

# Reliability of manufacturing equipment in complex environments

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**Abstract** We present two stochastic failure models for the reliability evaluation of manufacturing equipment that degrades due to its complex operating environment. The first model examines the case when the environment is a temporally nonhomogeneous continuous-time Markov chain, and the second assumes the environment is a temporally homogeneous semi-Markov process on a finite space. Derived are transform expressions for the lifetime distributions. A few examples are provided to illustrate the main results.

## 1 Introduction

Modern manufacturing organizations employ very complex, flexible machines that can be used to manufacture a variety of components using a wide range of material types. A critical factor in the economics of modern manufacturing facilities is the useful lifetime of machine tools that degrade over time under normal usage and fluctuating operating conditions that can be driven by external influences. It is well known that many factors can influence the rate at which machine tools degrade. Some of the most important factors include the work-piece material, the tool material, the cutting speed, and the depth of cut, to name only a few. Besides these obvious contributors to tool degradation, the ambient environment of the

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manufacturing equipment can significantly impact the rate at which tools (or other critical components) degrade. For instance, significant fluctuations in the ambient temperature or humidity can cause workpiece (or tool) expansion and contraction, and even premature corrosion. Because machine tools constitute a significant portion of a manufacturing organization's capital, and because their degradation or failure can significantly impact production and inventory levels, it is important to assess the useful lifetime of manufacturing equipment when it is subjected to complex dynamics.

In this paper we propose two stochastic models that can be used to derive mathematical expressions for the reliability of components that degrade dynamically due to their complex ambient environment or time-varying operating conditions. The rate of degradation is modulated by an external process (the random environment) which is modeled as a continuous-time, discrete-state stochastic process  $\{Z(t) : t \geq 0\}$ . At some initial time, say  $s$ , the machine tool (or component) begins its lifetime in perfect working order but degrades over time under the influence of its random environment until the cumulative degradation reaches a fixed threshold  $x$ , at which time it is no longer useful because it may not be able to maintain manufacturing specifications. Derived herein are transform expressions for the cumulative distribution function (c.d.f.) of the useful lifetime of the tool when the evolution of the operating environment assumes two distinct forms. First, we consider the case when the environment evolves as a *nonhomogeneous, continuous-time Markov chain* (NHCTMC) on a finite state space. Second, we consider the case of a *time-homogeneous semi-Markov process* (SMP) environment, also on a finite state space. For the first model, the main transient results are obtained by showing that the joint distribution of the cumulative degradation level and the environment state conditionally satisfies a nonhomogeneous partial differential equation. This equation leads to the Laplace-Stieltjes transform (LST) of the lifetime distribution function under certain mild conditions. For the second model, we employ a classical Markov renewal argument to derive the (double) Laplace transform of the component's lifetime distribution.

Many researchers have proposed stochastic models of machine tool reliability, primarily for the purpose of prescribing optimal tool replacement policies, or to minimize processing times. A sampling of these contributions can be found in Hirvikorpi et al. (2007), Hsu and Shu (2010), Jeang (1999), Makis (1999), Mazzuchi and Soyer (1989). Others have considered the control of either manufacturing or production/inventory systems using nonhomogeneous Markov chains to model the failure rate which is assumed to be dependent on the production rate of the system (cf. Liberopoulos and Caramanis 1994; Martinelli 2007; Martinelli and Piedimonte 2004, 2008). More generally, stochastic failure models that attempt to capture the impact of a randomly evolving environment have been examined extensively in the reliability and applied probability literature. An excellent survey due to Singpurwalla (1995) presents a number of models with various attributes. Much earlier work, due to Esary et al. (1973) provided several results for wear and shock processes. Many authors have considered the case when the environment evolves as a homogeneous, continuous-time Markov chain. For example, Çinlar (1977) demonstrated that the joint process of the unit's wear level and the state of its ambient environment may be considered as a Markov-additive process. He considered the case when the random environment is a general Markov process with a finite state space and the wear is assumed to increase as a Lévy process. Li and Luo (2005) considered a Markov-modulated shock process wherein the shock inter-arrival times and the random shock damage are both governed by a Markov chain. They obtain reliability bounds when the inter-arrival times have heavy- or light-tailed distributions; however, their degradation model does not include a continuous wear component. Kiessler et al. (2002) investigated the limiting average availability of a system whose time-varying wear rates are

governed by a continuous-time Markov chain. Kharoufeh et al. (2006) extended the model of Kiessler et al. (2002) by including damage-inducing shocks that arrive according to a time-homogeneous Poisson process and deriving the Laplace-Stieltjes transforms of a few transient reliability indices. Ebrahimi (2006) considered a system whose degradation is comprised of a continuous wear component as well as jumps. The properties of the model were investigated and bounds were established for the reliability function. However, the random environment in that model corresponds to a shock process that is superimposed on a gamma wear process.

Models that consider temporally nonhomogeneous Markov or homogeneous semi-Markov environments can also be found in the stochastic modeling and operations research literature. Platis et al. (1996, 2003), Platis (2006) have made extensive use of nonhomogeneous Markov models for assessing the performance and reliability (or performability) of complex, time-dependent systems, while Smotherman and Zemoudeh (1989) examined phased-mission reliability models using nonhomogeneous Markov models to reflect globally-time-dependent phase changes. Platis et al. (1998) analyzed hitting times for a discrete-time nonhomogeneous Markov chain and showed how these can be used for reliability evaluation.

The analysis we present here serves to extend the results of Kharoufeh (2003), Kharoufeh and Cox (2005), Kiessler et al. (2002) and others by considering environments that are either (i) temporally nonhomogeneous in their stochastic evolution, or (ii) non-Markovian but temporally homogeneous. Here, we solely focus our attention on transient reliability indices (i.e., when the critical degradation threshold and time are both finite). For the NHCTMC model, we provide a full characterization of the Laplace-Stieltjes transform of the lifetime distribution when the infinitesimal generator matrix entries satisfy some mild conditions. For the SMP model, we provide a double transform characterization of the lifetime distribution and illustrate the results assuming a few state sojourn time distributions.

The remainder of the paper is organized as follows. Section 2 provides a brief model description followed by the full analysis of the NHCTMC model in Sect. 3. Section 4 presents the results for the SMP model, while Sect. 5 provides a few concluding remarks and some directions for future research.

## 2 Model description

Here, we describe a general framework for modeling equipment degradation in the context of complex manufacturing environments. In the following discussion, we assume that all random variables are defined on a common and complete probability space denoted by  $(\Omega, \mathcal{A}, \mathbb{P})$ . We assume that the equipment (hereafter referred to as the unit) is placed into service at some initial time  $s$  ( $s \geq 0$ ) with no degradation (i.e., its condition is new initially). Due to normal usage and the influence of its environment, it accumulates degradation until a deterministic, critical threshold value  $x$  is reached or exceeded, at which time the system is considered to be failed. As a prototypical example, consider manufacturing operations involving metal cutting and material removal. Most cutting tools degrade in a manner that is dictated by the cutting speed, the workpiece material, cutting angle, use of cutting fluids, and a number of other critical factors. However, once the wear level of the tool reaches a critical level  $x$ , it may not be possible for it to maintain manufacturing specifications. Our primary concern is the development of a stochastic model for characterizing the random amount of time needed for the unit's cumulative degradation level to first reach this critical threshold.

We denote this time by a proper non-negative random variable  $T_x(s)$ . We include the dependence of this first passage time on the initial time  $s$ ; however, when the environment is homogeneous and we assume  $s = 0$ , we will suppress the variable  $s$ .

The degradation of the unit is induced by (i) the actual ambient environment in which it resides and operates (e.g., the ambient air temperature or humidity can induce degradation); and/or (ii) the various operational settings of the equipment. Degradation of the first type might occur due to temperature fluctuations that cause contraction and expansion of the workpiece and/or the cutting tool. For the second type, if different materials are processed according to an *a priori* schedule (e.g., according to a cyclic Markov chain), but the processing times are stochastic, the unit will experience different rates of degradation depending on the current workpiece. The key point here is that the *rate* of degradation, and the time spent in each degradation rate, are governed by a continuous-time stochastic process termed the random environment which shall be denoted throughout by  $\mathcal{Z} \equiv \{Z(t) : t \geq 0\}$ . The state space of this process is a finite set  $S = \{1, 2, \dots, \ell\}$  where each state of  $S$  corresponds to the current prevailing environment or operating condition.

The primary objective of this paper is to analyze two specific and distinct models for the governing environment  $\mathcal{Z}$ . For the first model, we assume that  $\mathcal{Z}$  is a nonhomogeneous continuous-time Markov chain (NHCTMC) whose transition functions (and infinitesimal generator matrix) depend explicitly on time. In the second case, we will examine a relaxation of the Markovian assumption by allowing  $\mathcal{Z}$  to evolve as a semi-Markov environment but insist that the environment be time-homogeneous for this latter case. For both models, whenever  $Z(t) = j \in S$ , the unit degrades at a constant rate  $r(j) > 0$ . More specifically, the rate of degradation evolves as a stochastic process and is modulated by the environment. For later use, we define the diagonal matrix containing these environment-dependent rates by  $\mathbf{R}_d \equiv \text{diag}(r(1), r(2), \dots, r(\ell))$ . The assumption of a constant degradation rate in each environment state amounts to assuming that the degradation process has samples paths that are piecewise linear and monotonically increasing almost surely (a.s.). This assumption is not restrictive since the degradation sample path can be approximated by a piecewise linear path as noted in Kharoufeh and Cox (2005). Furthermore, the approach is well-suited for modeling linear degradation paths which are common in many applications.

For both the NHCTMC and SMP models, given that the unit is placed into service at time  $s$  in perfect condition, we define the cumulative degradation of the unit up to time  $t$  by a random variable  $X(t)$ . For any  $t \geq s$ , this random variable is given by

$$X(t) = X(s) + \int_s^t r(Z(v)) \, dv \tag{1}$$

where we assume  $X(s) = 0$  a.s. and

$$\int_s^t |r(Z(v))| \, dv < \infty \quad \text{a.s.}$$

so that  $X(t)$  is well defined for each  $t \geq 0$ . Because we assume  $r(j) > 0$  for each  $j \in S$ , the sample paths of  $\{X(t) : t \geq 0\}$  are monotonically increasing a.s., and consequently, the events  $\{X(t) \leq x\}$  and  $\{T_x(s) > t\}$  are equivalent. The system’s random lifetime is the first passage time

$$T_x(s) = \inf\{t > 0 : X(t) \geq x\}, \tag{2}$$

or the first time the degradation process  $\{X(t) : t \geq 0\}$  reaches or exceeds  $x$ . Let  $F(x, s, t) \equiv \mathbb{P}(T_x(s) \leq t) = 1 - \mathbb{P}(X(t) \leq x)$  denote the unconditional c.d.f. of the unit’s lifetime, and

let its conditional counterpart be  $F_i(x, s, t) = \mathbb{P}(T_x(s) \leq t | Z(s) = i)$ ,  $i \in S$ , where  $i$  denotes the state of the environment at the initial time  $s$ . In the case when the environment is time-homogeneous, we drop the dependence on  $s$  and simply write  $T_x$ ,  $F(x, t)$ , and  $F_i(x, t)$  for the first passage time, its unconditional c.d.f., and its conditional c.d.f., respectively. The indicator function of an event  $A \in \mathcal{A}$  will be denoted by  $\mathbf{1}(A)$ , and it assumes the value 1 if  $A$  holds and 0 otherwise. Our primary aim is to derive transform expressions for the unconditional and/or conditional lifetime distributions for the NHCTMC and SMP models.

### 3 Time-nonhomogeneous Markov environment

In this section, we consider the case in which the unit’s environment evolves as a finite, nonhomogeneous continuous-time Markov chain (NHCTMC). These processes afford a great deal of modeling flexibility and have been applied in a variety of contexts including manufacturing and production systems (cf. Martinelli and Piedimonte 2004; Martinelli 2007) and remaining lifetime prognosis in health care (Pérez-Ocón et al. 2001). Here, we focus our attention on transient (as opposed to asymptotic) reliability indices in order to account for the environment’s explicit dependence on time. Our first main result shows that the case of a time-homogeneous, Markovian environment (cf. Kharoufeh and Cox 2005; Kharoufeh et al. 2006) can be extended very naturally to the time-nonhomogeneous case.

#### 3.1 Transient lifetime distribution

For each  $t \geq 0$ , let  $Z(t)$  denote the status of the environment at time  $t$ , and assume  $\mathcal{Z} \equiv \{Z(t) : t \geq 0\}$  is a NHCTMC with state space  $S = \{1, 2, \dots, \ell\}$ ,  $\ell \in \mathbb{Z}_+$ . For any  $s, t$  such that  $0 \leq s \leq t$ , the transition functions of  $\mathcal{Z}$  are given by

$$p_{ij}(s, t) = \mathbb{P}(Z(t) = j | Z(s) = i), \quad i, j \in S,$$

and the time-nonhomogeneous transition matrix is denoted by  $\mathbf{P}(s, t) = [p_{ij}(s, t)]_{i, j \in S}$ . Note that  $\mathbf{P}(s, s) = \mathbf{I}$ , where  $\mathbf{I}$  denotes the identity matrix. It is well known that the time-dependent infinitesimal generator matrix of  $\mathcal{Z}$ , denoted by  $\mathbf{Q}(s)$ ,  $s \geq 0$ , has elements

$$q_{ij}(s) = \lim_{h \rightarrow 0} \frac{p_{ij}(s, s+h) - \delta_{ij}}{h}, \quad i, j \in S, j \neq i,$$

and when  $j = i$ ,

$$-q_{ii}(s) = q_i(s) = \lim_{h \rightarrow 0} \frac{1 - p_{ii}(s, s+h)}{h}$$

so that  $q_i(s) = \sum_{j \neq i} q_{ij}(s)$ . These entries are interpreted as follows: When  $j \neq i$ ,  $q_{ij}(s)$  is the instantaneous rate at which the environment transitions from state  $i$  to state  $j$  at time  $s$ . When  $j = i$ ,  $q_i(s)$  is the total rate at which the process transitions out of state  $i$  at time  $s$ . (Note that  $q_{ii}(s) < 0$  for all  $s \geq 0$ .) In case the environment is time-homogeneous, then of course,  $\mathbf{P}(s, t) = \mathbf{P}(t - s)$  and  $\mathbf{Q}(s) = \mathbf{Q}$  for all  $s \geq 0$ .

For any  $s$  and  $t$  such that  $0 \leq s \leq t$ , define

$$V_{ij}(x, s, t) = \mathbb{P}(X(t) \leq x, Z(t) = j | Z(s) = i), \quad i, j \in S,$$

the joint probability that, at time  $t$ , the degradation of the system has not exceeded level  $x$ , and the environment process is in state  $j \in S$ , given that at that initial time  $s$  the environment was in state  $i \in S$ . For ease of notation, we also define the  $\ell \times \ell$  matrix

$V(x, s, t) = [V_{ij}(x, s, t)]$ . As noted in Sect. 2, because the degradation rates,  $\{r(j) : j \in S\}$ , are assumed to be strictly positive real numbers,  $\{X(t) : t \geq s\}$  is a continuous additive functional of  $\mathcal{Z}$  whose sample paths are monotone increasing almost surely. This monotonicity ensures that  $\{X(t) \leq x\} \equiv \{T_x(s) > t\}$ ; therefore, we can derive the c.d.f. (or complementary c.d.f.) of  $T_x(s)$  by analyzing the cumulative degradation  $X(t)$ . To this end, we next characterize the joint distribution functions  $V_{ij}(x, s, t)$ . This analysis is similar to the arguments presented in Kharoufeh (2003), Kharoufeh and Cox (2005), Kharoufeh et al. (2006), but requires some extra care to account for the time-nonhomogeneity of  $\mathcal{Z}$ . The following theorem shows that the matrix  $V(x, s, t)$  verifies a Chapman-Kolmogorov-type (partial) differential equation.

**Theorem 1** *If the operating environment,  $\{Z(t) : t \geq 0\}$ , is a nonhomogeneous CTMC with infinitesimal generator matrix  $Q(t)$ , then the matrix  $V(x, s, t)$  satisfies the partial differential equation*

$$\frac{\partial V(x, s, t)}{\partial t} + \frac{\partial V(x, s, t)}{\partial x} \mathbf{R}_d = V(x, s, t) \mathbf{Q}(t), \tag{3}$$

where  $\mathbf{R}_d = \text{diag}(r(1), \dots, r(\ell))$ .

*Proof* The proof of the result is standard for Markov processes and mirrors the proof of the time-homogeneous case presented in Kharoufeh and Cox (2005). Fix a small time increment  $\varepsilon > 0$  and suppose  $0 \leq s \leq t$ . Utilizing the Markov property of  $\mathcal{Z}$ , we obtain

$$\begin{aligned} V_{ij}(x, s, t + \varepsilon) &= \mathbb{P}(X(t + \varepsilon) \leq x, Z(t + \varepsilon) = j | Z(s) = i) \\ &= \sum_{k=1}^{\ell} \mathbb{P}(X(t + \varepsilon) \leq x, Z(t + \varepsilon) = j | Z(t) = k, Z(s) = i) \mathbb{P}(Z(t) = k) \\ &= \sum_{k=1}^{\ell} \mathbb{P}(Z(t + \varepsilon) = j | Z(t) = k, Z(s) = i) \mathbb{P}(X(t + \varepsilon) \leq x | Z(t) = k, Z(s) = i) \\ &= \sum_{k=1}^{\ell} \mathbb{P}(Z(t + \varepsilon) = j | Z(t) = k) \mathbb{P}(X(t + \varepsilon) \leq x, Z(t) = k | Z(s) = i) \\ &= \sum_{k=1}^{\ell} p_{kj}(t, t + \varepsilon) \mathbb{P}(X(t + \varepsilon) \leq x, Z(t) = k | Z(s) = i) \\ &= \sum_{k=1}^{\ell} p_{kj}(t, t + \varepsilon) V_{ik}(x - r(k)\varepsilon, s, t). \end{aligned} \tag{4}$$

Substituting the equality,

$$p_{kj}(t, t + \varepsilon) = \begin{cases} 1 + q_{jj}(t)\varepsilon + o(\varepsilon), & k = j, \\ q_{kj}(t)\varepsilon + o(\varepsilon), & k \neq j, \end{cases}$$

in (4) and rearranging terms, we obtain

$$\begin{aligned} V_{ij}(x, s, t + \varepsilon) &= [1 + q_{jj}(t)\varepsilon + o(\varepsilon)]V_{ij}(x - r(j)\varepsilon, s, t) \\ &\quad + \sum_{k \in S \setminus \{j\}} p_{kj}(t, t + \varepsilon)V_{ik}(x - r(k)\varepsilon, s, t) \\ &= V_{ij}(x - r(j)\varepsilon, s, t) + \varepsilon \sum_{k=1}^{\ell} q_{kj}(t)V_{ik}(x - r(k)\varepsilon, s, t) + o(\varepsilon). \end{aligned}$$

Therefore,

$$V_{ij}(x, s, t + \varepsilon) - V_{ij}(x - r(j)\varepsilon, s, t) = \varepsilon \sum_{k=1}^{\ell} q_{kj}(t)V_{ik}(x - r(k)\varepsilon, s, t) + o(\varepsilon). \tag{5}$$

Adding and subtracting  $V_{ij}(x, s, t)$  to the left-hand side of (5) gives

$$\begin{aligned} &[V_{ij}(x, s, t + \varepsilon) - V_{ij}(x, s, t)] + [V_{ij}(x, s, t) - V_{ij}(x - r(j)\varepsilon, s, t)] \\ &= \varepsilon \sum_{k=1}^{\ell} q_{kj}(t)V_{ik}(x - r(k)\varepsilon, s, t) + o(\varepsilon). \end{aligned} \tag{6}$$

Now, dividing each term of (6) by  $\varepsilon$  and allowing  $\varepsilon \downarrow 0$ , we obtain

$$\frac{\partial V_{ij}(x, s, t)}{\partial t} + r(j) \frac{\partial V_{ij}(x, s, t)}{\partial x} = \sum_k q_{kj}(t)V_{ik}(x, s, t), \tag{7}$$

which can be written in matrix form as

$$\frac{\partial \mathbf{V}(x, s, t)}{\partial t} + \frac{\partial \mathbf{V}(x, s, t)}{\partial x} \mathbf{R}_d = \mathbf{V}(x, s, t) \mathbf{Q}(t). \quad \square$$

We pause here to remark that the nonhomogeneous equation (3) extends the Chapman-Kolmogorov-type equation derived in Kharoufeh (2003), Kharoufeh and Cox (2005) to scenarios in which the dynamics of the environment’s evolution depend explicitly on time. While (3) is more general, its parametrization can be problematic in practice if the environment cannot be continuously observed. Nevertheless, the model can be used to capture these complex dynamics, particularly if the manufacturing equipment is heavily influenced by the effects of aging.

Note that (7), when expanded term-by-term, forms a system of partial differential equations that must be solved simultaneously. For example, if  $\mathcal{Z}$  has the state space  $S = \{1, 2\}$  (i.e., there are two distinct environment states), the resulting system has four equations and four unknown functions (namely  $V_{11}$ ,  $V_{12}$ ,  $V_{21}$ , and  $V_{22}$ ) with the obvious boundary condition  $V_{ij}(0, s, t) = 0$  and initial conditions  $V_{ii}(x, s, s) = 1$  and  $V_{ij}(x, s, s) = 0, j \neq i$ . This set of partial differential equations is given by

$$\begin{aligned} \frac{\partial V_{11}}{\partial t} + r(1) \frac{\partial V_{11}}{\partial x} &= q_{11}(t)V_{11} + q_{21}(t)V_{12}, \\ \frac{\partial V_{12}}{\partial t} + r(2) \frac{\partial V_{12}}{\partial x} &= q_{12}(t)V_{11} + q_{22}(t)V_{12}, \end{aligned}$$

$$\begin{aligned} \frac{\partial V_{21}}{\partial t} + r(1) \frac{\partial V_{21}}{\partial x} &= q_{11}(t)V_{21} + q_{21}(t)V_{22}, \\ \frac{\partial V_{22}}{\partial t} + r(2) \frac{\partial V_{22}}{\partial x} &= q_{12}(t)V_{21} + q_{22}(t)V_{22}. \end{aligned}$$

In general, if the environment has  $\ell$  ( $\ell < \infty$ ) states, then we must solve  $\ell^2$  partial differential equations with  $\ell^2$  unknown functions. Since the spatial and temporal variables ( $x$  and  $t$ , respectively) are weakly coupled, the method of separation of variables cannot be used to obtain  $V(x, s, t)$  in closed-form. Therefore, our aim is to characterize  $V(x, s, t)$  via a (Laplace) transform solution.

To that end, let  $u$  be a transform variable with  $\text{Re}(u) > 0$  and let  $0 \leq s \leq t$ . Define the  $\ell \times \ell$  matrix

$$V^*(u, s, t) = \int_0^\infty e^{-ux} V(x, s, t) dx, \tag{8}$$

the Laplace transform (LT) of the matrix  $V(x, s, t)$  with respect to the degradation threshold  $x$ . For later use, we also define

$$\tilde{V}(u, s, t) = \int_0^\infty e^{-ux} V(dx, s, t),$$

the Laplace-Stieltjes transform (LST) of  $V(x, s, t)$  with respect to  $x$  and note that

$$\tilde{V}(u, s, t) = uV^*(u, s, t). \tag{9}$$

Now, taking the LT of both sides of (3) with respect to  $x$  and using the well-known identity

$$\mathcal{L}\left(\frac{\partial V(x, s, t)}{\partial x}\right) = uV^*(u, s, t) - V(0, s, t),$$

we obtain

$$\frac{\partial V^*(u, s, t)}{\partial t} + (uV^*(u, s, t) - V(0, s, t))\mathbf{R}_d = V^*(u, s, t)\mathbf{Q}(t). \tag{10}$$

Note that  $s$  is a fixed initial time and  $u$  can be viewed as a constant. Therefore, by applying the boundary condition,  $V(0, s, t) = \mathbf{0}$  (the zero matrix), and rearranging the terms of (10), we obtain a linear first-order differential equation in  $t$ ,

$$\frac{dV^*(u, s, t)}{dt} = V^*(u, s, t)\mathbf{A}(u, t) \tag{11}$$

where  $\mathbf{A}(u, t) = \mathbf{Q}(t) - u\mathbf{R}_d$ ,  $t \geq 0$ . Our objective is to solve the following initial value problem (IVP) in the Laplace transform space:

$$\frac{dV^*(u, s, t)}{dt} = V^*(u, s, t)\mathbf{A}(u, t), \tag{12a}$$

$$V^*(u, s, s) = u^{-1}\mathbf{I} \tag{12b}$$

where (12b) follows from the initial condition

$$V_{ij}(x, s, s) = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$



In what follows,  $\exp(\mathbf{A})$  denotes matrix exponentiation of a square matrix  $\mathbf{A}$ , and the commutator of two square matrices  $\mathbf{A}$  and  $\mathbf{B}$  is defined to be

$$[\mathbf{A}, \mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA}.$$

Theorem 2 characterizes the solution of IVP (12) using a result due to Magnus (1954).

**Theorem 2** *For any fixed initial time  $s$ , the solution to the initial value problem (12) is the matrix  $\mathbf{V}^*(u, s, t)$  given by*

$$\mathbf{V}^*(u, s, t) = u^{-1} \exp[\mathbf{\Omega}(u, s, t)]$$

where

$$\mathbf{\Omega}(u, s, t) = \sum_{k=1}^{\infty} \mathbf{\Omega}_k(u, s, t)$$

and

$$\begin{aligned} \mathbf{\Omega}_1(u, s, t) &= \int_s^t \mathbf{A}(u, t_1) dt_1 \\ \mathbf{\Omega}_2(u, s, t) &= \frac{1}{2} \int_s^t \int_s^{t_1} [\mathbf{A}(u, t_1), \mathbf{A}(u, t_2)] dt_2 dt_1 \\ \mathbf{\Omega}_3(u, s, t) &= \frac{1}{6} \int_s^t \int_s^{t_1} \int_s^{t_2} ([\mathbf{A}(u, t_1), [\mathbf{A}(u, t_2), \mathbf{A}(u, t_3)]] \\ &\quad + [\mathbf{A}(u, t_3), [\mathbf{A}(u, t_2), \mathbf{A}(u, t_1)]]) dt_3 dt_2 dt_1 \\ &\quad \vdots \end{aligned}$$

*Proof* Note that (11) can be easily solved using an integrating factor if  $\mathbf{A}(u, v)$  and  $\mathbf{A}(u, w)$  commute for any pair of  $t$  values  $v, w \in \mathbb{R}_+$ . In general, these matrices do not commute; however, by invoking Magnus’ Theorem (see Blanes et al. 2009; Magnus 1954), we can assert that solutions of (11) are matrix exponentials of the form

$$\mathbf{C} \exp[\mathbf{\Omega}(u, s, t)]$$

where  $\mathbf{C}$  is a constant square matrix (with respect to  $t$ ),  $\mathbf{\Omega}(u, s, t)$  is the solution of the differential equation

$$\frac{d}{dt} \mathbf{\Omega} = \sum_{n=0}^{\infty} \frac{B_n}{n!} \text{ad}_{\mathbf{\Omega}}^n(\mathbf{A}(u, t)) = \mathbf{A}(u, t) - \frac{1}{2} [\mathbf{\Omega}, \mathbf{A}(u, t)] + \frac{1}{12} [\mathbf{\Omega}, [\mathbf{\Omega}, \mathbf{A}(u, t)]] + \dots$$

$B_n$  is the  $n$ th Bernoulli number, and  $\text{ad}_{\mathbf{\Omega}}$  is an operator acting on a square matrix defined by the commutator

$$\text{ad}_{\mathbf{\Omega}}(\mathbf{B}) = [\mathbf{\Omega}, \mathbf{B}] = \mathbf{\Omega}\mathbf{B} - \mathbf{B}\mathbf{\Omega}.$$

(This notation suppresses the symbols  $(u, s, t)$  in  $\mathbf{\Omega}$ .) The series expansion for the solution matrix  $\mathbf{\Omega}(u, s, t)$ ,

$$\mathbf{\Omega}(u, s, t) = \sum_{k=1}^{\infty} \mathbf{\Omega}_k(u, s, t),$$

determined by Magnus (1954) and commonly referred to as the *Magnus expansion*, is given by

$$\begin{aligned} \mathbf{\Omega}(u, s, t) = & \int_s^t \mathbf{A}(u, t_1) dt_1 + \frac{1}{2} \int_s^t \left[ \int_s^{t_1} \mathbf{A}(u, t_2) dt_2, \mathbf{A}(u, t_1) \right] dt_1 \\ & + \frac{1}{6} \int_s^t \int_s^{t_1} \int_s^{t_2} ([\mathbf{A}(u, t_1), [\mathbf{A}(u, t_2), \mathbf{A}(u, t_3)]] \\ & + [\mathbf{A}(u, t_3), [\mathbf{A}(u, t_2), \mathbf{A}(u, t_1)]]]) dt_3 dt_2 dt_1 + \dots \end{aligned}$$

which is known to commute with  $\frac{d}{dt}\mathbf{\Omega}(u, s, t)$ . Consequently, we may write

$$\frac{d}{dt} \exp[\mathbf{\Omega}(u, s, t)] = \exp[\mathbf{\Omega}(u, s, t)] \mathbf{A}(u, t),$$

so the solution to the initial value problem (12) is given by

$$\mathbf{V}^*(u, s, t) = u^{-1} \mathbf{I} \exp[\mathbf{\Omega}(u, s, t)] = u^{-1} \exp[\mathbf{\Omega}(u, s, t)]. \quad \square$$

By noting that  $\tilde{\mathbf{V}}(u, s, t) = u \mathbf{V}^*(u, s, t)$ , Theorem 2 allows us to write LST expressions for the unconditional and conditional c.d.f.s of  $T_x(s)$ , respectively. When  $\mathcal{Z}$  is a NHCTMC, it is well-known that the distribution of  $Z(s)$  is given by the row vector

$$\boldsymbol{\alpha}(s) = [\mathbb{P}(Z(s) = i)]_{i \in S} = \boldsymbol{\alpha}(0) \mathbf{P}(0, s) = \boldsymbol{\alpha}(0) \exp\left(\int_0^s \mathbf{Q}(\tau) d\tau\right). \quad (13)$$

To simplify notation in the results that follow, let us also define

$$\mathbf{e}'_i(s) = [\mathbf{1}(Z(s) = i)]_{i \in S},$$

a row vector of order  $\ell$  whose elements are all zero except for the  $i$ th entry which is unity if  $Z(s) = i$ , and let  $\mathbf{e} = [1, 1, \dots, 1]'$  be a column vector of ones of order  $\ell$ . Note that  $\boldsymbol{\alpha}$  and  $\mathbf{e}'_i$  both depend on the initial time  $s$  because, although the initial cumulative degradation  $X(s) = 0$  a.s., we must know the distribution of the state of the environment at time  $s$  to obtain the conditional and unconditional distributions of  $T_x(s)$  (via their respective transforms). The unconditional c.d.f. of  $T_x(s)$  is given by

$$\begin{aligned} F(x, s, t) = \mathbb{P}(T_x(s) \leq t) &= 1 - \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} V_{ij}(x, s, t) \mathbb{P}(Z(s) = i) \\ &= 1 - \boldsymbol{\alpha}(s) \mathbf{V}(x, s, t) \mathbf{e}, \end{aligned} \quad (14)$$

and its Laplace-Stieltjes transform will be denoted by

$$\tilde{F}(u, s, t) = \int_0^{\infty} e^{-ux} F(dx, s, t).$$

Likewise, for each  $i \in S$ , define the conditional c.d.f. of  $T_x(s)$ , given that  $Z(s) = i$ , by

$$\begin{aligned} F_i(x, s, t) = \mathbb{P}(T_x(s) \leq t | Z(s) = i) &= 1 - \sum_{j=1}^{\ell} V_{ij}(x, s, t) \\ &= 1 - \mathbf{e}'_i(s) \mathbf{V}(x, s, t) \mathbf{e}, \end{aligned} \quad (15)$$

and its LST

$$\tilde{F}_i(u, s, t) = \int_0^\infty e^{-ux} F_i(dx, s, t).$$

With these definitions, we can now state the following proposition.

**Proposition 1** *Suppose the unit operates in a time-nonhomogeneous CTMC environment,  $\{Z(t) : t \geq 0\}$  on state space  $S = \{1, 2, \dots, \ell\}$  with time-dependent infinitesimal generator matrix  $Q(t)$ . Then the LST of the unconditional lifetime c.d.f.,  $F(x, s, t)$ , is given by*

$$\tilde{F}(u, s, t) = 1 - \alpha(s) \exp[\Omega(u, s, t)] e, \quad \text{Re}(u) > 0, \tag{16}$$

and the LST of the conditional lifetime c.d.f.,  $F_i(x, s, t)$ , is

$$\tilde{F}_i(u, s, t) = 1 - e'_i(s) \exp[\Omega(u, s, t)] e, \quad \text{Re}(u) > 0, \tag{17}$$

where  $\Omega(u, s, t)$  is given by the Magnus expansion of Theorem 2.

*Proof* Equations (16) and (17) are obtained directly by taking, respectively, the LSTs of (14) and (15) with respect to  $x$  and substituting the result  $V(u, s, t) = \exp[\Omega(u, s, t)]$ .  $\square$

The results of Theorem 2 and Proposition 1 serve to generalize known results for homogeneous environments to a much broader class of temporally nonhomogeneous environments. Indeed, to ensure that the solution matrix  $V^*(u, s, t)$  is of the exponential form  $\exp(\Omega(u, s, t))$  we require only that the entries of  $Q(t)$  are Riemann integrable functions of  $t$  that satisfy

$$\int_s^t \|(\mathbf{Q}(\tau) - u\mathbf{R}_d)\|_2 \, d\tau < \pi$$

(see Moan 2002) where  $\|\cdot\|_p$  denotes the usual  $p$ -norm of a matrix. Magnus (1954) showed that the series  $\sum_{k=1}^\infty \Omega_k(u, s, t)$  will terminate at a finite index  $M$  if

$$\left[ \mathbf{A}(u, t), \int_s^t \mathbf{A}(u, \tau) \, d\tau \right] = \mathbf{0},$$

for all values of  $t$ . That is, if  $\mathbf{A}(u, t)$  and its integral commute for all  $t$ , then there exists a finite index  $M$  such that

$$\Omega(u, s, t) = \sum_{k=1}^M \Omega_k(u, s, t).$$

For the special case when  $Q(t) = Q$  for all  $t \geq 0$ , we can state the following corollary to Proposition 1 which confirms prior results for homogeneous Markovian environments.

**Corollary 1** *Suppose  $\{Z(t) : t \geq 0\}$  is temporally homogeneous, i.e.,  $Q(t) = Q$  for all  $t \geq 0$ . Then the LST of the unconditional lifetime distribution is given by*

$$\tilde{F}(u, s, t) = 1 - \alpha(s) \exp[(Q - u\mathbf{R}_d)(t - s)] e, \tag{18}$$

and the conditional version of the lifetime distribution is given by

$$\tilde{F}_i(u, s, t) = 1 - e'_i(s) \exp[(Q - u\mathbf{R}_d)(t - s)] e. \tag{19}$$

*Proof* If  $\mathbf{Q}(t) = \mathbf{Q}$  for all  $t \geq 0$ , then for  $u$  fixed and any  $v \in \mathbb{R}_+$ ,  $\mathbf{A}(u, v) = \mathbf{A}(u) \equiv \mathbf{Q} - u\mathbf{R}_d$ . Hence,

$$[\mathbf{A}(u, v), \mathbf{A}(u, w)] = \mathbf{0}$$

for any pair of times  $v, w \in \mathbb{R}_+$ . The terms of the Magnus expansion of Theorem 2 are given by

$$\mathbf{\Omega}_1(u, s, t) = \int_s^t \mathbf{A}(u, t_1) dt_1 = \int_s^t (\mathbf{Q} - u\mathbf{R}_d) dt_1 = (\mathbf{Q} - u\mathbf{R}_d)(t - s),$$

and  $\mathbf{\Omega}_k(u, s, t) = \mathbf{0}$  for  $k \geq 2$  since all commutators in the subsequent terms result in the zero matrix. Consequently,

$$\mathbf{\Omega}(u, s, t) = \mathbf{\Omega}_1(u, s, t) = (\mathbf{Q} - u\mathbf{R}_d)(t - s).$$

Equations (18) and (19) follow directly from Proposition 1. □

*Remark* It is worth noting that if the unit is placed into service in a time-homogeneous environment at time  $s = 0$ , then we can replace  $\alpha(s)$  and  $e'_i(s)$  by  $\alpha \equiv \alpha(0)$  and  $e'_i \equiv e'_i(0)$ , respectively, and  $\mathbf{\Omega}(u, s, t)$  by  $\mathbf{\Omega}(u, t) = (\mathbf{Q} - u\mathbf{R}_d)t$ . The corresponding result, expressed as a Laplace transform, is derived by Kulkarni et al. (1986). However, the methods used to obtain this one-dimensional transform of the lifetime distribution are quite different. Moreover, when  $s = 0$  in (18) and (19), the results are in agreement with equations (12) and (13) of Kharoufeh et al. (2006) when the effect of shocks are ignored, i.e., when  $\lambda = 0$ .

When  $\mathcal{Z}$  is a NHCTMC, it is not immediately clear how to obtain the  $n$ th moment of  $T_x(s)$  (or its LST) using (16) and (17) because the transforms and their inverse transforms are difficult to obtain (even through numerical inversion techniques). Suppose we let  $\widehat{F}(x, s, t)$  be the approximate lifetime c.d.f. obtained via numerical inversion of (16). Then by applying the method of integrating the tail, the approximate  $n$ th moment of  $T_x(s)$  is given by

$$\mathbb{E}[T_x^n(s)] \approx \int_s^\infty nt^{n-1}[1 - \widehat{F}(x, s, t)] dt.$$

An alternative approach, demonstrated in Kharoufeh and Cox (2005), Kharoufeh et al. (2006), is to derive the LST of  $\widetilde{F}(u, s, t)$  with respect to  $t$  and to evaluate its  $n$ th-order derivative at zero. The resulting expression is the LST of the  $n$ th moment which must subsequently be inverted numerically. However, a closed-form expression for this LST does not appear to be attainable unless  $\mathbf{A}(u, v)$  and  $\mathbf{A}(u, w)$  commute for any pair of times  $v$  and  $w$ . As shown in this section, the commutative property holds when the environment is temporally homogeneous; however, it does not hold in general. Further analysis is needed to examine the moments of the lifetime distribution when  $\mathbf{Q}(t)$  is not constant.

In the next subsection, we illustrate via a small numerical example how the distribution of  $T_x(s)$  can be approximated using (16).

### 3.2 A numerical illustration

In general it is a difficult task to compute or invert the Laplace-Stieltjes transforms of Proposition 1 due to the complexity of resolving the individual terms of the Magnus expansion of Theorem 2. However, approximate LSTs (and their inverse transforms) can be obtained by considering only the first term of the Magnus expansion ( $\mathbf{\Omega}_1(u, s, t)$ ). It is important to

note that the first term cannot give the whole solution of (12). Indeed, the remaining terms in the expansion provide the correction needed to maintain an exponential form of the solution as noted by Blanes et al. (2009). Nonetheless, we set out to illustrate the form of such an approximation via a specific numerical example. This will be achieved using numerical Laplace transform inversion.

Suppose the environment  $\mathcal{Z}$  is a NHCTMC with state space  $S = \{1, 2\}$ , time-dependent generator matrix

$$Q(\tau) = \begin{bmatrix} -3\tau & 3\tau \\ 2\tau & -2\tau \end{bmatrix},$$

and degradation rate matrix

$$R_d = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

Let us define  $\widehat{G}(u, s, t)$  as the approximate LST of  $T_x(s)$  obtained by evaluating  $\widetilde{F}(u, s, t)$  using only the first term of the Magnus expansion, i.e.,

$$\widehat{G}(u, s, t) = 1 - \alpha(s) \exp[\Omega_1(u, s, t)] e$$

where

$$\Omega_1(u, s, t) = \int_s^t A(u, \tau) d\tau = \int_s^t (Q(\tau) - uR_d) d\tau.$$

It is not difficult to show that for  $0 \leq s \leq t$ ,

$$\Omega_1(u, s, t) = \int_s^t \begin{bmatrix} -3\tau - u & 3\tau \\ 2\tau & -2\tau - 4u \end{bmatrix} d\tau = \begin{bmatrix} -(\frac{3}{2}t + \frac{3}{2}s + u) & \frac{3}{2}(t + s) \\ t + s & -(t + s + 4u) \end{bmatrix} (t - s).$$

Therefore, the approximate LST of  $F(x, s, t)$ , with respect to  $x$ , is

$$\widehat{G}(u, s, t) = 1 - \alpha(s) \exp \left( \begin{bmatrix} -(\frac{3}{2}t + \frac{3}{2}s + u) & \frac{3}{2}(t + s) \\ t + s & -(t + s + 4u) \end{bmatrix} (t - s) \right) e.$$

We arbitrarily chose a fixed initial time  $s = 1.0$ . To compute the vector  $\alpha(1.0)$ , we assume that  $Z(0) = 1$  with probability 0.25 and  $Z(0) = 2$  with probability 0.75 so that  $\alpha(0) = [0.25, 0.75]$ . Applying (13) we obtain

$$\alpha(1.0) = \alpha(0) \exp \left[ \int_0^1 Q(\tau) d\tau \right] \approx [0.3877, 0.6123].$$

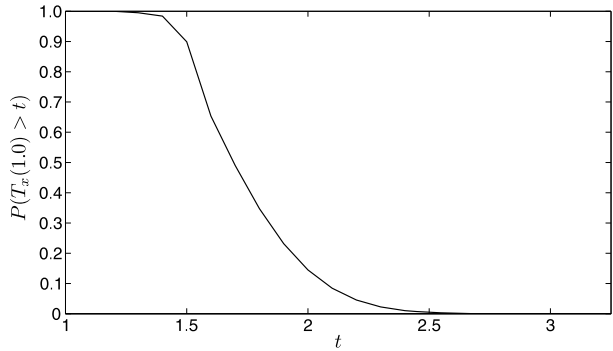
We computed the inverse Laplace transform with respect to  $u$  given by

$$\widehat{G}(x, s, t) = \mathcal{L}_u^{-1} (u^{-1} \widehat{G}(u, s, t))$$

using the numerical inversion algorithm due to Abate and Whitt (1995) for a range of  $t$  values ( $t \geq s = 1.0$ ) when  $x = 2.0$  units of degradation. The matrix computations and numerical inversion algorithm were coded in the MATLAB computing environment and executed on a personal computer.

For interpretive purposes, we plot the approximate survivor function  $\mathbb{P}(T_x(1.0) > t) \approx 1 - \widehat{G}(x, 1.0, t)$  which represents the probability that the unit’s lifetime exceeds  $t$ , given that

**Fig. 1** Approximate survivor function of  $T_x(1.0)$



it was placed into service at time 1.0 with no accumulated degradation. Figure 1 depicts the behavior of the survivor function which is bounded above by 1, bounded below by 0 and monotone decreasing in  $t$ , as expected.

In the next section, we consider an extension to the case of a semi-Markov environment whose evolution is time-homogeneous.

### 4 Homogeneous semi-Markov environment

In this section, we consider the case when  $\mathcal{Z} = \{Z(t) : t \geq 0\}$  is a temporally homogeneous semi-Markov process (SMP) with a Markov renewal process embedded at transition times (jump epochs). Let us define  $\tau_n$  as the  $n$ th jump epoch and let  $\xi_n \equiv Z(\tau_n^+)$  be the state of the environment just after the  $n$ th transition. The bivariate sequence of random variables,  $\{(\xi_n, \tau_n) : n \geq 0\}$  is a Markov renewal process with semi-Markov kernel matrix,  $\mathbf{G}(t) = [G_{ij}(t)]$  given by

$$\begin{aligned} G_{ij}(t) &= \mathbb{P}(\xi_{n+1} = j, \tau_{n+1} - \tau_n \leq t | \tau_n, \xi_n = i, \dots, \tau_0, \xi_0) \\ &= \mathbb{P}(\xi_{n+1} = j, \tau_{n+1} - \tau_n \leq t | \xi_n = i), \quad i, j \in S, t \geq 0, \end{aligned} \tag{20}$$

where  $\tau_0 = 0$  and  $\xi_n \in S$  w.p. 1 for each  $n \geq 0$ . Because we assume the SMP is time-homogeneous, (20) is equivalently

$$G_{ij}(t) = \mathbb{P}(\xi_{n+1} = j, \tau_{n+1} - \tau_n \leq t | \xi_n = i) = \mathbb{P}(\xi_1 = j, \tau_1 \leq t | \xi_0 = i)$$

for each  $n \geq 0$ . We assume that  $G_{ij}(t)$  is absolutely continuous and, therefore, possesses a probability density function (p.d.f.). The embedded Markov chain,  $\{\xi_n : n \geq 0\}$ , has one-step transition probabilities

$$p_{ij} \equiv \mathbb{P}(\xi_{n+1} = j | \xi_n = i), \quad i, j \in S,$$

and we let  $\mathbf{P} = [p_{ij}]$  denote its transition probability matrix. The matrix  $\mathbf{P}$  can be obtained by

$$\mathbf{P} = \lim_{t \rightarrow \infty} \mathbf{G}(t),$$

i.e.,  $p_{ij} = G_{ij}(\infty) \leq 1$ .

The environment’s holding time (or sojourn time) in state  $i \in S$  can also be obtained directly from the semi-Markov kernel. Specifically, if the c.d.f. of the holding time in state  $i$  is  $H_i$ , then

$$H_i(t) \equiv \mathbb{P}(\tau_1 \leq t | \xi_0 = i) = \sum_{j \in S} G_{ij}(t), \quad i \in S,$$

and its mean is given by

$$\mu_i = \int_0^\infty x H_i(dx), \quad i \in S.$$

For a SMP,  $G_{ij}(t) = p_{ij} H_i(t)$ . Assuming  $\mathbf{P}$  is irreducible and recurrent, it has an invariant probability vector  $\boldsymbol{\pi} = [\pi_i]$ . For our purposes, we assume the holding time in each state is absolutely continuous so that the p.d.f. associated with  $H_i$  is denoted by  $h_i$ , and we assume that  $\mu_i < \infty$  for each  $i \in S$ .

As before, we let  $X(t)$  denote the cumulative degradation up to time  $t$ , and let  $N(t)$  denote the total number of transitions of  $\mathcal{Z}$  up to time  $t$ . Define the set of integers,  $\mathcal{N} = \{n \geq 0 : \tau_{n+1} < t\}$ . Then we can express  $X(t)$  by

$$X(t) = \sum_{n \in \mathcal{N}} r(\xi_n)(\tau_{n+1} - \tau_n) + r(Z(t))\gamma(t) \tag{21}$$

where  $\gamma(t) = t - \tau_{N(t)}$  is the age process. Thus, we can view  $\{X(t) : t \geq 0\}$  as a semi-Markov reward process with positive, linear reward function  $r(\cdot)$ . While many authors have analyzed the asymptotic behavior of  $X(t)$  (cf. Glynn and Whitt 1993, 2002; Glynn and Hass 2004; Khorshidian 2008, 2009), the transient analysis of its distribution is notoriously difficult. However, by again exploiting the dual relationship between  $X(t)$  and  $T_x(s)$ , we provide a matrix Laplace transform expression for the unconditional transient lifetime of the unit. Because the SMP environment is temporally homogeneous, we assume (without loss of generality) that the unit enters service at time zero and drop the dependence of  $T_x$  on  $s$ . Likewise, we write  $F(x, t)$  and  $F_i(x, t)$  for the conditional and unconditional distribution functions of  $T_x$ .

#### 4.1 Laplace-Stieltjes transform of $T_x$

Our approach to deriving the transient lifetime distribution,  $F(x, t)$ , is in the spirit of techniques employed by Kulkarni et al. (1987). Specifically, we will use a Markov renewal argument to derive the Laplace-Stieltjes transform (LST) of  $T_x$ .

For any  $t \geq 0$ , whenever  $Z(t) = j \in S$ , the rate of degradation of the system is a positive number  $r(Z(t)) = r(j)$ . Define a column vector of these  $\ell$  degradation rates by

$$\mathbf{r} = (r(1), r(2), \dots, r(\ell))'$$

For each  $i, j \in S$  and  $t \geq 0$  let  $\psi_{ij}(t) = G'_{ij}(t)$ , and define the square matrix  $\boldsymbol{\psi}(t) \equiv [\psi_{ij}(t)]$ . Let the conditional lifetime distribution be

$$F_i(x, t) \equiv \mathbb{P}(T_x \leq t | Z(0) = i) = 1 - \mathbb{P}(X(t) \leq x | Z(0) = i),$$

and let its LST, with respect to  $t$ , be denoted by

$$\tilde{F}_i(x, s) \equiv \mathbb{E}(e^{-sT_x} | Z(0) = i) = \int_0^\infty e^{-st} F_i(x, dt)$$

so that

$$\tilde{F}(x, s) = \sum_{i=1}^{\ell} \tilde{F}_i(x, s) \mathbb{P}(Z(0) = i)$$

is the LST of the unconditional distribution function  $F(x, t) \equiv \mathbb{P}(T_x \leq t)$  with respect to  $t$ . (Here,  $s$  is a complex transform variable with  $\text{Re}(s) > 0$ .) Next, we define the Laplace transform (LT) of  $\tilde{F}(x, s)$  with respect to  $x$  by

$$\tilde{F}^*(u, s) \equiv \int_0^\infty e^{-ux} \tilde{F}(x, s) dx, \quad \text{Re}(u) > 0.$$

To simplify notation in the results that follow, let us also define the LT of  $\psi_{ij}(t)$  by

$$\psi_{ij}^*(\omega) = \int_0^\infty e^{-\omega t} \psi_{ij}(t) dt, \quad \text{Re}(\omega) > 0,$$

and for each  $i \in S$ , let

$$H_i^*(\omega) = \int_0^\infty e^{-\omega t} H_i(t) dt$$

denote the LT of the sojourn time distribution  $H_i$ . Theorem 3 provides a closed-form result for the (double) Laplace transform  $\tilde{F}^*(u, s)$ . Unfortunately, a one-dimensional transform result appears to be available only if  $\mathcal{Z}$  is a time-homogeneous CTMC.

**Theorem 3** *If the environment process  $\mathcal{Z}$  is a time-homogeneous semi-Markov process with finite state space  $S = \{1, 2, \dots, \ell\}$  and initial distribution  $\alpha$ , then*

$$\tilde{F}^*(u, s) = \alpha[\mathbf{I} - \boldsymbol{\psi}^*(u, s)]^{-1}[\mathbf{A}^*(u, s) - \mathbf{H}^*(u, s)]\mathbf{r} \tag{22}$$

where the matrix  $\boldsymbol{\psi}^*(u, s) = [\psi_{ij}^*(u, s)]$  has elements

$$\psi_{ij}^*(u, s) = p_{ij}h_i^*(s + r(i)u), \quad i, j \in S,$$

$\mathbf{A}^*(u, s) = \text{diag}[1/(s + r(1)u), 1/(s + r(2)u), \dots, 1/(s + r(\ell)u)]$ , and  $\mathbf{H}^*(u, s) = \text{diag}[H_1^*(s + r(1)u), H_2^*(s + r(2)u), \dots, H_\ell^*(s + r(\ell)u)]$ .

*Proof* The result will be proved using a Markov renewal argument by conditioning on the first transition time  $\tau_1$  and the initial state of the environment  $Z(0)$ . To this end, note that

$$\mathbb{E}(e^{-sT_x} | \tau_1 = h, Z(0) = i) = \begin{cases} e^{-sx/r(i)}, & \text{if } h \geq x/r(i), \\ e^{-sh} \sum_{j=1}^{\ell} p_{ij} \tilde{F}_j(x - r(i)h, s), & \text{if } h < x/r(i). \end{cases}$$

First, we uncondition with respect to the first sojourn time to obtain

$$\begin{aligned} \tilde{F}_i(x, s) &= \int_0^\infty \mathbb{E}(e^{-sT_x} | \tau_1 = h, Z(0) = i) dH_i(h) \\ &= \sum_{j=1}^{\ell} p_{ij} \int_0^{x/r(i)} e^{-sh} \tilde{F}_j(x - r(i)h, s) dH_i(h) + e^{-sx/r(i)} [1 - H_i(x/r(i))]. \end{aligned} \tag{23}$$



Now, taking the Laplace transform of both sides of (23) with respect to  $x$  and using the change of variable,  $y = r(i)h$  and  $dy = r(i) dh$ , we obtain

$$\begin{aligned} \tilde{F}_i^*(u, s) &= \int_0^\infty e^{-ux} \tilde{F}_i(x, s) dx \\ &= \sum_{j=1}^\ell \frac{p_{ij}}{r(i)} \int_0^\infty e^{-ux} \int_0^x \tilde{F}_j(x - y, s) e^{-sy/r(i)} h_i \left( \frac{y}{r(i)} \right) dy dx + \frac{r(i)}{s + r(i)u} \\ &\quad - \int_0^\infty e^{-ux} e^{-sx/r(i)} H_i(x/r(i)) dx. \end{aligned} \tag{24}$$

Using standard Laplace transform tables (cf. Oberhettinger and Badii 1973), it is known that

$$H_i^*(r(i)u - s) = \int_0^\infty e^{-ux} \frac{e^{sx/r(i)} H_i(x/r(i))}{r(i)} dx. \tag{25}$$

Therefore, the right-most integral of (24) can be simplified to obtain

$$\begin{aligned} \tilde{F}_i^*(u, s) &= \sum_j \frac{p_{i,j}}{r(i)} \int_0^\infty e^{-ux} \int_0^x \tilde{F}_j(x - y, s) e^{-sy/r(i)} h_i \left( \frac{y}{r(i)} \right) dy dx \\ &\quad + \frac{r(i)}{s + r(i)u} - r(i)H_i^*(s + r(i)u). \end{aligned} \tag{26}$$

To resolve the double integral of (26), we note that for some function  $g(y, s)$ ,

$$M(x, s) \equiv \int_0^x \tilde{F}_j(x - y, s) g(y, s) dy$$

is the convolution of the functions  $\tilde{F}_j(\cdot, s)$  and  $g(y, s)$ . It is well known that the Laplace transform of  $M(x, s)$ , denoted by  $M^*(u, s)$ , is given by

$$M^*(u, s) = \int_0^\infty e^{-ux} M(x, s) dx = \tilde{F}_j^*(u, s) g^*(u, s),$$

where  $\tilde{F}_j^*(u, s)$  is the LT of  $\tilde{F}_j(x, s)$  with respect to  $x$ , and  $g^*(u, s)$  is the LT of  $g(y, s)$ . From (26), we see that  $g(y, s) = \exp[-sy/r(i)]h_i(y/r(i))$ , and its LT with respect to  $y$  is  $g^*(u, s) = r(i)h_i^*(s + r(i)u)$ . Hence, (26) can be reduced to

$$\begin{aligned} \tilde{F}_i^*(u, s) &= \sum_{j=1}^\ell \frac{p_{ij}}{r(i)} \tilde{F}_j^*(u, s) r(i)h_i^*(s + r(i)u) + \frac{r(i)}{s + r(i)u} - r(i)H_i^*(s + r(i)u) \\ &= \sum_{j=1}^\ell p_{ij} \tilde{F}_j^*(u, s) h_i^*(s + r(i)u) + \frac{r(i)}{s + r(i)u} - r(i)H_i^*(s + r(i)u). \end{aligned} \tag{27}$$

Now, rearranging the terms of (27), we have

$$\begin{aligned} \tilde{F}_i^*(u, s) [1 - p_{ii}h_i^*(s + r(i)u)] - \sum_{j \neq i} p_{ij} \tilde{F}_j^*(u, s) h_i^*(s + r(i)u) \\ = r(i) \left[ \frac{1}{s + r(i)u} - H_i^*(s + r(i)u) \right]. \end{aligned} \tag{28}$$

Letting  $\mathbf{A}^*(u, s) = \text{diag}[1/(s + r(1)u), 1/(s + r(2)u), \dots, 1/(s + r(\ell)u)]$  and  $\mathbf{H}^*(u, s) = \text{diag}[H_1^*(s + r(1)u), H_2^*(s + r(2)u), \dots, H_\ell^*(s + r(\ell)u)]$ , and defining the  $\ell$ -dimensional column vector,  $\tilde{\mathbf{F}}^*(u, s) = (\tilde{F}_1^*(u, s), \tilde{F}_2^*(u, s), \dots, \tilde{F}_\ell^*(u, s))'$ , (28) can be expressed in the matrix form

$$[\mathbf{I} - \mathbf{P} \odot \mathbf{h}^*(u, s)]\tilde{\mathbf{F}}^*(u, s) = [\mathbf{A}^*(u, s) - \mathbf{H}^*(u, s)]\mathbf{r}, \tag{29}$$

where  $\mathbf{I}$  is the identity matrix,

$$\mathbf{h}^*(u, s) = \begin{bmatrix} h_1^*(s + r(1)u) & h_1^*(s + r(1)u) & \cdots & h_1^*(s + r(1)u) \\ h_2^*(s + r(2)u) & h_2^*(s + r(2)u) & \cdots & h_2^*(s + r(2)u) \\ \vdots & \vdots & \ddots & \vdots \\ h_\ell^*(s + r(\ell)u) & h_\ell^*(s + r(\ell)u) & \cdots & h_\ell^*(s + r(\ell)u) \end{bmatrix},$$

and  $\mathbf{P} \odot \mathbf{h}^*(u, s)$  is the Hadamard product of  $\mathbf{P}$  and  $\mathbf{h}^*(u, s)$  given by

$$\mathbf{P} \odot \mathbf{h}^*(u, s) = \begin{bmatrix} p_{11}h_1^*(s + r(1)u) & p_{12}h_1^*(s + r(1)u) & \cdots & p_{1\ell}h_1^*(s + r(1)u) \\ p_{21}h_2^*(s + r(2)u) & p_{22}h_2^*(s + r(2)u) & \cdots & p_{2\ell}h_2^*(s + r(2)u) \\ \vdots & \vdots & \ddots & \vdots \\ p_{\ell 1}h_\ell^*(s + r(\ell)u) & p_{\ell 2}h_\ell^*(s + r(\ell)u) & \cdots & p_{\ell \ell}h_\ell^*(s + r(\ell)u) \end{bmatrix}.$$

Because  $\text{Re}(u) > 0$  and  $\text{Re}(s) > 0$ , and  $h_i^*(s + r(i)u) = \tilde{H}_i(s + r(i)u) < 1$  for each  $i \in S$ ,  $\mathbf{I} - \mathbf{P} \odot \mathbf{h}^*(u, s)$  is invertible. To obtain the double transform,  $\tilde{\mathbf{F}}^*(u, s)$ , we uncondition with respect to the initial distribution of the environment

$$\tilde{\mathbf{F}}^*(u, s) = \boldsymbol{\alpha}[\mathbf{I} - \mathbf{P} \odot \mathbf{h}^*(u, s)]^{-1}[\mathbf{A}^*(u, s) - \mathbf{H}^*(u, s)]\mathbf{r}. \tag{30}$$

Now, since  $\psi_{ij}(t) = G'_{ij}(t) = p_{ij}H'_i(t) = p_{ij}h_i(t)$  for each  $i, j \in S$  and  $t \geq 0$  in an SMP, the  $(i, j)$ th element of  $\mathbf{P} \odot \mathbf{h}^*(u, s)$  is given by

$$\psi_{ij}^*(s + r(i)u) = p_{ij}h_i^*(s + r(i)u).$$

Therefore, we obtain the final result

$$\tilde{\mathbf{F}}^*(u, s) = \boldsymbol{\alpha}[\mathbf{I} - \boldsymbol{\psi}^*(u, s)]^{-1}[\mathbf{A}^*(u, s) - \mathbf{H}^*(u, s)]\mathbf{r} \tag{31}$$

where  $\boldsymbol{\psi}^*(u, s) = [\psi_{ij}^*(s + r(i)u)]$ . □

### 4.2 Illustrative examples

In this section, we illustrate the form of the double Laplace transform for the case when holding times are exponentially distributed, hyper-exponentially distributed, and Erlang distributed. These illustrations serve to emphasize the difficulty in computing the transient lifetime distribution (or cumulative degradation up to time  $t$ ).

*Example 1* (Exponential Sojourn Times) In case all of the sojourn times are exponentially distributed random variables with respective rates  $q_i, i \in S$ ,  $\mathcal{Z}$  is a time-homogeneous

CTMC on  $S$  with infinitesimal generator matrix  $\mathbf{Q} = [q_{ij}]$ . The sojourn time in state  $i$  has c.d.f.

$$H_i(t) = 1 - \exp(-q_i t),$$

and its Laplace transform evaluated at  $s + r(i)u$  is

$$H_i^*(s + r(i)u) = \frac{1}{s + r(i)u} - \frac{1}{s + r(i)u + q_i}. \tag{32}$$

Moreover, its p.d.f. is given by

$$h_i(t) = q_i \exp(-q_i t)$$

which shows that

$$h_i^*(s + r(i)u) = \frac{q_i}{(s + r(i)u + q_i)}. \tag{33}$$

It is worth mentioning that the transition probability matrix of the embedded chain,  $\{\xi_n : n \geq 0\}$ , is given by

$$P_{ij} = \begin{cases} q_{ij}/q_i, & \text{if } j \neq i, q_i \neq 0, \\ 0, & \text{if } j = i \end{cases}$$

where  $q_i = \sum_{j \neq i} q_{ij}$ . Substituting  $\mathbf{P}$  and (32) and (33) into (31), we obtain

$$\mathbf{I} - \boldsymbol{\psi}^*(u, s) = \begin{bmatrix} 1 & \frac{-q_{12}}{s+r(1)u+q_1} & \dots & \frac{-q_{1\ell}}{s+r(1)u+q_1} \\ \frac{-q_{21}}{s+r(2)u+q_2} & 1 & \ddots & \frac{-q_{2\ell}}{s+r(2)u+q_2} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{-q_{\ell 1}}{s+r(\ell)u+q_\ell} & \frac{-q_{\ell 2}}{s+r(\ell)u+q_\ell} & \dots & 1 \end{bmatrix} \tag{34}$$

and

$$\mathbf{A}^*(u, s) - \mathbf{H}^*(u, s) = \begin{bmatrix} \frac{1}{s+r(1)u+q_1} & 0 & \dots & 0 \\ 0 & \frac{1}{s+r(2)u+q_2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{s+r(\ell)u+q_\ell} \end{bmatrix}. \tag{35}$$

Using (34) and (35), the elements of the vector  $\tilde{\mathbf{F}}^*(u, s)$  are given by

$$\tilde{F}_i^*(u, s) = \frac{r(i)}{s + r(i)u + q_i} + \sum_{j \in S \setminus \{i\}} \frac{q_{i,j}}{s + r(i)u + q_i} \tilde{F}_j^*(u, s). \tag{36}$$

It is not difficult to show that this double-transform result is equivalent to the result given for the case of a continuous-time Markov chain in Kulkarni et al. (1986).

*Example 2* (Hyper-exponential Sojourn Times) A hyper-exponential random variable is a mixture of exponential random variables (see for example Whitt 1984). Suppose that the sojourn time in state  $i \in S$  is hyper-exponential with  $n_i$  phases such that the  $j$ th phase has rate parameter  $\lambda_{ij}$ ,  $j = 1, 2, \dots, n_i$ . Moreover, let  $\alpha_{ij} \in (0, 1)$  and  $\sum_j \alpha_{ij} = 1$  for each  $i \in S$ . For  $t \geq 0$ , the c.d.f. of this sojourn time is given by (cf. Kulkarni 1995)

$$H_i(t) = 1 - \sum_{j=1}^{n_i} \alpha_{ij} \exp(-\lambda_{ij}t),$$

which implies that its Laplace transform evaluated at  $s + r(i)u$  is given by

$$\begin{aligned} H_i^*(s + r(i)u) &= \int_0^\infty \exp(-(s + r(i)u)t) \left[ 1 - \sum_{j=1}^{n_i} \alpha_{ij} \exp(-\lambda_{ij}t) \right] dt \\ &= \frac{1}{s + r(i)u} - \sum_{j=1}^{n_i} \int_0^\infty \exp(-(s + r(i)u)t) \alpha_{ij} \exp(-\lambda_{ij}t) dt \\ &= \frac{1}{s + r(i)u} - \sum_{j=1}^{n_i} \frac{\alpha_{ij}}{s + r(i)u + \lambda_{ij}}. \end{aligned} \tag{37}$$

The p.d.f. of the sojourn time, for  $t \geq 0$ , is given by

$$h_i(t) = \sum_{j=1}^{n_i} \alpha_{ij} \lambda_{ij} \exp(-\lambda_{ij}t),$$

and its Laplace transform evaluated at  $s + r(i)u$  is

$$\begin{aligned} h_i^*(s + r(i)u) &= \int_0^\infty \exp(-(s + r(i)u)t) \sum_{j=1}^{n_i} \alpha_{ij} \lambda_{ij} \exp(-\lambda_{ij}t) dt \\ &= \sum_{j=1}^{n_i} \frac{\alpha_{ij} \lambda_{ij}}{s + r(i)u + \lambda_{ij}}. \end{aligned} \tag{38}$$

After substituting (37) and (38) into (31) and simplifying, the unconditional double Laplace transform,  $\tilde{F}^*(u, s)$ , is obtained by

$$\alpha \begin{bmatrix} \left(\sum_{j=1}^{n_1} \frac{\alpha_{1j} \lambda_{1j}}{s+r(1)u+\lambda_{1j}}\right)^{-1} - p_{11} & -p_{12} & \cdots & -p_{1\ell} \\ -p_{2,1} & \left(\sum_{j=1}^{n_2} \frac{\alpha_{2j} \lambda_{2j}}{s+r(2)u+\lambda_{2j}}\right)^{-1} - p_{22} & \ddots & -p_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{\ell 1} & -p_{\ell 2} & \cdots & \left(\sum_{j=1}^{n_\ell} \frac{\alpha_{\ell j} \lambda_{\ell j}}{s+r(\ell)u+\lambda_{\ell j}}\right)^{-1} - p_{\ell\ell} \end{bmatrix}^{-1} \mathbf{r}.$$

This result provides sufficient generality to allow for each state sojourn time to assume a distinct hyper-exponential distribution. Because the hyper-exponential belongs to the larger class of phase-type (PH) distributions, it can approximate a wide range of sojourn time distributions. Next, we consider the case of sums of i.i.d. exponential phases (i.e., Erlang distributed sojourn times).

*Example 3* (Erlang Sojourn Times) Suppose the sojourn time in state  $i \in S$  can be represented by  $n_i$  phases ( $n_i \geq 1$ ), each of which has the same distribution, namely exponential with mean  $1/\lambda_i$ . In such a case, the c.d.f. of the sojourn time is given by

$$H_i(t) = 1 - \sum_{j=0}^{n_i-1} \exp(-\lambda_i t) \frac{(\lambda_i t)^j}{j!}, \quad t \geq 0.$$

For an integer  $n \geq 1$  and real number  $a$ , we can apply the well-known Laplace transform identity

$$\mathcal{L}\left(\frac{t^{n-1} e^{at}}{(n-1)!}\right) = \frac{1}{(s-a)^n},$$

to establish that the Laplace transform of  $H_i(t)$  evaluated at  $s+r(i)u$  is

$$\begin{aligned} H_i^*(s+r(i)u) &= \frac{1}{s+r(i)u} - \frac{1}{s+r(i)u+\lambda_i} - \frac{\lambda_i}{(s+r(i)u+\lambda_i)^2} - \dots \\ &= \frac{1}{s+r(i)u} \left(\frac{\lambda_i}{s+r(i)u+\lambda_i}\right)^{n_i}. \end{aligned} \tag{39}$$

The p.d.f. of this Erlang( $n_i, \lambda_i$ ) random variable is

$$h_i(t) = \lambda_i \exp(-\lambda_i t) \frac{(\lambda_i t)^{n_i-1}}{(n_i-1)!}, \quad t \geq 0$$

so that its Laplace transform evaluated at  $s+r(i)u$  is

$$h_i^*(s+r(i)u) = \left(\frac{\lambda_i}{s+r(i)u+\lambda_i}\right)^{n_i}. \tag{40}$$

Substituting (39) and (40) into (31),  $\tilde{F}^*(u, s)$  reduces to

$$\alpha \begin{bmatrix} s+r(1)u + \frac{(1-p_{11})\lambda_1^{n_1}}{a_{1,n_1}} & \frac{(-p_{12})\lambda_1^{n_1}}{a_{1,n_1}} & \dots & \frac{(-p_{1\ell})\lambda_1^{n_1}}{a_{1,n_1}} \\ \frac{(-p_{21})\lambda_2^{n_2}}{a_{2,n_2}} & s+r(2)u + \frac{(1-p_{22})\lambda_2^{n_2}}{a_{2,n_2}} & \ddots & \frac{(-p_{2\ell})\lambda_2^{n_2}}{a_{2,n_2}} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{(-p_{\ell 1})\lambda_\ell^{n_\ell}}{a_{\ell,n_\ell}} & \frac{(-p_{\ell 2})\lambda_\ell^{n_\ell}}{a_{\ell,n_\ell}} & \dots & s+r(\ell)u + \frac{(1-p_{\ell\ell})\lambda_\ell^{n_\ell}}{a_{\ell,n_\ell}} \end{bmatrix}^{-1} \mathbf{r}$$

where the values  $a_{i,n_i}, i \in S$ , are obtained recursively by

$$a_{i,n_i} = a_{i,n_i-1}(s+r(i)u+\lambda_i) + \lambda_i^{n_i-1}$$

with  $a_{i,1} \equiv 1$ . Obviously, if  $n_i = 1$ , for all  $i \in S$ , then the semi-Markov environment reduces to a continuous-time Markov chain with  $q_i = \lambda_i, i \in S$ .

The illustrative examples in this section demonstrate the complexity involved in characterizing the reliability (or unit lifetime) of systems that are subject to complex environments. However, it is worth mentioning that other approaches can be used to approximate the SMP environment using a surrogate, time-homogeneous CTMC via phase-type sojourn

time distributions. This approach has been outlined and implemented by Kharoufeh et al. (2010). Alternatively, to circumvent the computational difficulties described herein, it may be possible to develop asymptotic results which lead to tractable solutions that are easy to implement. Khorshidian (2009) has recently derived a Functional Central Limit Theorem (FCLT) using an approximating martingale process. In the next section, we provide some concluding remarks and discuss future work in this area.

## 5 Conclusions

In this paper we have presented transient reliability indices for manufacturing equipment that operates in complex environments that induce stochastic degradation and ultimately lead to failure of the equipment. We specifically considered time-nonhomogeneous Markov environments and time-homogeneous semi-Markov environments. The mathematical frameworks presented here provide a great deal of modeling flexibility, particularly in scenarios in which the operating times in various states do not transition in a stationary way, or those times are distinctly non-exponential. We posit that these results can be very useful in scenarios where a direct mapping between the environmental conditions and the degradation can be effectively modeled. This is typically done by employing specific physics-of-failure models for the type of equipment under consideration.

Though the models are mathematically sound, they can be difficult to implement computationally. In particular, for the two-dimensional Laplace transform results, numerical inversion appears to be the best option. Some good algorithms for doing two-dimensional numerical Laplace transform inversion are described in Abate et al. (1998), Moorthy (1995). In the future, it will be instructive to consider asymptotic results, particularly for the SMP environment model, to obtain tractable reliability indices. Asymptotic results may provide an adequate representation of the reliability indices, particularly when the degradation threshold is very large. This may, for example, correspond to the case of highly-reliable equipment that experiences very little degradation over time. The asymptotic results may provide a starting point for applications requiring real-time updating of remaining lifetime distributions (as in a condition-based maintenance environment). Specifically, real-time observations of the evolution of the environment, or the degradation level, can be used to update the parameters of the modulating process and the state-dependent degradation rates. Tracking the occurrence of environment transitions over a (sufficiently long) time interval will provide reasonable statistical estimates, so long as the number of environment states is not too large.

Finally, a nontrivial task is the estimation of the degradation rates  $r(i)$  that describe the evolution of degradation as a function of the environment. Obviously, for this purpose, real degradation data is required, as is the guidance and experience of subject matter experts. Our approach provides a very macroscopic view of degradation whereas wear and degradation dynamics can be very detailed and complex. It will be instructive to consult with engineers and scientists to determine appropriate models for specific material types and certain applications.

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