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Robustness in stochastic programs with risk constraints

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Abstract This paper is a contribution to the robustness analysis for stochastic programs whose set of feasible solutions depends on the probability distribution P. For various reasons, probability distribution P may not be precisely specified and we study robustness of results with respect to perturbations of P. The main tool is the contamination technique. For the optimal value, local contamination bounds are derived and applied to robustness analysis of the optimal value of a portfolio performance under risk-shaping CVaR constraints. A new robust portfolio efficiency test with respect to the second order stochastic dominance criterion is suggested and the contamination methodology is exploited to analyze its resistance with respect to additional scenarios.

Keywords Expectation type constraints · Robustness analysis · Contamination technique · Risk-shaping with CVaR · Second order stochastic dominance · Robust SSD portfolio efficiency test

1 Introduction

In this paper we shall deal with robustness properties of risk constrained stochastic programs of the form

$$\min_{\mathbf{x}\in\mathcal{X}} F_0(\mathbf{x}, P)$$

subject to

$$F_j(\mathbf{x}, P) \le 0, \quad j = 1, \dots, J, \tag{1}$$

where

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– P is the probability distribution of a random vector ω with range $\Omega \subset \mathbb{R}^M$,
– $\mathcal{X} \subset \mathbb{R}^N$ is a fixed nonempty convex set,
– functions $F_j(\mathbf{x}, P),\ j = 0, \ldots, J$ may depend on P.

We shall denote $\mathcal{X}(P)$ the set of feasible solutions, $\mathcal{X}^*(P)$ the set of optimal solutions and $\varphi(P)$ the optimal value of the objective function in (1).

Probably the first paper formulating and analyzing risk constrained stochastic programs is due to Prékopa (1973) which includes joint probability constraints and constraints in the form of conditional expectations; see also Wets (1989) for the problem formulation and for properties of expectation functionals. Notice that *chance or probability constraints* are a special case of (1), however the set of feasible solutions $\mathcal{X}(P)$ is then convex only under special distributional and structural assumptions; consult Prékopa (2003).

Due to the tendency of an adequate treatment of risk, a growing interest in the risk constrained problems can be observed since 2000. It turns out that among others, the Sample Average Approximation technique, see e.g. Shapiro (2003), Pagoncelli et al. (2009), Wang and Ahmed (2008), and its asymptotics can be applied. This assumes that i.i.d. samples are drawn from a *fixed* (known, preselected) probability distribution P.

The wish is to apply reliable, robust or efficient decisions of (1) even in situations when the true probability distribution P has been approximated or when it is known only partly. Partial knowledge of P can be included into the model formulation, see e.g. Dentcheva and Ruszczyński (2010) for robust stochastic dominance constraints or Pflug and Wozabal (2007) for an inclusion of ambiguity of P into the model. In a similar vein a robust portfolio efficiency test will be developed in Sect. 3.2. A special case of robust portfolio efficiency was analyzed in Kopa (2010). Contrary to that, our new test allows probability distributions with nonequiprobable scenarios.

Another possibility is to rely on general quantitative stability results valid under suitable continuity assumptions for $F_j(\mathbf{x}, P),\ j = 0, \ldots, J$. Such results were proved by Römisch (2003) without convexity requirements and were detailed e.g. for chance constraints of a special structure and formulated also for risk measures *nonlinear* in P. Under modest assumptions they apply to the convex problem (1).

In Sect. 2, we shall follow the relatively simple ideas of output analysis based on the contamination technique, cf. Dupačová (1996, 2006), Dupačová and Polívka (2007). The considered special type of perturbations gets on with needs for what-if-analysis or stress testing. Robustness results with respect to contamination of P by another fixed probability distribution have been mainly developed for convex stochastic programs whose set of feasible decisions does not depend on P, an assumption which does not apply to problem (1), and for the objective function $F_0(\mathbf{x}, P)$ convex in $\mathbf{x}$ and linear or concave in P. To elaborate special techniques for stress testing and robustness analysis for problem (1) it is necessary to relax the assumption of a fixed set of feasible decisions and to allow its dependence on P. To this purpose, it is convenient if the constraints are *linear* in P being expectations of random convex functions. Even with the expectation type constraints the problem formulation (1) covers various known examples, e.g. CVaR constraints from Rockafellar and Uryasev (2002), Krokhmal et al. (2002) or the second order stochastic dominance constraints. This is the class of problems for which we shall detail our robustness analysis and provide numerical illustrations. The next example introduces the prototype form of the problem.

Example 1 (Risk-shaping with CVaR; Rockafellar and Uryasev 2002) Let $f(\mathbf{x}, \omega)$ denote the random loss caused by the decision $\mathbf{x} \in \mathcal{X}$ and $\alpha \in (0, 1)$ the selected confidence level. The *Conditional Value at Risk* at the confidence level α, CVaR_α, is defined as the mean of the α-tail distribution of $f(\mathbf{x}, \omega)$. According to the fundamental minimization formula by Rockafellar and Uryasev (2002) it can be evaluated by minimization of the auxiliary function

$$\Phi_\alpha(\mathbf{x}, v, P) := v + \frac{1}{1-\alpha} E_P(f(\mathbf{x}, \omega) - v)^+$$

with respect to $v \in \mathbb{R}$.

The auxiliary function $\Phi_\alpha(\mathbf{x}, v, P)$ is evidently linear in P and convex in v. Moreover, if $f(\mathbf{x}, \omega)$ is a *convex* function of $\mathbf{x}$, $\Phi_\alpha(\mathbf{x}, v, P)$ is convex jointly in $(v, \mathbf{x})$.

If P is a *discrete probability distribution* concentrated on $\omega^1, \ldots, \omega^S$, with probabilities $p_s > 0$, $s = 1, \ldots, S$, and $\mathbf{x}$ a *fixed* element of $\mathcal{X}$, then the optimization problem $\mathrm{CVaR}_\alpha(\mathbf{x}, P) = \min_v \Phi_\alpha(\mathbf{x}, v, P)$ has the form

$$\mathrm{CVaR}_\alpha(\mathbf{x}, P) = \min_v \left\{ v + \frac{1}{1-\alpha} \sum_{s=1}^{S} p_s (f(\mathbf{x}, \omega^s) - v)^+ \right\} \tag{2}$$

and can be written as

$$\mathrm{CVaR}_\alpha(\mathbf{x}, P) = \min_{v, z_1, \ldots, z_S} \left\{ v + \frac{1}{1-\alpha} \sum_{s=1}^{S} p_s z_s \,|\, z_s \geq 0,\ z_s + v \geq f(\mathbf{x}, \omega^s)\ \forall s \right\}. \tag{3}$$

Risk-shaping with CVaR handles several probability thresholds $\alpha_1, \ldots, \alpha_J$ and loss tolerances b_j, $j = 1, \ldots, J$. The problem is to minimize a performance function $F(\mathbf{x})$ subject to $\mathbf{x} \in \mathcal{X}$ and constraints $\mathrm{CVaR}_{\alpha_j}(\mathbf{x}, P) \leq b_j$, $j = 1, \ldots, J$. According to Theorem 16 of Rockafellar and Uryasev (2002), this problem is equivalent to

$$\min_{\mathbf{x}, v_1, \ldots, v_J} \{F(\mathbf{x}) \,|\, \mathbf{x} \in \mathcal{X},\ \Phi_{\alpha_j}(\mathbf{x}, v_j, P) \leq b_j,\ j = 1, \ldots, J\},$$

i.e. it is a problem of the form (1) with expectation type constraints.

2 Contamination bounds

Contamination means to model the perturbations of P by its contamination by another fixed probability distribution Q, i.e. to use $P_t := (1-t)P + tQ$, $t \in [0, 1]$ in stochastic program (1) at the place of P. Then the set of feasible solutions of (1) for the contaminated probability distribution P_t equals

$$\mathcal{X}(P_t) = \mathcal{X} \cap \{\mathbf{x} \,|\, F_j(\mathbf{x}, P_t) \leq 0, j = 1, \ldots, J\}. \tag{4}$$

We denote $\mathcal{X}(t)$, $\varphi(t)$, $\mathcal{X}^*(t)$ the set of feasible solutions, the optimal value $\varphi(P_t)$ and the set of optimal solutions $\mathcal{X}^*(P_t)$ of the contaminated problem

$$\text{minimize } F_0(\mathbf{x}, P_t) \text{ on the set } \mathcal{X}(P_t). \tag{5}$$

This is a *nonlinear parametric program* with a scalar parameter $t \in [0, 1]$ and a *parameter dependent set of feasible solutions* $\mathcal{X}(t) := \{\mathbf{x} \in \mathcal{X} \,|\, F_j(\mathbf{x}, t) \leq 0, j = 1, \ldots, J\}$.

The task is to construct computable lower and upper bounds for $\varphi(t)$. Such bounds were obtained for $\mathcal{X}$ fixed, independent of P and for objective function $F_0(\mathbf{x}, P)$ linear or concave in P, cf. Dupačová (1996, 1998). In this case, one can exploit the fact that the optimal value function $\varphi(t)$ is a concave function of the contamination parameter t. The derived bounds proved to be useful for testing the resistance with respect to a sample for scenario-based stochastic programs, e.g. Dupačová (1996), in stress testing of CVaR optimization problems, cf. Dupačová (2006), Dupačová and Polívka (2007), or for problems with polyhedral risk objectives, cf. Dupačová (2008). For the parameter dependent sets of feasible solutions the optimal value function $\varphi(t)$ is concave only under rather strict assumptions such as $F_j(\mathbf{x}, t),\ j = 1, \dots, J$ jointly concave on $\mathcal{X} \times [0, 1]$ (cf. Corollary 3.2 of Kyparisis and Fiacco 1987) which is not in agreement with our problem formulation.

We shall examine how to construct contamination bounds for SP of the type (5) whose constraints depend on the probability distribution. These bounds will be then applied in robustness analysis for risk-shaping with CVaR or for a stochastic dominance test with respect to inclusion of additional scenarios. We shall see that thanks to the assumed structure of perturbations the lower bound can be derived for $F_j(\mathbf{x}, P),\ j = 0, \dots, J$ linear or concave with respect to P without any smoothness or convexity assumptions with respect to $\mathbf{x}$. Convexity of the stochastic program (1) is essential for directional differentiability of the optimal value function, and further assumptions are needed for derivation of an upper bound.

2.1 Lower bound

Consider first only one constraint dependent on probability distribution P and an objective F_0 *independent* of P, i.e. the problem is

$$\min_{\mathbf{x}\in\mathcal{X}} F_0(\mathbf{x}) \quad \text{subject to} \quad F(\mathbf{x}, P) \le 0. \tag{6}$$

For probability distribution P contaminated by another fixed probability distribution Q, i.e. for $P_t := (1-t)P + tQ,\ t \in (0, 1)$ we get

$$\min_{\mathbf{x}\in\mathcal{X}} F_0(\mathbf{x}) \quad \text{subject to} \quad F(\mathbf{x}, t) := F(\mathbf{x}, P_t) \le 0. \tag{7}$$

Theorem 1 *Let $F(\mathbf{x}, t)$ be a concave function of $t \in [0, 1]$. Then the optimal value function of* (7)

$$\varphi(t) := \min_{\mathbf{x}\in\mathcal{X}} F_0(\mathbf{x}) \quad \textit{subject to} \quad F(\mathbf{x}, t) \le 0$$

is quasiconcave in $t \in [0, 1]$ with the lower bound

$$\varphi(t) \ge \min\{\varphi(0), \varphi(1)\}. \tag{8}$$

Proof For arbitrary $t_1,\ t_2 \in [0, 1]$ and $0 \le \lambda \le 1$ we have

$$\mathcal{X}((1-\lambda)t_1 + \lambda t_2) \subset \{\mathbf{x} \in \mathcal{X} \mid (1-\lambda)F(\mathbf{x}, t_1) + \lambda F(\mathbf{x}, t_2) \le 0\} \subset \mathcal{X}(t_1) \cup \mathcal{X}(t_2). \tag{9}$$

Hence, similarly as in Proposition 3.11 of Kyparisis and Fiacco (1987), the optimal value $\varphi(t)$ of (7) is quasiconcave which results in the lower bound (8). □

When also the objective function *depends* on the probability distribution, i.e. on the contamination parameter t, the problem is

$$\min_{\mathbf{x}\in\mathcal{X}} F_0(\mathbf{x}, t) := F_0(\mathbf{x}, P_t) \quad \text{subject to} \quad F(\mathbf{x}, t) \le 0. \tag{10}$$

For $F_0(\mathbf{x}, P)$ linear or concave in P, a lower bound can be obtained by application of the bound (8) separately to $F_0(\mathbf{x}, P)$ and $F_0(\mathbf{x}, Q)$:

$$\begin{aligned}\varphi(t) &= \min_{\mathbf{x}\in\mathcal{X}(t)} F_0(\mathbf{x}, (1-t)P + tQ) \geq \min_{\mathbf{x}\in\mathcal{X}(t)} [(1-t)F_0(\mathbf{x}, P) + tF_0(\mathbf{x}, Q)] \\ &\geq (1-t)\min\Big\{\varphi(0), \min_{\mathcal{X}(Q)} F_0(\mathbf{x}, P)\Big\} + t\min\Big\{\varphi(1), \min_{\mathcal{X}(P)} F_0(\mathbf{x}, Q)\Big\}. \end{aligned} \tag{11}$$

The bound is more complicated but still computable. It requires solution of 4 problems two of which are the non-contaminated programs for probability distributions P, Q and the other ones use both P and Q alternating in the objective function and constraints.

2.1.1 Comment

Of course, the lower bounds (8), (11) are loose, but for small values of t, say $t \leq t_0$ they can be improved to $\varphi(t) \geq \min\{\varphi(0), \varphi(t_0)\}$ when applied to P and to $\tilde{Q} := (1-t_0)P + t_0 Q$. Notice that no convexity assumption with respect to $\mathbf{x}$ is needed.

For *multiple constraints* and contaminated probability distribution it would be necessary to prove first the inclusion $\mathcal{X}(t) \subset \mathcal{X}(0) \cup \mathcal{X}(1)$ and then the lower bound (8) for the optimal value $\varphi(t) = \min_{\mathbf{x}\in\mathcal{X}(t)} F_0(\mathbf{x}, P_t)$ can be obtained as in the case of one constraint. As we shall see in Sect. 3.3, such inclusion holds true under special circumstances, otherwise we get only the following:

Denote $\mathcal{X}_j(t) = \{\mathbf{x}F \mid F_j(\mathbf{x}, P_t) \leq 0\}$. Then according to (9), $\mathcal{X}_j(t) \subset \mathcal{X}_j(0) \cup \mathcal{X}_j(1)$, hence

$$\mathcal{X}(t) \subset \mathcal{X} \cap \bigcap_j [\mathcal{X}_j(P) \cup \mathcal{X}_j(Q)] := \mathcal{X}_0.$$

To evaluate the corresponding lower bound $\min_{\mathbf{x}\in\mathcal{X}_0} F_0(\mathbf{x})$ would mean to solve a facial disjunctive program.

2.2 Directional derivative

Assume now that $F_j(\mathbf{x}, P)$, $j = 0, 1, \ldots, J$ in (1) are *convex* functions of $\mathbf{x}$. The directional derivative of the optimal value function can be obtained by the formula of Gol'shtein (1970), Theorem 17 applied to the Lagrange function

$$L(\mathbf{x}, \mathbf{u}, t) = F_0(\mathbf{x}, t) + \sum_j u_j F_j(\mathbf{x}, t)$$

provided that both the set of optimal solutions $\mathcal{X}^*(P) = \mathcal{X}^*(0)$ and the set of Lagrange multipliers $\mathcal{U}^*(P) = \mathcal{U}^*(0)$ are nonempty and bounded. If the functions F_j are *linear in* P, i.e. functions $F_j(\mathbf{x}, t)\ \forall j$ are linear in the contamination parameter t, then

$$\varphi'(0^+) = \min_{\mathbf{x}\in\mathcal{X}^*(0)} \max_{\mathbf{u}\in\mathcal{U}^*(0)} \frac{\partial}{\partial t} L(\mathbf{x}, \mathbf{u}, 0) = \min_{\mathbf{x}\in\mathcal{X}^*(0)} \max_{\mathbf{u}\in\mathcal{U}^*(0)} (L(\mathbf{x}, \mathbf{u}, Q) - L(\mathbf{x}, \mathbf{u}, P)). \tag{12}$$

Formula (12) simplifies substantially when $\mathcal{U}^*(0)$ is a singleton. When the constraints do not depend on P we get

$$\begin{aligned}\varphi'(0^+) &= \min_{\mathbf{x}\in\mathcal{X}^*(0)} \frac{\partial}{\partial t} F_0(\mathbf{x}, 0^+) = \min_{\mathbf{x}\in\mathcal{X}^*(0)} (F_0(\mathbf{x}, Q) - F_0(\mathbf{x}, P)) \\ &= \min_{\mathbf{x}\in\mathcal{X}^*(0)} F_0(\mathbf{x}, Q) - \varphi(0). \end{aligned} \tag{13}$$

These formulas can be exploited to construct an upper bound for the optimal value function $\varphi(t)$ of the form

$$\varphi(t) \le \varphi(0) + t\varphi'(0^+) \quad \forall t \in [0,1] \tag{14}$$

provided that $\varphi(t)$ is *concave*; see e.g. Dupačová (1996, 2006), Dupačová and Polívka (2007). The contaminated probability distribution P_t may also be understood as a result of contaminating Q by P and an alternative upper bound may be constructed in a similar way.

Under additional assumptions, Theorem 17 of Gol'shtein (1970) provides a formula for derivative of the optimal value function also in case of nonlinear dependence of functions F_j on t. See Dupačová (1990, 1996, 1998) for details and applications for problems with a fixed set $\mathcal{X}$ of feasible solutions. The general nonconvex case is treated e.g. in Theorems 4.25 and 4.26 of Bonnans and Shapiro (2000).

Example 2 (Upper contamination bound for CVaR) With reference to Rockafellar and Uryasev (2002), Example 1 and Dupačová (2006), Dupačová and Polívka (2007) we shall use the formula

$$\mathrm{CVaR}_\alpha(\mathbf{x}, P) = \min_v \Phi_\alpha(\mathbf{x}, v, P) := v + \frac{1}{1-\alpha} E_P(f(\mathbf{x}, \omega) - v)^+$$

and apply the contamination technique to get an upper bound. It is an unconstrained optimization problem, the set $\mathcal{V}^*(\mathbf{x}, P)$ of its optimal solutions is a nonempty compact interval of $\mathbb{R}$, for a fixed $\mathbf{x}$ the objective function is convex in v and linear in P. Formula (13) for $\mathrm{CVaR}_\alpha(\mathbf{x}, (1-t)P + tQ)$ reduces to

$$\frac{\partial}{\partial t}\mathrm{CVaR}_\alpha(\mathbf{x}, 0^+) = \min_{v \in \mathcal{V}^*(\mathbf{x}, P)} \Phi_\alpha(\mathbf{x}, v, Q) - \mathrm{CVaR}_\alpha(\mathbf{x}, P). \tag{15}$$

The optimal value function, now $\mathrm{CVaR}_\alpha(\mathbf{x}, t) := \mathrm{CVaR}_\alpha(\mathbf{x}, (1-t)P + tQ)$ is a concave function of t, hence, its lower bound is $(1-t)\mathrm{CVaR}_\alpha(\mathbf{x}, P) + t\mathrm{CVaR}_\alpha(\mathbf{x}, Q)$. For an arbitrary optimal solution $v^*(\mathbf{x}, P) \in \mathcal{V}^*(\mathbf{x}, P)$, the upper bound for the contaminated CVaR value at $\mathbf{x}$ follows by substitution to (14):

$$\mathrm{CVaR}_\alpha(\mathbf{x}, (1-t)P + tQ) \le (1-t)\mathrm{CVaR}_\alpha(\mathbf{x}, P) + t\Phi_\alpha(\mathbf{x}, v^*(\mathbf{x}, P), Q). \tag{16}$$

2.3 Upper bound

To derive an upper bound for the optimal value of the contaminated problem with probability dependent constraints we shall confine ourselves mostly to the *expectation type of the objective function and constraints*. Hence, all functions $F_j(\mathbf{x}, t)$, $j = 0, \dots, J$, are linear in t on the interval $[0, 1]$. Denote $F(\mathbf{x}, P_t) = F(\mathbf{x}, t) := \max_j F_j(\mathbf{x}, t)$. For convex $F_j(\bullet, P)\ \forall j$ the "max" function $F(\bullet, P)$ is convex as well. This allows to rewrite the set $\mathcal{X}(t)$ of feasible solutions of (5) in the form

$$\mathcal{X}(t) = \mathcal{X} \cap \{\mathbf{x} : F(\mathbf{x}, t) \le 0\}$$

with one linearly perturbed convex constraint.

Assume first that $F(\mathbf{x}^*(0), P) = 0$ for an optimal solution $\mathbf{x}^*(0) := \mathbf{x}^*(P)$ of (1) and $F(\mathbf{x}^*(0), Q) \leq 0$. Then at least one of the constraints is active at the optimal solution. Moreover, $\mathbf{x}^*(0) \in \mathcal{X}(t), \forall t \in [0, 1]$:

$$F(\mathbf{x}^*(0), t) = \max_j [(1-t)F_j(\mathbf{x}^*(0), P) + tF_j(\mathbf{x}^*(0), Q)] \\ \leq (1-t)F(\mathbf{x}^*(0), P) + tF(\mathbf{x}^*(0), Q) \leq 0.$$

It means that there is a *trivial global upper bound*

$$\varphi(t) \leq F_0(\mathbf{x}^*(0), t) \quad \forall t \in [0, 1]. \tag{17}$$

When $F_0(\mathbf{x}, \bullet)$ is linear, a more convenient form of (17) follows:

$$\varphi(t) \leq F_0(\mathbf{x}^*(0), t) = (1-t)\varphi(0) + tF_0(\mathbf{x}^*(0), Q) \quad \forall t \in [0, 1] \tag{18}$$

otherwise one may apply suitable numerically tractable upper bounds for $F_0(\mathbf{x}^*(0), t)$; see Example 3.

If the above assumption $F(\mathbf{x}^*(0), P) = 0$ and $F(\mathbf{x}^*(0), Q) \leq 0$ is not fulfilled, to get at least a local upper bound for $\varphi(t)$ valid for small t we shall switch to stability results for nonlinear parametric programming. Let $J_0 := \{j : F_j(\mathbf{x}^*(0), P) = 0\}$ be the set of indexes of active constraints of (1) at $\mathbf{x}^*(0)$.

In the *convex case*, it is possible to analyze the optimal value function by the first order methods. Various results in this direction can be mentioned: For example, according to Robinson (1987) the perturbed problem with a fixed convex polyhedral set $\mathcal{X}$ in (4) reduces locally to a problem with a parameter independent set of feasible solutions if $\mathbf{x}^*(0)$ is a nondegenerate point and the strict complementarity conditions hold true. In particular, $\mathbf{x}^*(0)$ is a nondegenerate point of (1) iff gradients $\nabla_x F_j(\mathbf{x}^*(0), P)$, $j \in J_0$ are linearly independent, i.e. under the linear independence condition; cf. Bonnans and Shapiro (2000), Example 4.78. Then for t small enough, $t \leq t_0$, $t_0 > 0$, the optimal value function $\varphi(t)$ is concave and its upper bound equals

$$\varphi(t) \leq \varphi(0) + t\varphi'(0^+) \quad \forall t \in [0, t_0]. \tag{19}$$

A more detailed insight can be obtained if there is a continuous trajectory $[\mathbf{x}^*(t), \mathbf{u}^*(t)]$ of optimal solutions and Lagrange multipliers of the perturbed problem (5) emanating from the unique optimal solution $\mathbf{x}^*(0)$ and unique Lagrange multipliers $u_j^*(0)$, $j = 1, \ldots, J$ of (1). Such result follows usually by the implicit function theorem applied to the first order necessary conditions. In addition to the nondegeneracy and the strict complementarity conditions it requires also nonsingularity of the Hessian matrix of the Lagrange function on the tangent space to the active constraints, i.e. the second order sufficient condition valid at $\mathbf{x}^*(0), \mathbf{u}^*(0)$; see e.g. Bonnans and Shapiro (2000) or Fiacco (1983). At this point, convexity with respect to $\mathbf{x}$ is not needed and the trajectory $[\mathbf{x}^*(t), \mathbf{u}^*(t)]$ satisfies the first order optimality conditions also for $0 < t \leq t_0$:

$$F_j(\mathbf{x}^*(t), P_t) \leq 0,\ u_j^*(t) \geq 0,\ F_j(\mathbf{x}^*(t), P_t)u_j^*(t) = 0, \quad j = 1, 2, \ldots, J$$

$$\nabla_x F_0(\mathbf{x}^*(t), P_t) + \sum_j u_j^*(t)\nabla_x F_j(\mathbf{x}^*(t), P_t) = 0.$$

Moreover, for convex expectation type functionals F_j, $j = 0, \dots, J$, the derivative (12) of the optimal value function reduces to

$$\varphi'(0^+) = \frac{\partial}{\partial t} L(\mathbf{x}^*(0), \mathbf{u}^*(0), 0) = L(\mathbf{x}^*(0), \mathbf{u}^*(0), Q) - L(\mathbf{x}^*(0), \mathbf{u}^*(0), P)$$
$$= F_0(\mathbf{x}^*(0), Q) + \sum_j u_j^*(0) F_j(\mathbf{x}^*(0), Q) - F_0(\mathbf{x}^*(0), P). \tag{20}$$

If no constraint is active at $\mathbf{x}^*(0)$, we face a locally unconstrained optimization problem and the optimal value function $\varphi(t)$ is concave on a right neighborhood of 0, say for $t \in [0, t_0]$, $t_0 > 0$, hence, for $t \le t_0$, the upper bound (19) applies.

In the opposite case, the strict complementarity conditions imply that for small $t \in [0, t_0]$, $t_0 > 0$ the set J_0 of indexes of active constraints remains fixed and for a local analysis, constraints $F_j(\mathbf{x}, P) \le 0$ with $j \notin J_0$ need not be considered. Then $\mathcal{X}(t)$ reduces locally to the set of solutions of the system of equations $F_j(\mathbf{x}, t) = 0$, $j \in J_0$ which can be replaced locally by a parameter independent set.

To summarize – there exists $t_0 > 0$ such that for $0 \le t \le t_0$ the optimal value $\varphi(t)$ of the contaminated problem (5) can be obtained as $\varphi(t) = \min_{\mathbf{x} \in \mathcal{X}_0} F_0(\mathbf{x}, t)$ where the set of feasible solutions $\mathcal{X}_0$ does not depend on t. Hence, $\varphi(t)$ is concave on $[0, t_0]$, $t_0 > 0$ which opens the possibility of constructing local upper contamination bounds (19). Accordingly, the following theorem holds true:

Theorem 2 *Let* (1) *be a twice differentiable program*, $\mathbf{x}^*(P) = \mathbf{x}^*(0)$ *its optimal solution and* $\varphi(P) = \varphi(0)$ *its optimal value. Assume that at* $\mathbf{x}^*(0)$ *linear independence*, *the strict complementarity and the second order sufficient conditions are satisfied. Then there exists* $t_0 > 0$ *such that for all* $t \in [0, t_0]$ *the optimal value function* $\varphi(t)$ *is concave and the* local *upper contamination bound is given by*

$$\varphi(t) \le \varphi(0) + t\varphi'(0^+) \quad \forall t \in [0, t_0]. \tag{21}$$

Moreover, for convex expectation type problems (1) *the directional derivative is given by* (20).

2.3.1 Comment

Except for the form of the directional derivative, Theorem 2 applies also to problems with nonconvex functions $F_j(\bullet, P)\ \forall j$.

2.4 Illustrative examples

Consider $S = 50$ equiprobable scenarios of monthly returns $\boldsymbol{\varrho}$ of $N = 9$ assets (8 European stock market indexes: AEX, ATX, FCHI, GDAXI, OSEAX, OMXSPI, SSMI, FTSE and a risk free asset) in period June 2004–August 2008. The scenarios can be collected in the matrix

$$R = \begin{pmatrix} \mathbf{r}^1 \\ \mathbf{r}^2 \\ \vdots \\ \mathbf{r}^S \end{pmatrix}$$

Table 1 Descriptive statistics and the additional scenario of returns of 8 European stock indexes and of the risk free asset

Index	Country	Mean	Max	Min	A.S.
AEX	Netherlands	0.00456	0.07488	−0.14433	−0.19715
ATX	Austria	0.01358	0.13247	−0.14869	−0.23401
FCHI	France	0.0044	0.0615	−0.13258	−0.1005
GDAXI	Germany	0.01014	0.07111	−0.15068	−0.09207
OSEAX	Norway	0.01872	0.12176	−0.19505	−0.23934
OMXSPI	Sweden	0.00651	0.08225	−0.14154	−0.12459
SSMI	Switzerland	0.00563	0.05857	−0.09595	−0.08065
FTSE	England	0.00512	0.06755	−0.08938	−0.13024
Risk free		0.002	0.002	0.002	0.002

where $\mathbf{r}^s = (r_1^s, r_2^s, \ldots, r_N^s)$ is the s-th scenario. We will use $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)'$ for the vector of portfolio weights and the portfolio possibilities are given by

$$\Lambda = \{\boldsymbol{\lambda} \in \mathbb{R}^N | \mathbf{1}'\boldsymbol{\lambda} = 1,\ \lambda_n \geq 0,\ n = 1, 2, \ldots, N\},$$

that is, the short sales are not allowed. The historical data comes from pre-crisis period. The data is contaminated by a scenario $\mathbf{r}^{S+1}$ from September 2008 when all indexes strongly fell down. The additional scenario can be understood as a stress scenario or the worst-case scenario. It can be seen in Table 1 presenting basic descriptive statistics of the original data and the additional scenario (A.S.).

We will apply the contamination bounds to mean-risk models with CVaR as a measure of risk. Two formulations are considered: In the first one, we are searching for a portfolio with minimal CVaR and at least the prescribed expected return, see e.g. Dupačová (2006) or Kilianová and Pflug (2009). Secondly, we minimize the expected loss of the portfolio under the condition that CVaR is below a given level, a special case of Example 1.

Example 3 (Minimizing CVaR) Mean-CVaR model with CVaR minimization is a special case of the general formulation (1) when $F_0(\mathbf{x}, P) = \mathrm{CVaR}(-\boldsymbol{\varrho}'\boldsymbol{\lambda})$ and $F_1(\mathbf{x}, P) = E_P(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) - \mu(P)$; $\mu(P)$ is the maximal allowable expected loss. We choose

$$\mu(P) = -E_P\boldsymbol{\varrho}'\left(\frac{1}{9}, \frac{1}{9}, \ldots, \frac{1}{9}\right)' = \frac{1}{50}\sum_{s=1}^{50} -\mathbf{r}^s\left(\frac{1}{9}, \frac{1}{9}, \ldots, \frac{1}{9}\right)'.$$

It means that the minimal required expected return is equal to the average return of the equally diversified portfolio. The significance level $\alpha = 0.95$ and Λ is a fixed convex polyhedral set representing constraints that do not depend on P. Since P is a discrete distribution with equiprobable scenarios $\mathbf{r}^1, \mathbf{r}^2, \ldots, \mathbf{r}^{50}$, using (3), the mean-CVaR model can be formulated as the following linear program:

$$\begin{aligned} \varphi(0) = & \min_{\lambda \in \Lambda, v \in \mathbb{R}, z_s \in \mathbb{R}^+} v + \frac{1}{50 * 0.05}\sum_{s=1}^{50} z_s \\ \text{s.t.} \quad & z_s \geq -\mathbf{r}^s\boldsymbol{\lambda} - v, \quad s = 1, 2, \ldots, 50 \\ & \frac{1}{50}\sum_{s=1}^{50} -\mathbf{r}^s\boldsymbol{\lambda} - \mu(P) \leq 0. \end{aligned} \tag{22}$$

By analogy, for the additional scenario we have:

$$\begin{aligned}\varphi(1) &= \min_{\boldsymbol{\lambda}\in\Lambda, v\in\mathbb{R}, z\in\mathbb{R}^+} v + \frac{1}{0.05}z \\ \text{s.t.} &\quad z \geq -\mathbf{r}^{51}\boldsymbol{\lambda} - v,\ -\mathbf{r}^{51}\boldsymbol{\lambda} - \mu(Q) \leq 0\end{aligned} \tag{23}$$

or, equivalently:

$$\varphi(1) = \min_{\boldsymbol{\lambda}\in\Lambda}\{-\mathbf{r}^{51}\boldsymbol{\lambda} \mid -\mathbf{r}^{51}\boldsymbol{\lambda} - \mu(Q) \leq 0\} \tag{24}$$

where $\mu(Q) = -\mathbf{r}^{51}(\frac{1}{9}, \frac{1}{9}, \dots, \frac{1}{9})'$.

First, we compute for $t \in [0, 1]$ the optimal value function of the contaminated problem.

$$\begin{aligned}\varphi(t) &= \min_{\boldsymbol{\lambda}\in\Lambda, v\in\mathbb{R}, z_s\in\mathbb{R}^+} v + \frac{1}{0.05}\left(\sum_{s=1}^{50}\frac{1}{50}(1-t)z_s + tz_{51}\right) \\ \text{s.t.} &\quad z_s \geq -\mathbf{r}^s\boldsymbol{\lambda} - v, \quad s = 1, 2, \dots, 51 \\ &\quad -\sum_{s=1}^{50}\frac{1}{50}(1-t)\mathbf{r}^s\boldsymbol{\lambda} - t\mathbf{r}^{51}\boldsymbol{\lambda} - \mu((1-t)P + tQ) \leq 0\end{aligned} \tag{25}$$

where $\mu((1-t)P + tQ) = -\sum_{s=1}^{50}\frac{1}{50}(1-t)\mathbf{r}^s(\frac{1}{9}, \frac{1}{9}, \dots, \frac{1}{9})' - t\mathbf{r}^{51}(\frac{1}{9}, \frac{1}{9}, \dots, \frac{1}{9})'$.

Secondly, applying (11), we derive a *lower bound* for $\varphi(t)$. Note that now

$$\begin{aligned}\min_{\mathcal{X}(Q)} F_0(\mathbf{x}, P) &= \min_{\boldsymbol{\lambda}\in\Lambda, v\in\mathbb{R}, z_s\in\mathbb{R}^+} v + \frac{1}{50 * 0.05}\sum_{s=1}^{50} z_s \\ \text{s.t.} &\quad z_s \geq -\mathbf{r}^s\boldsymbol{\lambda} - v, \quad s = 1, 2, \dots, 50 \\ &\quad -\mathbf{r}^{51}\boldsymbol{\lambda} - \mu(Q) \leq 0\end{aligned}$$

and

$$\min_{\mathcal{X}(P)} F_0(\mathbf{x}, Q) = \min_{\boldsymbol{\lambda}\in\Lambda}\left\{-\mathbf{r}^{51}\boldsymbol{\lambda} \mid \frac{1}{50}\sum_{s=1}^{50} -\mathbf{r}^s\boldsymbol{\lambda} - \mu(P) \leq 0\right\}.$$

Finally, we construct an *upper bound* for $\varphi(t)$. Since the optimal solution $\boldsymbol{\lambda}^*$ of (22) is a feasible solution of (23) we can apply (17) with $\mathbf{x}^*(0) = \boldsymbol{\lambda}^*$ as a trivial upper bound for all $t \in [0, 1]$:

$$\begin{aligned}\varphi(t) \leq F_0(\mathbf{x}^*(0), t) &= \min_{v\in\mathbb{R}, z_s\in\mathbb{R}^+} v + \frac{1}{0.05}\left(\sum_{s=1}^{50}\frac{1}{50}(1-t)z_s + tz_{51}\right) \\ \text{s.t.} &\quad z_s \geq -\mathbf{r}^s\boldsymbol{\lambda}^* - v, \quad s = 1, 2, \dots, 51.\end{aligned}$$

The disadvantage of this trivial bound is the fact, that it would require evaluation of the CVaR for $\boldsymbol{\lambda}^*$ for each t. Linearity with respect to t does not hold true, but we may apply the bound (16). This yields an upper estimate for $F_0(\mathbf{x}^*(0), t)$ which is a convex combination of $\varphi(0)$ and $\Phi_\alpha(\mathbf{x}^*(0), v^*(\mathbf{x}^*(0), P), Q)$. The optimal value $\varphi(0)$ is given by (22) and

$$\Phi_\alpha(\mathbf{x}^*(0), v^*(\mathbf{x}^*(0), P), Q) = v^* + \frac{1}{0.05}(-\mathbf{r}^{51}\boldsymbol{\lambda}^* - v^*)^+$$

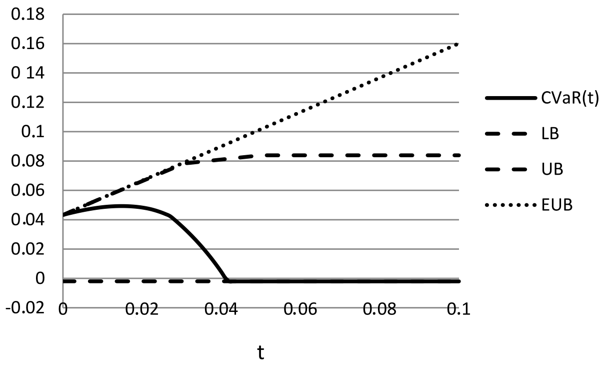

Fig. 1 Comparison of minimal (CVaR(t)) value of mean-CVaR model with lower bound (LB), upper bound (UB) and the estimated upper bound (EUB)

where v^* and $\boldsymbol{\lambda}^*$ are optimal solutions of (22). The graphs of $\varphi(t)$, its lower bound and two upper bounds (trivial one and its upper estimate) for small contamination $t \in [0, 0.1]$ are presented in Fig. 1. Since all original scenarios have probability 0.02, the performance for $t > 0.1$ is not of much interest. For $t > 0.04$, $\varphi(t)$ in (25) coincides with its lower bound because the optimal portfolios consist only of risk free asset. The upper bound is piecewise linear in t and for small values of t it coincides with the estimated upper bound.

Example 4 (Minimizing expected loss) As the second example, consider the mean-CVaR model minimizing the expected loss subject to a constraint on CVaR. This corresponds to (1) with $F_0(\mathbf{x}, P) = E_P(-\boldsymbol{\varrho}'\boldsymbol{\lambda})$ and $F_1(\mathbf{x}, P) = \text{CVaR}(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) - c$ where $c = 0.19$ is the maximal accepted level of CVaR. For simplicity, this level does not depend on the probability distribution. Similarly to the previous example, we compute the optimal value $\varphi(t)$ and its lower and upper bound. Using Theorem 16 of Rockafellar and Uryasev (2002), the minimal CVaR-constrained expected loss is obtained for $t \in [0, 1]$ as

$$\varphi(t) = \min_{\boldsymbol{\lambda}\in\Lambda, v\in\mathbb{R}} -\sum_{s=1}^{50} \frac{1}{50}(1-t)\mathbf{r}^s\boldsymbol{\lambda} - t\mathbf{r}^{51}\boldsymbol{\lambda} \tag{26}$$

$$\text{s.t.}\quad v + \frac{1}{0.05}\left(-\sum_{s=1}^{50}\frac{1}{50}(1-t)\mathbf{r}^s\boldsymbol{\lambda} - t\mathbf{r}^{51}\boldsymbol{\lambda} - v\right)^+ - c \le 0 \tag{27}$$

and equals thus the optimal value function of the parametric linear program

$$\begin{aligned}
\varphi(t) = &\min_{\boldsymbol{\lambda}\in\Lambda, v\in\mathbb{R}, z_s\in\mathbb{R}^+} -\sum_{s=1}^{50}\frac{1}{50}(1-t)\mathbf{r}^s\boldsymbol{\lambda} - t\mathbf{r}^{51}\boldsymbol{\lambda}\\
&\text{s.t.}\quad v + \frac{1}{0.05}\left(\sum_{s=1}^{50}\frac{1}{50}(1-t)z_s + tz_{51}\right) - c \le 0\\
&\qquad z_s \ge -\mathbf{r}^s\boldsymbol{\lambda} - v, \quad s = 1, 2, \ldots, 51
\end{aligned} \tag{28}$$

for $t \in [0, 1]$. In particular, for $t = 1$ we have

$$\begin{aligned}
\varphi(1) = &\min_{\boldsymbol{\lambda}\in\Lambda, v\in\mathbb{R}, z_s\in\mathbb{R}^+} -\mathbf{r}^{51}\boldsymbol{\lambda}\\
&\text{s.t.}\quad v + \frac{1}{0.05}z_{51} - c \le 0,\ z_{51} + v \ge -\mathbf{r}^{51}\boldsymbol{\lambda},
\end{aligned}$$

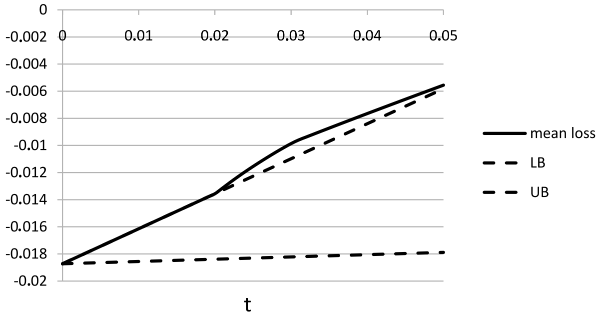

Fig. 2 Comparison of minimal mean loss value with its lower bound (LB) and upper bound (UB)

what is equivalent to

$$\varphi(1) = \min_{\boldsymbol{\lambda}\in\Lambda}\{-\mathbf{r}^{51}\boldsymbol{\lambda} \mid -\mathbf{r}^{51}\boldsymbol{\lambda} - c \le 0\};$$

compare with (24). Using (11), we can evaluate the *lower bound* for $\varphi(t)$ with

$$\min_{\mathcal{X}(Q)} F_0(\mathbf{x}, P) = \min_{\boldsymbol{\lambda}\in\Lambda}\left\{-\sum_{s=1}^{50}\frac{1}{50}\mathbf{r}^s\boldsymbol{\lambda} \mid -\mathbf{r}^{51}\boldsymbol{\lambda} - c \le 0\right\}$$

and

$$\min_{\mathcal{X}(P)} F_0(\mathbf{x}, Q) = \min_{\boldsymbol{\lambda}\in\Lambda, v\in\mathbb{R}, z_s\in\mathbb{R}^+} -\mathbf{r}^{51}\boldsymbol{\lambda}$$

$$\text{s.t.} \quad v + \frac{1}{0.05}\sum_{s=1}^{50}\frac{1}{50}z_s - c \le 0, \quad z_s \ge -\mathbf{r}^s\boldsymbol{\lambda} - v, \quad s = 1, 2, \dots, 50.$$

Finally, we compute an *upper bound* for $\varphi(t)$. Contrary to the previous example, the optimal solution $\mathbf{x}^*(0)$ of the noncontaminated problem is not a feasible solution of the fully contaminated problem. Therefore, the trivial global upper bound (17) cannot be used. We apply instead the local upper bound (21) with the directional derivative (20). In this example, the value of multiplier $\mathbf{u}^*(0)$ corresponding to (27) for $t = 0$ is equal to zero, the CVaR constraint (27) is not active and for sufficiently small t, the upper bound reduces to:

$$\varphi(t) \le (1 - t)\varphi(0) + tF_0(\mathbf{x}^*(0), Q). \tag{29}$$

Figure 2 depicts the graph of $\varphi(t)$ given by (28) and its lower and upper bound. The upper bound coincides with $\varphi(t)$ for $t \le 0.02$. It illustrates the fact that the local upper bound is meaningful if the probability of the additional scenario is not too large, i.e. no more than probabilities of the original scenarios for our example.

3 Robustness in portfolio efficiency testing

3.1 Portfolio efficiency test

In this section, we shall study robustness of portfolio efficiency tests with respect to the second-order stochastic dominance relation. Consider N assets and a random vector of their returns $\boldsymbol{\varrho}$. Since all existing portfolio efficiency tests have been derived for a discrete probability distribution P of returns we assume that $\boldsymbol{\varrho}$ takes S values $\mathbf{r}^s = (r_1^s, r_2^s, \ldots, r_N^s)$, called scenarios, with probabilities $p_1, p_2, \ldots, p_S$. Contrary to all former tests, e.g. Kopa and Chovanec (2008) or Kopa (2010), *we do not assume equiprobable scenarios.* Again, the scenarios are collected in the matrix

$$R = \begin{pmatrix} \mathbf{r}^1 \\ \mathbf{r}^2 \\ \vdots \\ \mathbf{r}^S \end{pmatrix}$$

and the portfolio possibilities are given by

$$\Lambda = \{\boldsymbol{\lambda} \in \mathbb{R}^N | \mathbf{1}'\boldsymbol{\lambda} = 1,\ \lambda_n \geq 0,\ n = 1, 2, \ldots, N\}.$$

Alternatively, one can consider any bounded polytope: $\Lambda' = \{\boldsymbol{\lambda} \in R^N | A\boldsymbol{\lambda} \geq \mathbf{b}\}$.

For any portfolio $\boldsymbol{\lambda} \in \Lambda$, let $(-R\boldsymbol{\lambda})^{[k]}$ be the k-th smallest element of $(-R\boldsymbol{\lambda})$, i.e. $(-R\boldsymbol{\lambda})^{[1]} \leq (-R\boldsymbol{\lambda})^{[2]} \leq \cdots \leq (-R\boldsymbol{\lambda})^{[S]}$ and let $I(\boldsymbol{\lambda})$ be a permutation of the index set $I = \{1, 2, \ldots, S\}$ such that $-\mathbf{r}^{i(\boldsymbol{\lambda})}\boldsymbol{\lambda} = (-R\boldsymbol{\lambda})^{[i]}$. Accordingly, we can order the corresponding probabilities and we denote $p_i^{\boldsymbol{\lambda}} = p_{i(\boldsymbol{\lambda})}$. Hence, $p_i^{\boldsymbol{\lambda}} = P(-\boldsymbol{\varrho}\boldsymbol{\lambda} = (-R\boldsymbol{\lambda})^{[i]})$. The same notation is applied for the tested portfolio $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_N)'$.

Let $F_{\boldsymbol{\varrho}'\boldsymbol{\lambda}}(x)$ denote the cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$. The twice cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$ is defined as

$$F^{(2)}_{\boldsymbol{\varrho}'\boldsymbol{\lambda}}(y) = \int_{-\infty}^{y} F_{\boldsymbol{\varrho}'\boldsymbol{\lambda}}(x)\,dx. \tag{30}$$

Following Ruszczyński and Vanderbei (2003), Kuosmanen (2004), Kopa and Chovanec (2008) and Kopa (2010), we define the second-order stochastic dominance relation in the strict form in the context of SSD portfolio efficiency.

Definition 1 Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by the second-order stochastic dominance ($\boldsymbol{\varrho}'\boldsymbol{\lambda} \succ_{SSD} \boldsymbol{\varrho}'\boldsymbol{\tau}$) if and only if

$$F^{(2)}_{\boldsymbol{\varrho}'\boldsymbol{\lambda}}(y) \leq F^{(2)}_{\boldsymbol{\varrho}'\boldsymbol{\tau}}(y) \quad \forall y \in \mathbb{R}$$

with strict inequality[1] for at least one $y \in \mathbb{R}$.

As in Ogryczak and Ruszczyński (2002) or Kopa and Chovanec (2008), we express the SSD relation using the conditional value at risk (CVaR).

Lemma 1 *Let* $\boldsymbol{\lambda}, \boldsymbol{\tau} \in \Lambda$. *Then* $\boldsymbol{\varrho}'\boldsymbol{\lambda} \succ_{SSD} \boldsymbol{\varrho}'\boldsymbol{\tau}$ *if and only if*

$$\mathrm{CVaR}_\alpha(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) \leq \mathrm{CVaR}_\alpha(-\boldsymbol{\varrho}'\boldsymbol{\tau}) \quad \textit{for all } \alpha \in [0, 1] \tag{31}$$

with strict inequality for at least one α.

[1] This type of SSD relation is sometimes referred to as the strict second-order stochastic dominance. If no strict inequality is required then the relation can be called the weak second-order stochastic dominance.

Since we limit our attention to a discrete probability distribution of returns, the inequality of CVaRs need not be verified in all $\alpha \in [0, 1]$, but only in at most $S + 1$ particular points.

Theorem 3 *Let* $q_s^{\boldsymbol{\lambda}} = \sum_{i=1}^{s} p_i^{\boldsymbol{\lambda}}$ *and* $q_s^{\boldsymbol{\tau}} = \sum_{i=1}^{s} p_i^{\boldsymbol{\tau}}$, $s = 1, 2, \ldots, S$. *Let* $q_0^{\boldsymbol{\lambda}} = q_0^{\boldsymbol{\tau}} = 0$. *Then* $\boldsymbol{\varrho}'\boldsymbol{\lambda} \succ_{SSD} \boldsymbol{\varrho}'\boldsymbol{\tau}$ *if and only if* $\mathrm{CVaR}_{q_s^{\boldsymbol{\lambda}}}(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) \le \mathrm{CVaR}_{q_s^{\boldsymbol{\lambda}}}(-\boldsymbol{\varrho}'\boldsymbol{\tau})$ *for all* $s = 0, 1, 2, \ldots, S$ *with strict inequality for at least one* $q_s^{\boldsymbol{\lambda}}$.

Proof Assume $\alpha > 0$. Following Rockafellar and Uryasev (2002), Proposition 8, let $s(\alpha)$ be the unique index such that $q_{s(\alpha)}^{\boldsymbol{\lambda}} \ge \alpha > q_{s(\alpha)-1}^{\boldsymbol{\lambda}}$. Then

$$\mathrm{CVaR}_\alpha(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) = \frac{1}{1-\alpha}\left[(q_{s(\alpha)}^{\boldsymbol{\lambda}} - \alpha)(-R\boldsymbol{\lambda})^{[s(\alpha)]} + \sum_{i=s(\alpha)+1}^{S} p_i^{\boldsymbol{\lambda}}(-R\boldsymbol{\lambda})^{[i]}\right].$$

Consider $\mathrm{LC}_\alpha(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) := (1-\alpha)\mathrm{CVaR}_\alpha(-\boldsymbol{\varrho}'\boldsymbol{\lambda})$. Since $1 - q_{s(\alpha)}^{\boldsymbol{\lambda}} = \sum_{i=s(\alpha)+1}^{S} p_i^{\boldsymbol{\lambda}}$ we have:

$$\begin{aligned}
\mathrm{LC}_\alpha(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) &= q_{s(\alpha)}^{\boldsymbol{\lambda}}(-R\boldsymbol{\lambda})^{[s(\alpha)]} - \alpha(-R\boldsymbol{\lambda})^{[s(\alpha)]} + \sum_{i=s(\alpha)+1}^{S} p_i^{\boldsymbol{\lambda}}(-R\boldsymbol{\lambda})^{[i]} \\
&= (1-\alpha)(-R\boldsymbol{\lambda})^{[s(\alpha)]} - (-R\boldsymbol{\lambda})^{[s(\alpha)]}(1 - q_{s(\alpha)}^{\boldsymbol{\lambda}}) + \sum_{i=s(\alpha)+1}^{S} p_i^{\boldsymbol{\lambda}}(-R\boldsymbol{\lambda})^{[i]} \\
&= (1-\alpha)(-R\boldsymbol{\lambda})^{[s(\alpha)]} + \sum_{i=s(\alpha)+1}^{S} p_i^{\boldsymbol{\lambda}}\big((-R\boldsymbol{\lambda})^{[i]} - (-R\boldsymbol{\lambda})^{[s(\alpha)]}\big).
\end{aligned}$$

A similar analysis can be done for portfolio $\boldsymbol{\tau}$. Since both $\mathrm{LC}_\alpha(-\boldsymbol{\varrho}'\boldsymbol{\lambda})$ and $\mathrm{LC}_\alpha(-\boldsymbol{\varrho}'\boldsymbol{\tau})$ are concave piecewise linear functions in α, Lemma 1 implies that $\boldsymbol{\varrho}'\boldsymbol{\lambda} \succ_{SSD} \boldsymbol{\varrho}'\boldsymbol{\tau}$ if and only if $\mathrm{LC}_\alpha(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) \le \mathrm{LC}_\alpha(-\boldsymbol{\varrho}'\boldsymbol{\tau})$ for all $\alpha = q_s^{\boldsymbol{\lambda}}$, $s = 0, 1, \ldots, S$, with strict inequality for at least one $q_s^{\boldsymbol{\lambda}}$. Passing back to CVaR expressions completes the proof. □

Following Ruszczyński and Vanderbei (2003), Kuosmanen (2004), Kopa and Chovanec (2008) and Kopa (2010) we define portfolio efficiency with respect to the second order stochastic dominance.

Definition 2 A given portfolio $\boldsymbol{\tau} \in \Lambda$ is SSD inefficient if there exists portfolio $\boldsymbol{\lambda} \in \Lambda$ such that $\boldsymbol{\varrho}'\boldsymbol{\lambda} \succ_{SSD} \boldsymbol{\varrho}'\boldsymbol{\tau}$. Otherwise, portfolio $\boldsymbol{\tau}$ is SSD efficient.

This definition classifies portfolio $\boldsymbol{\tau} \in \Lambda$ as SSD efficient if and only if no other portfolio is better (in the sense of the SSD relation) for all risk averse and risk neutral decision makers. Inspired by Kopa and Chovanec (2008) we consider the following measure:

$$\begin{aligned}
\xi(\boldsymbol{\tau}, R, \mathbf{p}) &= \min_{a_s, \boldsymbol{\lambda}} \sum_{s=0}^{S} a_s \\
\text{s.t.} \quad & \mathrm{CVaR}_{q_s^{\boldsymbol{\lambda}}}(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) - \mathrm{CVaR}_{q_s^{\boldsymbol{\lambda}}}(-\boldsymbol{\varrho}'\boldsymbol{\tau}) \le a_s, \quad s = 0, 1, \ldots, S \\
& a_s \le 0, \quad s = 0, 1, \ldots, S \\
& \boldsymbol{\lambda} \in \Lambda.
\end{aligned} \tag{32}$$

The objective function of (32) represents the sum of differences between CVaRs of a portfolio $\boldsymbol{\lambda}$ and CVaRs of the tested portfolio $\boldsymbol{\tau}$. The differences are considered in points $q_s^{\boldsymbol{\lambda}}$, $s = 0, 1, \ldots, S$. All differences must be non-positive and at least one negative to guarantee that portfolio $\boldsymbol{\lambda}$ dominates portfolio $\boldsymbol{\tau}$. Moreover, minimizing these differences, we find portfolio $\boldsymbol{\lambda}^*$ that cannot be dominated by any other one. On the other hand, if no dominating portfolio exists, that is, portfolio $\boldsymbol{\tau}$ is SSD efficient, then $\xi(\boldsymbol{\tau}, R, \mathbf{p}) = 0$ because the only feasible solutions of (32) are $\boldsymbol{\tau}$ and portfolios $\overline{\boldsymbol{\lambda}}$ satisfying $R\overline{\boldsymbol{\lambda}} = R\boldsymbol{\tau}$. Summarizing, Theorem 3 implies the following necessary and sufficient SSD portfolio efficiency test:

Theorem 4 *A given portfolio $\boldsymbol{\tau}$ is SSD efficient if and only if $\xi(\boldsymbol{\tau}, R, \mathbf{p}) = 0$. If $\xi(\boldsymbol{\tau}, R, \mathbf{p}) < 0$ then the optimal portfolio $\boldsymbol{\lambda}^*$ in (32) is SSD efficient and it dominates portfolio $\boldsymbol{\tau}$ by SSD.*

Until now, perfect information about the probability distribution of returns was assumed and portfolio $\boldsymbol{\tau}$ was tested with respect to this distribution. However, in many practical applications, the probability distribution of returns is not perfectly known. And therefore, we will study robust versions of SSD efficiency.

3.2 Portfolio efficiency with respect to ϵ-SSD relation

Assume that the probability distribution $\bar{P}$ of random returns $\bar{\boldsymbol{\varrho}}$ takes again values $\mathbf{r}^s$, $s = 1, 2, \ldots, S$ but with other probabilities $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_S)$. We define the distance between P and $\bar{P}$ as $d(\bar{P}, P) = \max_i |\bar{p}_i - p_i|$.

Definition 3 A given portfolio $\boldsymbol{\tau} \in \Lambda$ is ϵ-SSD inefficient if there exists portfolio $\boldsymbol{\lambda} \in \Lambda$ and $\bar{P}$ such that $d(\bar{P}, P) \leq \epsilon$ with $\bar{\boldsymbol{\varrho}}'\boldsymbol{\lambda} \succ_{SSD} \bar{\boldsymbol{\varrho}}'\boldsymbol{\tau}$. Otherwise, portfolio $\boldsymbol{\tau}$ is ϵ-SSD efficient.

The introduced ϵ-SSD efficiency is a robustification of the classical SSD portfolio efficiency. It guarantees stability of the SSD efficiency classification with respect to small changes (prescribed by parameter ϵ) in probability vector $\mathbf{p}$. A given portfolio $\boldsymbol{\tau}$ is ϵ-SSD efficient if and only if no portfolio $\boldsymbol{\lambda}$ SSD dominates $\boldsymbol{\tau}$ neither for the original probabilities $\mathbf{p}$ nor for arbitrary probabilities $\bar{\mathbf{p}}$ from ϵ-neighborhood of the original vector $\mathbf{p}$. For testing ϵ-SSD efficiency of a given portfolio $\boldsymbol{\tau}$ we modify (32) in order to introduce a new measure of ϵ-SSD efficiency:

$$
\begin{aligned}
\xi_\epsilon(\boldsymbol{\tau}, R, \mathbf{p}) = \min_{a_s, \boldsymbol{\lambda}, \bar{\mathbf{p}}} & \sum_{s=0}^{S} a_s \\
\text{s.t.} \quad & \mathrm{CVaR}_{\bar{q}_s^{\boldsymbol{\lambda}}}(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) - \mathrm{CVaR}_{\bar{q}_s^{\boldsymbol{\lambda}}}(-\boldsymbol{\varrho}'\boldsymbol{\tau}) \leq a_s, \quad s = 0, 1, \ldots, S \\
& \bar{q}_s^{\boldsymbol{\lambda}} = \sum_{i=1}^{s} \bar{p}_i^{\boldsymbol{\lambda}}, \quad s = 1, \ldots, S \\
& \bar{q}_0^{\boldsymbol{\lambda}} = 0 \\
& \sum_{i=1}^{S} \bar{p}_i = 1 \\
& -\epsilon \leq \bar{p}_i - p_i \leq \epsilon, \quad i = 1, 2, \ldots, S \\
& \bar{p}_i \geq 0, \quad i = 1, 2, \ldots, S
\end{aligned}
$$

$$\begin{aligned} & a_s \le 0, \quad s = 0, 1, \dots, S \\ & \boldsymbol{\lambda} \in \Lambda. \end{aligned} \tag{33}$$

Theorem 5 *Portfolio* $\boldsymbol{\tau} \in \Lambda$ *is* ϵ*-SSD efficient if and only if* $\xi_\epsilon(\boldsymbol{\tau}, R, \mathbf{p})$ *given by* (33) *is equal to zero.*

Proof The proof directly follows from Theorem 4 because (33) is obtained from (32) by an additional minimization over $\bar{\mathbf{p}}$ from ϵ-neighborhood of the original probability vector $\mathbf{p}$. □

3.3 Resistance of SSD portfolio efficiency with respect to additional scenarios

In the previous sections, we assumed a fixed set of scenarios. In many practical applications, an additional scenario may be of interest. Therefore, the aim of this section is to analyze the robustness of SSD portfolio efficiency with respect to the additional scenario denoted by $\mathbf{r}^{S+1}$. For a contamination parameter $t \in [0, 1]$, we assume that the random return $\tilde{\boldsymbol{\varrho}}(t)$ takes values $\mathbf{r}^1, \mathbf{r}^2, \dots, \mathbf{r}^{S+1}$ with probabilities $\tilde{\mathbf{p}}(t) = ((1-t)p_1, (1-t)p_2, \dots, (1-t)p_S, t)$. The cumulative probabilities for portfolio $\boldsymbol{\lambda}$ are

$$\tilde{q}_s^{\boldsymbol{\lambda}} = \sum_{i=1}^{s} \tilde{p}_i^{\boldsymbol{\lambda}} = \sum_{i=1}^{s} P(-\tilde{\boldsymbol{\varrho}}(t)\boldsymbol{\lambda} = (-\tilde{R}\boldsymbol{\lambda})^{[i]}), \quad s = 1, 2, \dots, S+1, \quad \tilde{q}_0^{\boldsymbol{\lambda}} = 0$$

and the same notation is used for portfolio $\boldsymbol{\tau}$. We denote the extended scenario matrix by $\tilde{R}$, that is,

$$\tilde{R} = \begin{pmatrix} R \\ \mathbf{r}^{S+1} \end{pmatrix}.$$

Definition 4 A given portfolio $\boldsymbol{\tau} \in \Lambda$ is directionally SSD inefficient with respect to $\mathbf{r}^{S+1}$ if it exists $t_0 > 0$ such that for every $t \in [0, t_0]$ there is a portfolio $\boldsymbol{\lambda}(t) \in \Lambda$ satisfying $\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\lambda}(t) \succ_{SSD} \tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\tau}$.

Definition 5 A given portfolio $\boldsymbol{\tau} \in \Lambda$ is directionally SSD efficient with respect to $\mathbf{r}^{S+1}$ if there exists $t_0 > 0$ such that for arbitrary $t \in [0, t_0]$ there is no portfolio $\boldsymbol{\lambda}(t) \in \Lambda$ satisfying $\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\lambda}(t) \succ_{SSD} \tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\tau}$.

According to these definitions, a given portfolio is classified as directionally SSD efficient (inefficient) with respect to scenario $\mathbf{r}^{S+1}$ if it is SSD efficient (inefficient) and a sufficiently small contamination of the original probability distribution of returns by the additional scenario does not change the SSD efficiency classification, that is, the SSD efficient (inefficient) portfolio remains SSD efficient (inefficient). Applying (32) to contaminated data, portfolio $\boldsymbol{\lambda}(t) \in \Lambda$ satisfying $\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\lambda}(t) \succ_{SSD} \tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\tau}$ exists if and only if $\xi(\boldsymbol{\tau}, \tilde{R}, \tilde{\mathbf{p}}(t)) < 0$, where

$$\begin{aligned} \xi(\boldsymbol{\tau}, \tilde{R}, \tilde{\mathbf{p}}(t)) &= \min_{a_s, \boldsymbol{\lambda}} \sum_{s=0}^{S} a_s \\ \text{s.t.} \quad & \mathrm{CVaR}_{\tilde{q}_s^{\boldsymbol{\lambda}}}(-\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\lambda}) - \mathrm{CVaR}_{\tilde{q}_s^{\boldsymbol{\lambda}}}(-\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\tau}) \le a_s, \quad s = 0, 1, \dots, S \\ & a_s \le 0, \quad s = 0, 1, \dots, S \\ & \boldsymbol{\lambda} \in \Lambda. \end{aligned} \tag{34}$$

Example 5 (a) Consider the following three assets and three scenarios example:

$$R = \begin{pmatrix} 0 & 3 & 2 \\ 2 & 2 & 2 \\ 4 & 1 & 2 \end{pmatrix}.$$

Assume that scenarios are equiprobable. It can be shown that portfolio $\boldsymbol{\tau} = (\frac{1}{3}, \frac{2}{3}, 0)$ is SSD efficient. Let the additional scenario $\mathbf{r}^4 = (0, 0, 2)$ and consider portfolio $\boldsymbol{\lambda} = (0, 0, 1)$. Then $\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\lambda} \succ_{SSD} \tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\tau}$ for any contamination parameter $t > 0$. Hence, portfolio $\boldsymbol{\tau}$ is SSD efficient but not directionally SSD efficient with respect to scenario $\mathbf{r}^4$.

(b) Consider another three assets and three scenarios example:

$$R = \begin{pmatrix} 0 & 3 & 2 \\ 2 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix}.$$

Assume again that scenarios are equiprobable. It can be shown that portfolio $\boldsymbol{\tau} = (\frac{1}{3}, \frac{2}{3}, 0)$ is SSD inefficient, because portfolio $\boldsymbol{\lambda} = (0, 0, 1)$ SSD dominates portfolio $\boldsymbol{\tau}$. Let the additional scenario $\mathbf{r}^4 = (2, 2, 0)$. Then no portfolio SSD dominates $\boldsymbol{\tau} = (\frac{1}{3}, \frac{2}{3}, 0)$ for any contamination parameter $t > 0$. Hence, portfolio $\boldsymbol{\tau}$ is SSD inefficient but not directionally SSD inefficient with respect to scenario $\mathbf{r}^4$.

Example 5 shows that there are situations where an arbitrarily small contamination of the original probability distribution of returns leads to the opposite SSD classification. Using contamination bounds we will derive a sufficient condition for directional SSD efficiency and directional SSD inefficiency with respect to additional scenario $\mathbf{r}^{S+1}$.

Theorem 6 *Let $\boldsymbol{\tau} \in \Lambda$ be an SSD efficient portfolio for the noncontaminated distribution P. Let*

$$\mathbf{r}^{S+1}\boldsymbol{\tau} \geq \mathbf{r}^{S+1}\boldsymbol{\lambda} \quad \text{for all } \boldsymbol{\lambda} \in \Lambda. \tag{35}$$

Then $\boldsymbol{\tau} \in \Lambda$ is directionally SSD efficient with respect to $\mathbf{r}^{S+1}$.

Proof The SSD efficiency of $\boldsymbol{\tau}$ implies that $\xi(\boldsymbol{\tau}, R, \mathbf{p}) = 0$. Condition (35) gives $\xi(\boldsymbol{\tau}, \mathbf{r}^{S+1}, 1) = 0$. Since the objective function of (32) does not depend on probability distribution, verification of (9) for $t_1 = 0$, $t_2 = 1$ will imply the lower bound (8). Consequently, $\xi(\boldsymbol{\tau}, \tilde{R}, \tilde{\mathbf{p}}(t))$ will necessarily be equal to zero for all $t \in [0, 1]$ what yields directional SSD efficiency with respect to $\mathbf{r}^{S+1}$ of $\boldsymbol{\tau}$. Hence, it suffices to show, that any feasible solution $\boldsymbol{\lambda}$ of (34) with an arbitrary parameter $t \in (0, 1)$ is a feasible solution of (32). Let $F^{(2)}_{\boldsymbol{\lambda},S+1}(z)$ be a cumulative distribution function of returns of portfolio $\boldsymbol{\lambda}$ for the contaminated distribution taking $S + 1$ scenarios with probabilities $\tilde{\mathbf{p}} = ((1 - t)p_1, (1 - t)p_2, \ldots, (1 - t)p_S, t)$. Similarly, let $F^{(2)}_{\boldsymbol{\lambda},S}(z)$ correspond to the original distribution with S scenarios. Then

$$\begin{aligned} F^{(2)}_{\boldsymbol{\lambda},S+1}(z) &= \int_{-\infty}^{z} F_{\boldsymbol{\lambda}}(y)\, dy = \int_{-\infty}^{z} \sum_{s=1}^{S+1} \tilde{p}_s 1_{(\mathbf{r}^s\boldsymbol{\lambda} \leq y)}\, dy \\ &= \sum_{s=1}^{S+1} \tilde{p}_s (z - \mathbf{r}^s\boldsymbol{\lambda}) 1_{(\mathbf{r}^s\boldsymbol{\lambda} \leq z)} = \sum_{s=1}^{S+1} \tilde{p}_s (z - \mathbf{r}^s\boldsymbol{\lambda})^+. \end{aligned} \tag{36}$$

The same notation and analysis is applied to portfolio $\boldsymbol{\tau}$.

Since $\boldsymbol{\lambda}$ is a feasible solution of (34), Theorem 3 implies that $\boldsymbol{\varrho}(t)'\boldsymbol{\lambda} \succ_{SSD} \boldsymbol{\varrho}(t)'\boldsymbol{\tau}$. Hence, directly from Definition 1, one obtains

$$F^{(2)}_{\boldsymbol{\lambda},S+1}(z) \leq F^{(2)}_{\boldsymbol{\tau},S+1}(z) \quad \forall z \in \mathbb{R}. \tag{37}$$

Applying (36) to (37)

$$\sum_{s=1}^{S}(1-t)p_s(z-\mathbf{r}^s\boldsymbol{\lambda})^+ + t(z-\mathbf{r}^{S+1}\boldsymbol{\lambda})^+ \leq \sum_{s=1}^{S}(1-t)p_s(z-\mathbf{r}^s\boldsymbol{\tau})^+ + t(z-\mathbf{r}^{S+1}\boldsymbol{\tau})^+. \tag{38}$$

Note that according to (35) $(z-\mathbf{r}^{S+1}\boldsymbol{\lambda})^+ \geq (z-\mathbf{r}^{S+1}\boldsymbol{\tau})^+$. Combining it with (38) implies that $\sum_{s=1}^{S} p_s(z-\mathbf{r}^s\boldsymbol{\lambda})^+ \leq \sum_{s=1}^{S} p_s(z-\mathbf{r}^s\boldsymbol{\tau})^+$. Therefore

$$F^{(2)}_{\boldsymbol{\lambda},S}(z) \leq F^{(2)}_{\boldsymbol{\tau},S}(z) \quad \forall z \in \mathbb{R}.$$

According to Definition 1, $\boldsymbol{\varrho}'\boldsymbol{\lambda} \succ_{SSD} \boldsymbol{\varrho}'\boldsymbol{\tau}$ and the rest of the proof directly follows from Theorem 3. □

In Example 5(a), $\xi(\boldsymbol{\tau}, \mathbf{r}^4, 1) = -2$ and $\xi(\boldsymbol{\tau}, \tilde{R}(t), \tilde{\mathbf{p}}(t)) < 0$ for all $t \in (0, 1]$ because $\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\lambda} \succ_{SSD} \tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\tau}$ for all $t \in (0, 1]$.

Theorem 7 *Let $\boldsymbol{\tau} \in \Lambda$ be an SSD inefficient portfolio for the noncontaminated distribution P. If there exists a portfolio $\boldsymbol{\lambda} \in \Lambda$ such that*

$$\mathrm{CVaR}_{q_s^{\lambda}}(-\boldsymbol{\varrho}'\boldsymbol{\lambda}) - \mathrm{CVaR}_{q_s^{\lambda}}(-\boldsymbol{\varrho}'\boldsymbol{\tau}) < 0, \quad s = 0, 1, \ldots, S \tag{39}$$

$$\mathbf{r}^{S+1}\boldsymbol{\lambda} \geq \min((R\boldsymbol{\tau})^{[1]}, \mathbf{r}^{S+1}\boldsymbol{\tau}) \tag{40}$$

then $\boldsymbol{\tau}$ is directionally SSD inefficient with respect to $\mathbf{r}^{S+1}$.

Proof Let $j(\boldsymbol{\tau})$ be such index that $(-\tilde{R}\boldsymbol{\tau})^{[j(\boldsymbol{\tau})]} = -\mathbf{r}^{S+1}\boldsymbol{\tau}$ and similarly let $j(\boldsymbol{\lambda})$ be such that $(-\tilde{R}\boldsymbol{\lambda})^{[j(\boldsymbol{\lambda})]} = -\mathbf{r}^{S+1}\boldsymbol{\lambda}$. If $j(\boldsymbol{\lambda}) \geq 2$ then continuity of CVaR and assumptions (39) imply that there exists a sufficiently small t_0 such that for all $t \in [0, t_0]$

$$\mathrm{CVaR}_{\tilde{q}_s^{\lambda}(t)}(-\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\lambda}) - \mathrm{CVaR}_{\tilde{q}_s^{\lambda}(t)}(-\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\tau}) < 0, \quad s = 0, 1, \ldots, S$$

$$\mathrm{CVaR}_{\tilde{q}_s^{\tau}(t)}(-\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\lambda}) - \mathrm{CVaR}_{\tilde{q}_s^{\tau}(t)}(-\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\tau}) < 0, \quad s = 0, 1, \ldots, S$$

holds true. Hence, $\tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\lambda} \succ_{SSD} \tilde{\boldsymbol{\varrho}}(t)'\boldsymbol{\tau}$ and therefore $\boldsymbol{\lambda}$ is a feasible solution of (34) for all $t \in [0, t_0]$. The directional SSD inefficiency with respect to $\mathbf{r}^{S+1}$ of $\boldsymbol{\tau}$ follows.

If $j(\boldsymbol{\lambda}) = 1$ then (40) implies that $(\tilde{R}\boldsymbol{\lambda})^{[1]} \geq (\tilde{R}\boldsymbol{\tau})^{[1]}$ and the rest of the proof is similar to the previous case. □

Condition (40) is needed to guarantee that even in the contaminated case the smallest return of portfolio $\boldsymbol{\lambda}$ is larger than or equal to that of portfolio $\boldsymbol{\tau}$ what is a necessary condition of SSD relation. For data in Example 5(b), none of the conditions (39)–(40) is fulfilled.

4 Conclusions

The contamination technique was extended to construction of bounds for the optimal value function of perturbed stochastic programs whose set of feasible solutions depends on the probability distribution. In spite of the local character of these bounds their usefulness was illustrated for analysis of resistance with respect to additional scenarios in stochastic programs with risk constraints and in a new SSD portfolio efficiency test. Unlike the former portfolio efficiency tests, neither this test nor its robust version assume equiprobable scenarios.

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