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Properties of a method for polyhedral approximation of the feasible criterion set in convex multiobjective problems

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Abstract The paper describes new results in the field of multiobjective optimization techniques. The Interactive Decision Maps (IDM) technique is based on approximation of Feasible Criterion Set (FCS) and subsequent visualization of the Pareto frontier of FCS by interactive displaying the bi-criteria slices of FCS. The Estimation Refinement (ER) method is now the main method for approximating convex FCS in the framework of IDM. The properties of the ER method are studied. We prove that the number of facets of the approximation constructed by ER and the number of the support function calculations of an approximated set are asymptotically optimal. These results are important from the point of view of real-life applications of ER.

Keywords Multicriteria optimization · Polyhedral approximation · Estimation refinement method · Feasible goals method

1 Introduction

After Gass and Saaty (1955) proposed to compute and display the Pareto frontier for bi-criteria linear models, approximating the Pareto frontier of the set of feasible criterion vectors of a problem, so-called Feasible Criterion Set (FCS), became an efficient approach in multicriteria optimization. In the framework of this approach, the Pareto frontier is approximated by various techniques that depend upon the problem under consideration. Then, the decision maker (DM) is provided with the Pareto frontier, represented graphically, and has to identify the most preferable criterion point at the frontier. However, such a convenient form of informing decision makers for several decades was restricted to the bi-criteria problems. In the case of more than two criteria, the standard approach was to approximate the

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Pareto frontier by a large number of criterion points and provide the DM with the list of points. However, selecting from the large lists of multiobjective points is recognized to be too complicated for a human being, see Larichev (1992).

The approximation and visualization of the Pareto frontier for the multicriteria optimization problem with relatively large (till 7–8) number of criteria was introduced in the framework of the group of methods that have got the name of the Interactive Decision Maps (IDM) technique, see Lotov et al. (2004a). In the IDM technique, FCS (or the Edgeworth-Pareto Hull of FCS, which is the maximal set in criterion space with the same Pareto frontier as FCS) is approximated and the Pareto frontier is provided to the decision maker in the form of decision maps, i.e. collections of superimposed bi-criterion slices of FCS. To present more that three criteria, animation of decision maps is used.

The combination of IDM and single-shot goal identification on one of decision maps, so-called Feasible Goals Method (FGM), turned out to be a convenient decision support. The Estimation Refinement (ER) method, proposed in Bushenkov and Lotov (1982), has become a reliable method for approximation of FCS in the framework of IDM. Chernykh (1988, 1995) developed a numerical scheme of ER that is computationally stable to round-off errors. Kamenev (1994) studied ER and proved its asymptotic optimality. The ER method is implemented in various software, see www.ccas.ru/mmes/mmeda/soft/.

In the IDM technique, approximation is separated from the human-computer exploration of decision maps. At the same time, bi-criterion slices of an approximation can be computed very fast. This feature of IDM facilitates implementation on computer networks, where decision maps may be depicted and animated on-line. It is based on using a simple web client-server architecture: the approximation is accomplished on a server side, while the exploration of the Pareto frontier is carried out by means of Java applets on the user's computer, see Lotov et al. (2004b) for details of the architecture and Lotov et al. (2004b), Dietrich et al. (2006) and Efremov et al. (2006) as some examples of the IDM-based decision aid tools, implemented in Web.

Nevertheless, the practice showed, that the approximation process requires up to 99% of the computing efforts. Having this in a view, we want to be sure beforehand that the approximation will not exceed practical time limits and will generally be solved. That is why theoretical and experimental studies of approximation methods have always been an important task. The theoretical study of the ER method gave rise to the concept of *Hausdorff methods* for polyhedral approximation of compact convex bodies (CCB), see Kamenev (1992). As it was proven in Schneider and Wieacker (1981), the optimal order of convergence of approximating polyhedrons for a smooth CCB equals to $2/(d-1)$ where d is the dimension of the space. Kamenev (1992) showed that Hausdorff methods approximate a smooth CCB with the optimal order of convergence with respect to the number of iterations and vertices of approximating polytopes. Since the ER method belongs to the class of Hausdorff methods, see Kamenev (1994), it has the optimal order of convergence with respect to the number of iterations and vertices. Here we study the order of the number of facets of approximating polyhedrons and the related problem of the number of calculations of the support function by the ER method.

The outline of the paper is as follows. In Sect. 2 we give more formal description the FGM/IDM technique and the ER method. Section 3 is devoted to our theoretical results. We prove that, for the Hausdorff methods, the order of the number of facets of approximating polytopes is also optimal. In addition, we prove that the order of the number of support function calculations in ER method for a class of smooth CCB is optimal, too. We end up with some discussion.

2 The ER method in the framework of FGM/IDM

2.1 The FGM/IDM method

Let X be the *feasible decision set* of a problem and let $f: X \to \mathbf{R}^d$ be a mapping from X to the criterion space $\mathbf{R}^d$: the performance of each feasible decision $x \in X$ is described by a d-dimensional vector $y = f(x)$. Here, $Y := f(X)$ is the FCS of the problem. We shall assume here Y to be compact and convex and f to be linear. With no loss of generality, we shall assume that the criteria must be maximized. This defines a Pareto order in the criterion space: y dominates y' (in the Pareto sense) if, and only if, $y \geq y'$ and $y \neq y'$. The Pareto frontier of the set Y is defined as $P(Y) := \{y \in Y : \{y' \in Y : y' \geq y, y \neq y'\} = \emptyset\}$. Let $\mathbf{R}^d_-$ be the non-positive orthant in $\mathbf{R}^d$. The set $H(Y) = Y + \mathbf{R}^d_-$ is known as the Edgeworth–Pareto Hull of Y. $H(Y)$ is the maximal set that satisfies $P(H(Y)) = P(Y)$.

The key feature of IDM consists of displaying the Pareto frontier for more than two criteria through interactive display of bi-criterion slices of $H(Y)$. A bi-criterion slice is defined as follows. Let (y_1, y_2) designates two specified criteria, the so-called "axis" criteria, and z denotes the remaining criteria, which we shall fix at $z^* \in \mathbf{R}^{d-2}$. A bi-criterion slice of $H(Y)$, parallel to the criterion plane (y_1, y_2) and related to z^*, is defined as $G(H(Y), z^*) = \{(y_1, y_2) : (y_1, y_2, z^*) \in H(Y)\}$. Note that a slice of $H(Y)$ contains all feasible combinations of values for the specified criteria when the values of the remaining criteria are not worse than z^*. Bi-criterion slices of $H(Y)$ are used in the IDM technique for displaying *decision maps*. To define a particular decision map, the user has to choose a "third", or colour-associated, criterion. Then, a decision map is a collection of superimposed slices, for which the values of the colour-associated criterion change, while the values of the remaining criteria are fixed. Note that slices are monotonic: a slice contains all slices with better values of the colour-associated criterion.

The FGM method provides single-shot identification of a goal on a decision map. Once the goal $y \in P(Y)$ is chosen, an associated decision can be computed by using y as the reference point, see Wierzbicki (1981).

2.2 The ER method

Let $\mathcal{C}$ be the class of CCB in the Euclidean space $\mathbf{R}^d$ with the scalar product $\langle \cdot, \cdot \rangle$ and the Hausdorff metric $\delta(C_1, C_2) := \max\{\sup\{\rho(x, C_2) : x \in C_1\}, \sup\{\rho(x, C_1) : x \in C_2\}\}$, $C_1, C_2 \in \mathcal{C}$. Let $\mathcal{P}$ be the class of convex polytopes. Let $P \in \mathcal{P}$. Let $C \in \mathcal{C}$ and ∂C be its boundary. Denote the set of the vertices of P by $M^t(P)$ and the set of $(d-1)$-dimensional facets of P by $M^f(P^n)$. Let $M^f(P^n)$ be given in the form of a system of linear inequalities. Let $\mathcal{P}(C) := \{P \in \mathcal{P} : M^t(P) \subset \partial C\}$. For $P \in \mathcal{P}(C)$ let $U(P)$ be the list of unit outer normals to its facets. Let $g(u, C)$ be the support function of C. Let conv$\{\cdot\}$ means the convex hull of a set. Let $C \in \mathcal{C}$ to be approximated.

The ER method The ER method Prior to the $(n+1)$-th iteration of the method, $M^f(P^n)$ must be constructed. Each facet must be given along with the list of vertices that belong to it. Then, the following steps are carried out.

Step 1. (a) Find $u_n \in U(P^n)$ that solves $\max\{g(u, C) - g(u, P^n): u \in U(P^n)\}$. If $|g(u_n, C) - g(u_n, P^n)| \leq \varepsilon$, then stop the method, otherwise proceed to (b).
(b) Select a point $p_n \in \partial C$ such that $\langle u_n, p_n \rangle = g(u_n, C)$.
Step 2. Find $M^f(P^{n+1})$ for $P^{n+1} := \text{conv}\{p_n \cup P^n\}$.

A stable implementation of the method is discussed in Chernykh (1988, 1995). The stability is provided due to the form of the facet description in the ER method. Furthermore, the implementation of the second step, which is based on the "beneath-beyond" method Preparata and Shamos (1985) of constructing of convex hull of a polytope and a point, see Preparata and Shamos (1985), does not generate redundant inequalities, that is, on every iteration n, inequalities that belong to $M^f(P^n)$ and only these inequalities are generated, see Chernykh (1988) for details. Thus, to estimate the complexity of ER we need to know the convergence rate of ER with regard to the number of facets, since, virtually, it may be very big, and with regard to the number of optimization problem to solve, since the optimization problem solving may be time consuming.

We sketch the results we have obtained for these characteristics and prove them for a special case in the next section. However, the proofs of these results exceed the scope of this paper.

3 Optimal estimates of convergence rates of methods for polyhedral approximation

To start with, let us introduce some notions and results of the general theory of polyhedral approximation of CCB we will need to formulate our results.

3.1 Hausdorff methods for internal polyhedral approximation

Kamenev (1992) proposed the *augmentation scheme* that is the general algorithmic scheme for the internal polyhedral approximation of CCB with increasing number of vertices. Let $P^0 \in \mathcal{P}(C)$. Given $P^n \in \mathcal{P}(C)$, the $(n+1)$-th iteration consists of two steps: choose $p_n \in \partial C$ and construct the new polytope $P^{n+1} := \text{conv}\{p_n \cup P^n\}$. The augmentation method for polyhedral approximation of $C \in \mathcal{C}$ is called Hausdorff with constant $\gamma > 0$ if it generates the sequence of polytopes $\{P^n\}_{n=0,1,\ldots}$ such that $\delta(P^n, P^{n+1}) \geq \gamma\delta(P^n, C)$ holds for any $n > 0$. Such a sequence itself is called the $H(\gamma, C)$-(*augmentation*) *sequence*.

Let us denote by $\mathcal{C}^2$ the class of CCB with the two times continuously differentiated boundary and the positive principal curvatures and $C \in \mathcal{C}^2$. In Schneider (1983) and Gruber (1993) it was proven that

$$\inf_{\{P^n\}, P^n \in \mathsf{P}(C)} \limsup_{n\to\infty} \delta(C, P^n)\big[m^t(P^n)\big]^{2/(d-1)} = A(C), \tag{1}$$

where $m^t(P)$ is the number of vertices of P and $A(C)$ is a constant that depends on the body C.

Efremov and Kamenev (2002) proved that the following estimate cannot be improved for the class of Hausdorff methods:

$$\limsup_{n\to\infty} \delta(C, P^n)\big[m^t(P^n)\big]^{2/(d-1)} \leq \frac{2}{\big(1-\sqrt{1-\gamma}\big)^2} A(C). \tag{2}$$

On the basis of this result and a theorem that describes the facial structure of a convex hull of a polytope and a point, see Preparata and Shamos (1985) and McMullen and Shephard (1971), we have found the similar estimate for the convergence rate with respect to the number of facets.

3.2 Main results for general case

Let $k_{\partial C}^{\max}$ and $k_{\partial C}^{\min}$ be the maximal and minimal principal curvatures of a surface ∂C. Let $\mathbf{C}(n,d)$ be the number of combinations from n by d. By $m^f(P)$ denote the number of facets of P. Let $f(n) := m^f(P^{n+1}) - m^f(P^n)$. By $B(r,z)$ denote the ball of radius r centred in z; let $S^{d-1} := \partial B(1,0)$.

Theorem 1 *Let $C \in \mathcal{C}^2$ and let $\{P^n\}_{n=0,1,\dots}$ be the $H(\gamma, C)$-sequence. Then*

$$\limsup_{n\to\infty} f(n) \le \overline{f}(C,\gamma),$$

where

$$\overline{f}(C,\gamma) = \mathbf{C}\big(6^{d-1}\big(1-(1-\gamma)^{1/2}\big)^{(1-d)/2}\big(k_{\partial C}^{\max}/k_{\partial C}^{\min}\big)^{5(d-1)/2}, d-1\big).$$

Theorem 1 and (2) result in the following estimate for the order of convergence of $H(\gamma, C)$-sequences with respect to the number of facets:

Corollary 1

$$\limsup_{n\to\infty} \delta(C, P^n)\big[m^f(P^n)\big]^{2/(d-1)} \le \frac{2\overline{f}(C,\gamma)}{\big(1-\sqrt{1-\gamma}\big)^2} A(C).$$

As can be seen from Corollary 1 and (1)–(2), the order of convergence of $H(\gamma, C)$-sequences with respect to the number of facets is the same as with respect to the number of vertices and is optimal.

We will present *the proof of the Theorem* 1 in case of $C \equiv B(1,0)$ and $\gamma = 1$. This will give the main idea of the full proof, which is much more complicated.

Proof of Theorem 1 Let $\{P^n\}_{n=0,1,\dots}$ be an $H(1, B(1,0))$-sequence. Let $P \in \mathcal{P}$ and $p \notin P$. Let $P' := \operatorname{conv}\{p \cup P\}$.

Let

$$\partial P_p^+ := \{F \in M^f(P)\text{: if } q \in F \text{ then } \langle u_F, p-q\rangle > 0\},$$

$$\partial P_p^0 := \{F \in M^f(P) : p \in F\},$$

$$\partial P_p^- := M^f(P)\backslash\{\partial P_p^+ \cup \partial P_p^0\}.$$

Let

$$T(n) := \big\{t \in M^t(P^n) : \exists F_1, F_2 \in M^f(P),\ t \subset F_1 \cap F_2 \text{ and } F_1 \in \partial P_{pn}^+,\ F_2 \in \partial P_{pn}^-\big\}.$$

Let $|T(n)|$ be the number of elements in $T(n)$. From McMullen and Shephard (1971) it follows that

$$\overline{f}(C,\gamma) \le \mathbf{C}(|T(n)|, d-1). \tag{3}$$

Let

$$K(p,S) := \{x \in \mathbf{R}^d : x = \lambda(z-p), z \in S, \lambda \ge 0\},$$

$$\delta(n) := \delta(B^d(1,0), P^n).$$

Let $\Omega(n) := S^{d-1} \backslash [K(p_n, B^d(1-\delta(n))) \cap S^{d-1}]$. We will prove that

$$T(n) \subset \Omega(n). \tag{4}$$

From Kamenev (1992) it follows that $B^d(1-\delta(n), 0) \subset P^n$. Then $K(p, B^d(1-\delta(n), 0)) \subseteq K(p, P^n)$. But $T(n) \subset \Omega_1(n)$, where $\Omega_1(n) := S^{d-1} \backslash (K(p, P^n) \cap S^{d-1})$; moreover, $\Omega_1(n) \subset \Omega(n)$. So we prove inclusion (4).

Let $x, y \in S^{d-1}$ and let $\mu(x, y)$ be geodetic distance between x and y on S^{d-1} (that is the shortest arc of the big circle that connects x with y). In the case considered $\mu(x, y)$ is equal to the angle value between vectors x and y. Let

$$D^\mu(p, \nu) := \{x \in S^{d-1} : \mu(p, x) \leq \nu\},$$

$$\mu_n := \frac{1}{2} \min\{\mu(p_1, p_2) : p_1, p_2 \in M^t(P^n),\ p_1 \neq p_2\}, \quad n \geq 0,$$

$$\mu_n^* := \frac{1}{2} \min\{\mu(p_n, p) : p \in M^t(P^n)\}, \quad n \geq 1.$$

Note, that

$$\mu_{N+1} = \min\{\mu_0, \min\{\mu_n^*,\ 1 \leq n \leq N\}\}. \tag{5}$$

Moreover, for any $n_1 > n_2 \geq 1$ holds $\mu_{n_1}^* \subseteq \mu_{n_2}^*$. Then, from (5) it follows that

$$\mu_{n+1} = \min\{\mu_0, \mu_n^*\}. \tag{6}$$

Now we will prove that there exists $N > 0$ (depending on P^0) that for any $n \geq N$ valid

$$\mu_{n+1} = \mu_n^*. \tag{7}$$

Let $n > 0$ and $p_n \in S^{d-1}$ such that $P^{n+1} = \mathrm{conv}\{p_n \cup P^n\}$. For $H(1, B^d(1, 0))$-sequence for any n we have

$$g(p_n, B^d(1, 0)) - g(p_n, P^n) = \delta(n). \tag{8}$$

Note that $g(p_n, P^n) = \max\{< p_n, p >: p \in M^t(P^n)\}$. Let $p(n) \in M^t(P^n)$ be such that $\langle p_n, p(n)\rangle = \max\{\langle p_n, p\rangle : p \in M^t(P^n)\}$. From (8) it follows that

$$\langle p_n, p(n)\rangle = 1 - \delta(n). \tag{9}$$

As far as $\cos(\mu(p_n, p(n))) = \langle p_n, p(n)\rangle$, from (9) it follows that

$$\mu(p_n, p(n)) = \arccos(1 - \delta(n)). \tag{10}$$

Let $l_{(n)} := \{x \in \mathbf{R}^d : \langle x, p_n\rangle = g(p_n, P^n)\}$. Hyperplane $l_{(n)}$ separate p_n from P^n. Then from $\langle p_n, p(n)\rangle = g(p_n, P^n)$ follows $p(n) \in l_{(n)}$ and

$$\mu(p_n, x) = \mu(p_n, p(n)) \quad \text{for any } x \in B^d(1) \cap l_{(n)}. \tag{11}$$

So, for any point $p \in M^t(P^n)$ holds $\mu(p_n, p) \geq \mu(p_n, p(n))$. From this and (10) follows

$$\mu_n^* = \frac{1}{2}\mu(p_n, p(n)) = \frac{1}{2}\arccos(1 - \delta(n)). \tag{12}$$

We have $\lim_{n\to\infty} \delta(n) = 0$ (see for example (2)). Then from (6) and (12) follows (7).

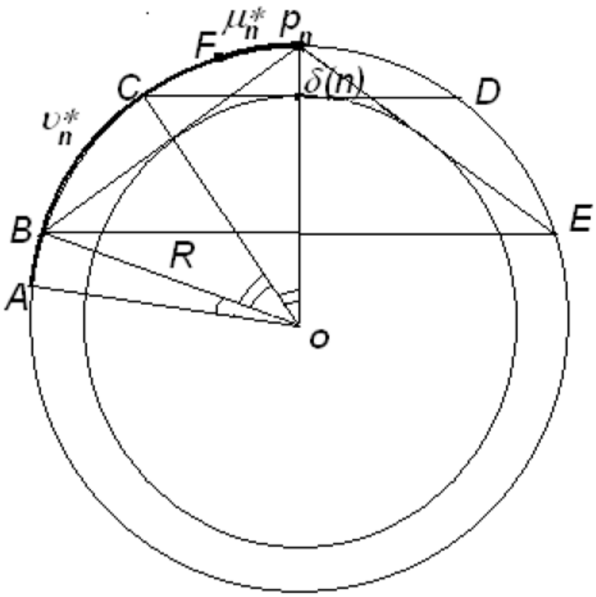

Fig. 1 To the proof of the Theorem 1

Now let $\Omega^*(n)$ be the "spherical" cap ($d-1$ ball on S^{d-1}), containing caps which are centered in points of $T(n)$ with radii of μ_{n+1}. Let $\Omega^*(n) = D^\mu(p_n, \nu_n^*)$, where p_n is its center and ν_n^*—its geodetic radius. From (11) and (12) for any $x \in B^d(1) \cap l_{(n)}$ holds $\mu(p_n, x) = 2\mu_n^*$. Then $\nu_n^* \le 5\mu_n^*$ (see Fig. 1).

Now from (7) it follows that there exists $N > 0$ that for any $n \ge N$ holds

$$\nu_n^* \le 5\mu_{n+1}. \tag{13}$$

For all $p_1, p_2 \in M^t(P^{n+1})$, $p_1 \ne p_2, n \ge 0$, holds

$$D^\mu(p_1, \mu_{n+1}) \cap D^\mu(p_2, \mu_{n+1}) = \emptyset. \tag{14}$$

From the definition of $\Omega^*(n)$ follows

$$\bigcup \left\{ D^\mu(t, \mu_{n+1}) : t \subset \Omega(n) \cap M^t(P^{n+1}) \right\} \subset D^\mu(p_n, \nu_n^*). \tag{15}$$

Let $\sigma(\cdot)$ be the surface area ($d-1$ volume) on S^{d-1}. From (14), (15) and (4) follows

$$|T(n)| \le \frac{\sigma(D^\mu(p_n, \nu_n^*))}{\sigma(D^\mu(p_n, \mu_{n+1}))}. \tag{16}$$

For S^{d-1} we have $\sigma(D^\mu(p, \mu)) = \pi \cdot (\sin\mu)^{d-1} V_{d-2}$, where V_{d-2} is the unit ball volume in $\mathbf{R}^{d-2}$. Then from (13) and (16) follows

$$|T(n)| \le \left(\frac{\sin(5\mu_{n+1})}{\sin \mu_{n+1}} \right)^{d-1} = \left(16(\cos(\mu_{n+1}))^4 - 12(\cos(\mu_{n+1}))^2 + 1\right)^{d-1} \le 5^{d-1}. \tag{17}$$

Finally from (3) and (17) follows the statement of the Theorem 1 for $C \equiv B(1, 0)$ and $\gamma = 1$. □

3.3 Applications to ER

It is shown in Kamenev (1994) that the ER method is Hausdorff; namely, if $\{P^n\}_{n=0,1,\ldots}$ is generated for $C \in \mathcal{C}^2$ by ER, there exists such $N \geq 0$ that $\{P^n\}_{n=N,N+1,\ldots}$ is $H(1-\varepsilon, C)$-augmentation sequence for any ε, $0 < \varepsilon < 1$. From this fact and the theorem 1 follows:

Theorem 2 *Let $\{P^n\}_{n=0,1,\ldots}$ be generated by ER for $C \in \mathcal{C}^2$. Then, it holds*:

$$\limsup_{n\to\infty} \delta(C, P^n)\big[m^f(P^n)\big]^{2/(d-1)} \leq 2\overline{f}(C)A(C),$$

where $\overline{f}(C) = C(6^{d-1}(k_{\partial C}^{\max}/k_{\partial C}^{\min})^{5(d-1)/2}, d-1)$ and $C(\cdot,\cdot)$—the binomial coefficient.

Theorem 2 and (2) result in the following estimate for the order of convergence of ER-generated sequences with respect to the number of facets:

Corollary 2 $\limsup_{n\to\infty} \delta(C, P^n)[m^f(P^n)]^{2/(d-1)} \leq 2\overline{f}(C)A(C)$.

Denote by $s(n)$ the number of computations of the support function at the iterations up to n.

Theorem 3 *Let $\{P^n\}_{n=0,1,\ldots}$ be generated by the ER method for $C \in \mathcal{C}^2$. Then, it holds*:

$$\limsup_{n\to\infty} s(n) \leq m^f(P^0) + \overline{f}(C) \cdot n,$$

where $\overline{f}(C)$ is the same as in Theorem 2.

Thus, the value of $s(n)$ is bounded from above by a linear function of n. It follows then that its order of convergence is optimal.

4 Conclusions

It was shown theoretically that the order of convergence of the Hausdorff methods for internal polyhedral approximation and compact convex bodies from $\mathcal{C}^2$ are optimal with respect to the number of facets. For the ER method, it was additionally shown that the number of support function calculations up to the iteration n is bounded from above by a linear function of n. The results for the number of facets and the number of support function calculations are important for ER applications in the framework of the IDM technique, when approximating EPH, since every support function calculation may be time-consuming, as well as when constructing (animating) decision maps, since the complexity of the algorithm for constructing bi-criterion slices is directly proportional to the number of facets in the approximation of EPH.

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