Ann Oper Res (2009) 167: 327–336
DOI 10.1007/s10479-008-0336-z

On the convergence of the generalized Weiszfeld algorithm

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Published online: 24 September 2008

Abstract In this paper we consider Weber-like location problems. The objective function is a sum of terms, each a function of the Euclidean distance from a demand point. We prove that a Weiszfeld-like iterative procedure for the solution of such problems converges to a local minimum (or a saddle point) when three conditions are met. Many location problems can be solved by the generalized Weiszfeld algorithm. There are many problem instances for which convergence is observed empirically. The proof in this paper shows that many of these algorithms indeed converge.

Keywords Location · Weber problem · Weiszfeld

1 Introduction

The Weber problem (Weber 1909) is to find the location for a facility which minimizes the sum of weighted distances from a set of n demand points. An iterative "Weiszfeld" procedure for its solution was proposed by Weiszfeld (1937) and re-discovered by Kuhn (1967). A convergence proof was offered by Weiszfeld (1937) and Ostresh (1978). A Proof for a convergence of a more general type problem was proposed by Morris (1981), Drezner and Drezner (2004). In this paper we prove convergence of a "Generalized Weiszfeld Algorithm" for a family of Weber-like problems. The proof follows the outline given in Morris (1981) which was used also for similar problems by Brimberg and Love (1993) and Puerto and Rodriguez-Chia (1999).

Let a set of demand points $A_1, A_2, \ldots, A_n$ be given in $\mathcal{R}^m$. Suppose that an optimization problem in $\mathcal{R}^m$ can be formulated as minimizing (or maximizing) an objective function which is a sum of individual terms, each a function of the Euclidean distance between the facility and a demand point:

$$f(X) = \sum_{i=1}^{n} f_i(d_i(X)) \tag{1}$$

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where $d_i(X)$ is the Euclidean distance between demand point A_i and the facility located at $X \in \mathcal{R}^m$.

The "Generalized Weiszfeld Algorithm" (GWA) is an extension of the original Weiszfeld algorithm proposed for $f_i(d) = w_i d$ (Drezner and Drezner 1998).

The conditions $\frac{\partial f}{\partial X} = 0$ lead to (for all points in $\mathcal{R}^m$ except the set of demand points):

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial d_i} \frac{X - A_i}{d_i(X)} = 0 \tag{2}$$

which leads to the recursive relationship (which is similar to the Weiszfeld 1937 iteration):

$$X = \frac{\sum_{i=1}^{n} \frac{1}{d_i(X)} \frac{\partial f_i(d_i(X))}{\partial d_i(X)} A_i}{\sum_{i=1}^{n} \frac{1}{d_i(X)} \frac{\partial f_i(d_i(X))}{\partial d_i(X)}} \tag{3}$$

In a GWA iteration we substitute the location X of the present iteration in the right hand side of (3). The left hand side is the next iteration. Let X^k be the k'th iterate. Then (3) can be written as

$$X^{k+1} = X^k - \frac{1}{\sum_{i=1}^{n} \frac{1}{d_i(X^k)} \frac{\partial f_i(d_i(X^k))}{\partial d_i(X^k)}} \left(\sum_{i=1}^{n} \frac{1}{d_i(X^k)} \frac{\partial f_i(d_i(X^k))}{\partial d_i(X^k)} (X^k - A_i) \right) \tag{4}$$

which is a gradient method with a step size of $\frac{1}{\sum_{i=1}^{n} \frac{1}{d_i(X^k)} \frac{\partial f_i(d_i(X^k))}{\partial d_i(X^k)}}$.

Many location problems can be solved by the generalized Weiszfeld algorithm. There are many problem instances for which convergence is observed empirically. The proof in this paper shows that many of these algorithms indeed converge.

2 A convergence proof

We prove that, under certain conditions, the GWA iterations converge to a local minimum or a saddle point of $f(X)$.

We define the function $\phi_i(D) = f_i(d)$ where $D = d^2$. For example, for $f_i(d) = w_i d$, $\phi_i(D) = w_i\sqrt{D}$. This definition leads to:

$$f(X) = \sum_{i=1}^{n} \phi_i(d_i^2(X)) \tag{5}$$

The sufficient assumptions for the convergence proof presented in this paper are that the functions $\phi_i(D)$ are twice differentiable (except, possibly at the demand points), concave, and monotonically non-decreasing with the distance. These assumptions lead to:

$$\frac{\partial \phi_i(D)}{\partial D} \geq 0 \tag{6}$$

$$\frac{\partial^2 \phi_i(D)}{\partial D^2} \leq 0 \tag{7}$$

Since $D \geq 0$, (6) and (7) are equivalent to:

$$\frac{\partial f_i(d)}{\partial d} \geq 0 \tag{8}$$

$$d\frac{\partial^2 f_i(d)}{\partial d^2} \leq \frac{\partial f_i(d)}{\partial d} \tag{9}$$

Note that if $f_i(d)$ is concave, then (9) is automatically fulfilled once (8) is fulfilled. However, (9) may hold for non-concave $f_i(d)$ (but concave $\phi_i(D)$). For example, $f_i(d) = d^{1.5}$ is a convex and not concave function but $\phi_i(D) = D^{0.75}$ is concave. Condition (9) for this function is $0.75\sqrt{d} \leq 1.5\sqrt{d}$.

First we prove that the value of the objective function $f(X)$ decreases every iteration of the GWA.

For any two squared distances D_1, D_2 we have by the Taylor expansion:

$$\phi_i(D_2) = \phi_i(D_1) + \frac{\partial \phi_i(D_1)}{\partial D}(D_2 - D_1) + \frac{\partial^2 \phi_i(\zeta)}{\partial D^2}\frac{(D_2 - D_1)^2}{2} \tag{10}$$

where ζ is between D_1 and D_2.

Lemma 1 $\phi_i(D_2) \leq \phi_i(D_1) + \frac{\partial \phi_i(D_1)}{\partial D}(D_2 - D_1)$.

Proof Follows (7) and (10). □

Let X, Z be two points in $\mathcal{R}^m$. We define $\Phi(Z)$ in the following Lemma.

Lemma 2 $f(Z) \leq \Phi(Z) = f(X) + \frac{1}{2}\sum_{i=1}^{n} \frac{1}{d_i} \times \frac{\partial f_i(d_i(X))}{\partial d_i}[d_i^2(Z) - d_i^2(X)]$

Proof By definition, $\frac{\partial \phi_i(D)}{\partial D} = \frac{1}{2d} \times \frac{\partial f_i(d)}{\partial d}$, and the Lemma follows Lemma 1 by substituting $D_1 = d_i^2(X)$, $D_2 = d_i^2(Z)$, and adding up the individual terms in (5). □

Lemma 3 $\Phi(X) = f(X)$.

Proof Follows Lemma 2 because the second term vanishes for $Z = X$. □

Minimizing $\Phi(Z)$ (for a given X) is equivalent to minimizing (all terms dependent on X are constant):

$$\sum_{i=1}^{n} \frac{1}{d_i} \times \frac{\partial f_i(d_i(X))}{\partial d_i} d_i^2(Z) \tag{11}$$

Equation (11) is the squared Euclidean distance Weber problem whose solution is the center of gravity (Love et al. 1988). Therefore, the minimum of $\Phi(Z)$ for a given X is at the center of gravity which is (3).

Let $T(X)$ be the next GWA iteration of X. We proved the following lemma:

Lemma 4 *Let X be the current GWA iteration, then the next GWA iteration $T(X)$ is the Z which minimizes $\Phi(Z)$.*

This Lemma leads to the following theorem:

Theorem 1 *Let X be the present GWA iteration. Then, $f(T(X)) \leq f(X)$ with a sharp inequality if $T(X) \neq X$.*

Proof By Lemmas 2, 3 and 4

$$f(T(X)) \leq \Phi(T(X)) = \min_Z\{\Phi(Z)\} \leq \Phi(X) = f(X) \tag{12}$$

with equality holding only if $T(X) = X$ because the optimal solution point for the squared Euclidean Weber problem is unique. □

Corollary 1 *If $f(T(X)) = f(X)$, then $T(X) = X$.*

Proof $f(T(X)) \leq f(X)$ by Theorem 1. If $T(X) \neq X$, then by Theorem 1 $f(T(X)) < f(X)$. Therefore, $T(X) = X$. □

Let $X_1, X_2, \ldots$ be the sequence of the GWA iterations ($X_{k+1} = T(X_k)$).

Lemma 5 *The limit* $\lim_{k\to\infty}\{f(X_k)\}$ *exists.*

Proof The limit exists as a monotonically non-increasing sequence bounded from below by the global optimum. □

Corollary 2 $\lim_{k\to\infty}\{f(X_{k+1}) - f(X_k)\} = 0$.

Lemma 6 *The series $X_1, X_2, \ldots$ are in the convex hull of the demand points.*

Proof Each iteration is a convex combination of the demand points. □

Lemma 7 *If $T(X) = X$, then X is a stationary point* (*which means that the gradient of $f(X)$ vanishes*).

Proof If $T(X) = X$, then the same X is used on both sides of (3) which means that (2) is fulfilled. □

Lemma 8 *The function $T(X)$ is continuous.*

Proof Follows directly from the assumption that $f_i(d)$ are twice differentiable. □

Theorem 2 *The sequence $X_1, X_2, \ldots$ converges to either a local minimum of $f(X)$, or a saddle point of $f(X)$ unless one of the points in the sequence is a local maximum.*

Proof The sequence $X_1, X_2, \ldots$ is in the convex hull of demand points which is a compact set. By the Bolzano-Weierstrass Theorem, there exists a subsequence $X_{k_1}, X_{k_2}, \ldots$ converging to a point $\overline{X}$. By Corollary 2 $f(T(\overline{X})) = f(\overline{X})$. By Corollary 1 $T(\overline{X}) = \overline{X}$. Since the function $T(X)$ is continuous by Lemma 8, if X_k is close to $\overline{X}$, then X_{k+1} is close to $T(\overline{X}) = \overline{X}$ and thus the sequence X_k converges to $\overline{X}$. By Lemma 7, $\overline{X}$ is a stationary point. However, the sequence $f(X_k)$ is monotonically decreasing (by Theorem 1), so it cannot be a local maximum (unless the very first iteration is at a local maximum). □

2.1 Comments

1. When the objective is to maximize $f(X)$, then it is equivalent to minimizing $-f(X)$ which means that (6) and (7) are reversed.
2. $f(X)$ can be a sum of various $f_i(X)$ as long as each of them satisfies (6) and (7).
3. This proof does not address the issue of starting or getting a GWA iteration on a non-optimal demand point (Chandrasekaran and Tamir 1989). However, such instances are very rare.
4. The case of converging to a saddle point cannot be eliminated if $f(X)$ does have saddle points. If the iterations start (or one of the iteration lands) at a saddle point, then $T(X) = X$, and this saddle point is the result of the algorithm.

2.2 Examples

Restriction (8) holds in most practical applications because the objective function is usually increasing with the distance. However, restriction (9) may not hold for some common applications. The following are examples for which both conditions hold and Theorem 2 can be applied.

- In the "regular" Weber problem $\phi_i(D) = w_i\sqrt{D}$ for $w_i \geq 0$, and (6) and (7) are fulfilled.
- $f_i(d) = w_i d^\lambda$ fulfills (6) and (7) for $0 < \lambda \leq 2$.
- When d^2 is replaced by $d^2 + \epsilon$ (Eyster et al. 1973; Drezner and Drezner 1997), then $\phi(D)$ is replaced by $\phi(D + \epsilon)$ and conditions (6) and (7) are fulfilled if they are fulfilled for $\phi(D)$.
- The maximization problem using $f_i(d_i) = \frac{b_i}{1+h_i d_i^\lambda}$ (Drezner 1994, 1995; Drezner and Drezner 2004) fulfills (6) and (7) for $0 < \lambda \leq 2$.
- In the Logit model (Drezner et al. 1998) the objective is maximization, and $f_i(d) = \frac{1}{1+e^{\psi(d)}}$. Condition (6) leads to $\psi(d)$ being monotonically increasing. Condition (7) leads to a messy expression involving the second derivative of $\psi(d)$.
- The obnoxious facility formulation by using $f_i(d_i) = \frac{w_i}{d_i^2}$ will actually converge to a maximum which is at a demand point and thus the GWA is not suitable for finding local minima.
- An objective based on $f_i(d) = w_i e^{-\alpha_i d_i^\lambda}$ finds local maxima by the GWA for $0 < \lambda \leq 2$. Conversely, an objective based on $f_i(d) = w_i[1 - e^{-\alpha_i d_i^\lambda}]$ finds local minima by the GWA for $0 < \lambda \leq 2$.
- In Drezner et al. (2003) the GWA was proposed for a minimization problem where

  $$f_i(d_i) = \alpha_i d_i + w_i\sqrt{A_i d_i^2 + B_i d_i + C_i}$$

  where all coefficients are nonnegative, and $B_i^2 \geq 4A_iC_i$. It is shown there that $f_i(d_i)$ is monotonically increasing and concave. By the proof presented here, the GWA converges to a local minimum.

2.3 Relaxing the restrictions

The restrictions imposed by (6) and (7) may be relaxed and a convergence proof can still be constructed.

A simple relaxation of (8) and (9) can be obtained if the starting solution is in the convex hull of the demand points (such as the center of gravity). In this case, once X is in the convex hull, the distance cannot exceed the diameter of the convex hull and thus conditions (8) and (9) are not required for distances greater than this diameter. If (8) holds, all subsequent iterations will be in the convex hull. A maximum required distance can be calculated for each demand point and (8) is required only up to that distance. For example, consider the minimization problem based on (the index i removed for convenience) $f(d) = \alpha + \beta d + \gamma d^2 + \delta d^3$ with a diameter R of the convex hull of demand points. Equation (8) is

$$\beta + 2\gamma d + 3\delta d^2 \geq 0 \tag{13}$$

Equation (9) leads to

$$3\delta d^2 \leq \beta \tag{14}$$

Equation (13) for $d = 0$ yields $\beta \geq 0$ (and if $\beta = 0$ we must have $\gamma \geq 0$). In addition, to verify that it holds for $0 \leq d \leq R$, we require that both roots (if there are any roots) of the quadratic equation (13) are not in the interior of this segment. The second condition leads to

$$\delta \leq \frac{\beta}{3R^2}$$

There are two main results that must hold for the proof: (i) Theorem 1 and (ii) Lemma 6 (or any proof that the GWA iterations are in a compact set).

Condition (6) or (8) is required for the proofs of Lemmas 4 and 6. It is not necessary for the proof of Theorem 1. Careful assessment of the proofs reveals that replacing (6) by the conditions for any values $K, L > 0$:

$$\sum_{i=1}^{n} \frac{1}{d_i} \frac{\partial f_i(d_i)}{\partial d_i} \geq K \tag{15}$$

$$\frac{1}{d_i} \frac{\partial f_i(d_i)}{\partial d_i} \geq -L \quad \forall i \tag{16}$$

for any possible values of d_i's suffices for the proofs. These conditions confirm that the sequence of the GWA iterations is in a bounded set. Also, (15) is necessary for the proof of Lemma 4.

Restriction (7) or (9) is needed *only* for the proof of Theorem 1. An equivalent proof can be constructed when the "majority" of the functions are concave and the sum of the second derivatives (each multiplied by the appropriate square of the differences), must be negative for all possible values of D_i.

It is definitely easier to check whether (6) and (7) are fulfilled rather than establishing the more relaxed conditions which may yield a convergence proof when (6) or (7) are not satisfied.

3 On a demand point being a local minimum

Let $A_i = (x_{1i}, x_{2i}, \ldots, x_{mi})$. In the regular Weber problem, demand point k is a local minimum if and only if (Love et al. 1988):

$$w_k^2 \geq \sum_{l=1}^{m} \left(\sum_{i \neq k} w_i \frac{x_{lk} - x_{li}}{d_{ik}} \right)^2 \tag{17}$$

We develop here a similar condition for a demand point being a local minimum of the generalized Weber problem (1). Consider demand point k and a small sphere of radius ϵ centered at demand point k. A point on the circumference of the sphere is $X(\epsilon,\theta) = X_k + \epsilon\theta$ where $\theta = \{\theta_l\}$ is a vector of length m with the condition $\sum_{l=1}^m \theta_j^2 = 1$. By substitution in (1) and using first order approximation:

$$\begin{aligned} &f(X_k+\epsilon\theta) - f(X_k) \\ &\quad = f_k(\epsilon) - f_k(0) + \sum_{i\neq k}\left\{ f_i\left(d_{ik} + \epsilon\sum_{l=1}^m \frac{x_{li}-x_{lk}}{d_{ik}}\theta_l\right) - f_i(d_{ik})\right\} + o(\epsilon^2) \end{aligned}$$

The first term ($+o(\epsilon^2)$) represents the difference for the function $f_k(X)$ because the distance between X_k and $X_k+\epsilon\theta$ is ϵ. The second term ($+o(\epsilon^2)$) represents the difference between the two values of the rest of the functions in the sum excluding term k.

$$\begin{aligned} &\frac{f(X_k+\epsilon\theta) - f(X_k)}{\epsilon} \\ &\quad = \left(\frac{\partial f_k(0)}{\partial d} + o(\epsilon)\right) + \left(\sum_{i\neq k}\frac{\partial f_i(d_{ik})}{\partial d}\sum_{l=1}^m\left\{\frac{x_{li}-x_{lk}}{d_{ik}}\theta_l\right\} + o(\epsilon)\right) + o(\epsilon) \\ &\quad = \frac{\partial f_k(0)}{\partial d} + \sum_{l=1}^m\left\{\sum_{i\neq k}\frac{\partial f_i(d_{ik})}{\partial d}\frac{x_{li}-x_{lk}}{d_{ik}}\right\}\theta_l + o(\epsilon) \end{aligned} \tag{18}$$

Lemma 9 *For any value of* θ:

$$-\sqrt{\sum_{l=1}^m \lambda_l^2} \le \sum_{l=1}^m \lambda_l\theta_l \le \sqrt{\sum_{l=1}^m \lambda_l^2}$$

and there exists a vector θ *such that the right hand side of the inequality is an equality. The vector* $-\theta$ *yields an equality for the left hand side of the inequality.*

Proof By the Cauchy-Schwarz inequality:

$$\left[\sum_{l=1}^m \lambda_l\theta_l\right]^2 \le \sum_{l=1}^m \lambda_l^2 \sum_{l=1}^m \theta_l^2$$

and the Lemma follows because $\sum_{l=1}^m \theta_l^2 = 1$ by the definition of θ. The equality is obtained for $\theta_l = \pm\frac{\lambda_l}{\sqrt{\sum_{s=1}^m \lambda_s^2}}$. □

Suppose that $\frac{\partial f_k(0)}{\partial d} > 0$ (a similar condition for a local maximum exists for a negative derivative).

Theorem 3 *For* $\frac{\partial f_k(0)}{\partial d} > 0$:

(a) *If* $[\frac{\partial f_k(0)}{\partial d}]^2 > \sum_{l=1}^m [\sum_{i\neq k}\frac{\partial f_i(d_{ik})}{\partial d}\{\frac{x_{li}-x_{lk}}{d_{ik}}\}]^2$, *then demand point* k *is a local minimum.*
(b) *If* $[\frac{\partial f_k(0)}{\partial d}]^2 < \sum_{l=1}^m [\sum_{i\neq k}\frac{\partial f_i(d_{ik})}{\partial d}\{\frac{x_i-x_k}{d_{ik}}\}]^2$ *then* X_k *is not a local minimum.*

Proof

(a) By Lemma 9 (substituting $\lambda_l = \sum_{i\neq k} \frac{\partial f_i(d_{ik})}{\partial d} \frac{x_{li}-x_{lk}}{d_{ik}}$) and condition (a) of the Theorem 3, we get by (18) that for a small enough ϵ, $f(X(\epsilon,\theta)) > f(X_k)$ which proves that X_k is a local minimum.

(b) If $[\frac{\partial f_k(0)}{\partial d}]^2 < \sum_{l=1}^{m}[\sum_{i\neq k} \frac{\partial f_i(d_{ik})}{\partial d}\{\frac{x_i-x_k}{d_{ik}}\}]^2$, then X_k is not a local minimum because there exists a small ϵ and a vector θ (that fulfills the left inequality in Lemma 9 as an equality) such that $f(X(\epsilon,\theta)) < f(X_k)$ for this ϵ and all smaller ϵ's. □

These conditions reduce to (17) for $f_i(d) = w_i d$. The case of equality cannot be determined for general functions. It can be proven that when equality holds, the demand point is either a local minimum or a saddle point because the directional derivatives are positive except in one direction for which they vanish. Therefore, it can be proven that, in case of equality, the demand point is a local minimum (and thus it is the global minimum) for convex functions because convex functions do not have saddle points.

4 An example

Consider the minimization problem based on $f_i(d) = w_i d^\lambda$ for $\lambda > 0$. As an example consider $n = 2$ in $\mathcal{R}^1$, $w_i = 1$ and the two points at $A_1 = 0$ and $A_2 = 1$.

$$f(X) = |X|^\lambda + |1 - X|^\lambda$$

For $\lambda \geq 1$ the problem is convex leading to an optimal solution at $X = 0.5$. For $0 < \lambda < 1$ the minimum is obtained for $X = 0$ or $X = 1$. The GWA iterations ($0 \leq X \leq 1$ from the second iteration) are

$$X_{k+1} = \frac{1}{1 + |\frac{X_k}{1-X_k}|^{\lambda-2}}$$

For $0 < \lambda \leq 2$ convergence is to the local minimum (which is the global minimum for this particular problem). For $\lambda > 2$, if $X_k < 0.5$, then $X_{k+1} > 0.5$ and vice versa, i.e., the GWA iterations oscillate around the global optimum of $X = 0.5$. For $2 < \lambda < 3$ the iterations converge to 0.5. $\lambda = 3$ is an interesting case. The GWA iterations are $X_{k+1} = 1 - X_k$, and there is no convergence. For $\lambda > 3$, the iterations get further and further from 0.5, and are eventually oscillating between the neighborhood of $X = 0$ and the neighborhood of $X = 1$. A subsequence (odd or even terms) converges to $X = 0$ and the other subsequence converges to $X = 1$. In conclusion, for $\lambda \geq 3$ the GWA iterations may not converge; for $0 < \lambda \leq 2$, the GWA iterations converge to a local minimum (or Saddle point), by Theorem 2. The case $2 < \lambda < 3$ requires further investigation.

We coded a Fortran program and ran the GWA for randomly generated problems in the plane (points in a unit square, w_i uniformly generated in (0,1), and $\lambda = 3, 3.5$, and 4). If the solution failed to be within 10^{-5} of the previous iteration in 10,000 iterations, we consider the procedure non-convergent. Each n was run 10 times. The results are summarized in Table 1. Examining Table 1 reveals that, in our experiments, GWA diverges for problems with $\lambda = 4$. The GWA also diverges for problems with only two demand points for $\lambda \geq 3$ (as in the example). In conclusion, when one is interested in a minimization of a Weber problem based on the distance to the fourth power, then GWA will probably fail even though the problem is convex and there is only one local (which is the global) minimum. A different approach needs to be designed for such problems.

Table 1 GWA Results

n	# Diverged			# Converged			Ave. iterations		
	$\lambda = 3$	$\lambda = 3.5$	$\lambda = 4$	$\lambda = 3$	$\lambda = 3.5$	$\lambda = 4$	$\lambda = 3$	$\lambda = 3.5$	$\lambda = 4$
2	10	10	10	0	0	0	–	–	–
3	1	8	10	9	2	0	182	75	–
5	0	9	10	10	1	0	109	52	–
10	0	4	10	10	6	0	27	120	–
100	0	0	10	10	10	0	18	59	–
1000	0	0	10	10	10	0	16	40	–
10000	0	0	10	10	10	0	16	37	–

5 Summary

The Weiszfeld algorithm is widely used because of its simplicity and effectiveness. The algorithm can be generalized to other location problems where the cost is a function of the Euclidean distance rather than just being proportional to the distance. In many papers convergence of the algorithm is claimed empirically. It is useful to know whether the algorithm is guaranteed to converge.

We proved the convergence of the generalized Weiszfeld algorithm under certain conditions (which are sufficient but not necessary).

1. The cost function of the distance is twice differentiable (except, possibly for $d = 0$).
2. The cost function of the distance is a non-decreasing function of the distance.
3. A condition on the second derivative (9) which holds for all concave functions.

The first two conditions are very common in applications. The third condition may be restrictive. If these conditions do not hold, one can attempt to prove the following results for a specific problem and thus obtain a complete proof. The proof is based on two main results:

1. The sequence of the Weiszfeld iterates is in a compact set (for example, in the convex hull of the demand points). The proof of this condition just requires the non-decreasing property of the function and thus holds for most practical applications.
2. The value of the objective function is non-increasing every iteration. The proof of this property is based, in this paper, on (9).

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