Ann Oper Res (2009) 166: 243–260
DOI 10.1007/s10479-008-0422-2

On the existence of a minimum integer representation for weighted voting systems

Josep Freixas · Xavier Molinero

Published online: 24 September 2008

Abstract A basic problem in the theory of simple games and other fields is to study whether a simple game (Boolean function) is weighted (linearly separable). A second related problem consists in studying whether a weighted game has a minimum integer realization. In this paper we simultaneously analyze both problems by using linear programming.

For less than 9 voters, we find that there are 154 weighted games *without* minimum integer realization, but *all* of them have minimum normalized realization. Isbell in 1958 was the first to find a weighted game without a minimum normalized realization, he needed to consider 12 voters to construct a game with such a property. The main result of this work proves the existence of weighted games with this property with less than 12 voters.

Keywords Simple games · Weighted voting games · Minimal realizations · Minimum realization · Realizations with minimum sum

1 Introduction

Simple games can be viewed as models of voting systems in which a single alternative, such as a bill or an amendment, is pitted against the status quo.

This research was partially supported by Grant MTM 2006-06064 of "Ministerio de Ciencia y Tecnología y el Fondo Europeo de Desarrollo Regional" and SGRC 2005-00651 of "Generalitat de Catalunya", and by the Spanish "Ministerio de Ciencia y Tecnología" programmes ALINEX (TIN2005-05446 and TIN2006-11345).

J. Freixas
Dept. de Matemàtica Aplicada 3, Escola Politècnica Superior d'Enginyeria de Manresa, Universitat Politècnica de Catalunya, 08242 Manresa, Spain
e-mail: josep.freixas@upc.edu

X. Molinero (✉)
Dept. de Llenguatges i Sistemes Informàtics, Escola Politècnica Superior d'Enginyeria de Manresa, Universitat Politècnica de Catalunya, 08242, Manresa, Spain
e-mail: molinero@lsi.upc.edu

Definition 1.1 A simple game G is a pair (N, W) in which $N = \{1, 2, \dots, n\}$ and W is a collection of subsets of N that satisfies: $N \in W$, $\emptyset \notin W$ and, the monotonicity property, $S \in W$ and $S \subseteq R \subseteq N$ implies $R \in W$.

Any set of voters is called a *coalition*, the set N is called the *grand coalition*, and the empty set $\emptyset$ is called the *empty coalition*. Members of N are called *players* or *voters*, and the subsets of N that are in W are called *winning coalitions*. Our definition of simple game demands that the grand coalition is winning and the empty coalition is losing. The gain, albeit small, in the exclusion of these two innocuous examples is in the interpretation of simple games as voting systems. However, in other fields like, for example, circuits theory or Boolean algebra, these two restrictions are not required.

The intuition here is that a set S is a winning coalition *if and only if* the bill or amendment passes when the players in S are precisely the ones who voted for it. A subset of N that is not in W is called a *losing coalition*, let $L = 2^N \setminus W$. A *minimal winning coalition* is a winning coalition all of whose proper subsets are losing. Analogously, a *maximal losing coalition* is a losing coalition all of whose proper supersets are winning. Because of monotonicity, any simple game is completely determined by its set of minimal winning coalitions or by its set of maximal losing coalitions, which are denoted by W^m and L^M, respectively.

Monotonicity leads to an efficiency in describing (N, W): we need list only the minimal winning coalitions (or the maximal losing coalitions). The entire collection of winning (losing) coalitions can now be obtained by closing out under the operation of adding new elements to the minimal winning coalitions. Thus (N, W^m) $((N, L^M))$ is enough information to describe the game.

A voter $a \in N$ is *null* if a does not belong to any minimal winning coalition. A voter $a \in N$ is *winner* if $\{a\}$ is a winning coalition.

Of fundamental importance to us is the class of weighted simple games.

Definition 1.2 A simple game $G = (N, W)$ is said to be weighted if there exists a "weight function" $w : N \to \mathbb{R}$ and a real number "quota" $q \in \mathbb{R}$ such that a coalition S is winning exactly when the sum of the weights of the players in S meets or exceeds quota.

Any specific example of such a weight function $w : N \to \mathbb{R}$ and quota q as in Definition 1.2 are said to *realize* G as a weighted game. A particular realization of a weighted simple game is denoted as $(q; w_1, \dots, w_n)$, or briefly $(q; w)$, where w represents $(w_1, \dots, w_n)$. The weight $\sum_{i \in S} w_i$, of a non-empty coalition S, is denoted by $w(S)$, and 0 is assigned to $w(\emptyset)$. Three parameters can rbe associated to any realization $(q; w)$ of a weighted simple game (briefly, *weighted game*) (N, W):

$$T = w(N), \qquad a = \min_{S \in W} w(S) \quad \text{and} \quad b = \max_{S \in L} w(S).$$

Some real-world examples of weighted games can be found in some voting systems: the United Nations Security Council seen as a simple game (see Freixas and Zwicker 2003 for a weighted representation of this example with abstention), the first European Economic Community (1958), and many national and supranational European Parliaments. All these examples use integer weights and quota. On the other hand, examples of non-weighted games are: the United States Federal System, the System to Amend the Canadian Constitution, the current European Economic Community, etc. See Taylor (1995) for a thorough presentation of these examples.

A basic problem in the theory of simple games, and other fields, consists in studying whether a simple game is weighted. The only results giving necessary and sufficient conditions can be found under one of the next three topics (see Taylor and Zwicker 1999 for a more detailed explanation on them):

(i) Geometric approach based on separating hyperplanes;
(ii) Approach based on trading transforms;
(iii) Algebraic approach based on systems of linear inequalities.

The geometric approach requires translating the question of weightedness into one of separability via a hyperplane of the convex hulls associated to winning coalitions and losing coalitions respectively in $\mathbb{R}^n$. Such characterization attaches an attractive geometric intuition to the notion of weightedness, but it is not clear how one might actually apply it to check whether a given simple game is weighted.

The approach based on trades offers a natural, albeit quite complex, procedure. Weighted games are characterized as those for which is not possible to convert an arbitrarily sequence of winning coalitions to a sequence of losing coalitions by leaving invariant the number of times each player appears in both sequences. This approach is proposed in Taylor and Zwicker (1992) (see also Taylor and Zwicker 1999 for more details). In Freixas and Molinero (2008) a new test is provided, which only uses a subset of minimal winning coalitions.

Theorems on the existence of solutions for systems of linear inequalities go back to the early 1900s. An important result mentioned in Von Neumann and Morgenstern (1944), and extensively treated in Fishburn (1973), is the *Theorem of the Alternative*. This third procedure is the one chosen in this paper (see for example Chavátal 1983; Shapiro 1979 to deal with linear programming in depth). From linear programs we will get simple games that verify some relevant properties.

The organization of this paper is as follows. The main issue of Sect. 2 is to provide an efficient system of inequalities which allow to study whether a simple game is weighted. Section 3 is devoted to classify several types of integer realizations for weighted games and to state an *efficient* linear programming problem to find minimum (sum) integer realizations. Section 4 makes use of classical computational packages to tackle the main problems stated in the previous sections. It contains a full classification for weighted games with less than 9 voters. In particular, we list all 154 weighted games without minimum integer realization. Section 5 is devoted to find weighted games without minimum integer realization and with less than 12 voters. Finally, the conclusions are stated in Sect. 6.

2 An efficient system of inequalities

The most natural way to find out if a simple game (N, W) is weighted consists in determining the consistency of the following system of inequalities:

$$w(S) > w(R) \quad \text{for all } S \in W^m,\ R \in L^M \tag{1}$$

where $w = (w_1, \dots, w_n)$ are the unknowns.

If system (1) is consistent then any solution $w = (w_1, \dots, w_n)$ provides a realization $(q; w)$ for (N, W) where q is any number belonging to the interval $(b, a]$. This system has $|W^m| \cdot |L^M|$ inequalities (or constraints), which is a smaller number than $|W| \cdot |L|$, which is the number of constraints for the innocuous equivalent system obtained by considering all winning and all losing coalitions.

In this paper we will propose two additional simplifications for system (1). Although they were already pointed out in Carreras and Freixas (1996), no further analysis have been done on them. We will use the most simplified version of them to classify all games with less than 9 voters (see system (3) below).

2.1 Real and integer realizations

Previously to establish the two alternative "equivalent" systems of inequalities, we need to introduce some preliminaries on realizations of weighted games.

Proposition 2.1 *Let* (N, W) *be a weighted game with* $(q; w)$ *as a realization. Then*:

(i) $T \geq q > 0$,
(ii) $w_i \leq 0$ *for some* $i \in N$ *implies that* i *is a null voter,*
(iii) $(c \cdot q; c \cdot w)$ *for every* $c > 0$ *is a realization for* (N, W).

Proof

(i) If $q \leq 0$ the empty set would be a winning coalition, and if $T < q$ the grand coalition would be losing. Both facts contradict the definition of simple game (cf. Definition 1.1).
(ii) Assume by the way of contradiction that there exists i such that $w_i \leq 0$. So, there is at least some $S \in W^m$ with $i \in S$, $w(S \setminus \{i\}) = w(S) - w_i \geq w(S) \geq q$. Hence, $S \setminus \{i\}$ would be a winning coalition which is a contradiction with the minimality of S.
(iii) It easily follows from the fact that $w(S) \geq q$ *if and only if* $(c \cdot w)(S) \geq c \cdot q$ for every $c > 0$, since $(c \cdot w)(S) = c \cdot w(S)$. □

So we may restrict our attention to real realizations with positive quota, $0 < q \leq T$, and non-negative weights, $w_i \geq 0$; that is, weights in $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$ and quota in $\mathbb{R}_{++} = \{x \in \mathbb{R}, x > 0\}$. Now, $\mathcal{R}(N, W)$ denotes the set of real realizations for the weighted game (N, W), that is, pairs $(q; w) \in \mathbb{R}_{++} \times \mathbb{R}^n_+$. Proposition 2.1-(iii) shows that the set $\mathcal{R}(N, W)$ is a *cone*.

Proposition 2.2 (Integer realizations) *Every weighted game* (N, W) *admits an integer realization.*

Proof Let $(q; w)$ be a real realization for (N, W). As $a = \min_{S \in W} w(S)$ and $b = \max_{S \notin W} w(S)$ it holds $b < q \leq a$. If $q \notin \mathbb{Q}$ we can adjust q by the subtraction of a small irrational to convert q to a rational number q'. Next, if some of the weights are irrational numbers, we can also adjust each of them, w_i, by adding a small irrational to convert it to a rational number w'_i ($w'_i > w_i$). The remaining weights, w_i which are rationals, are left invariant ($w'_i = w_i$). As the changes in weights or quota are small enough the "wiggle room" between b and a for $(q; w)$ guarantee that $b \leq b' < q' \leq q \leq a \leq a'$, thus $(q; w)$ and $(q'; w')$ are equivalent realizations. Then, we may consider the integer realization $(q''; w'')$ which is obtained from $(q'; w')$ by multiplying (cf. Proposition 2.1-(iii)) all numbers by the great common divisor ($g.c.d.$, for short) of the denominators of q' and w'_i (with $w'_i \neq 0$) which allows to obtain an equivalent integer realization. □

On account to this proposition, from now on, we will just consider integer realizations: $\mathcal{I}(N, W)$ denotes the set of integer realizations for the weighted game (N, W); that is, pairs $(q; w) \in \mathbb{N}_{++} \times \mathbb{N}^n_+$. If $(q; w) \in \mathcal{I}(N, W)$ then $(c \cdot q; c \cdot w) \in \mathcal{I}(N, W)$ whenever $c \in \mathbb{N}$.

Thus, $\mathcal{I}(N, W)$ is unbounded and so $\mathcal{R}(N, W)$ is. Note that the set $\mathcal{I}(N, W)$ is a *cone of integer values*. When there is no confusion we write simply $\mathcal{I}$ instead of $\mathcal{I}(N, W)$. If integer realizations are used one may ask for the "smallest" realization(s) within the cone of integer values. This point will be tackled in Sect. 3, we now previously need to define a relation of importance among voters.

2.2 A natural relation

To define a natural relation between voters, we start by introducing the *desirability relation* which goes back at least to Isbell (1958), and later generalized in Maschler and Peleg (1966) (see also Muroga 1971).

Definition 2.3 Let (N, W) be a simple game.

(i) Player i is more desirable than j ($i \succeq j$, for short) in (N, W) if

$$S \cup \{j\} \in W \;\Rightarrow\; S \cup \{i\} \in W, \quad \text{for all } S \subseteq N \setminus \{i, j\}.$$

(ii) Players i and j are equally desirable ($i \sim j$, for short) in (N, W) if

$$S \cup \{i\} \in W \Leftrightarrow S \cup \{j\} \in W, \quad \text{for all } S \subseteq N \setminus \{i, j\}.$$

(iii) Player i is strictly more desirable than player j ($i \succ j$, for short) in (N, W) if i is more desirable than j, but i and j are not equally desirable.

If the desirability relation is complete for (N, W), then the game is called *complete*. In the field of Boolean algebra, complete games correspond to 2-monotonic positive Boolean functions, which were already considered in Hu (1965). The problem of identifying this type of functions by using polynomial-time recognition have been treated in Boros et al. (1991, 1997).

It is clear that if a weighted game admits $(q; w_1, \ldots, w_n)$ as a realization then $w_i \geq w_j$ implies $i \succeq j$. Thus, for a weighted game the desirability relation is *complete*. In fact non-complete games are not weighted thus, from now on, we will just deal with complete simple games. Note also that $w_i = w_j$ implies $i \sim j$. Although, $i \sim j$ does not necessarily imply $w_i = w_j$. However, $i \succ j$ implies $w_i > w_j$. These comments suggest the following definition.

Definition 2.4 Let $(q; w)$ be a realization of a weighted game (N, W), $(q; w)$ is said to preserve types if $w_i = w_j$ whenever $i \sim j$.

From now on, whenever we consider a realization $(q; w)$ of a weighted game (N, W) it will be assumed that $i \succeq i + 1$ for each $i = 1, \ldots, n - 1$, that is, voter i is more desirable than voter $i + 1$. Thus the set of voters admits a partition in t classes, $N_1, N_2, \ldots, N_t$ such that $i \succ j$ and $i \in N_p$, $j \in N_{p'}$ implies $p < p'$, i.e. voters in N_1 are the most desirable, and voters in N_t are the least desirable. The extreme cases for this partition arise when $t = 1$ which means that all voters are equally desirable; and $t = n$ which means that each class reduces to a singleton.

Given a simple game (N, W) one can study the desirability relation among voters so this information allows another improvement to describe (N, W): we need list only the winning coalitions that are minimal in this sense. A *shift-minimal winning coalition* (*shift-maximal losing coalition*) is a coalition S that is minimal (maximal), among minimal winning coalitions (maximal losing coalitions), in the $\succeq$-relation (or $\succeq$-preorder). Formally,

Definition 2.5 Let (N, W) be a simple game and $\succeq$ be its desirability relation. A coalition $S \in W$ is shift-minimal if for every $i \in S$ and $j \notin S$ such that $i \succ j$ it holds $(S \setminus \{i\}) \cup \{j\} \in L$.

Definition 2.6 Let (N, W) be a simple game and $\succeq$ be its desirability relation. A coalition $S \in L$ is shift-maximal if for every $i \notin S$ and $j \in S$ such that $i \succ j$ it holds $(S \setminus \{j\}) \cup \{i\} \in W$.

From now on, the set of shift-minimal winning coalitions (shift-maximal losing coalitions) will be denoted by W^h (L^H).

This *shift-ordering* has been rediscovered a number of times, and in more than one context. For example, in the context of simple games, it appears in Ostmann (1985), Krohn and Sudhölter (1995), Carreras and Freixas (1996), Taylor and Zwicker (1999); in Boolean functions, it appears in Hammer et al. (2000); in social choice, it occurs in Fishburn (1969); in the study of fair division, it occurs in Brams et al. (1979); and in voting theory, it appears in Brams and Fishburn (1976).

If (N, W) is a complete simple game (i.e., the desirability relation is complete), we may think of the linearly ordered set of $\succeq$-equally desirable classes as being lined up from the most influential on the left to the least influential on the right. Thus the coalition $(S \setminus \{i\}) \cup \{j\}$ described in Definition 2.5 is obtained by "shifting a one to the right".

A second useful way to check whether a simple game is weighted consists in studying the consistency of the following system of linear inequalities:

$$w(S) > w(R) \quad \text{for all } S \in W^h,\ R \in L^H \tag{2}$$

where $w_1, \ldots, w_n$ are the unknowns.

The inclusions $W^h \subseteq W^m$ and $L^H \subseteq L^M$ mean that the inequalities in system (1) but not in system (2) are redundant. To clarify the notion introduced in Definitions 2.5 and 2.6 let us consider an example.

Example 2.7 Consider a game with 5 voters whose set of minimal winning coalitions is

$$W^m = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}\}.$$

Then one can generate the remaining winning coalitions, the losing coalitions and among these the maximal ones. The reader may easily check that there are 19 winning coalitions and 13 losing coalitions. The list of maximal losing coalitions is

$$L^M = \{\{1\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4,5\}\}.$$

The desirability relation is complete since $1 \succ 2 \succ 3 \sim 4 \sim 5$, thus the game is complete and, therefore the game could be weighted. Coalition $\{1,2\}$ is minimal winning but is not shift-minimal because coalitions of the form $\{1, j\}$ also win for $j = 3, 4, 5$. It is not difficult to check that the remaining minimal winning coalitions are also shift-minimal. Thus,

$$W^h = \{\{1,3\}, \{1,4\}, \{1,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}\}.$$

Analogously, it can be checked that every maximal losing coalition is shift-maximal. Thus,

$$L^H = \{\{1\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4,5\}\}.$$

System (1) has $7 \cdot 5 = 35$ inequalities (whereas considering all winning and all losing coalitions would involve $19 \cdot 13 = 247$ inequalities). System (2) still allows a better reduction with respect to system (1) since it only takes $6 \cdot 5 = 30$ inequalities. Systems (1) and (2) are consistent and admit as a solution the weights $(w_1, w_2, w_3, w_4, w_5) = (5, 4, 1, 1, 1)$, therefore a realization for this game is $(6; 5, 4, 1, 1, 1)$.

2.3 A more efficient system of inequalities

We start this section with an example which suggests a further simplification for system (2).

Example 2.8 A company is formed by three owners, nine team leaders and ninety workers. Each team leader is in charge of ten workers. Some internal decisions in the company are taken by the following rule: a decision is passed, *if and only if*, a majority of the owners (at least 2-out-of-3) vote in favor, a majority of owners and team leaders (at least 7-out-of-12) votes in favor, and one third of all the members of the company (at least 34-out-of-102) votes in favor. That is to say, from the viewpoint of voting there are three types of members in the company whose respective cardinalities can be gathered in vector $(3, 9, 90)$. Each shift-minimal coalition has exactly 2 owners, 5 team leaders and 27 workers, so it can be represented by vector $(2, 5, 27)$. Note that in this voting game, an owner is strictly more desirable than a team leader, and a team leader is strictly more desirable than a worker, so that the game is complete. Once we have identified the vector of types and the models of shift-minimal winning coalitions one may easily identify the models of shift-maximal losing coalitions, which are the vectors: $(3, 9, 22)$, $(3, 3, 90)$ and $(1, 9, 90)$. Indeed, if we add a single element in one of these vectors or move one element from the right to the left, we get a vector corresponding to winning coalitions.

System (2) has much less inequalities than systems (1), although the number of shift-minimal coalitions is $\binom{3}{2} \cdot \binom{9}{5} \cdot \binom{90}{27} \approx 2,6 \cdot 10^{25}$, and the number of shift-maximal coalitions is $\binom{3}{3} \cdot \binom{9}{9} \cdot \binom{90}{22} + \binom{3}{3} \cdot \binom{9}{3} \cdot \binom{90}{90} + \binom{3}{1} \cdot \binom{9}{9} \cdot \binom{90}{90} \approx 5,3 \cdot 10^{20}$, which makes system (2) too large to study its consistency. Now, one may consider the alternative system of inequalities obtained by considering “models” of shift-minimal and shift-maximal coalitions instead of shift-minimal winning and shift-maximal losing coalitions, respectively. We consider “representatives” for each element in the partition N_1, N_2, N_3 of N instead of voters. Thus, the system is

$$\begin{aligned} 2\overline{w}_1 + 5\overline{w}_2 + 27\overline{w}_3 &> 3\overline{w}_1 + 9\overline{w}_2 + 22\overline{w}_3 \\ 2\overline{w}_1 + 5\overline{w}_2 + 27\overline{w}_3 &> 3\overline{w}_1 + 3\overline{w}_2 + 90\overline{w}_3 \\ 2\overline{w}_1 + 5\overline{w}_2 + 27\overline{w}_3 &> \overline{w}_1 + 9\overline{w}_2 + 90\overline{w}_3 \end{aligned}$$

where $\overline{w}_1, \overline{w}_2, \overline{w}_3$ are the unknowns.

Clearly, this system has not solution, if it had been consistent, $\overline{w}_i$, for $i = 1, 2, 3$, would have represented the *common* weight for all voters in N_i, for $i = 1, 2, 3$.

This example gives the idea on how to solve the underlying problem. The following theorem arises from the equivalence of systems (1) and (2), and it establishes the vectorial formulation of systems (2).

Theorem 2.9 *A complete simple game* (N, W) *is a weighted game* if and only if *there is a vector* $\overline{w} = (\overline{w}_1, \overline{w}_2, \dots, \overline{w}_t)$ *such that* $\overline{w}_1 > \overline{w}_2 > \cdots > \overline{w}_t \geq 0$, *which satisfies the system of inequalities*:

$$(\overline{m}_i - \overline{l}_j) \cdot \overline{w} > 0 \quad \textit{for } i = 1, \dots, r, \ j = 1, \dots, s; \tag{3}$$

where $\overline{m}_i$, *for* $i = 1, \dots, r$, *are the models of shift-minimal winning coalitions and* $\bar{l}_j$, *for* $j = 1, \dots, s$, *are the models of shift-maximal losing coalitions*; *and* t *stands for the number of classes* $N_1, \dots, N_t$ *in the partition of* N *for game* (N, W).

Now, we enumerate some remarks from which we will develop most of our experiments below in Sects. 4 and 5.

a. When a simple game is complete we may represent it by a pair $(\overline{n}, \mathcal{M})$ where $\overline{n} = (n_1, n_2, \dots, n_t)$ and $n_i = |N_i|$; being voters in N_1 the most desirable, voters in N_2 the second most desirable, and so on. Each row $\overline{m}_i = (m_{i1}, \dots, m_{it})$ in matrix $\mathcal{M}$ represents a model of shift-minimal winning coalitions S with the relationship $m_{i1} = |S \cap N_1|, \dots, m_{it} = |S \cap N_t|$. Repetitions of rows in $\mathcal{M}$ are not allowed. Hence a complete simple game may be represented by the pair $(\overline{n}, \mathcal{M})$. Analogously, one may use a pair $(\overline{n}, \mathcal{L})$ where $\overline{n} = (n_1, n_2, \dots, n_t)$ and each row $\bar{l}_j = (l_{j1}, \dots, l_{jt})$ in matrix $\mathcal{L}$ represents a model of shift-maximal losing coalitions S with the relationship $l_{j1} = |S \cap N_1|, \dots, l_{jt} = |S \cap N_t|$.
b. System (3) is consistent *if and only if* system (2) is consistent. Moreover, each model $\overline{m}_i$ represents $\binom{n_1}{m_{i,1}} \cdots \binom{n_t}{m_{i,t}}$ shift-minimal winning coalitions. Analogously, each model $\bar{l}_j$ represents $\binom{n_1}{l_{j,1}} \cdots \binom{n_t}{l_{j,t}}$ shift-maximal losing coalitions. Hence, the simplification in passing from system (2) to system (3) is usually a good improvement.
c. If system (3) is consistent one may easily obtain a solution for system (2) by assigning weight $\overline{w}_i$ to each voter belonging to N_i. Reciprocally, if system (2) is consistent one may easily obtain a solution for system (3) by assigning to the representative of N_i the average of the weights for all voters belonging to N_i.
d. There are no games "rare enough" so that systems (2) and (3) have the same number of inequalities. That is, we will always get some kind of reduction by using system (3) instead of system (2).

Our goal in Sect. 4 is to classify weighted games with less than 9 voters, so it will be very useful to know what properties must fulfill any allowed pair $(\overline{n}, \mathcal{M})$ representing a complete simple game (N, W). The following theorem gives the answer and will be extensively used later on.

Theorem 2.10 (Carreras and Freixas 1996)

Part A *Let* $G = (N, W)$ *be a complete simple game with non-empty classes* $N_1 > N_2 > \cdots > N_t$, *let* $\overline{n}$ *be the vector defined by their cardinalities and let* $\mathcal{M}$ *be the matrix with* r *rows that are the profiles corresponding to shift-minimal coalitions. If* $\mathcal{M} = (m_{i,j})$, *with* $1 \le i \le r$ *and* $1 \le j \le t$, *the pair* $(\overline{n}, \mathcal{M})$ *satisfies the four conditions below*:

(1) $m_{i,j} \in \mathbb{N} \cup \{0\}$ *and* $0 \le m_{i,j} \le n_j$ *for all* i, j *with* $1 \le i \le r$ *and* $1 \le j \le t$;
(2) *for every pair of different rows of* $\mathcal{M}$, $\overline{m}_i, \overline{m}_h$, *it holds neither* $\sum_{j=1}^{k} m_{i,j} \ge \sum_{j=1}^{k} m_{h,j}$ *for all* $1 \le k \le t$, *nor* $\sum_{j=1}^{k} m_{h,j} \ge \sum_{j=1}^{k} m_{i,j}$ *for all* $1 \le k \le t$;
(3) *if* $t = 1$ *then* $m_{1,1} > 0$; *if* $t > 1$ *then for every* $k < t$ *there exists some* h *such that* $m_{h,k} > 0$ *and* $m_{h,(k+1)} < n_{k+1}$; *and*
(4) *the rows of* $\mathcal{M}$ *are lexicographically ordered by partial sums*; *that is, let* k *be the lowest number* $1 \le k \le t$ *such that* $\sum_{j=1}^{k} m_{ij} \ne \sum_{j=1}^{k} m_{hj}$, *then it is* $\sum_{j=1}^{k} m_{ij} > \sum_{j=1}^{k} m_{hj}$ *whenever* $i < h$.

Part B (*Uniqueness*) *Two complete simple games* $G = (N, \mathcal{W})$ *and* $G' = (N', \mathcal{W}')$ *are isomorphic* if and only if $\overline{n} = \overline{n}'$ *and* $\mathcal{M} = \mathcal{M}'$.

Part C (*Existence*) *Given a vector* $\overline{n}$ *and a matrix* $\mathcal{M}$ *satisfying the conditions of Part A, there exists a complete simple game* $G = (N, \mathcal{W})$ *associated to vector* $\overline{n}$ *and matrix* $\mathcal{M}$.

We note that Theorem 2.10 is a *parameterization theorem* because of it allows one to enumerate all complete simple games *up to isomorphism* by listing the possible values of certain invariants $(\overline{n}, \mathcal{M})$. We refer the interested reader to Carreras and Freixas (1996) for more details.

3 Minimal integer normalized realizations

In this section we only deal with weighted games with integer realizations. Let us consider two different concepts for the term "smallest" realization of a weighted game.

Definition 3.1 A realization $(q; w)$ of a weighted game (N, W) is called *minimum* if $w' \geq w$ (that is, $w'_i \geq w_i$ for all $1 \leq i \leq n$) for all realization $(q'; w') \in \mathcal{I}(N, W)$ with $w' \neq w$. Let $M\mathcal{I}$ be the set of minimum realizations.

For instance, one may easily check that the realization (6; 5, 4, 1, 1, 1) for the game defined in Example 2.7 is minimum.

Definition 3.2 A realization $(q; w)$ of a weighted game (N, W) has *minimum sum* if $w'(N) \geq w(N)$ for all $(q'; w') \in \mathcal{I}(N, W)$. Let $sM\mathcal{I}$ be the set of all minimum sum realizations.

In the following we seek properties that minimum sum realizations must fulfill.

Proposition 3.3 *Let* $(q; w) \in sM\mathcal{I}$ *be a realization for the weighted game* (N, W), *then*:

(i) $q = a$ *and* $q = b + 1$,
(ii) $w_i = 0$ *if* i *is a null voter*,
(iii) $w_i = q$ *if* i *is a winner*,
(iv) $g.c.d.(q; w_1, \ldots, w_k) = 1$, *where* $\{1, \ldots, k\}$ *denotes the set of non-null voters in* N.

Proof

(i) Let $(q; w) \in sM\mathcal{I}$, recall that $a = \min_{S \in W} w(S)$, $b = \max_{S \in L} w(S)$ and $T = w(N)$. If $a - b \geq 2$ it means that there are no coalitions S with $w(S) = a - 1$. We distinguish between two cases:
  - If $q = a$, then we may consider $(q'; w') \in \mathcal{I}$ with $q' = q - 1$ and w' is obtained from w by decreasing one unit a positive weight and leaving the remaining weights with the same value. Clearly, both realizations define the same game, which is a contradiction with the assumption $(q; w) \in sM\mathcal{I}$ because $w'(N) < w(N)$. Thus, $a - b = 1$ and $q = a$.
  - If $q \leq a - 1$, we just have to take $q' = q$ and proceeding as before.

  Thus, $(q; w) \in sM\mathcal{I}$ implies both: $a = b + 1$ and $q = a$.
(ii) It is a consequence of Proposition 2.1-(ii).
(iii) Let $(q; w) \in sM\mathcal{I}$ and assume that voter i is a winner with $w_i > q$, then we may consider a new representation $(q'; w') \in \mathcal{I}$ in which w_i is transformed to q, the remaining components in w are left invariant in w' and $q' = q$. The inequality $w \geq w'$ together with $w \neq w'$, is a contradiction with $(q; w) \in sM\mathcal{I}$.

(iv) Let $(q; w) \in s\mathcal{MI}$ and assume that $g.c.d.(q; w_1, \ldots, w_k) = c \neq 1$, where $\{1, \ldots, k\}$ denotes the set of non-null voters in N. By Part (ii) the weights of null voters (if any) in w are zero. Therefore, we may consider the realization $(\frac{1}{c} \cdot q; \frac{1}{c} \cdot w)$ which belongs to $\mathcal{I}$ and preserves winning coalitions. Thus a contradiction with $(q; w) \in s\mathcal{MI}$. □

We note that Proposition 3.3-(i) implies that the quota q for minimum sum realizations is forced to be $q = a$. This justifies why there is no any restriction for the quota in Definitions 3.1 and 3.2.

Now, from Definition 3.2 and system (2) we can directly establish the following theorem.

Theorem 3.4 *Assuming the previous notations and definitions, minimum sum realizations for game (N, W), $s\mathcal{MI}$, are the solutions of the linear program*:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} w_i \\ \text{s.t.} \quad & w(S) > w(R) \quad \textit{for all } S \in W^h,\ R \in L^H. \end{aligned} \tag{4}$$

Note that the constraints in system (4) are exactly the inequalities in system (2).

Proposition 3.5 *Let (N, W) be a weighted game, $\mathcal{MI}$ and $s\mathcal{MI}$ be the sets defined above.*

(i) *$\mathcal{MI}$ has at most one element.*
(ii) *$s\mathcal{MI}$ is never empty.*
(iii) *$\mathcal{MI} \subseteq s\mathcal{MI}$. Moreover, the equality arises* if and only if *$\mathcal{MI}$ reduces to a singleton.*

Proof

(i) Let $(q; w)$, $(q'; w')$ be two elements in $\mathcal{MI}$, $w' \geq w$ and $w \geq w'$ implies $w = w'$. Part (i) in Proposition 3.3 implies $q = q'$ and so the uniqueness.
(ii) The number of voters is finite as well as each weight and quota for any realization. Thus $w(N)$ is a positive integer for every realization and it follows that $s\mathcal{MI}$ is never empty.
(iii) Firstly, we prove that $\mathcal{MI} \subseteq s\mathcal{MI}$ by the way of contradiction: let $(q; w) \in \mathcal{MI} \setminus s\mathcal{MI}$, then it exists $(q'; w')$ with $w(N) > w'(N)$ but $w'_i \geq w_i$ for all $i \in N$, so we get the contradiction because of $w'(N) \geq w(N)$.

Secondly, we proof that $\mathcal{MI} = s\mathcal{MI}$ *if and only if* $|\mathcal{MI}| = 1$:

($\Rightarrow$) Assume $\mathcal{MI} = s\mathcal{MI}$. By Part (ii), $s\mathcal{MI}$ is never empty, and by Part (i), $\mathcal{MI}$ has at most one element. Thus, $\mathcal{MI} = s\mathcal{MI}$ implies that $\mathcal{MI}$ reduces to a singleton.

($\Leftarrow$) Let $(q; w) \in \mathcal{MI}$, if $\mathcal{MI}$ reduces to a singleton, then it satisfies that $w' \geq w$ $(w' \neq w)$ for any other realization $(q'; w')$. Thus there is only a unique realization with minimum sum. □

According to Proposition 3.5-(iii), system (2) is consistent *if and only if* the linear program described in system (4) has solution.

Definition 3.6 A realization $(q; w) \in \mathcal{I}$ for a weighted game (N, W) is called normalized if it preserves types and meets the four conditions in Proposition 3.3.

Let $\mathcal{N}(N, W)$ denotes the set of normalized realizations, briefly $\mathcal{N}$. Let $M\mathcal{N}$ and $sM\mathcal{N}$ be, respectively, the sets of minimum normalized realizations and normalized realizations with minimum sum.

Proposition 3.7 *Let (N, W) be a weighted game and $\mathcal{N}$ be the set of all normalized realizations. Then*:

(i) *$M\mathcal{N}$ has at most one element.*
(ii) *$sM\mathcal{N}$ is never empty.*
(iii) *$M\mathcal{N} \subseteq sM\mathcal{N}$. Moreover, the equality arises* if and only if *$M\mathcal{N}$ reduces to a singleton.*
(iv) *$M\mathcal{I} \subseteq M\mathcal{N}$. Moreover, the equality arises* if and only if *$M\mathcal{N} = \emptyset$ or $M\mathcal{I}$ reduces to a singleton.*

Proof The proofs of parts (i), (ii) and (iii) are, *mutatis mutandis*, equivalent to the proofs of parts (i), (ii) and (iii) given in Proposition 3.5.

To proof (iv), firstly, we assume $M\mathcal{I}$ has just one element. Then a realization preserves types and properties of Proposition 3.3. Hence, $M\mathcal{I}$ and $M\mathcal{N}$ coincide. Secondly, if $M\mathcal{I} = \emptyset$ the inclusion $M\mathcal{I} \subseteq M\mathcal{N}$ is obvious. The second part is clear from Part (i) and Proposition 3.3-(i). □

Now, as a consequence of system (4) and Theorem 2.10, we are able to establish the main result of this work to realize the outcomes in Sect. 4.

Theorem 3.8 *Assuming the previous notations and definitions, minimum normalized sum realizations for game (N, W), $sM\mathcal{N}$, are the solution of the linear program*:

$$\begin{aligned} \min \quad & \overline{n} \cdot \overline{w} \\ \text{s.t.} \quad & A\overline{w}^{tr} > \overline{0}^{tr} \end{aligned} \tag{5}$$

where $\overline{n} = (n_1, \ldots, n_t)$, "$\cdot$" stands for the inner product, $\overline{w} = (\overline{w}_1, \ldots, \overline{w}_t)$ and $\overline{w}^{tr}$ is its transposition, the matrix A has $r \cdot s$ rows and t columns, the rows of A are $\overline{m}_i - \overline{l}_j$, i.e., the subtraction for all the pairs of models of shift-minimal coalitions (r models) with models of shift-maximal coalitions (s models); and $\overline{0}$ stands for the null vector of $r \cdot s$ components $(0, \ldots, 0)$.

Note that the constraints in the linear program (5) are the inequalities in system (3). Thus, according to Proposition 3.7-(iii), system (3) is consistent *if and only if* the linear program (5) has solution. In the following sections we will solve the linear programs (5) for all games with less than 9 voters and some linear programs for games with more than 8 voters with the goal of finding games without a minimum normalized integer realization.

In Bohosian and Bruck (2003) two methods for identifying if a particular realization of a simple game has minimum sum are provided. These methods are based on determining first, all coalitions with weight sum equal to q and to $q - 1$, constructing an associated matrix with them and proving a matricial equality with a degree of freedom. As the exact number of complete and weighted games for less than 9 voters is known, our approach is exhaustive solving linear program (5) whenever system (3) has solution. However, the approach followed to prove the existence of games without a minimum normalized realization for more than 8 voters could be applied by generating simple games at random (maybe without uniform probability) and using the method proposed in Bohosian and Bruck (2003) to check the minimality.

The statements above will let us:

- compute how many simple games are weighted (Definition 1.2 using system (3));
- compute how many weighted games have minimum integer realization (Definition 3.1 using system (4));
- provide minimum integer realizations whenever they exist (Definition 3.1 using system (4));
- compute how many weighted games have minimum sum integer realization (Definition 3.2 using system (4));
- provide minimum sum integer realizations whenever they exist (Definition 3.2 using system (4));
- compute how many weighted games have minimum sum integer normalized realization (Definition 3.6 using system (5)); and
- provide some new weighted games without a unique minimum integer normalized realization (Definition 3.6 using system (5)).

4 Algorithms and experiments to classify weighted games

We have done all our experiments with a processor AMD64X2 4400 (two cores at 2.2 GHz) with 4 Gb of DDR memory with ECC and we have essentially been able to deal with simple games with less than 9 voters because of there is a huge number of games with more than 8 voters (see Table 1), and it requires to much time to compute *all* of them. We have basically used the *GLPK (GNU Linear Programming Kit)* package (2005) to solve our (integer) linear programs. However, we have also done some experiments with the *Optimization* package of MAPLE (Maple 2005) and with Matlab (2005), but GLPK is clearly more useful for our purpose, specially for the integer restrictions.

4.1 Weighted complete simple games

The first necessary step in our experiments is to compute which complete simple games are weighted: firstly, for each complete simple game we have computed the corresponding vector $\overline{n}$ and the matrices $\mathcal{M}$ (following the steps given in Theorem 2.10) and $\mathcal{L}$ (in a similar way); secondly, for each pair $(\mathcal{M}, \mathcal{L})$, the system of inequalities (3) has solution *if and only if* the corresponding game is weighted (see Theorem 2.9).

Table 1, obtained by programming Theorem 2.10, provides how many complete simple games *up to isomorphism* are weighted depending on the number of voters (see also Krohn and Sudhölter 1995). Let us note that in Muroga et al. (1962) are listed all weighted games with less than 7 voters. In our approach GLPK gives the solution to linear program (5) for *all* weighted games with less than nine voters.

Table 1 Number of Complete Simple Games (CG), number of Weighted Complete Simple Games (WG) and CPU time (in seconds) needed to compute all of them, with n voters

n	1	2	3	4	5	6	7	8
CG	1	3	8	25	117	1171	44313	16175188
WG	1	3	8	25	117	1111	29373	2730164
CPU time	<1	<1	<1	<1	<1	<1	3	66532 (≈18.5 h.)

4.2 Complete classification for weighted games with less than 9 voters

Once we have determined all weighted complete simple games with the corresponding vector $\overline{n}$ and pair $(\mathcal{M}, \mathcal{L})$, we are able to give a full classification of them according to whether they have a minimum realization or not.

We have got that *all weighted games with less than* 8 *voters have minimum realization*, $|\mathcal{MI}| = 1$, *but there are* 154 *weighted games with* 8 *voters and two minimum sum realizations*, $\mathcal{MI} = \emptyset$ and $|s\mathcal{MI}| = 2$. In Tables 3 and 4 of Appendix we list all of them.

We remark here that all these 154 games have exactly *two* minimum sum realizations. However, none of these 154 games is *self-dual* (that is, $S \in W$ *if and only if* $N \setminus S \notin W$, for each $S \subseteq N$), hence the quota for all of them is different of $\frac{T+1}{2}$. As a consequence, once the weights and quota are determined we get a new solution by duality, considering the same weights and the *new* quota $q' = T + 1 - q$.

Finally, for all these 154 weighted games with minimum sum realization but without minimum realization, we have solved the linear program described in system (5) to compute the corresponding minimum sum normalized realization. We have checked that *all these weighted games have minimum normalized realization*; that is, even thought there are 154 weighted games with 8 voters such that $\mathcal{MI} = \emptyset$, all of them verify $\mathcal{MN} \neq \emptyset$.

For instance, the weighted game $(12; 7, 6, 6, 4, 4, 4, 3, 2)$ without minimum realization (#2 in Table 3 with q') has minimum normalized realization $(14; 8, 7, 7, 5, 5, 5, 3, 3)$ obtained solving linear program (5). Of course, its dual game defined by $(25; 7, 6, 6, 4, 4, 4, 3, 2)$ (#2 in Table 3 with q) has minimum normalized realization $(30; 8, 7, 7, 5, 5, 5, 3, 3)$.

5 Games without minimum sum normalized realizations

We have seen in the previous subsection that *all* weighted games with less than 9 voters have a minimum normalized realization, $\mathcal{MN} = \emptyset$. In Isbell (1959) it is exhibited a remarkable example with 12 voters in which the two affected voters with different weight are not equi-desirable, getting therefore an example with $\mathcal{MN} \neq \emptyset$. This example has been very important in game theory since (Dubey and Shapley 1979) mentioned it. As far as we know nobody has claimed on the existence of games with $\mathcal{MN} = \emptyset$ for less than 12 voters, even though some scholars are currently devoting efforts in this issue.

Now, we are going to study what occurs with some weighted games with more than 8 voters. Unfortunately, it has been not possible to do an exhaustive study for weighted games with more than 8 voters because there are too many complete simple games to be checked and it requires too much CPU time. So, we have just applied linear program (5) for some thousand weighted games generated at random with 9, 10 and 11 voters, but all of them had just one minimum sum normalized realization.

Then, we have considered the weighted game with 12 voters and two minimum normalized sum realizations given in Isbell (1959), i.e., $(99; 38, 31, 31, 28, 23, 12, 11, 8, 6, 5, 3, 1)$ and $(99; 37, 31, 31, 28, 23, 12, 11, 8, 7, 5, 3, 1)$. For the weights of these games we have considered *all* proper weighted games (i.e., all weighted games with quota strictly greater than $T/2$) that can be generated from the weights by removing either 1 or 2 or 3 weights to get weighted games of either 11 or 10 or 9 voters respectively. For each one of these weighted games we have calculated matrix A and solved linear program (5). Table 2 contains the 10 proper weighted games we have obtained *without* a minimum integer normalized sum realization. By duality, see e.g. Carreras and Freixas (1996), ten additional games are obtained taking as a quota the number $T - q + 1$ instead of q.

See Table 2.

Table 2 Twenty weighted games with 11 voters and two weighted games with 10 voters such that have non-minimum normalized realization, $M\mathcal{N} = \emptyset$, but have minimum sum normalized realization, $sM\mathcal{N} \neq \emptyset$

#	T	q	q'	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}
2	166	99	68	(38)	31	28	23	12	11	8	(6)	5	3	1	(37)	31	28	23	12	11	8	(7)	5	3	1
4	169	99	71	(38)	31	31	23	12	11	8	(6)	5	3	1	(37)	31	31	23	12	11	8	(7)	5	3	1
6	174	99	76	(38)	31	31	28	12	11	8	(6)	5	3	1	(37)	31	31	28	12	11	8	(7)	5	3	1
8	185	99	87	(38)	31	31	28	23	11	8	(6)	5	3	1	(37)	31	31	28	23	11	8	(7)	5	3	1
10	185	118	68	(38)	31	31	28	23	11	8	(6)	5	3	1	(37)	31	31	28	23	11	8	(7)	5	3	1
12	186	118	69	(38)	31	31	28	23	12	8	(6)	5	3	1	(37)	31	31	28	23	12	8	(7)	5	3	1
14	189	99	91	(38)	31	31	28	23	12	11	(6)	5	3	1	(37)	31	31	28	23	12	11	(7)	5	3	1
16	192	99	94	(38)	31	31	28	23	12	11	8	(6)	3	1	(37)	31	31	28	23	12	11	8	(7)	3	1
18	194	99	96	(38)	31	31	28	23	12	11	8	(6)	5	1	(37)	31	31	28	23	12	11	8	(7)	5	1
20	196	99	98	(38)	31	31	28	23	12	11	8	(6)	5	3	(37)	31	31	28	23	12	11	8	(7)	5	3
22	154	87	68	(38)	31	28	23	11	8	(6)	5	3	1	–	(37)	31	28	23	11	8	(7)	5	3	1	–

Realizations with integer minimum sum realization are important in the voting literature. These realizations are connected with some solution concepts if the game is self-dual. In Krohn and Sudhölter (1995) some connections appear linking minimal integer realizations with the least core and the nucleolus of self-dual games. For instance, they compute these two solution concepts for the self-dualized game #2 and #44 in Table 3 and for the Isbell's example. One might desire to extend their analysis for the lists of games provided in our Tables 2, 3 and 4. It also becomes an interesting open problem characterizing games for which the minimum integer realization coincides with some particular solution concept like the Penrose–Banzhaf–Coleman, the Shapley–Shubik, the Johnston, the Deegan-Packel or the Holler indices.

Minimal integer realizations might also be useful to study the occurrence of paradoxes of measures of relative power.

6 Conclusions

We have provided some new results to determine whether a complete simple game is weighted (Theorem 2.9), whether a weighted complete simple game has minimum sum realization (Theorem 3.4), and whether a weighted complete simple game has minimum normalized sum realization (Theorem 3.8). From these results we have computed:

- all realizations without minimum sum for weighted games with less than 9 voters (this full classification for weighted games with 8 voters is new); and
- some new normalized realizations of weighted simple games without minimum sum. In particular, we have found some of such realizations with 10 and 11 voters.

The main contribution of our work proves the existence of weighted games without a minimum integer realization for less than 12 voters reducing, therefore, the number of voters, 12, that Isbell needed in his example (Isbell 1959).

We leave open to determine the exact number of weighted games for more than 8 voters using the techniques presented here or another ones. From our work we know this number is whether 9 or 10.

Another problem we are working on is to determine the minimum number of voters needed to get games without a minimum normalized realization.

From a computational viewpoint, it is also interesting to study the required CPU time to solve the linear programs (4) and (5) of Theorems 3.4 and 3.8, respectively. Our experiments establish that the *GLPK (GNU Linear Programming Kit)* package is more efficient.

Another open problem is: given a fix number of voters n, to generate a random (and maybe with uniform probability) game and study whether it is weighted and if so, determining whether it has a minimum (normalized) realization. Many real-world voting systems has a huge number of voters, so that dealing with weightedness and integer minimality for them is also an interesting problem.

Finally, it becomes interesting to complete existent works in the theory of simple games linking minimal integer realizations with some solution concepts. In this line of research our results in Tables 2, 3 and 4 could be helpful to establish some new contributions.

Acknowledgements The authors thank the referees for carefully reading an earlier version of this paper and for pointing out useful comments and references.

Appendix

Table 3 Pairs of weighted games with 8 voters with quotas q and q' such that $\mathcal{MI} = \emptyset$ and $q' = T - q + 1$. The other minimum sum normalized realization is obtained by interchanging encircled weights. *It continues in Table 4*

#	T	q	q'	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
2	36	25	12	7	6	6	4	4	4	③	②
4	36	24	13	9	6	6	4	4	4	②	①
6	38	24	15	11	6	6	4	4	4	②	①
8	39	27	13	9	7	6	5	5	3	②	②
10	41	24	18	11	9	6	4	4	4	②	①
12	42	28	15	9	7	7	6	4	4	③	②
14	42	30	13	9	8	⑦	⑥	5	3	2	2
16	42	28	15	11	7	7	5	5	4	②	①
18	42	28	15	11	8	6	6	4	4	②	①
20	42	25	18	13	6	6	4	4	4	③	②
22	43	24	20	11	9	6	6	4	4	②	①
24	43	25	19	13	7	6	4	4	4	③	②
26	44	28	17	13	7	7	5	5	4	②	①
28	44	28	17	13	8	6	6	4	4	②	①
30	44	27	18	14	⑦	⑥	5	5	3	2	2
32	45	25	21	13	7	6	6	4	4	③	②
34	46	28	19	13	7	7	6	4	4	③	②
36	47	28	20	13	11	6	6	4	4	②	①
38	48	33	16	12	8	8	⑦	⑥	3	2	2
40	48	28	21	13	9	7	6	4	4	③	②
42	48	32	17	13	9	7	7	5	4	②	①
44	48	32	17	13	10	8	6	4	4	②	①
46	48	28	21	13	11	7	5	5	4	②	①
48	48	27	22	14	9	⑦	⑥	5	3	2	2
50	49	28	22	13	9	7	7	4	4	③	②
52	49	28	22	13	11	8	6	4	4	②	①
54	50	28	23	13	11	7	7	5	4	②	①
56	50	27	24	14	9	⑦	⑥	5	5	2	2
58	50	32	19	15	9	7	7	5	4	②	①
60	50	32	19	15	10	8	6	4	4	②	①
62	50	30	21	17	8	⑦	⑥	5	3	2	2
64	51	28	24	13	9	7	7	6	4	③	②
66	51	28	24	13	11	7	7	5	5	②	①
68	51	28	24	13	11	8	6	6	4	②	①
70	51	27	25	14	9	⑦	⑥	5	5	3	2
72	51	34	18	14	11	9	6	4	4	②	①
74	51	30	22	17	9	⑦	⑥	5	3	2	2
76	52	33	20	11	10	8	6	6	⑤	④	2
78	53	32	22	15	13	8	6	4	4	②	①

Table 4 Pairs of weighted games with 8 voters with quotas q and q' such that $M\mathcal{I}=\emptyset$ and $q'=T-q+1$. The other minimum sum normalized realization is obtained by interchanging encircled weights. *It comes from Table 3*

#	T	q	q'	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
80	53	34	20	16	11	9	6	4	4	②	①
82	53	33	21	17	8	8	⑦	⑥	3	2	2
84	54	33	22	13	10	8	6	6	⑤	④	2
86	54	36	19	15	11	9	7	5	4	②	①
88	54	32	23	15	13	7	7	5	4	②	①
90	54	30	25	17	9	8	⑦	⑥	3	2	2
92	55	33	23	13	11	8	6	6	⑤	④	2
94	55	32	24	15	13	10	6	4	4	②	①
96	56	32	25	15	13	9	7	5	4	②	①
98	56	34	23	16	14	9	6	4	4	②	①
100	56	30	27	17	9	8	⑦	⑥	5	2	2
102	56	36	21	17	11	9	7	5	4	②	①
104	57	33	25	13	11	10	6	6	⑤	④	2
106	57	32	26	15	13	10	8	4	4	②	①
108	57	38	20	16	12	10	7	5	4	②	①
110	57	30	28	17	9	8	⑦	⑥	5	3	2
112	57	33	25	17	12	8	⑦	⑥	3	2	2
114	58	32	27	15	13	9	7	7	4	②	①
116	58	34	25	16	14	11	6	4	4	②	①
118	59	33	27	13	11	10	8	6	⑤	④	2
120	59	32	28	15	13	9	7	7	5	②	①
122	59	32	28	15	13	10	8	6	4	②	①
124	59	38	22	18	12	10	7	5	4	②	①
126	60	36	25	17	15	9	7	5	4	②	①
128	61	34	28	16	14	11	9	4	4	②	①
130	62	33	30	17	12	8	8	⑦	⑥	2	2
132	62	36	27	17	15	11	7	5	4	②	①
134	63	33	31	13	11	10	8	6	6	⑤	④
136	63	34	30	16	14	11	9	6	4	②	①
138	63	33	31	17	12	8	8	⑦	⑥	3	2
140	63	38	26	18	16	10	7	5	4	②	①
142	64	36	29	17	15	11	9	5	4	②	①
144	65	38	28	18	16	12	7	5	4	②	①
146	66	36	31	17	15	11	9	7	4	②	①
148	67	36	32	17	15	11	9	7	5	②	①
150	68	38	31	18	16	12	10	5	4	②	①
152	70	38	33	18	16	12	10	7	4	②	①
154	71	38	34	18	16	12	10	7	5	②	①

References

Bohosian, V., & Bruck, J. (2003). Algebraic techniques for constructing minimal weights threshold functions. *SIAM Journal on Discrete Mathematics*, *16*, 114–126.

Brams, S. J., & Fishburn, P. C. (1976). Approval voting. *American Political Science Review*, *72*, 831–847.

Boros, E., Hammer, P. L., Ibaraki, T., & Kawakawi, K. (1991). Identifying 2-monotonic positive boolean functions in polynomial time. In W. L. Hsu & R. C. T. Lee (Eds.), *LNCS: ISA'91 Algorithms* (Vol. *557*, pp. 104–115). Berlin: Springer.

Boros, E., Hammer, P. L., Ibaraki, T., & Kawakawi, K. (1997). Polynomial time recognition of 2-monotonic positive functions given by an oracle. *SIAM Journal of Computing*, *26*, 93–109.

Brams, S. J., Jr., & Straffin, P.D. (1979). Prisoners' dilemma and the professional sports drafts. *American Mathematical Monthly*, *80*, 80–88.

Carreras, F., & Freixas, J. (1996). Complete simple games. *Mathematical Social Sciences*, *32*, 139–155.

Chavátal, V. (1983). *Linear programming*. New York: Freeman.

Dubey, P., & Shapley, L. S. (1979). Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research*, *4*, 99–131.

Fishburn, P. C. (1969). Preference, summation, and social welfare functions. *Management Science*, *16*, 179–186.

Fishburn, P. C. (1973). *The theory of social choice*. Princeton: Princeton University Press.

Freixas, J., & Molinero, X. (2008, accepted). Simple games and weighted games: a theoretical and computational viewpoint. *Discrete Applied Mathematics*.

Freixas, J., & Zwicker, W. S. (2003). Weighted voting, abstention, and multiple levels of approval. *Social Choice and Welfare*, *21*, 399–431.

Glpk (gnu linear programming kit) package. (2005). URL:http://www.gnu.org/software/glpk/.

Hammer, P. L., Kogan, A., & Rothblum, U. G. (2000). Evaluation, strength and relevance of boolean functions. *SIAM Journal on Discrete Mathematics*, *13*, 302–312.

Hu, S. T. (1965). *Threshold logic*. USA: Univ. of California Press.

Isbell, J. R. (1958). A class of simple games. *Duke Mathematics Journal*, *25*, 423–439.

Isbell, J. R. (1959). On the enumeration of majority games. *Mathematical Tables and Other Aids to Computation*, *13*, 21–28.

Krohn, I., & Sudhölter, P. (1995). Directed and weighted majority games. *ZOR-Mathematical Methods of Operations Research*, *42*, 189–216.

Maple. (2005). URL: http://www.maplesoft.com/.

Matlab. (2005). URL: http://www.mathworks.com/products/matlab/.

Maschler, M., & Peleg, B. (1966). A characterization, existence proof, and dimension bounds for the kernel of a game. *Pacific Journal of Mathematics*, *18*, 289–328.

Muroga, S., Toda, I., & Kondo, M. (1962). Majority decision functions of up to six variables. *Mathematics of Computation*, *16*, 459–472.

Muroga, S. (1971). *Threshold logic and its applications*. New York: Wiley.

Von Neumann, J., & Morgenstern, O. (1944). *Theory of games and economic behavior*. Princeton: Princeton University Press.

Ostmann, A. (1985). Decisions by players of comparable strength. *Journal of Economics*, *45*, 267–284.

Shapiro, J. F. (1979). *Mathematical programming: structures and algorithms*. New York: Wiley.

Taylor, A. D. (1995). *Mathematics and politics*. New York: Springer.

Taylor, A. D., & Zwicker, W. S. (1992). A characterization of weighted voting. *Proceedings of the American Mathematical Society*, *115*, 1089–1094.

Taylor, A. D., & Zwicker, W. S. (1999). *Simple games: desirability relations, trading, and pseudoweightings*. New Jersey: Princeton University Press.