results of the deterministic model. In the next subsections we present the results related with how the uncertainty affects optimality, feasibility and structure of the solution. The robust counterpart considers 3,972 decisions variables and 13,776 constraints. This is larger than the nominal problem, but it is still a linear program and solves very quickly. Larger instances will, of course, generate much larger linear programs.

5.3.1 Optimality

We computed the difference between the optimal value of the deterministic model and the robust model. Figure 2 shows the optimal values in each case for the different uncertainty scenarios of relative variation of the yield coefficient and as a function of the uncertainty budget Γ.

As expected, the graph shows that the reduction in the objective function value is higher in the cases with high variability. Moreover, if all yield parameters reach their worst case value simultaneously, we are precisely in the solution provided by Soyster's approach. Table 4 shows the optimal value achieved with the maximum value of the uncertainty budget (Γ equal to 104) for each variability range in the yield parameters. Another important conclusion is that the worst case is not very different with respect to the optimal value of the robust model when Γ is around 60. This means that the worst case is achieved with an intermediate range of variability budget.

5.3.2 Feasibility

An important issue is the analysis of the computed solutions in terms of a posteriori analysis of feasibility. Robust solutions are 100% guaranteed feasible only when Γ achieves its

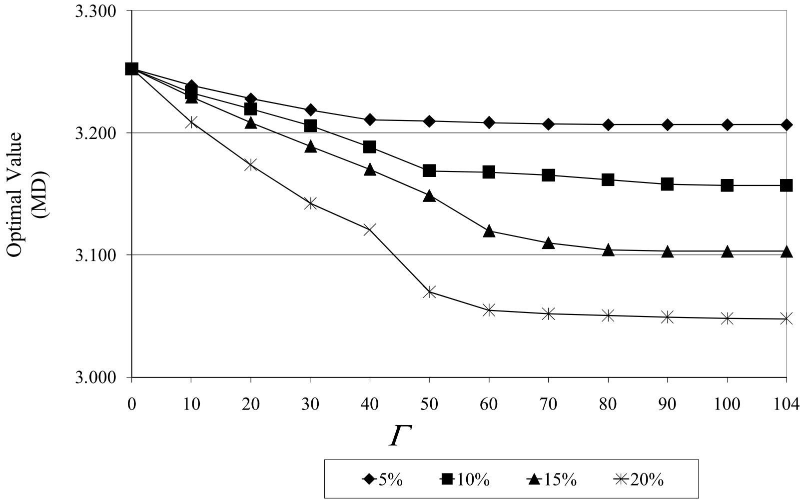

Fig. 2 Optimal values

Table 4 Worst case

Variability (%)	Optimal value (US$)	Reduction (%)
0	3,252,270	–
5	3,206,530	1.41
10	3,156,790	2.94
15	3,102,940	4.59
20	3,047,810	6.29

maximum value of 104. In any intermediate case, there is still the possibility that for some scenarios the solution might be infeasible. We have evaluated this by means of Monte-Carlo simulation and we observed the behavior of the solutions for different values of Γ and different variability. For the simulation, we consider 600 scenarios for the parameter subject to uncertainty, R_{jekm}. 300 of those scenarios were generated in accordance with a uniform distribution within the interval of variation, and for each of the cases of 5%, 10%, 15% and 20%. The other 300 corresponds to scenarios out of a normal distribution defined in such a way that 95% of the probability lays within the parameter interval.

Figures 3 and 4 show graphs with the results of the simulation for both distributions. The numbers indicate the % of scenarios in which the computed robust solution in each case is feasible. Feasibility is considered as the full satisfaction of the constraints.

The graphs in the figure are very useful when combined with the graphs in Fig. 2. We can use them to determine which will be an acceptable value of Γ for computing robust solutions. The experience of the decision maker and his or her degree of risk aversion will affect the selection of an specific value for Γ, but for both distributional cases we can see that if $\Gamma = 50$, the estimated probability of infeasibilities is less than 65%, but in the order of 90% for less variable parameters. In any case, the loss of optimality is never more that 7%. With a higher value of Γ, the estimated feasibility of the computed solutions approaches 100%, specially for the normal case, as would be expected given that the normal distribution is more concentrated around the mean. The conclusion is that a very acceptable solution

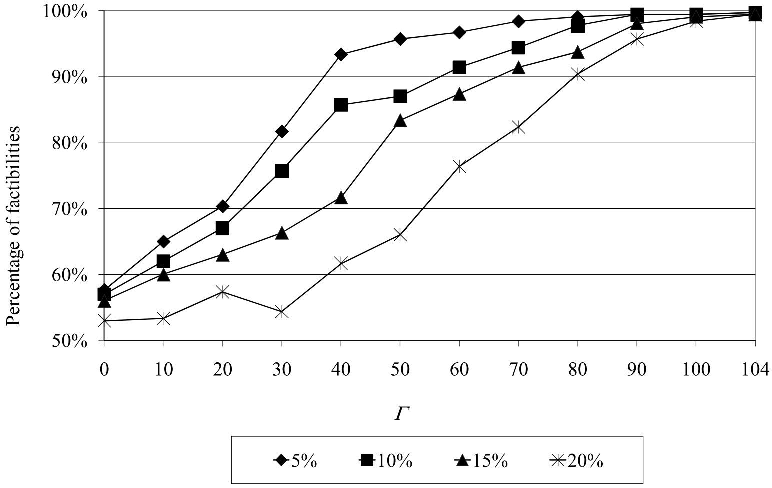

Fig. 3 Simulation results under uniform distribution

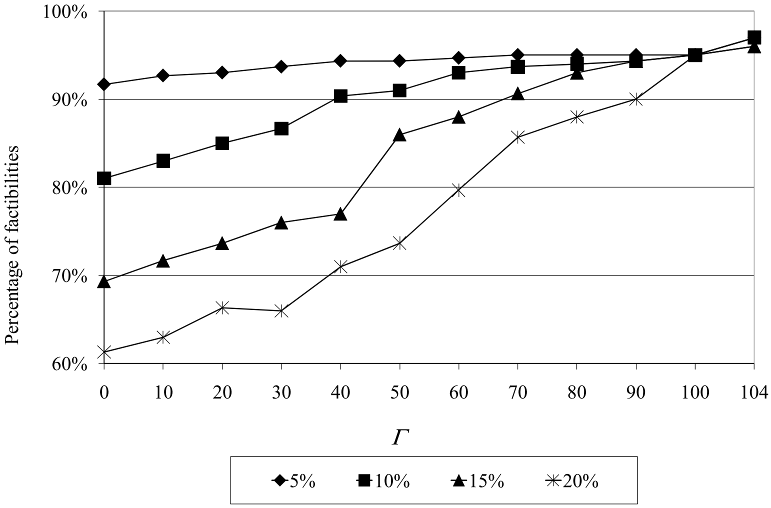

Fig. 4 Simulation results under normal distribution

can be obtained already with a value of Γ in the middle range, without being excessively conservative.

In addition, it is important to quantify in a more precise way the percentage of demand which is not actually satisfied when the solution is infeasible for one scenario. Table 5 shows, for three values of Γ, the percentage of infeasibility in the demand constraints. This is calculated adding the demand not satisfied and computing the proportion with respect to the overall total demand. The table shows the results for the scenarios generated using the normal distribution; the results for the uniform distribution are similar.

For high variability and low values of Γ, we can see an important percentage of infeasibility. This result can be expected as a low value of the uncertainty budget provides low

Table 5 Percentage of non-satisfaction of demand for some variation levels

Γ	Variability			
	5%	10%	15%	20%
10	35%	36%	40%	40%
60	17%	20%	20%	23%
100	10%	14%	15%	18%

Table 6 Percentage of more than 5% of non-completion demand using the normal distribution

	Variability			
Γ	5%	10%	15%	20%
10	23%	22%	25%	27%
60	7%	9%	10%	14%
100	2%	3%	5%	8%

protection for the solutions. However, an intermediate range of uncertainty generates robust solutions which perform reasonably well in terms of feasibility. We recall here that the scenarios came from a normal distribution which covers only 95% of the range so, by design, there could be some infeasibility.

A deeper analysis can be obtained if we classify the infeasibilities according to the percentage of demand violation. We can consider the infeasibility as significant if the percentage of demand violation is more than 5% of the total demand. This is considered reasonable in practice as demand stock out below 5% can usually be handled by management at the operational level by procuring boards form other sawmills or accepting a penalty for not fully satisfying demand. Table 6 shows the percentage of scenarios which are feasible within a 5% of the total demand.

The results are consistent with the previous conclusions as an increase in the level of protection generates lower infeasibilities.

5.3.3 Structure of the solution

Another important property to analyze is the structure of the solutions in terms of its robustness. It is not desirable in a management context that the production plan changes too much when uncertainty is considered. Robust solutions are meant to get a stable production plan despite the variability in the data.

The main decision variables mentioned before are y_{jkt} and s_{jekt}, which represent the volume of logs demanded and the volume processed in different sawmills. Table 7 shows, for the variable y_{jkt}, the comparison between the production plan based on the nominal solution (without robustness) and the plan generated using the robust problem considering 10% of variability and $\Gamma = 60$. There is a mark in the combination of logs and sawmill which present positive values.

This behavior is similar as in the others solutions. The decisions variables that take positive values in both scenarios are repeated in at least 51% of the combinations of logs and sawmills in the worst case, being much higher in several intermediate cases.

Of course, the decision variables values changes in the different scenarios analyzed. Sawmills demand more logs and process more volume to fulfill demand, but the specific logs which are demanded to the Forest Division by sawmills are, in general, of the same

Table 7 Comparison between the production plan of the deterministic and the robust model

Sawmill	Type of Log											
	Non-robust solution VOF: 3,252,270						Robust solution VOF: 3,138,840					
	L1	L2	L3	L4	L5	L6	L1	L2	L3	L4	L5	L6
1				⊗	⊗	⊗				⊗	⊗	⊗
2	⊗		⊗				⊗		⊗			
3	⊗						⊗			⊗		

Table 8 Comparison between cutting patterns in the deterministic and in the robust model

	Pattern of cut	Type of Log											
		Non-robust solution VOF: 3,252,270						Robust solution VOF: 3,138,840					
		L1	L2	L3	L4	L5	L6	L1	L2	L3	L4	L5	L6
Sawmill 1	e1					⊗	⊗					⊗	⊗
	e2				⊗						⊗		
	e3												
Sawmill 2	e1	⊗						⊗					
	e2			⊗						⊗			
	e3			⊗						⊗			
Sawmill 3	e1	⊗						⊗			⊗		
	e2												
	e3												

type as in Table 5. More specifically, of the 6 types of logs, 5 are demanded in any uncertainty scenario and for any value of the parameter Γ. This is because logs that have high yield levels can be used to produce more types of boards (more flexibility).

Otherwise, the processed logs volume in sawmills depend strongly of the logs that came from the forest. The interesting thing is that cutting patterns utilized for each log are independent of the uncertainty scenario used. This is shown in Table 8 where we can see that cutting patterns e_1 and e_2 are the most used in the planning problem.

6 Conclusions

In this paper we have shown how Robust Optimization Approach can be used to improve the performance and reliability of the solutions obtained from a linear programming model to support production planning in a sawmill operation. The approach allowed us to consider the natural variability induced by the variation in the yields of the various cutting patterns used in the operations. The main advantage of the approach is that it allows studying the tradeoff between robustness requirements and the lost in optimality. The approach we have followed, the one by Bertsimas and Sim, also preserves the original linear structure of the problem. The main conclusion is that the "Price of Robustness" for this particular problem is not very high as the objective function did not deteriorated in more 7% in every scenario that we considerer

(5%, 10%, 15% and 20% overall variability and considering the most conservative budget of uncertainty). This is, of course, a conclusion based on the particular instance we tested for this problem but it might reflect a property of the model as a whole. The specific instance presents high flexibility in getting boards from the logs. Even more, the solution obtained guarantees certain immunity with respect to the variations on the yields, variations which corresponds approximately to what is observed in practice. It is also interesting to notice that the worst case deterioration is already obtained for values of the budget of uncertainty which are in the middle range, so it does not generates more loses to be overall conservative.

Those conclusions are supported by the simulation results, since the robust solutions computed preserved feasibility in a high percentage. In fact, the percentage of feasibility is always more than 50% and increase quickly when the level of protection increases. When feasibility is measured in terms of a relaxed measure which allows up to a 5% violation of the demand constraints, the previous percentage is more than 70%, increasing quickly. This is important as we can take as practical robust solution one computed, for instance, with an intermediate range of the uncertainty budget, guaranteeing a good performance of the solution with a price of robustness which is never greater than 7% of the overall objective function value.

This suggests that the methodology of Robust Optimization would be a good tool to improve the current deterministic model, in order to allow the consideration of some variability and uncertainty in the model, without a significant computational cost or the use of any other specialized software than the standard linear programming solvers used in many companies. As we have pointed out, this research was motivated by a particular situation in a forest company in Chile which uses a linear programming model to plan sawmill operations. Incorporating robustness in the planning process should not be a difficult task following this approach as the linear structure of the problem is preserved and the same software implementation could continue being used. The use of the model will allow the company to compute robust solutions but, at the same time, asses the sensitivity of their tactical decision model with respect to the uncertainty in the yield parameters.

Another important conclusion is that the principal decisions on the optimal robust solutions do not present significant changes in their structure, i.e., the types of logs demanded by sawmills and the types of board to be produced, when compared with the nominal solution. Moreover, the cutting patterns used tend to be the same in different variability scenarios. This is important as it indicates that the model is relatively robust and can be used with confidence in a real management environment, especially if robust solutions area used. This further reinforces the applicability of this methodology in problems associated to the Chilean Forest Supply Chain.

7 Future challenges

Some of the future challenges that we have identified for this research corresponds to further computational testing of the approach using other data representing different business structures. Also the extension of the model to a rolling horizon setting in which decisions are made over several periods, but incorporating the possibility of modifying decisions as uncertainty disappears when the present time finally comes. In this way, an action plan can be created so that it is adjusted in every period in order to incorporate the actual yields. Thus, only the decision relative to the optimal solution for the first period would be adopted, and then it would be modified according to what occurs in reality. This is, of course, an approximation to introduce some adaptability in the decisions, but does not optimize over time as a stochastic dynamic programming formulation will do. However, using a decision model over a rolling horizon is common in practice.

On the other hand, the uncertain data in our model (yield coefficients) are also present in the objective function as well as in several constraints. Moreover, the same coefficient is repeated in several constraints. This means that there is an implicit relation between the constraints and the objective function. In this paper we ignored that and our solution is probably more conservative than it is really needed. Recall also that the model used in this paper is a tactical one and the parameters are an aggregate of detailed operational data. The question is whether it is actually relevant to consider these relations between the coefficients. Further research in this issue is being conducted and some extensions to the Robust Optimization methodology to handle this situation are being considered.

Finally, is important to evaluate the applicability of this methodology to the whole forest supply chain, integrating solutions that connect different problems in each division (sawmill, forest, etc.).

One final comment regarding the approach of the paper. The model presented in (16) is static, which means that all the decisions must be made before the actual realization of the uncertain data (Chen and Zhang 2009). This assumption might produce solutions that are too conservative and restrictive. Some of the decision, like inventory of products, should adapt to the decisions regarding actual processing. A two stage stochastic problem will allow handling that, but we have preferred in this work to consider the robust model as presented, as it is easier to work with. However, Ben Tal et al. (2004) have developed extensions to the Robust Optimization approach, which considers adaptive decisions. They introduce the concept of Adjustable Robust Counterpart (ARC) to include decisions that must be taken after the realization of the uncertain events. ARC generates less conservative solutions but a price is paid as the resulting problems are computationally intractable (NP-hard). However, some simplifications can be made and the model we present might be tractable in an ARC approach. This will the subject of further research, which will allow to compare the robust adaptive approach with the more traditional two stage stochastic programming approach.

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