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An optimization model and a solution algorithm for the many-to-many car pooling problem

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Abstract Car pooling is one method that can be easily instituted and can help to resolve a variety of problems that continue to plague urban areas, ranging from energy demands and traffic congestion to environmental pollution. Although car pooling is becoming more common, in practice, participant matching results are still being obtained by an inefficient manual approach, which may possibly result in an inferior solution. In the past, when car pooling studies have been done the problem has been treated as either a to-work problem (from different origins to the same destination) or return-from-work problem (from the same origin to different destinations). However, in this study we employ a time-space network flow technique to develop a model for the many-to-many car pooling problem with multiple vehicle types and person types. The model is formulated as an integer multiple commodity network flow problem. Since real problem sizes can be huge, it could be difficult to find optimal solutions within a reasonable period of time. Therefore, we develop a solution algorithm based on Lagrangian relaxation, a subgradient method, and a heuristic for the upper bound solution, to solve the model. To test how well the model and the solution algorithm can be applied to real world, we randomly generated several examples based upon data reported from a past study carried out in northern Taiwan, on which we performed numerical tests. The test results show the effectiveness of the proposed model and solution algorithm.

Keywords Car pooling · Many-to-many · Time-space network · Multiple commodity network flow problem · Lagrangian relaxation

1 Introduction

In the past few years, the number of private automobiles on the roads has grown significantly, gasoline has become more and more expensive, and parking space in urban areas more and

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more difficult to find. Although public transport services are becoming more developed, they are often incapable of effectively servicing non-urban areas where cost-effective transportation systems cannot be set up. It is in this type of situation that carpooling could be an effective solution. We define car pooling as the sharing of a private vehicle by more than one user who needs to reach their destination following a semi-common route, between the points of origin and destination (Ferrari et al. 2003). There are many benefits to be derived from car pooling such as the sharing of driving costs and parking fees. From the environmental perspective, the practice results in less traffic congestion, less pollution, less stress, and fewer accidents. In addition, it should be mentioned that not only does car pooling reduce the traffic volume in urban areas, but it can also make up for the lack of public transportation system in outlying non-urban areas. In Taiwan, there is a particularly serious lack of adequate public transportation system in non-urban areas, plus most of trips in these areas are of necessity longer than those in more urban areas. Therefore, our study is focused on non-urban carpool planning.

Although carpool is a good and old idea, it has not often been applied, at least in Taiwan. There are many reasons for this, the major one being that it is difficult to group people together and find a plan that is mutually satisfactory. Some car pooling organizations have set up web sites for consulting or exchanging information about partners' travel routes, departure/arrival times, departure/arrival location or requests, however, users can only find matches by exchanging data or by simple logic program. This means that the matching results obtained are not very efficient, possibly resulting in an inferior solution.

There have actually been only a few studies of the carpool optimization problem. Ferrari et al. (2003) did design several different automatic and heuristic data processing routines based on savings functions to support efficient matching in carpool schemes. The problem involved locating users at their origin locations who needed to reach the same destination point given temporal and topographical constraints. Baldacci et al. (2004) discussed car pooling as a transportation service organized by a large company which encouraged its employees to pick up colleagues while driving to/from work, to minimize the number of private cars traveling to/from the company site. Their model was formulated as a dial-a-ride problem/pickup and delivery vehicle routing problem. To solve this model they also proposed both an exact and a heuristic method, based on two integer programming formulations. The exact method is based on a bounding procedure that combines three lower bounds derived from different relaxations of the problem. A valid upper bound is obtained by the heuristic method, which transforms the Lagrangian lower bound solution into a feasible solution. Wolfler-Calvo et al. (2004) presented an integrated system for the organization of a car pooling service, using several current information and communication technologies: i.e., the web, GIS and SMS. At the core of the system is an optimization module which solves the daily car pooling problem heuristically. The service is supported by a database of potential users (company employees that commute daily from their home to the workplace).

It should be mentioned that Fagin and Williams (1983) presented a simple carpool scheduling algorithm where driver fairness is considered, in which no penalty is assessed for a carpool member who does not ride on any given day. Naor (2005) and Coppersmith et al. (2005) also solved the same problem with a different algorithm. However, they did not decide on vehicle routes or schedules, which is different from our study.

As mentioned above, car pooling is an activity based on the shared use of private vehicles by participants who need to reach their destination by a semi-common route. In practical terms this most often leads to many-origins-to-many-destinations (henceforth called many-to-many) car pooling problems. For example, a transportation service can be organized by the government designed to encourage commuters to pick up colleagues while driving to/from work, especially when the group company has many branches or sub-companies.

This car pooling problem can be generalized as a many-to-many problem, which includes the many-origins-to-one-destination (henceforth called many-to-one) and one-origin-to-many-destinations (henceforth called one-to-many) car pooling problems.

There has been some research on the "dial-a-ride" problem, which is closely related to the car pooling problem. The dial-a-ride problem consists of designing vehicle routes and schedules for n users who specify pickup and delivery requests between origins and destinations. In the standard version, transport is supplied by a fleet of m identical vehicles from the same depot (Cordeau and Laporte 2007). A number of studies have been carried out dealing with these types of problems. For example, Teodorovic and Radivojevic (2000), Colorni and Righini (2001), Cordeau and Laporte (2003a, 2003b, 2007), Diana and Dessouky (2004), Rekiek et al. (2005), Xiang et al. (2006), Wong and Bell (2006), Cordeau (2006), Coslovich et al. (2006), Ropke et al. (2007), Melachrinoudis et al. (2007), Wolfler-Calvo and Colorni (2007), and Jørgensen et al. (2007). The major difference between the dial-a-ride problem and our problem is the vehicle ownership. In the former type of problem, it is assumed that there are full-time drivers that service all of the passengers. In our problem, the vehicles belong to the participants themselves, who share the use of their cars and take turns as drivers to pick up the others. In other words, the drivers are also subject to a transportation demand.

It is also found that individuals were matched in past car pooling studies. However, in reality, groups may also need to be matched in the car pooling problem. Each group may contain more than one individual. In addition, in most past studies (whether on car pooling or dial-a-ride problems) only a single vehicle type and a single person type have been considered. However, we consider multiple vehicle types and person types so as to conform more closely to reality, although this does make our problem more complicated and difficult to solve. As a result, it is difficult to apply past approaches to our problem, which is a many-to-many car pooling problem with multiple vehicle types and person types.

The many-to-many car pooling problem, with its multiple vehicle and person types, involves complicated analysis among numerous time-window and space constraints which are highly correlated to each other, coupled with many choice constraints between vehicle and person types. It is difficult to use the traditional integer programming techniques (e.g., traditional vehicle routing models) to formulate and efficiently solve this type of problem. On the other hand, the time-space network method has been popularly employed to solve conveyance scheduling/routing problems, because it provides a natural and efficient way to represent multiple conveyance routings with multiple ODs (origins-destinations) in the dimensions of time and space. Although the resulting model scale is generally enlarged due to extension in the dimension of time, complicated time-related constraints can normally be easily modeled for realistic problems, particularly in comparison with the space network models. Coupled with the development of efficient algorithms, the time-space models (usually formulated as multiple commodity network flow problems or network flow problems with side constraints) can be effectively and efficiently solved. For examples, see Yan et al. (2006b) and Yan and Shih (2009). Based on these characteristics, the time-space network technique could be a suitable way for solving many-to-many car pooling problems, although, to the best of our knowledge, there has not yet been any model formulated using this technique to solve this type of problem. Therefore, in this study we employ the time-space network flow technique to develop a model designed to help the authorities solve the many-to-many car pooling problem with multiple vehicle types and person types. Certainly, the development of other models using other methods for solving this type of problem and comparison with our model could be a direction of future research.

The model is formulated as an integer multiple commodity network flow problem that is characterized as NP-hard (Garey and Johnson 1979). Since real problem sizes are expected to be huge, the model is difficult to optimally solve within a reasonable time. To efficiently solve the model for practical-sized problems, we develop a solution algorithm based on Lagrangian relaxation, a sub-gradient method and a heuristic for the upper bound solution. The development of this model, together with the Lagrangian relaxation-based algorithm, is the focus of this study. The model and the solution algorithm are anticipated to function as an effective planning tool. The rest of this paper is organized as follows: The problem is outlined in Sect. 2, the model is introduced in Sect. 3, the development of the solution algorithm is elucidated in Sect. 4, and numerical tests are performed in Sect. 5. Finally, we conclude in Sect. 6.

2 Problem description

This study is aimed at developing a car pooling model and a solution algorithm suitable for practical applications. It is very complicated to simultaneously and optimally determine every participant's role (driver or passenger), driver schedules, and passenger deliveries, as well as to match several participants in a car. This involves complicated movements of drivers (or vehicles) and passengers in terms of time and space. The objective is to minimize the sum of the route cost, time cost and the penalty for unserved carpool members, while satisfying the participant related constraints. To simplify the description of this complicated problem, we first define the terminologies used.

Carpool member group: A carpool member can request to share the same trip with his/her friends, in which case they are treated as a group. Each carpool member group (CG) is associated with a data set. The number of persons in a CG can be greater than one, but cannot exceed the vehicle capacity (or a planning capacity, less than the true vehicle capacity, which might be set by the authorities due to operational considerations). CGs can be further divided into CVGs and CNGs, which are explained below.

CNG: A CNG is a CG which cannot provide a vehicle. A CNG will be assigned to act as passengers or determined not to be serviced.

CVG: A CVG is a CG which can provide a vehicle with a driver. A CVG could be assigned to act as a driver with passengers or have only passengers (similar to a CNG) utilizing the optimization process to be described later. In order to discriminate between a CNG and a pure passenger CVG, a new term "CVPG" is used to represent a CVG when it is assigned to act as passengers.

CNG characteristic: CNGs can be divided into four types according to sex and smoking status: non-smoking female, smoking female, non-smoking male and smoking male. If there is more than one member in a CNG and their characteristics are different, they should be identified by one characteristic. It should be mentioned that in Taiwan non-smokers and females are disadvantaged in car pooling. In order to protect them, if there are females and males in a CNG, then the CNG characteristic is regarded as male; if there are non-smokers and smokers in a CNG, then the CNG characteristic is regarded as a smoker.

CNG request: This indicates the CNG characteristic of partners requested by a CNG to participate in the carpool plan. CNG requests can be divided into nine different types: (1) riding with non-smoking females; (2) riding with smoking females; (3) riding with females (do not care smoking); (4) riding with non-smoking males; (5) riding with smoking males; (6) riding with males (do not care about smoking); (7) riding with non-smokers (do not care about sex); (8) riding with smokers (do not care about sex); (9) non-requester (no preference). However, the categories can be simplified (according to Taiwan customs) by removing requests for riding with males or smokers only, because these kinds of CNGs

Table 1 CNG characteristic and request analysis

Characteristics	Request			
	Non-smoking female	Female	Non-smoking	No request
Non-smoking female	(1)	(2)	(3)	(4)
Smoking female	–	(5)	–	(6)
Non-smoking male	–	–	(7)	(8)
Smoking male	–	–	–	(9)

Note: "–" means the corresponding CNG type is illegitimate

can be matched with non-requester CNGs, after excluding requests for riding with females or non-smokers. As a result, CNG requests can be compressed into four types: (1) riding with non-smoking females; (2) riding with females; (3) riding with non-smokers; (4) non-requesters. Note that passengers within a CNG can make only one request, which should comply with the protection of non-smokers and females only, as mentioned above.

CNG type: This leaves us with four characteristics and four request types. After removing illegitimate requests (such as a smoker requesting to ride with non-smokers, or males request to ride with females only), we are left with nine different CNG types: (1) non-smoking females who request to ride with non-smoking females; (2) non-smoking females who request to ride with only females; (3) non-smoking females who request to ride with non-smokers; (4) non-smoking females who have no request; (5) smoking females who request to ride with females; (6) smoking females who have no request; (7) non-smoking males who request to ride with non-smokers; (8) non-smoking males who have no request; (9) smoking males who have no request; see Table 1.

CVG/Vehicle type: Based on the four CNG requests, CVGs/vehicles (which are provided by the CVGs) can be divided into four types: "non-smoking female"; "female"; "non-smoking" and "general." Note that if the members in a CVG agree to allow smoking, then this CVG cannot provide a "non-smoking female" or "non-smoking" type of vehicle. Similarly, if a CVG contains male members, then it cannot provide the "non-smoking female" or "female" type of vehicle.

CNG/CVG types and vehicle types are shown in Fig. 1. The non-smoking female vehicles can service CNGs of types (1), (2), (3), (4), and CVGs of type (1); the female vehicles can service CNGs of types (2), (4), (5), (6), and CVGs of type (2); the female vehicles can service CNGs of types (3), (4), (7), (8), and CVGs of type (3); and the general vehicles can service CNGs of types (4), (6), (8), (9), and CVGs of type (4).

In this study, the data set provided for each CG must be known beforehand including:

- The CG's origin and destination.
- Time window: Time window is composed of a CG's earliest departure time and its last arrival time. The length of the time window indicates the longest travel time that can be tolerated by the CG, which can influence the matching success and efficiency. If the length of time window is too short, then matching is difficult. To better understand the influence of the length of time window, we perform a sensitivity test in Sect. 5.
- The number of people: The number of members requesting the same trip in the CG.
- Vehicle or no vehicle: If a vehicle can be provided, then it will be labeled a CVG; otherwise, it will be labeled a CNG.
- Vehicle capacity and vehicle type: These are provided by the CVG.
- CNG type.

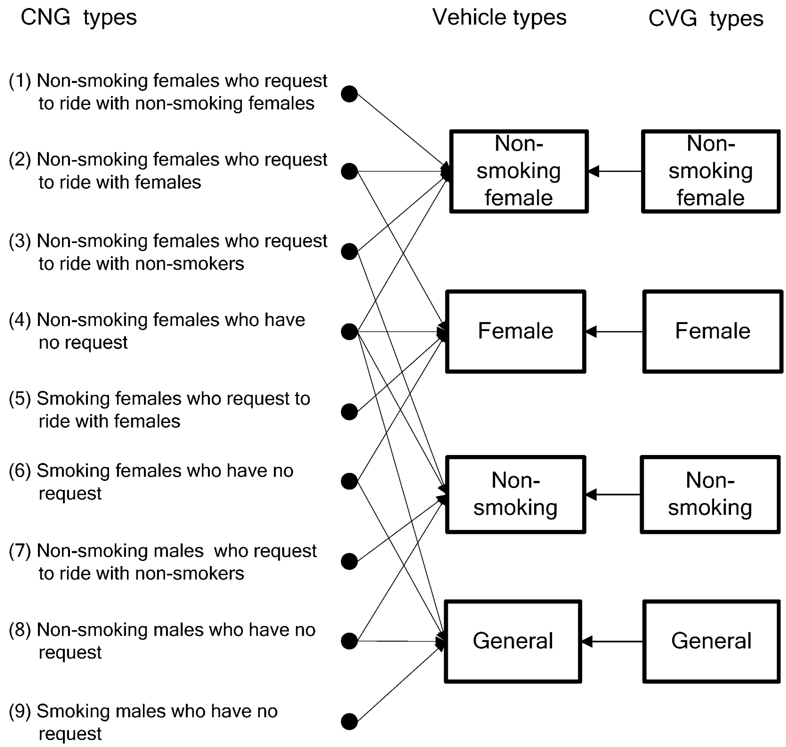


Fig. 1 Relations between CNG/CVG types and vehicle types

In conclusion, the many-to-many car pooling problem involves selecting the CVGs to drive and to be carried (i.e., as CVPGs), matching all CNGs and CVGs and simultaneously identifying the routes of all CVGs and CNGs. The goal is to minimize the service costs, subject to the CG (whether CNG or CVG) time window, CNG type, vehicle type, CVPG selection, and vehicle capacity constraints. There is a penalty for unserved CNGs. Note that the objective function is adjustable and can be modified. Since most vehicles for carpool actives in Taiwan are four-passenger vehicles, the vehicle capacity is assumed to be five (including the driver and four passengers) in this study for simplicity. The car pooling model can be extended to include vehicles with different capacity in future.

3 The model

A time-space network technique is applied to construct a car pooling model for the purpose of minimizing the system cost. This model demands the optimal management of CVG and CNG movements/matching within the network. The major elements in the modeling include the CVG vehicle-flow time-space networks, the CVG passenger-flow time-space networks, the CNG passenger-flow time-space networks, other relational constraints, and the mathematical formulation. Note that a CVG could be assigned to act as a driver with passengers or as pure passengers (a CVPG). The CVG vehicle-flow time-space networks and the CVG passenger-flow time-space networks are used to formulate the potential movements of the two types of CVGs.

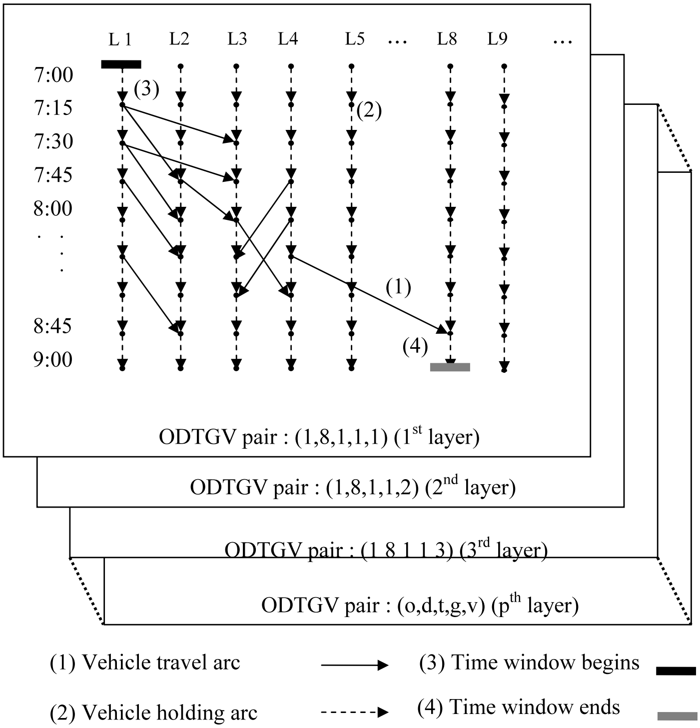


Fig. 2 CVG vehicle-flow time-space networks

3.1 The CVG vehicle-flow time-space networks

As shown in Fig. 2 each CVG vehicle-flow time-space network represents potential movements for CVGs and vehicle within a certain time period between certain locations. A layer in the CVG vehicle-flow time-space networks is associated with a specific ODTGV pair (origin, destination, time-window, group, and vehicle type), as shown in Fig. 2. In other words, the number of network layers is equal to the number of ODTGV pairs. Note that our problem allows a carpool member to share the same trip with his/her friends, which are then treated as a group. To avoid mixed flows of groups with different numbers of members, different networks are built for groups with a different numbers of members. In other words, groups are classified into one-member groups, two-member groups, etc. For example, in Fig. 2, the ODTGV pair of (1, 8, 1, 1, 2) in the second layer of network indicates the ODTGV pair of one-member CVGs providing type 2 vehicles, form L1 to L8, within the type 1 time window. The horizontal axis represents the location; the vertical axis the time duration. “Nodes” and “arcs” are the two major components in the network. A node stands for a location at a specific time, while an arc designates an activity for the vehicles. The arc flows express the flow of CVGs (each being associated with a vehicle) in the network. The length of the network is set according to the length of the time window specified for each CVG. The starting time for the time window (the network begins) is the earliest time at which the CVG leaves the original location; the ending time for the time window (the network ends) is the last time at which the CVG can arrive at the destination. The length of each CVG vehicle-flow network may be different. In addition, the node of origin, when the network begins, is a supply node.

The destination node, when the network ends, is a demand node. The flow unit for the CVG vehicle-flow network is a "group." The node supply/demand denotes the amount of associated CVGs that will flow in the network. Since a CVG may be assigned as a CVPG, the node supply/demand is set as a variable. This will be addressed in more detail in the CVG passenger-flow time-space networks. There are two types of arcs.

(1) Vehicle travel arc

A vehicle travel arc represents a CVG trip (associated with a vehicle) from one time-space point (location and time) to another. In practice, after CVGs have departed from their origin point, they do not return to the origin to service other CGs. Similarly, when CVGs have arrived in their destination, they do not depart to service other CGs. Therefore, at each time-space point associated with the CVGs' origin, we need only build a travel arc from their origin to the other zones. Similarly, at each time-space point associated with their destination, we need only build a travel arc from the other zones to the destination. All possible vehicle travel arcs within a reasonable block of time are installed into the network. Each arc contains information about the departure time, the departure location, the arrival time, the arrival location, and the travel cost of a CVG. The time block for a vehicle travel arc is calculated as from the time when a vehicle is prepared for travel to the time when this travel is finished. The arc cost is the vehicle operating cost plus a time cost for the members of a CVG to travel between locations. The arc flow's upper bound is infinity. The arc flow's lower bound is zero, implying that no CVG travels on this arc.

(2) Vehicle holding arc

A vehicle holding arc represents the holding of CVGs (each being associated with a vehicle) at a location in a time period. The arc cost denotes the time cost incurred for holding a CVG at this location, except the origin/destination location, in the corresponding time period. Note that CVGs are not really held at their origin/destination, since they can be notified in advance. Therefore, the vehicle holding arc cost associated with the CVGs' origin/destination location is set to zero. Naturally, the arc costs can be modified to match the planner's needs. The arc flow's upper bound is infinity. The arc flow's lower bound is zero, implying that no CVG is held on this arc.

3.2 CVG passenger-flow time-space networks

Each CVG passenger-flow time-space network represents the potential movement of CVPGs, when the associated CVGs are assigned to be pure passenger CVGs, within a certain time period and at certain locations, as shown in Fig. 3. To facilitate problem solving, these networks are designed to be symmetrical to the CVG vehicle-flow time-space networks; the number of network layers and network lengths are the same as that of the CVG vehicle-flow networks. In addition, side constraints should be used to ensure that each CVG only can flow in only one network between the CVG vehicle-flow and the associated CVG passenger-flow networks. The horizontal and vertical axes are the same as those described in the CVG vehicle-flow networks. The node of origin, where the network begins, is a supply node. The node of destination, where the network ends, is a demand node. The flow unit for the CVG passenger-flow network is a "group." Since a CVG could be assigned to having a car or a CVPG during the optimization process, the supply/demand in each CVG vehicle-flow time-space network is set as a supply/demand variable. Hence, the node supply/demand in each CVG passenger-flow time-space network is set to be the amount of CVGs associated with the corresponding ODTGV pair, minus the node supply/demand variable of the corresponding CVG vehicle-flow network. There are two types of arcs.

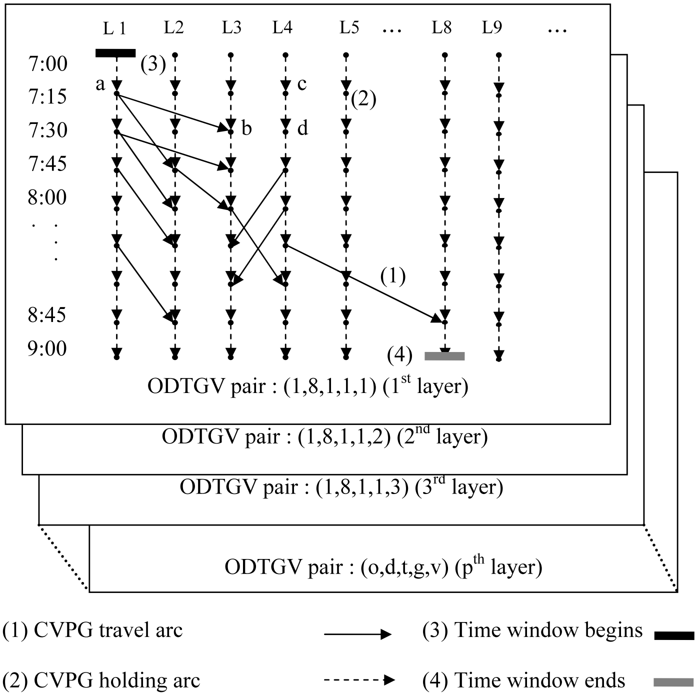


Fig. 3 CVG passenger-flow time-space networks

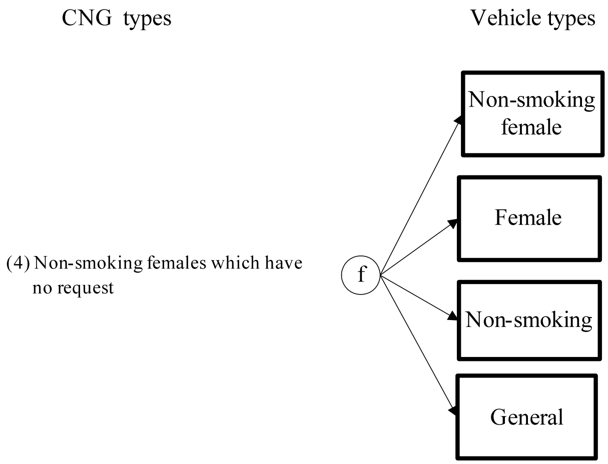
(1) CVPG travel arc
A CVPG travel arc represents the transport of CVPGs from one time-space point (location and time) to another; for example, if arc (a, b) has 2 units of flow, it means that 2 CVPGs are transported from L1 at 7:15 to L3 at 7:30. The transportation time is the same as the corresponding time block for the associated CVG trip in the corresponding CVG vehicle-flow network. The arc cost is a variable time cost for transporting a CVPG between locations. The arc flow's upper bound is infinity. The arc flow's lower bound is zero.

(2) CVPG holding arc
A CVPG holding arc indicates the holding of CVPGs at some location in some time period; for example, if arc (c, d) has 1 unit of flow, it means that a CVPG is held at L4 from 7:15 to 7:30. The arc cost is the time cost incurred for holding a CVPG at this location in the corresponding time period. Similar to the design of holding arcs in the CVG vehicle-flow networks, CVPGs do not need to arrive before their departure time or stay after their arrival time. Therefore, the CVPG holding arc cost associated with the corresponding origin and destination locations is set to zero. The arc flow's upper bound is infinity. The arc flow's lower bound is set to be zero.

3.3 CNG passenger-flow time-space networks

To facilitate problem solving, these networks are designed to be similar to the CVG vehicle-flow time-space networks; see Fig. 5. The horizontal and vertical axes are the same as those

Fig. 4 Example of a CNG passenger-flow time-space network



in the CVG vehicle-flow network. However, each CNG passenger-flow time-space network associated with a specific ODTGV pair represents a potential movement of CNG*s* within a certain time period and certain locations. Unlike a CVG which is associated with a given vehicle type, a CNG associated with an ODTG pair (origin, destination, time-window, and group) could be assigned to different vehicle types. Thus, a CNG with an ODTG pair could be built associated with different vehicle type networks. For example, for a non-requesting non-smoking female CNG we build a four-layer CNG passenger-flow time-space network, to differentiate whether they ride in the non-smoking female vehicle, female vehicle, non-smoking vehicle, or the general vehicle; see Fig. 4.

The networks for each CNG are designed similarly to those in the CVG vehicle/passenger-flow time-space networks. To illustrate the node supply/demand specified in each network for a CNG we can use Fig. 4 as an example. First assume that the number of non-requesting non-smoking female CNGs is f (a constant) and the node supply/demand in the CNG passenger-flow network associated with the non-smoking female vehicle is set to be γ (a variable); associated with the female vehicle it is set to be ε (a variable); and associated with the non-smoking vehicle it is set to be ϕ (a variable). Since the sum of CNGs assigned to all of the four vehicle types is f, the supply/demand in the CNG passenger-flow network associated with the general vehicle is then set to be $f-\gamma-\varepsilon-\phi$. There are three types of arcs.

(1) CNG travel arc

A CNG travel arc represents the transport of CNGs from one time-space point (location and time) to another. The transportation time is the same as the corresponding time block for the associated CVG trip in the CVG vehicle-flow time-space network. The arc cost is a variable time cost for transporting a CNG between locations. The arc flow's upper bound is infinity. The arc flow's lower bound is zero.

(2) CNG holding arc

A CNG holding arc indicates the holding of CNGs at some location in some time period. The arc cost is the time cost incurred for holding a CNG at this location in the corresponding time period. Similar to the design of holding arcs in the CVG passenger-flow networks, CNGs do not need to arrive early before their departure time or stay after their arrival time. Therefore, the CNG holding arc cost is set to zero associated

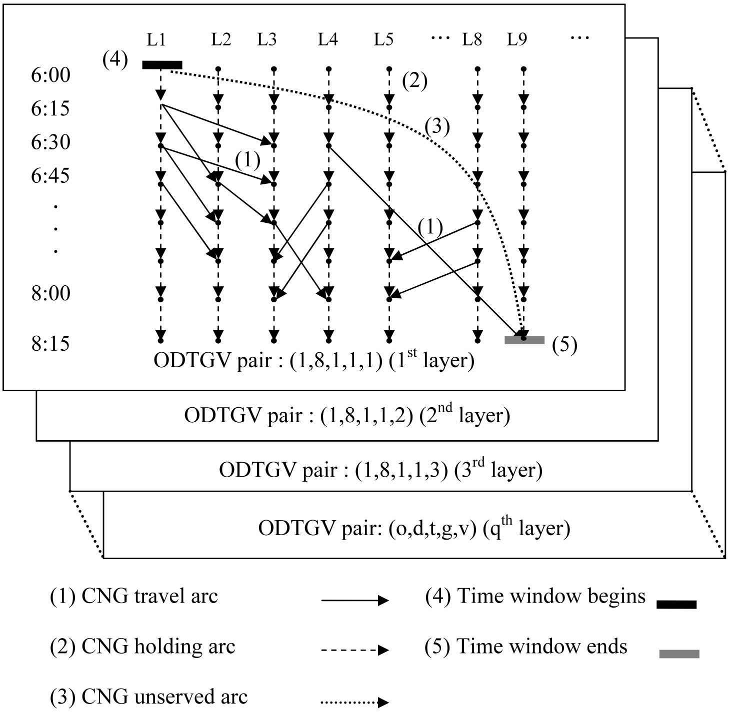


Fig. 5 CNG passenger-flow time-space networks

with their corresponding origin and destination locations. The arc flow's upper bound is infinity. The arc flow's lower bound is set to be zero.

(3) CNG unserved arc

A CNG unserved arc indicates CNGs which cannot be serviced for this ODTGV pair. A CNG unserved arc connects the origin supply node to the destination demand node. The arc cost is set as a large penalty per person multiplied by the number of people in the CNG. The arc flow's upper bound is infinity and the arc flow's lower bound is zero, indicating that all CNGs from the corresponding ODTGV pair are serviced.

3.4 Other relational constraints

In addition to the flow conservation constraints for all nodes in the above three types of networks, there are several relational constraints between/among different networks that need to be considered in the modeling.

(1) Supply/demand constraints between the CVG vehicle and passenger-flow networks

The sum of supply/demand in the paired CVG vehicle and passenger-flow networks must be equal to the number of CVGs participating in the carpool plan, for each ODTGV pair.

(2) Supply/demand constraints for the CNG passenger-flow networks

The sum of supply/demand among the CNG passenger-flow networks associated with all possible vehicle types must be equal to the number of CNGs participating in the carpool plan, for each ODTG pair.

(3) Vehicle capacity constraints

The sum of all members, with respect to all associated CVGs, CVPGs and CNGs, in each vehicle must not exceed the vehicle capacity.

3.5 The model formulation

Given the CVG vehicle-flow, CVG passenger-flow, and the CNG passenger-flow time-space networks, as well as the relational constraints, the model can now be formulated as an integer program. The objective is to simultaneously "flow" the CVGs and the CNGs in all networks, at a minimum cost. We now list the notations and symbols used in the model formulation:
Sets:

VA^m/VN^m: the set of all arcs/nodes in the mth CVG vehicle-flow network, which is associated with a ODTGV pair;
VPA^m/VPN^m: the set of all arcs/nodes in the mth CVG passenger-flow network, which is associated with the mth CVG vehicle-flow network;
PA^{kn}/PN^{kn}: the set of all arcs/nodes in the (k, n)th (namely the kth ODTG pair of CNG rides the nth type of vehicle) CNG passenger-flow network; since the CNGs in an ODTG pair can choose different vehicle types, for ease of model formulation, we use two indexes (k and n) to represent a ODTGV pair.
M: the set of all CVG vehicle-flow networks;
H: the set of all CVG passenger-flow networks;
K: the set of all CNG passenger-flow networks;
$ODTG$: the set of all ODTG pairs (origin, destination, time-Window, and group);
VS^m: the set of all supply nodes in the mth CVG vehicle-flow network;
VD^m: the set of all demand nodes in the mth CVG vehicle-flow network;
VT^m: the set of all transfer nodes in the mth CVG vehicle-flow network;
VPS^m: the set of all supply nodes in the mth CVG passenger-flow network;
VPD^m: the set of all demand nodes in the mth CVG passenger-flow network;
VPT^m: the set of all transfer nodes in the mth CVG passenger-flow network;
PS^{kn}: the set of all supply nodes in the (k, n)th CNG passenger-flow network;
PD^{kn}: the set of all demand nodes in the (k, n)th CNG passenger-flow network;
PT^{kn}: the set of all transfer nodes in the (k, n)th CNG passenger-flow network;
FNV: the set of all CNG passenger-flow networks which contain the CNGs that can ride the "female and non-smoking" type of vehicle;
FV: the set of all CNG passenger-flow networks which contain the CNGs that can ride the "female" type of vehicle;
NV: the set of all CNG passenger-flow networks which contain the CNGs that can ride the "non-smoking" type of vehicle;
GV: the set of all CNG passenger-flow networks which contain the CNGs that can ride the "general" type of vehicle;
VTA: the set of all routes between every two-location pair;
QV: the set of all vehicle types;
PCH^n: the set of all CNG passenger-flow networks which contain the CNGs that can ride the nth type of vehicle;
VCH^n: the set of all CVG vehicle-flow networks which contain the nth vehicle type.

Parameters:

v^m: the number of CVGs supplied to the mth CVG vehicle-flow network;
c_{ij}^m: the arc(i, j) cost in the mth CVG vehicle-flow network;

t_{ij}^{m}: the arc(i, j) cost in the mth CVG passenger-flow network;
t_{ij}^{kn}: the arc(i, j) cost in the (k, n)th CNG passenger-flow network;
a^k: the number of non-smoking female CNGs which request to ride with non-smoking females for the kth ODTG pair;
b^k: the number of non-smoking female CNGs which request to ride with females for the kth ODTG pair;
d^k: the number of non-smoking female CNGs which request to ride with non-smokers for the kth ODTG pair;
f^k: the number of non-smoking female CNGs which have no request for the kth ODTG pair;
g^k: the number of smoking female CNGs which request to ride with females for the kth ODTG pair;
h^k: the number of smoking female CNGs which have no request for the kth ODTG pair;
l^k: the number of non-smoking male CNGs which request to ride with non-smokers for the kth ODTG pair;
o^k: the number of non-smoking male CNGs which have no request for the kth ODTG pair;
p^k: the number of smoking male CNGs which have no request for the kth ODTG pair;
e^m: the number of people per group in the mth CVG passenger-flow network;
e^{kn}: the number of people per group in the (k, n)th CNG passenger-flow network;
q^m: the vehicle's remaining capacity in the mth CVG vehicle-flow network; which equals the vehicle capacity excluding the number of people in a CVG.

Variables:

x_{ij}^{m}: the flow of arc(i, j) in the mth CVG vehicle-flow network;
z_{ij}^{m}: the flow of arc(i, j) in the mth CVG passenger-flow network;
y_{ij}^{kn}: the flow of arc(i, j) in the (k, n)th CNG passenger-flow network;
ω^m: the supply variable in the mth CVG vehicle-flow network, representing the number of CVGs driving a vehicle;
λ^k: the supply variable in the CNG passenger-flow network associated with the kth ODTG pair and the non-smoking female vehicle, representing the number of non-smoking female CNGs which request to ride with females for the kth ODTG pair, that are assigned to non-smoking female vehicles;
β^k: the supply variable in the CNG passenger-flow network associated with the kth ODTG pair and the non-smoking female vehicle, representing the number of non-smoking female CNGs which request to ride with non-smokers for the kth ODTG pair, that are assigned to non-smoking female vehicles;
γ^k: the supply variable in the CNG passenger-flow network associated with the kth ODTG pair and the non-smoking female vehicle, representing the number of non-smoking female CNGs which have no request, that are assigned to non-smoking female vehicles;
ε^k: the supply variable in the CNG passenger-flow network associated with the kth ODTG pair and the female vehicle, representing the number of non-smoking female CNGs which have no request, that are assigned to female vehicles;
ϕ^k: the supply variable in the CNG passenger-flow network associated with the kth ODTG pair and the non-smoking vehicle, representing the number of non-smoking female CNGs which have no request, that are assigned to non-smoking vehicles;
θ^k: the supply variable in the CNG passenger-flow network associated with the kth ODTG pair and the female vehicle, representing the number of smoking female CNGs which have no request, that are assigned to female vehicles;

δ^k: the supply variable in the CNG passenger-flow network associated with the kth ODTG pair and the non-smoking vehicle, representing the number of non-smoking male CNGs which have no request, that are assigned to non-smoking vehicles.

The model is formulated as an integer multiple commodity network flow problem as follows:

Model (A)

$$\text{Min} \quad \sum_{m\in M}\sum_{(i,j)\in VA^m} c_{ij}^m x_{ij}^m + \sum_{m\in H}\sum_{(i,j)\in VPA^m} e^m t_{ij}^m z_{ij}^m + \sum_{(k,n)\in K}\sum_{(i,j)\in PA^{kn}} e^{kn} t_{ij}^{kn} y_{ij}^{kn} \tag{1}$$

The objective function (1) is to minimize the system cost, which includes CVG traveling cost (including vehicle operating cost and CVG time cost), CVPG time cost, and CNG time cost including the penalty for unserved CNGs.

S.t.

$$\sum_{j\in VN^m} x_{ij}^m = \omega^m \qquad \forall i\in VS^m,\ \forall m\in M \tag{2}$$

$$\sum_{j\in VN^m} x_{hj}^m - \sum_{i\in VN^m} x_{ih}^m = 0 \qquad \forall h\in VT^m,\ \forall m\in M \tag{3}$$

$$-\sum_{i\in VN^m} x_{ij}^m = -\omega^m \qquad \forall j\in VD^m,\ \forall m\in M \tag{4}$$

$$\sum_{j\in VPN^m} z_{ij}^m = v^m - \omega^m \qquad \forall i\in VPS^m,\ \forall m\in H \tag{5}$$

$$\sum_{j\in VPN^m} z_{hj}^m - \sum_{i\in VPN^m} z_{ih}^m = 0 \qquad \forall h\in VPT^m,\ \forall m\in H \tag{6}$$

$$-\sum_{i\in VPN^m} z_{ij}^m = -(v^m - \omega^m) \qquad \forall j\in VPD^m,\ \forall m\in H \tag{7}$$

Constraint (2) ensures flow conservation at the supply node in each CVG vehicle-flow network. Constraint (3) ensures flow conservation at every transfer node in each CVG vehicle-flow network. Constraint (4) ensures flow conservation at the demand node in each CVG vehicle-flow network. Constraint (5) ensures flow conservation at the supply node in each CVG passenger-flow network. Note that $(v^m - \omega^m)$ represents the CVPG supply in the mth CVG passenger-flow network. Constraint (6) ensures flow conservation at every transfer in each CVG passenger-flow network. Constraint (7) ensures flow conservation at the demand node in each CVG passenger-flow network.

$$\sum_{j\in PN^{kn}} y_{ij}^{kn} = a^k + \lambda^k + \beta^k + \gamma^k \qquad \forall i\in PS^{kn},\ \forall k\in ODTG,\ \forall n\in FNV \tag{8}$$

$$\sum_{j\in PN^{kn}} y_{ij}^{kn} = (b^k - \lambda^k) + \varepsilon^k + g^k + \theta^k \qquad \forall i\in PSi^{kn},\ \forall k\in ODTG,\ \forall n\in FV \tag{9}$$

$$\sum_{j\in PN^{kn}} y_{ij}^{kn} = (d^k - \beta^k) + \phi^k + l^k + \delta^k \qquad \forall i\in PS^{kn},\ \forall k\in ODTG,\ \forall n\in NV \tag{10}$$

$$\sum_{j \in PN^{kn}} y_{ij}^{kn} = (f^k - \gamma^k - \varepsilon^k - \phi^k) + (h^k - \theta^k) + (o^k - \delta^k) + p^k$$

$$\forall i \in PS^{kn},\ \forall k \in ODTG,\ \forall n \in GV \tag{11}$$

$$-\sum_{i \in PN^{kn}} y_{ij}^{kn} = -(a^k + \lambda^k + \beta^k + \gamma^k) \quad \forall j \in PD^{kn},\ \forall k \in ODTG,\ \forall n \in FNV \tag{12}$$

$$-\sum_{i \in PN^{kn}} y_{ij}^{kn} = -((b^k - \lambda^k) + \varepsilon^k + g^k + \theta^k)$$

$$\forall j \in PD^{kn},\ \forall k \in ODTG,\ \forall n \in FV \tag{13}$$

$$-\sum_{i \in PN^{kn}} y_{ij}^{kn} = -((d^k - \beta^k) + \phi^k + l^k + \delta^k)$$

$$\forall j \in PD^{kn},\ \forall k \in ODTG,\ \forall n \in NV \tag{14}$$

$$-\sum_{i \in PN^{kn}} y_{ij}^{kn} = -((f^k - \gamma^k - \varepsilon^k - \phi^k) + (h^k - \theta^k) + (o^k - \delta^k) + p^k)$$

$$\forall j \in PD^{kn},\ \forall k \in ODTG,\ \forall n \in GV \tag{15}$$

$$\sum_{j \in PN^{kn}} y_{hj}^{kn} - \sum_{i \in PN^{kn}} y_{ih}^{kn} = 0 \quad \forall h \in PT^{kn},\ \forall (k,n) \in K \tag{16}$$

Constraints (8)–(11) ensure flow conservation at the supply node in the CNG passenger-flow networks, for CNGs taking the non-smoking female, female, non-smoking and general types of vehicle. To better understand these constraints, from Fig. 1 we know that the amount of CNGs which ride in "non-smoking female" vehicles = all of type 1 CNGs (a constant) + part of type 2 CNGs (a variable) + part of type 3 CNGs (a variable) + part of type 4 CNGs (a variable) for the same ODTG pair. Constraint (8) can now be transformed into $\sum_{j \in PN^{kn}} y_{ij}^{kn} = a^k + \lambda^k + \beta^k + \gamma^k$. Constraints (9)–(11) can be generated similarly. Similar to Constraints (8)–(11), Constraints (12)–(15) ensure flow conservation at the demand node in the CNG passenger-flow networks, for CNGs taking the non-smoking female, female, non-smoking and general types of vehicle. Constraint (16) ensures flow conservation at every transfer node in each CNG passenger-flow network.

$$\sum_{kn \in PCH^p} e^{kn} \times y_{ij}^{kn} + \sum_{m \in PCH^p} e^m \times z_{ij}^m \le \sum_{m \in VCH^p} q^m \times x_{ij}^m$$

$$\forall (i,j) \in VTA,\ \forall p \in QV \tag{17}$$

$$b^k - \lambda^k \ge 0 \qquad \forall k \in ODTG \tag{18}$$

$$d^k - \beta^k \ge 0 \qquad \forall k \in ODTG \tag{19}$$

$$f^k - \gamma^k - \varepsilon^k - \phi^k \ge 0 \qquad \forall k \in ODTG \tag{20}$$

$$h^k - \theta^k \ge 0 \qquad \forall k \in ODTG \tag{21}$$

$$o^k - \delta^k \ge 0 \qquad \forall k \in ODTG \tag{22}$$

$$v^m - \omega^m \ge 0 \qquad \forall m \in M \tag{23}$$

Constraint (17) ensures that the sum of all passengers, with respect to all associated CVPGs and CNGs, does not exceed the vehicle's remaining capacity on every route between two locations for every vehicle type. It should be mentioned that the arc flow unit in the CNG passenger-flow time-space network is the "group," but the vehicle capacity is the "person." Therefore, each arc's flow must be multiplied by a corresponding parameter e^{kn} which represents the number of people per CNG. For example, if the network is associated with a CNG of two people, then e^{kn} is 2. Similarly, each arc's flow in the CVG passenger networks must be multiplied by a corresponding parameter e^m which represents the number of people per CVG. Constraints (18)–(22) ensure that the CNG supply in each CNG passenger-flow network is nonnegative. Constraint (23) ensures that the CVPG supply in each CVG vehicle-flow network is nonnegative.

$$x_{ij}^m \geq 0, \text{integer} \quad \forall (i, j) \in VA^m, \ \forall m \in M \tag{24}$$

$$y_{ij}^{kn} \geq 0, \text{integer} \quad \forall (i, j) \in PA^{kn}, \ \forall (k, n) \in K \tag{25}$$

$$z_{ij}^m \geq 0, \text{integer} \quad \forall (i, j) \in PA^m, \ \forall m \in H \tag{26}$$

$$\omega^m \geq 0, \text{integer} \quad \forall m \in M \tag{27}$$

$$\lambda^k \geq 0, \text{integer} \quad \forall k \in ODTG \tag{28}$$

$$\beta^k \geq 0, \text{integer} \quad \forall k \in ODTG \tag{29}$$

$$\gamma^k \geq 0, \text{integer} \quad \forall k \in ODTG \tag{30}$$

$$\varepsilon^k \geq 0, \text{integer} \quad \forall k \in ODTG \tag{31}$$

$$\phi^k \geq 0, \text{integer} \quad \forall k \in ODTG \tag{32}$$

$$\theta^k \geq 0, \text{integer} \quad \forall k \in ODTG \tag{33}$$

$$\delta^k \geq 0, \text{integer} \quad \forall k \in ODTG \tag{34}$$

Constraints (24)–(34) ensure the integrality and non-negativity of all variables. It should be mentioned that our model is designed for the many-to-many car pooling problem with multiple vehicle types and person types. With suitable reduction of networks or modification of parameters, the model can be simplified to solve the many-to-one, one-to-many, single vehicle type, or single person type car pooling problems. Moreover, the model is designed to be flexible enough to be modified to match the planner's needs. For example, if the objective is to minimize the number of vehicles used, then the objective function can be modified to be $\sum_{m\in M}\sum_{i\in VS^m}\sum_{j\in VN^m} c_{ij}^m x_{ij}^m$, where c_{ij}^m is modified to be 1. Finally, to ensure the correctness of the model, several small examples are designed and solved using the mathematical programming solver, CPLEX. Since these examples are so small, the obtained solutions can be manually verified to be correct. One of the examples is shown in the Appendix.

4 Solution algorithm

The model is formulated as an integer multiple commodity network flow problem which is characterized as NP-hard (Garey and Johnson 1979). We first tried using the mathematical programming solver, CPLEX 11.0, to directly solve the problem. Note that the solution method used in CPLEX is a branch and bound algorithm, coupled with the simplex method to solve the linear relaxation problem in each sub-problem. It was found that after 5 hours we could not find an optimal solution (or even a feasible solution) for many problem instances. We thus developed a heuristic algorithm to solve the problem.

Lagrangian relaxation with subgradient methods (LRS) is known for its fast convergence and efficient allocation of memory space when solving large-scale integer linear programs (Fisher 1981). Therefore, we suggest using LRS for the approximation of near-optimal solutions. The solution process is described below. We first relax constraint (17) to construct a Lagrangian problem, which is then solved to procure a lower bound for the optimal solution. Second, a Lagrangian heuristic is developed to solve for the upper bound of the optimal solution. A sub-gradient method is then utilized to revise the Lagrangian multipliers, by iterating the lower and upper bounds, until an acceptable convergence result is reached, or until the number of iterations exceeds a preset number. The algorithm includes the computation of lower bound of the optimal solution, the computation of upper bound of the optimal solution, the sub-gradient method and the solution process.

4.1 The lower bound of the optimal solution

The steps for searching for the lower bound are:

Step 1: Side constraints (17) are relaxed with the corresponding non-negative Lagrangian multipliers π_{ij}^p and are added to the objective function of Model (A), resulting in Model (B). The optimal objective value for Model (B) becomes the lower bound of Model (A).

Step 2: Model (B) is a pure network flow problem. The problem sizes are also reduced, so that they can be directly solved using the mathematical programming solver, CPLEX.

Step 3: The lower bound of the optimal solution is obtained.

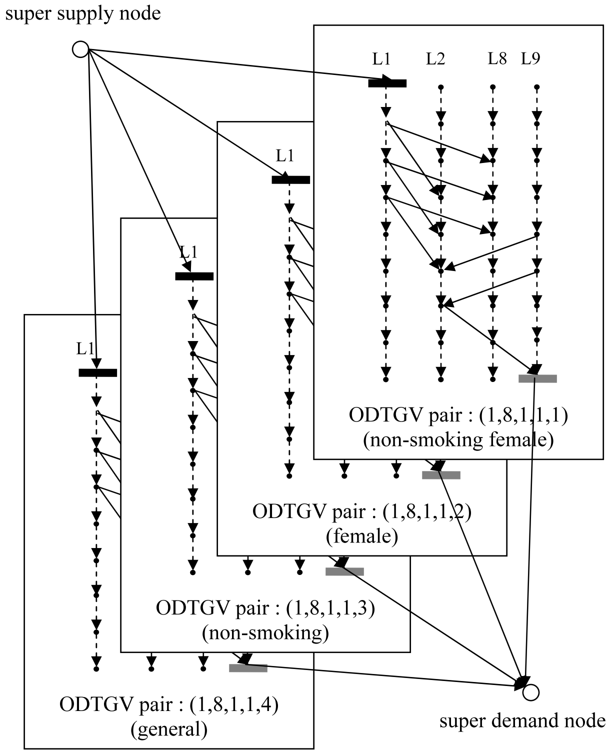
Model (B):

$$\begin{aligned}\text{Min} \quad & \sum_{m\in M}\sum_{(i,j)\in VA^m} e^m c_{ij}^m x_{ij}^m + \sum_{m\in H}\sum_{(i,j)\in VPA^m} e^m c_j^m z_{ij}^m + \sum_{(k,n)\in K}\sum_{(i,j)\in PA^{kn}} e^{kn} c_{ij}^{kn} y_{ij}^{kn} \\ & + \sum_{(i,j)\in VTA}\sum_{p\in QV} \pi_{ij}^p \Bigg(\sum_{(k,n)\in PCH^p} e^{kn}\times y_{ij}^{kn} + \sum_{m\in PCH^p} e^m \times z_{ij}^m \\ & - \sum_{m\in VCH^p} q^m \times x_{ij}^m \Bigg) \end{aligned} \tag{35}$$

subject to constraints (2)–(16) and (18)–(34).

It should be mentioned that constraints (8)–(15) may look like side constraints, rather than node conservation constraints. However, they are the same as node conservation constraints. For example, as shown in Fig. 6, suppose that we add a super supply node with four starting arcs and a super demand node with four ending arcs to the CNG networks associated with the four vehicle types for an ODTG pair. Each starting arc goes from the super node to the original supply node in each network and each ending arc goes from the original demand node in each network to the super demand node. Let us set the super node supply/demand to be $f/-f$. Now constraints (8)–(15) can be substituted by two conservation constraints, one for the super supply node and the other for the super demand node. Similarly, the other supply/demand constraints can be transferred into node conservation constraints. Because the three networks in Model (B) are independent, Model (B) can be divided into three independent and pure network flow problems. As a result, Model (B) is a pure network flow problem.

Fig. 6 Example illustrating the supply/demand constraints



4.2 The upper bound of the optimal solution

Since our model contains three types of networks: CVG vehicle-flow networks; CVG passenger-flow networks; and CNG passenger-flow networks with many side constraints across several networks, it is not simple to adjust a good feasible network flow from an infeasible lower bound solution. The searching process is outlined in Fig. 7, and the steps are listed below. We first define the following symbols that are used in the Lagrangian heuristic:

CVGNs, CVPGNs, CNGNs: the CVG vehicle-flow networks, the CVG passenger-flow networks and the CNG passenger-flow networks, respectively.

$cvgflow_i$, $cvpgflow_i$, $cngflow_i$: the CVG flows in the CVG vehicle-flow networks, the CVPG flows in the CVG passenger-flow networks, and the CNG flows in the CNG passenger-flow networks, in the ith step.

sv_i: the supply variable values in the ith step.

sol_i: the upper bound solution in the ith step, including the $cvgflow_i$, $cvpgflow_i$ and $cngflow_i$.

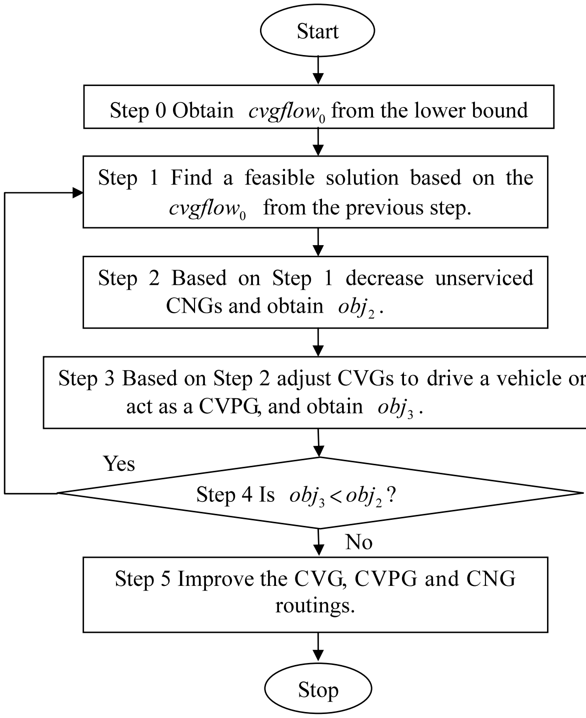
obj_i: the objective value of sol_i.

The steps are listed below:

Step 0: Obtain $cvgflow_0$ from the lower bound solution.

Step 1: Let the vehicle flows from the lower bound (Step 0) or Step 3 solution be $cvgflow_0$. Now, construct a modified network from the original network. For every arc in each CVGN, the flow lower bound is reset to be the arc flow obtained from $cvgflow_0$. Finally, use CPLEX to solve the modified network to find

Fig. 7 Upper bound search process



$cvgflow_1$, $cvpgflow_1$, $cngflow_1$ and obj_1. With this step, we obtain the initial feasible solution sol_1.

Step 2: Find a new sol_2 based on $cvgflow_1$ and $cngflow_1$ as obtained from Step 1. First, we construct a modified network from the original network. For every vehicle/CNG holding arc and travel arc in CVGNs/CNGNs, the flow lower bound is then reset to be the arc flow obtained from $cvgflow_1/cngflow_1$. Note that we do not reset the lower bound of the unserved arc in CNGNs in order to decrease the number of unserved CNGs. Finally, we use CPLEX to solve the modified network to find $cvgflow_2$, $cvpgflow_2$, $cngflow_2$ and obj_2. In this step, the number of unserved CNGs could be decreased, and obj_2 will not be worse than obj_1.

Step 3: Find a new sol_3 based on $cngflow_2$ as obtained from Step 2. First, we fix the values of variables obtained from $cngflow_2$ in CNGNs to construct a modified model from Model (A). Now, use CPLEX to solve the modified model to find a new sol_3 (including $cvgflow_3$, $cvpgflow_3$ and $cngflow_3$) and obj_3. In this step, the CVGs are assigned again to driving a vehicle or acting as a CVPG, and obj_3 will not be worse than obj_2.

Step 4: If obj_3 is better than obj_2, then set $cvgflow_0 = cvgflow_3$ obtained from Step 3 and go to Step 1; else, go to Step 5.

Step 5: Find a new sol_5 based on sv_3 from Step 3. We first fix sv_3 to construct a modified model from Model (A). Note that since sv_3 is fixed, the supply/demand constraints are removed from the modified model, which can be then divided into four independent sub-models according to the four vehicle types. Each sub-model associated with a vehicle type contains several CVGNs, CVPGNs, and CNGNs. Finally, use CPLEX to solve these four sub-models, and add up the four objective values to obtain obj_5. The CVG, CVPG and CNG routings can be improved in this step, and obj_5 will not be worse than obj_3.

4.3 Sub-gradient method and the solution process

Yan and Young's (1996) sub-gradient method for adjusting Lagrangian multipliers is applied in this study due to its good performance in actual experience. The steps of the Lagrangian relaxation-based algorithm are:

Step 1: Set iteration $i = 0$; the initial Lagrangian multipliers are set to be the dual variables of constraint (17) for the optimal solution of the linear relaxation problem of Model (A).

Step 2: Use CPLEX to solve Model (B) and get a lower bound $Z^L(\pi^{\mathrm{i}})$. If the solution is feasible and also satisfies the condition,

$$\sum_{(i,j)\in VTA} \sum_{p\in QV} \pi_{ij}^{p} \left(\sum_{(k,n)\in PCH^p} e^{kn} \times y_{ij}^{kn} + \sum_{m\in PCH^p} e^{m} \times \alpha_{ij}^{m} - \sum_{m\in VCH^p} q^{m} \times x_{ij}^{m} \right) = 0,$$

then we have found an optimal solution and the solution process can be stopped. Otherwise, update the lower bound Z^L.

Step 3: Apply the upper bound heuristic to find an upper bound $Z^U(\pi^i)$, then update the upper bound Z^U.

Step 4: If the gap between the lower bound Z^L, and the upper bound Z^U falls within a specified tolerance θ (i.e., $|(Z^U - Z^L)/Z^U| \le \theta$), or the solution time reaches a preset limit, stop the algorithm.

Step 5: Adjust π^i to help improve the convergence by applying the sub-gradient method developed in Yan and Yang (1996).

Step 6: Set $i = i + 1$. Go to Step 2.

It should be mentioned that, generally, the initial Lagrangian multipliers (π_0) are set to be zero. We first used Yan and Young's (1996) sub-gradient method to revise the Lagrangian multipliers from 0, but found that the lower bound was not good (i.e., good Lagrangian multipliers could not be found) and the convergence was slow. Hence, we had to relax the integer constraints in Model (A), and then use CPLEX to solve for the optimal solution to this linear relaxation problem. We use the dual variables for constraint (17), found based on the linear optimal solution, as the initial Lagrangian multipliers. In theory, these π values are the optimal Lagrangian multipliers for the linear relaxation problem (Yan 1996). That is, the optimal objective of the Lagrangian problem relaxed with these optimal Lagrangian multipliers (i.e., the initial lower bound solution) is the same as the objective value of the linear relaxation problem. Thus, the sub-gradient method mainly functions for the adjustment of the Lagrangian multipliers in the search for various good lower bound solutions that are used for searching for potentially good upper bound solutions, rather than improving the lower bound solutions for this algorithm.

The complexity of the algorithm is now discussed. During each iteration, a lower bound solution and an upper bound solution are solved after which a subgradient method is used to revise the Lagrangian multipliers. To search for the lower bound solution, the simplex algorithm is used to solve the relaxation problem, Model (B), whose complexity is $O(2^{\psi})$ (Papadimitriou and Steiglitz 1982), where $^{\psi}$ is the number of total variables. To search for the upper bound solution, the simplex algorithm, coupled with the branch and bound technique, is used to solve a series of models modified from the original, whose complexity is $O(\sigma 2^{\psi})$, assuming that σ is the number of modified models that need to be solved (i.e., until the solution cannot be improved). The Lagrangian multipliers are revised by the subgradient method which contains a simple calculation whose complexity is $O(\psi^2)$, according to Yan and Young (1996). Suppose that the number of iterations in the solution process is τ, then the algorithm complexity is $\tau \times (O(2^{\psi}) + O(\sigma 2^{\psi}) + O(\psi^2))$, which is approximately $O(\kappa 2^{\psi})$, where $\kappa = \tau \times \sigma$. Although the algorithm complexity is exponential, the algorithm is efficient in practice as shown in the numerical tests to be described later.

Finally, it should be noted that the CVG/CVPG/CNG flows obtained above cannot yet be directly put into practice without identifying each CVG/CVPG/CNG path in the networks. Therefore, we use a flow decomposition method (Yan and Yang 1996) to decompose the arc flows, in every CVG/CVG/CNG vehicle/passenger-flow time-space network, into arc chains. Each arc chain represents a CVG/CVPG/CNG's route/schedule in the planning period.

4.4 Another heuristic method

Based on real practices, we develop another heuristic method (herein called the AH method) to suitably evaluate our algorithm. The steps of the AH method are as follows.

Step 1: Sort CVGs and CNGs, separately, by their time windows and then their origins/destinations. Note that we have tried to sort CVGs and CNGs, separately, by their origins/destinations, and then by their time windows. However, after testing, we found the results were worse.

Step 2: Assign a CVG to sequentially service the CNGs according to the sequence obtained from Step 1, provided that the CVG/CNG/vehicle type, time window and vehicle capacity constraints are satisfied. The process is repeated until every CVG has been examined or all CNGs have been assigned to a CVG. Note that this step is used to assign a CVG to act as a driver.

Step 3: Assign a pending CVG (which has not been examined in Step 2), to be serviced by a driver (i.e., a CVG which has already been examined in Step 2), provided that all constraints (including CVG/CNG/vehicle type, time window and capacity constraints) are satisfied. The process is repeated until the pending CVGs are all examined and either assigned to a driver or not (meaning that they will act as a driver). Note that this step is used to assign a CVG to act as a CVPG (passenger) and to decrease the number of used vehicles.

5 Numerical tests

To test the applicability of the network model and the solution algorithm in the real world, we randomly generated several instances after referring to data from Guo's (2003) study of northern Taiwan. The C++ computer language, coupled with the mathematical programming solver, CPLEX 11.0, was used to build the model and to develop the solution algorithm. The tests were performed on a Core2 Quad Q6600 2.4 GHz with 3 GB of RAM in the environment of Microsoft Windows XP. Finally, we analyzed the test results and carried out a number of sensitivity analyses.

5.1 Test results

To evaluate the efficiency of the proposed model and solution algorithm for different problems, we randomly generated and tested 30 problem instances having five different network

Table 2 Problem instances for tests

#Network layers (CVGNs/CVPGNs/CNGNs)	Problem instance	# Variables	# Constraints	# CVGs (groups)	# CNGs (groups)
45 (10/10/25)	P1-1, Q1-1	175,457	54,080	158	376
	P1-2, Q1-2	162,392	53,549	159	397
	P1-3, Q1-3	144,885	51,612	132	312
70 (15/15/40)	P2-1, Q2-1	264,945	62,081	257	608
	P2-2, Q2-2	202,963	58,119	260	556
	P2-3, Q2-3	239,070	60,248	288	496
95 (20/20/55)	P3-1, Q3-1	325,198	66,763	355	681
	P3-2, Q3-2	281,273	63,703	323	728
	P3-3, Q3-3	345,707	68,908	460	743
120 (25/25/70)	P4-1, Q4-1	380,732	72,787	550	838
	P4-2, Q4-2	428,480	75,467	651	1,152
	P4-3, Q4-3	443,295	74,928	701	1,206
145 (30/30/85)	P5-1, Q5-1	537,659	79,895	823	1,399
	P5-2, Q5-2	537,584	78,302	798	1,349
	P5-3, Q5-3	453,307	76,266	808	1,439

scales. In the real world, if the number of people within a CG is three or four, then they would not participate in the carpool plan, but would likely coordinate their own plan. The most common CGs are those which contain one or two people. Our 30 problem instances were divided into two classes, P and Q, each consisting of 15 instances of 5 different network scales, varying from 45 to 145 layers, with an increment of 25 layers. Each scale is composed of 3 different problem instances, as shown in Table 2. All CGs in the P-class problem instances contain one person while in the Q-class, 20% of contain two people and 80% contain one person. Note that the number of CGs does not affect the network scale.

The time interval for constructing the time-space points in each network is 15 minutes, based on the configuration and average travel speed in this area of northern Taiwan. Theoretically, the higher the density of the nodes (i.e., the shorter the time interval associated with a node), the more precise the solution; however, the larger the problem size. In practice, the planner would select the most suitable time interval associated with a node to meet their own actual requirements. It should be mentioned that when applied to real operations the model solution could be slightly and manually adjusted by the planner, after optimization, to meet the actual operating requirements (usually called a post-optimization analysis, for example, see Yan et al. 2006a), if the model solution is not fully satisfactory due to the model design. For example, if the time interval associated with a node is set too large, then the obtained vehicle departure or arrival schedule can be slightly modified by a couple of minutes, to conform to real practices. The adjustment should not affect the optimality or feasibility of the model solution. Based on post-optimization analysis, some inaccuracy due to the model design or solution efficiency can be resolved when handling real problems. There were 30 carpool stations in the carpool plan. All CG origins and destinations were located at these stations. The planning time length was assumed to be 4 hours (17 time points). The average vehicle speed was assumed to be 60 km/hr. All the cost parameters were set according to Guo's (2003) survey in north Taiwan and are listed as follows:

- Vehicle operating cost: NT $5.49/km.
- CG's traveling time cost: NT $1.13/min·person.
- CG's holding time cost: NT $2.34/min·person.
- The penalty cost of a person who cannot be serviced: NT $1000/person.

The generation process of problem instances is now outlined. For the P-class: given the number of total network layers, we first set the number of network layers for the CVGNs, CVPGNs and CNGNs. Since a CVGN and a CVPGN should be built for each ODTGV pair of CVGs, the number of CVGNs and of CVPGNs are the same. Several CNGNs were built for every ODTG pair with its possible vehicle types for riding. All the CNGNs generated for every ODTV pair were then summed up to match the designated number of CNGNs. Next, we randomly generated the number of CVGs/CNGs for each ODTGV/ODTG pair. The randomly generated number of CVGs for each ODTGV pair and that of CNGs for each ODTG pair ranged from 1 to 35. The ratio of the number of CNGs to that of CVGs was between 1.5 and 3.

To generate the detailed attributes for each ODTGV/ODTG pair of CVGs/CNGs, the 30 carpool stations were first used to generate the origin and destination for each ODTGV/ODTG pair. Then, the time-window length for each CG was set according to the direct travel time plus a randomly chosen time between 2 and 6 time intervals (i.e., 30 minutes and 90 minutes). The CVG type was randomly generated from the four CVG types (which are the same as the four vehicle types); the CNG type was randomly generated from the nine CNG types. Finally, every arc cost was calculated according to the given parameters. For the generation of Q-class problem instances: for simplicity we modified the 15 P-class problem instances to generate another 15 instances. In particular, for each P-class problem instance we randomly choose 20% of CVGs and 20% of CNGs and set these groups to contain two people. Note that in all P-class problem instances, every CVG/CNG contains only one person. All in all as shown in Table 2, 30 problem instances of substantially large size, up to 537,659 variables and 79,895 constraints, were generated.

To evaluate how well the problem instances could be optimally solved using the exact method (the branch and bound method), we first used the mathematical programming solver, CPLEX 11.0, to directly solve all the problem instances. It was found that after 5 hours we could not find a feasible solution for many problem instances (e.g., P3-1, P4-1, and P5-1). Therefore, we used the proposed solution algorithm to solve the problem instances. The results are shown in Tables 3 and 4. We found that all problem instances were solved to within a gap of 3% convergence, which denotes the maximum error for the feasible solution (the upper bound solution) from the truly optimal solution. We also found that the computation time increased as the problem size (mainly the number of network layers) increased. The most time-consuming instance (P5-3) was still solvable within 2 hours, which is generally efficient enough for real practices, typically one day before the operating day. If necessary, better computer equipment could be used to speed the solution process in real operations. The number of vehicles used, the number of people unserved, the vehicle use ratios ($V/C =$ the number of vehicle used / the number of CVGs), the service ratios ($S/P =$ (the number of participants – the number of unserved people) / the number of participants), and the average number of persons served by a vehicle ($T/V =$ (the number of participants – the number of unserved people) / the number of vehicles used), are shown in Tables 3 and 4. We found that the vehicle use ratios (V/Cs) ranged from 41.96% to 60.68% (on average 49.19%) for class P, and from 48.04% to 71.83% (on average 58.70%) for class Q. Most people were serviced in each instance, with S/Ps ranging from 62.92% to 93.01% (on average 82.91%) for class P, and from 56.16% to 93.52% (on average 81.19%) for class Q. The average number of persons served by a vehicle (T/V) ranged from 4.39 to 5.37 (on average 4.95) for class P,

Table 3 Test results for P-class problem instances

Problem instance	Upper bound	Lower bound	Gap	CPU (s)	# vehicles	# unserved people	# participants	V/C (%)	S/P (%)	T/V
P1-1	236,681.40	233,257.77	0.014	3.28	73	198	534	46.20	62.92	4.60
P1-2	168,797.85	166,010.63	0.017	6.33	84	117	556	52.83	78.96	5.23
P1-3	177,169.55	174,937.32	0.013	18.50	61	154	444	46.21	65.32	4.75
P2-1	180,010.10	176,499.53	0.020	324.28	127	125	784	49.42	84.06	5.19
P2-2	242,702.65	239,344.95	0.014	29.77	144	184	816	55.38	77.45	4.39
P2-3	288,737.30	283,004.63	0.020	36.48	129	218	865	44.79	74.80	5.02
P3-1	171,435.95	166,663.53	0.028	819.86	182	92	1,036	51.27	91.12	5.19
P3-2	242,326.60	235,607.27	0.028	863.12	196	160	1,051	60.68	84.78	4.55
P3-3	426,462.10	418,857.04	0.018	1,063.59	193	331	1,203	41.96	72.49	4.52
P4-1	200,254.75	194,794.76	0.027	1,120.61	267	97	1,388	48.55	93.01	4.84
P4-2	305,586.20	301,065.32	0.015	4,316.63	307	155	1,803	47.16	91.40	5.37
P4-3	337,898.30	327,783.73	0.030	4,713.72	330	173	1,907	47.08	90.93	5.25
P5-1	363,035.15	353,014.98	0.028	6,498.02	388	173	2,222	47.14	92.21	5.28
P5-2	361,573.10	350,952.16	0.029	6,302.28	397	167	2,147	49.75	92.22	4.99
P5-3	379,371.30	368,223.18	0.029	6,885.83	400	180	2,247	49.50	91.99	5.17
		Average	0.023				Average	49.19%	82.91%	4.95

Table 4 Test results for Q-class problem instances

Problem instance	Upper bound	Lower bound	Gap	CPU (s)	# vehicles	# unserved people	# participants	V/C (%)	S/P (%)	T/V
Q1-1	322,764.35	313,623.18	0.028	1.0	77	281	641	48.73	56.16	4.68
Q1-2	228,292.25	224,850.44	0.015	7.58	105	164	667	66.04	75.41	4.79
Q1-3	226,076.05	223,272.38	0.012	7.03	69	199	532	52.27	62.59	4.83
Q2-1	251,267.50	243,845.44	0.030	9.78	150	187	941	58.37	80.13	5.03
Q2-2	325,159.65	315,878.64	0.029	93.61	165	255	979	63.46	73.95	4.39
Q2-3	375,350.60	368,071.07	0.019	26.28	154	287	1,038	53.47	72.35	4.88
Q3-1	243,501.00	239,984.90	0.014	1,347.23	224	144	1,243	63.10	88.42	4.91
Q3-2	313,160.25	307,661.14	0.018	2,789.24	232	216	1,262	71.83	82.88	4.51
Q3-3	539,522.25	529,289.25	0.019	655.00	221	423	1,444	48.04	70.71	4.62
Q4-1	263,723.70	256,199.88	0.029	637.28	320	138	1,666	58.18	91.72	4.78
Q4-2	355,093.45	345,039.40	0.028	4,328.20	380	161	2,163	58.37	92.56	5.27
Q4-3	414,110.35	402,912.41	0.027	6,459.50	428	202	2,288	61.06	91.17	4.87
Q5-1	430,313.55	422,602.15	0.018	3,300.78	474	186	2,667	57.59	93.03	5.23
Q5-2	403,811.30	392,205.68	0.029	2,320.70	477	167	2,577	59.77	93.52	5.05
Q5-3	434,868.45	425,147.79	0.022	1,987.98	487	180	2,697	60.27	93.33	5.17
		Average	0.023				Average	58.70%	81.19%	4.87

Table 5 Test results and differences from the AH method for all problem instances

Problem instance	OBJ by AH	OBJ Diff. (%)[a]	# vehicles by AH	Diff. (%)[b] # vehicles	# unserved people by AH	Diff. (%)[c] # unserved people	CPU time (s) by AH
P1-1	349,040.64	47.47	104	42.27	283	42.93	1.54
P1-2	283,667.08	68.05	143	69.66	187	59.83	2.24
P1-3	259,655.20	46.56	90	47.75	222	44.16	1.70
P2-1	338,842.81	88.24	205	61.53	202	61.60	3.36
P2-2	467,049.79	92.44	254	76.62	353	91.85	3.04
P2-3	466,689.58	61.63	163	26.08	391	79.36	2.73
P3-1	432,164.71	152.09	223	22.64	325	253.26	3.97
P3-2	608,806.18	151.23	269	37.09	435	171.88	3.54
P3-3	787,044.68	84.55	401	107.65	569	71.90	5.01
P4-1	666,977.93	233.06	393	47.01	500	415.46	6.49
P4-2	819,901.50	168.30	383	24.91	613	295.48	6.32
P4-3	955,436.10	182.76	570	72.80	642	271.10	5.70
P5-1	1,210,231.45	233.36	688	77.42	691	299.42	9.06
P5-2	1,234,125.26	241.32	668	68.24	730	337.13	7.89
P5-3	1,196,521.35	215.40	796	99.04	760	322.22	9.17
Q1-1	390,626.87	21.03	152	97.27	300	6.76	0.97
Q1-2	426,598.23	86.86	136	29.88	298	81.71	1.71
Q1-3	300,245.93	32.81	119	72.06	249	25.13	1.88
Q2-1	659,947.06	162.65	241	92.93	469	150.80	3.18
Q2-2	523,969.42	61.00	232	40.64	401	57.25	2.74
Q2-3	478,953.27	28.00	211	36.80	346	20.56	3.40
Q3-1	845,245.00	247.00	333	63.31	595	313.19	5.40
Q3-2	792,479.89	153.00	303	30.64	597	176.39	4.98
Q3-3	983,652.24	82.00	424	92.07	639	51.06	3.77
Q4-1	877,533.50	233.00	510	59.46	657	376.09	6.47
Q4-2	1,137,424.39	220.00	563	48.21	821	409.94	5.95
Q4-3	1,308,902.62	216.00	684	75.69	735	263.86	7.79
Q5-1	1,201,140.05	179.00	806	70.10	744	300.00	9.16
Q5-2	1,306,252.00	223.00	758	74.74	669	300.60	9.73
Q5-3	1,248,084.23	187.00	801	64.46	787	337.22	8.36

[a]Diff. = (OBJ by AH − OBJ by Proposed algorithm)/OBJ by Proposed algorithm

[b]Diff. = (#vehicles by AH − # vehicles by Proposed algorithm)/# vehicles by Proposed algorithm

[c]Diff. = (#unserved people by AH − # unserved people by Proposed algorithm)/# unserved people by Proposed algorithm

and from 4.39 to 5.27 (on average 4.87) for class Q. Note that since the goal of car pooling is to fully utilize the vehicle occupancy within allowable travel time windows so as to decrease the system cost, in some cases the average number of persons served by a vehicle (T/V) could exceed 5. For example, assume that the route for a CVG is $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E$, and that the CVG carries 4 people from A to B to C and 2 other people from C to D to E, excluding the driver. Thus, the T/V for this route is 7, and the vehicle capacity constraint is satisfied for the four segments of this route. To sum up, our solution algorithm could solve all the problem instances to within a reasonable toleration gap (on average 2.3% for class P and for class Q, respectively) in a reasonably short computation time, as well as reduce the number of vehicles used, service most of the participants, and increase the vehicle usage. From this we conclude that our model and solution algorithm are both efficient and effective and could be a useful planning tool for car pooling practices. The planner should perform similar tests on their own operations to evaluate the model and solution algorithm and to understand its limitations, before putting them to practical use.

To suitably evaluate the proposed algorithm, we also compare the results obtained from the proposed algorithm and the AH method. As shown in Table 5, the proposed algorithm consumes more computation time but has better objective values than the AH method by an average of 139.99%. In addition, the proposed algorithm requires fewer vehicles and there are fewer unserved people than the AH method, on average by 60.97% and 189.60%, respectively. Note that the AH method, which is a greedy based algorithm, requires a shorter computation time to solve the problems, but cannot ensure solution effectiveness. On the contrary, the proposed algorithm, developed from a systematic optimization perspective, can solve the problems comparatively effectively and steadily. Consequently, since the computation time is not a major concern in the planning stage, the proposed algorithm is a significant improvement over the AH method and is therefore more effective to use in actual operations.

5.2 Case illustration with sensitivity analyses

Here, we choose P3-1 to illustrate the detailed solution results shown in Table 6. Case illustrations of other examples can be similarly performed in future. In P3-1 there are 355 CVGs and 681 CNGs. The average time-window is 95.75 minutes for CVGs and 85.68 minutes for CNGs. As shown in Table 6, there were 182 CVGs assigned to drive a vehicle in 355 CVGs. That is, 182 vehicles were used in this car pooling plan. The total CVG driving time (including traveling and holding times) was 14,985 minutes; the average CVG driving time was 82.34 minutes/CVG. 173 groups were assigned to CVPG in 355 CVGs. The total CVPG riding time was 9,495 minutes; the average CVPG riding time was 54.88 minutes/CVPG. 92 CNGs (or people) could not be serviced in 681 CNGs, due to the small time-windows

Table 6 Detailed results for P3-1

# Vehicles used	# CVPGs	# CNGs unserved
182	173	92
Total CVG driving time (min.)	Total CVPG riding time (min.)	Total CNG riding time (min.)
14,985	9,495	21,135
Avg. CVG driving time (min./vehicle)	Avg. CVPG riding time (min./CVPG)	Avg. CNG riding time (min./CNG)
82.34	54.88	35.88
V/C	S/P	T/V
51.27 %	91.12%	5.19

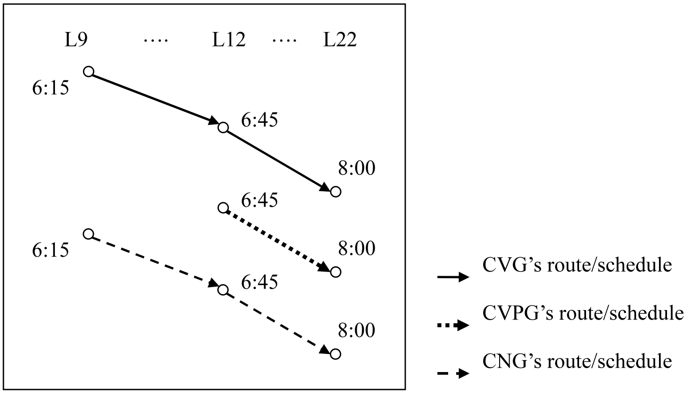


Fig. 8 Example of a CVG/CVPG/CNG's route/schedule

required. This will be discussed further in later sensitivity analyses. The total CNG riding time was 21,135 minutes; the average CNG riding time was 35.88 minutes. We found that the CVG's average driving time was significantly longer than the CVPG's and the CNG's. There could be two reasons for this. The first is that the CVG's time window is longer than that of the CNG's (i.e., 95.75 minutes $\geq$ 85.68 minutes). The second is that CVGs fully utilize time to service more passengers (whether CVPGs or CNGs) so as to decrease total system cost. We also found that the vehicles could be efficiently used when the V/C was equal to 51.27%. Most people could be serviced when the S/P was equal to 91.12%; and each vehicle could efficiently service people when the T/C was equal to 5.19. Note that since we focused on carpool plans in non-urban areas, each vehicle had a greater chance to carry CVPGs or CNGs.

Finally, we use the flow decomposition method to decompose all the arc flows in each network into arc chains, each representing a CVG/CVPG/CNG's route/schedule. To save space, only an example of a CVG's route/schedule, coupled with the CVPG and CNG it served, is shown in Fig. 8. This example shows that a CVG carries a CNG from Station L9 at 6:15, and then travels to L12 to carry a CVPG at 6:45. Finally, they arrive together at Station L22 at 8:00.

To more clearly understand how well the model would perform in different situations, we performed sensitivity analyses of the CG holding cost, vehicle operating cost, time-window length, percentage of two-people groups, and unserved CNG penalty cost which are essential inputs. We chose instance P3-1 and the proposed solution algorithm to perform the sensitivity analyses. Note that the sensitivity results may be slightly disturbed by the heuristic algorithm at optimality, but the overall trends are reasonable and are thus useful for interpreting the parameter effects. Sensitivity analyses of other factors may be similarly performed in future.

(1) CG holding cost

Here, the CG holding cost is set to the average participant time cost. Since the time cost varies in practice (Guo 2003), we tested four scenarios with different CG holding costs to examine its effect on the results. The results displayed in Table 7 show that the objective value increased as the CG holding cost increased. We further analyzed the solution in detail and found that the value of the total CG holding time decreased as the holding cost increased. When the holding cost was 0, 1.13, 2.34, and 3.55, the number

Table 7 Results due to changing the CG holding cost

Holding cost (NT $/min)	Obj.	Total CG holding time (min)	# Vehicles used
0	169,504.10	14,580	178
1.13	171,020.00	8,715	182
2.34	171,435.95	8,550	182
3.55	172,333.15	8,460	187

Table 8 Results due to changing the vehicle operating cost

Percentage of vehicle operating cost (%)	Obj.	Total vehicle operating time (min.)	# Vehicles used
50	150,486.95	11,370	196
75	160,374.50	11,115	191
100	171,020.00	10,920	182
125	179,105.00	10,680	179
150	189,131.45	10,635	178

of vehicles used was 178, 182, 182, and 187, respectively. In other words, as the holding costs increased, more vehicles had to be used to decrease the total holding time.

(2) Vehicle operating cost

The vehicle operating cost is an important factor in car pooling. It is affected by many operating issues, for example the price of gasoline. To examine its effect on the results, we tested five scenarios, 50%, 75%, 100%, 125% and 150% of the original vehicle operating cost. The results displayed in Table 8 show that the objective value increased as the vehicle operating cost increased. We also found that the total vehicle operating time (excluding the holding time) and the number of vehicles used decreased as the vehicle operating cost increased, meaning that fewer vehicles had to be used in order to reduce the total vehicle operating time and therefore the objective value.

(3) Time-window length

The length of the time-window required by each participant has an effect on successful matching and efficiency. In order to better understand its influence on the results we tested three scenarios, −15 minutes, +0 minutes and +15 minutes of the original time-window length. The results displayed in Table 9 show that the objective value decreased as the time-window length increased, meaning the less restricted the time-window required by the participants, the lower the system cost. Moreover, when the length of the time-windows increased, more CNGs could be served and fewer vehicles needed to be used. Note that when the original time-window lengths were 15 minutes larger, all CNGs could be serviced.

(4) Percentage of two-people CGs

Differences in the number of two-person and one-person CGs would have different impacts on the car pooling results. To examine this issue, we tested five scenarios, 0%, 10%, 20%, and 30% of the percentage of two-people groups (whose e^m or $e^{kn} = 2$) (denoted as p_{g2}). The results displayed in Table 10 show that the objective value increased as p_{g2} increased. We also found that the total CVG driving time (including the holding time) and the number of vehicles used increased as p_{g2} increased. This is because, as

Table 9 Results due to changing the length of the time-window

Addition to length of time-window	Obj.	# Unserved CNGs	# Vehicles used
−15 min	227,744.05	154	183
+0 min	171,020.00	92	182
+15 min	118,745.60	0	173

Table 10 Results due to changing the percentage of two-people groups

Percentage of two-people groups (p_{g2})	Obj.	Total CVG driving time (min.)	Total CVPG & CNG riding time (min.)	# Vehicles used
0%	171,020.00	14,985	30,630	182
10%	198,775.60	16,290	30,570	194
20%	243,501.00	19,080	29,085	224
30%	251,879.55	20,805	28,650	242

Table 11 Results due to changing the unserved CNG penalty cost

Penalty cost (per person)	Obj.	# CNGs unserved
1000	171,020.00	92
800	150,748.90	92
600	130,432.82	92
400	113,664.30	93
200	95,342.15	106

p_{g2} increased, the vehicle capacity was correspondingly reduced. The number of vehicles had to be increased, and, consequently, the total CVG driving time increased. In contrast to the total CVG driving time, the total CVPG and CNG riding time (include the holding time) decreased as p_{g2} increased. This is because as p_{g2} increased, vehicle capacity was correspondingly reduced, so the number of CVPGs or CNGs for vehicle service was reduced. As a result, the CVG routing complexity decreased and therefore the riding time of a CVPG or a CNG on average was reduced.

(5) Unserved CNG penalty cost

The unserved CNG penalty cost in the model is not a real cost, but rather a policy-making cost used in car pooling to reduce the number of CNGs unserved. To examine the influence of its value on the results, we tested five scenarios, 1000, 800, 600, 400 and 200 per person. As shown in Table 11, as the unserved CNG penalty cost decreased, the objective value decreased, as expected. We found that the number of CNGs unserved was not changed until the penalty cost was less than 600 per person. Therefore, if we want as many CNGs to be serviced as possible, then the penalty cost should be set at least as 600 per person.

6 Conclusion

In this study we extend the conventional many-to-one and one-to-many car pooling problem to become a many-to-many car pooling problem, with multiple vehicle types and person types. We employ the time-space network flow technique to develop a model for this problem. The model includes multiple CVG vehicle-flow networks, CVG passenger-flow networks, CNG passenger-flow networks and a set of side constraints across the networks. Mathematically, the model is formulated as an integer multiple commodity network flow problem. An algorithm, based on Lagrangian relaxation, a subgradient method and a heuristic for the upper bound solution, are developed to solve the problem. Numerical tests, utilizing the data from Guo's (2003) study on northern Taiwan, were performed to demonstrate and to preliminarily evaluate the model and the solution algorithm. Specifically, we generated 30 random instances of substantially large size (up to 537,659 variables and 79,895 constraints) to evaluate the model and solution algorithm in simulated real-word operations. The average convergence gap was about 2.30%. The most time-consuming problem was still solvable within 2 hours, which is efficient enough for the matching/scheduling decision in practice, typically one day before the operating day. We also found that the model could help reduce the number of vehicles used, service most of the participants, and increase the vehicle usage. To demonstrate the detailed model results and to better understand the model's performance in different situations, we provided a case illustration and performed sensitivity analyses of several essential model parameters. All of the results show that our model and solution algorithm are both efficient and effective enough to solve the many-to-many car pooling problem and have the potential to be a useful planning tool for authorities in designing car pooling plans.

Although the results show that the model and the solution algorithm could be useful, more tests should be conducted, to allow the authorities to better grasp the model's limitations, before putting it to practical use. It should be mentioned that, when the same test problem instances are used but the time interval associated with a node is set to 5 or 10 minutes, it will result in huge sized problems, up to 896,098 variables and 119,842 constraints or 1,182,849 variables and 239,685 constraints, which cannot be solved using the algorithm within one day. This means that the algorithm needs to be improved before solving larger-size problems in future. How to develop suitable algorithms for solving larger-size problems could be a direction of future research. Finally, we used an average travel time between any two locations for ease of modeling. However, in real practice, the travel time between two locations is generally stochastic, which could affect the car pooling/scheduling results. Thus, how to incorporate the stochastic travel times into the many-to-many car pooling problem to develop a more realistic and useful model could be a future research topic.

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Appendix

In this example, it is assumed that three groups (two CVGs and one CNG) participate the carpool plan. As shown in Table 12, two CVGs (each containing one person) with the same ODT from L1 at 7:00 to L3 at 8:00 can provide a car with a driver. One CNG (containing two people) with the ODT from L2 at 7:00 to L3 at 8:00 does not provide a car. The traveling costs per vehicle/person are 198.6/33.9, 297.9/50.85 and 99.3/16.95 between L1 and L2, between L1 and L3, and between L2 and L3, respectively. The holding cost per vehicle/person per 15 minutes at an intermediate station is 35.1/35.1. Note that since there is no extra operating cost except for the driver time cost for holding a vehicle still at a station, the holding costs per vehicle and per person are assumed to be the same.

Table 12 OD data

NO.	CVG or CNG	e	Group	Type	Origin	tw-begins	Destination	tw-ends
1	CVG	1	2	Non-smoking only	L1	7:00	L 3	8:00
2	CNG	2	1	CNG type (3)	L 2	7:00	L 3	8:00

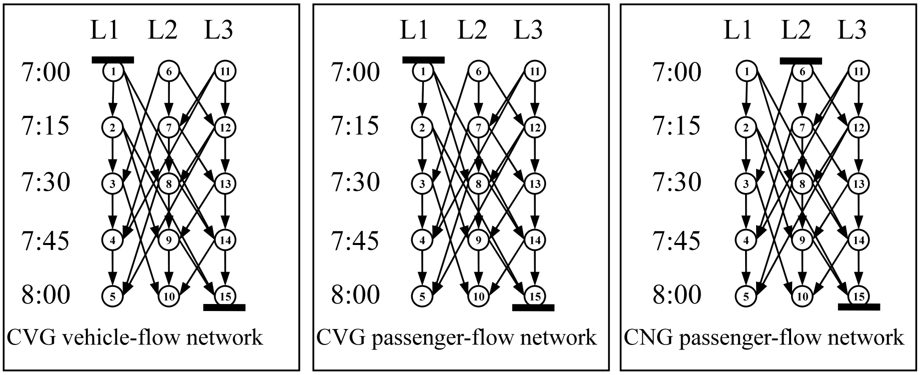

Fig. 9 Time-space networks

Based on these data, three networks are created, one CVG vehicle-, one CVG passenger- and one CNG passenger-flow time-space network (see Fig 9).

Denoting VA/VPA/PA as the set of all arcs in the CVG vehicle-/CVG passenger-/CNG passenger-flow networks, the problem can be formulated as follows.

Formulation

Min

$$
\begin{aligned}
&1 \times 35.1 \times (x_{0607} + x_{0708} + x_{0809} + x_{0910}) \\
&\quad + 1 \times 198.6 \times (x_{0108} + x_{0209} + x_{0310} + x_{0603} + x_{0704} + x_{0805}) \\
&\quad + 1 \times 297.9 \times (x_{0114} + x_{0215} + x_{1104} + x_{1205}) \\
&\quad + 1 \times 99.3 \times (x_{0612} + x_{0713} + x_{0814} + x_{0915} + x_{1107} + x_{1208} + x_{1309} + x_{1410}) \\
&\quad + 1 \times 35.1 \times (z_{0607} + z_{0708} + z_{0809} + z_{0910}) \\
&\quad + 1 \times 33.9 \times (z_{0108} + z_{0209} + z_{0310} + z_{0603} + z_{0704} + z_{0805}) \\
&\quad + 1 \times 50.85 \times (z_{0114} + z_{0215} + z_{1104} + z_{1205}) \\
&\quad + 1 \times 16.95 \times (z_{0612} + z_{0713} + z_{0814} + z_{0915} + z_{1107} + z_{1208} + z_{1309} + z_{1410}) \\
&\quad + 2 \times 35.1 \times (y_{0102} + y_{0203} + y_{0304} + y_{0405}) \\
&\quad + 2 \times 33.9 \times (y_{0108} + y_{0209} + y_{0310} + y_{0603} + y_{0704} + y_{0805}) \\
&\quad + 2 \times 50.85 \times (y_{0114} + y_{0215} + y_{1104} + y_{1205}) \\
&\quad + 2 \times 16.95 \times (y_{0612} + y_{0713} + y_{0814} + y_{0915} + y_{1107} + y_{1208} + y_{1309} + y_{1410})
\end{aligned}
$$

S.t.

$$x_{0102} + x_{0108} + x_{0114} = \omega$$

$$- x_{0102} + x_{0203} + x_{0209} + x_{0215} = 0$$

$$- x_{0203} - x_{0603} + x_{0304} + x_{0310} = 0$$

$$- x_{0304} - x_{0704} - x_{1104} + x_{0405} = 0$$

$$- x_{0405} - x_{0805} - x_{1205} = 0$$

$$x_{0603} + x_{0607} + x_{0612} = 0$$

$$- x_{0607} - x_{1107} + x_{0704} + x_{0708} + x_{0713} = 0$$

$$- x_{0108} - x_{0708} - x_{1208} + x_{0805} + x_{0809} + x_{0814} = 0$$

$$- x_{0209} - x_{0809} - x_{1309} + x_{0910} + x_{0915} = 0$$

$$- x_{0310} - x_{0910} - x_{1410} = 0$$

$$x_{1107} + x_{1104} + x_{1112} = 0$$

$$- x_{0612} - x_{1112} + x_{1218} + x_{1205} + x_{1213} = 0$$

$$- x_{0713} - x_{1213} + x_{1309} + x_{1314} = 0$$

$$- x_{0114} - x_{0814} - x_{1314} + x_{1410} + x_{1415} = 0$$

$$- x_{0215} - x_{0915} - x_{1415} = -\omega$$

$$z_{0102} + z_{0108} + z_{0114} = 2 - \omega$$

$$- z_{0102} + z_{0203} + z_{0209} + z_{0215} = 0$$

$$- z_{0203} - z_{0603} + z_{0304} + z_{0310} = 0$$

$$- z_{0304} - z_{0704} - z_{1104} + z_{0405} = 0$$

$$- z_{0405} - z_{0805} - z_{1205} = 0$$

$$z_{0603} + z_{0607} + z_{0612} = 0$$

$$- z_{0607} - z_{1107} + z_{0704} + z_{0708} + z_{0713} = 0$$

$$- z_{0108} - z_{0708} - z_{1208} + z_{0805} + z_{0809} + z_{0814} = 0$$

$$- z_{0209} - z_{0809} - z_{1309} + z_{0910} + z_{0915} = 0$$

$$- z_{0310} - z_{0910} - z_{1410} = 0$$

$$z_{1107} + z_{1104} + z_{1112} = 0$$

$$- z_{0612} - z_{1112} + z_{1218} + z_{1205} + z_{1213} = 0$$

$$- z_{0713} - z_{1213} + z_{1309} + z_{1314} = 0$$

$$- z_{0114} - z_{0814} - z_{1314} + z_{1410} + z_{1415} = 0$$

$$- z_{0215} - z_{0915} - z_{1415} = -(2 - \omega)$$

$$y_{0102} + y_{0108} + y_{0114} = 0$$

$$- y_{0102} + y_{0203} + y_{0209} + y_{0215} = 0$$

$$- y_{0203} - y_{0603} + y_{0304} + y_{0310} = 0$$

$$- y_{0304} - y_{0704} - y_{1104} + y_{0405} = 0$$

$$- y_{0405} - y_{0805} - y_{1205} = 0$$

$$y_{0603} + y_{0607} + y_{0612} = 1$$

$$- y_{0607} - y_{1107} + y_{0704} + y_{0708} + y_{0713} = 0$$

$$- y_{0108} - y_{0708} - y_{1208} + y_{0805} + y_{0809} + y_{0814} = 0$$

$$- y_{0209} - y_{0809} - y_{1309} + y_{0910} + y_{0915} = 0$$

$$- y_{0310} - y_{0910} - y_{1410} = 0$$

$$y_{1107} + y_{1104} + y_{1112} = 0$$

$$- y_{0612} - y_{1112} + y_{1218} + y_{1205} + y_{1213} = 0$$

$$- y_{0713} - y_{1213} + y_{1309} + y_{1314} = 0$$

$$- y_{0114} - y_{0814} - y_{1314} + y_{1410} + y_{1415} = 0$$

$$- y_{0215} - y_{0915} - y_{1415} = -1$$

$$2 \times y_{0108} + z_{0108} \leq 4 \times x_{0108}$$

$$2 \times y_{0114} + z_{0114} \leq 4 \times x_{0114}$$

$$2 \times y_{0209} + z_{0209} \leq 4 \times x_{0209}$$

$$2 \times y_{0215} + z_{0215} \leq 4 \times x_{0215}$$

$$2 \times y_{0310} + z_{0310} \leq 4 \times x_{0310}$$

$$2 \times y_{0603} + z_{0603} \leq 4 \times x_{0603}$$

$$2 \times y_{0612} + z_{0612} \leq 4 \times x_{0612}$$

$$2 \times y_{0704} + z_{0704} \leq 4 \times x_{0704}$$

$$2 \times y_{0713} + z_{0713} \leq 4 \times x_{0713}$$

$$2 \times y_{0805} + z_{0805} \leq 4 \times x_{0805}$$

$$2 \times y_{0814} + z_{0814} \leq 4 \times x_{0814}$$

$$2 \times y_{0915} + z_{0915} \leq 4 \times x_{0915}$$

$$2 \times y_{1104} + z_{1104} \leq 4 \times x_{1104}$$

$$2 \times y_{1107} + z_{1107} \leq 4 \times x_{1107}$$

$$2 \times y_{1205} + z_{1205} \leq 4 \times x_{1205}$$

$$2 \times y_{1208} + z_{1208} \leq 4 \times x_{1208}$$

$$2 \times y_{1309} + z_{1309} \leq 4 \times x_{1309}$$

$$2 \times y_{1410} + z_{1410} \leq 4 \times x_{1410}$$

$$x_{ij} \geq 0, \text{integer} \quad \forall (i, j) \in VA$$

$$y_{ij} \geq 0, \text{integer} \quad \forall (i, j) \in PA$$

$$z_{ij} \geq 0, \text{integer} \quad \forall (i,j) \in VPA$$

$$\omega \geq 0, \text{integer}$$

Solution CPLEX can be used to optimally solve the problem with the objective of 382.65 and $\omega = 1$, $x_{0108} = 1$, $x_{0814} = 1$, $x_{1415} = 1$, $z_{0108} = 1$, $z_{0814} = 1$, $z_{1415} = 1$, $y_{0607} = 1$, $y_{0708} = 1$, $y_{0814} = 1$, $y_{1415} = 1$ with the other variables being 0. This solution can be manually checked to be correct.

Extension The formulation needs to be modified if more CVG types and CNG types are considered. For example, if two one-person CVG groups of type 1 (i.e., non-smoking female only) from L1 at 7:00 to L3 at 8:00 and a two-person CNG group of type 4 (i.e. non-smoking female with no request) from L2 at 7:00 to L3 at 8:00 join the carpool plan, then the formulation is modified as follows: Firstly, three networks similar to the previous three are added, i.e., an additional CVG and CVPG network for the additional two CVGs, and an additional CNG network for the additional CNG. Secondly, the decision variables are similarly defined. Thirdly, a similar operating cost for the three new networks is added to the objective. Fourthly, retaining all the original constraints, the network conservation constraints for the three new networks and the variable domain constraints are similarly added. Finally, the carrying constraints for the travel arcs in the additional CVG network are added. Note that these carrying constraints are different from the previous ones, because the additional CVGs can carry the previous CVGs/CNG and the additional CVGs and CNG. To clearly describe these new carrying constraints, assume that xx_{ij}, zz_{ij}, and yy_{ij} represent the variables associated with arc (i, j) in the three additional networks, respectively. The carrying constraint associated with arc (01, 08) in the additional CVG network can now be represented as $2 \times y_{0108} + z_{0108} + 2 \times yy_{0108} + zz_{0108} \leq 4 \times xx_{0108}$. Other carrying constraints can be similarly constructed. The formulation is now extended to include two types of CVGs and two types of CNGs. Finally, the formulation can be similarly modified to extend to other problems indicated in the problem description (Sect. 2).

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