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The single-machine earliness-tardiness scheduling problem with due date assignment and resource-dependent processing times

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Abstract We study the earliness-tardiness scheduling problem on a single machine with due date assignment and controllable processing times. We analyze the problem with three different due date assignment methods and two different processing time functions. For each combination of these, we provide a polynomial-time algorithm to find the optimal job sequence, due date values and resource allocation minimizing an objective function which includes earliness, tardiness, due date assignment, makespan and total resource consumption costs.

Keywords Single-machine scheduling · Due date assignment · Controllable processing times · Resource allocation · Earliness-tardiness scheduling

Introduction

Meeting due dates has always been one of the most important objectives in scheduling and supply chain management. Customers demand that suppliers meet contracted delivery dates or face large penalties. For example, Slotnick and Sobel (2005) cite contracts from the aerospace industry, which may impose tardiness penalties as high as one million dollars per day on subcontractors for aircraft components. The widespread use of Just-in-Time systems in industry made the early delivery of products also undesirable. This led to the introduction of earliness penalties, which may reflect additional storage or insurance costs, or costs of product deterioration over time. While traditional scheduling models considered due dates as given by exogenous decisions (see Baker and Scudder 1990 for a survey), in an integrated

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system they are determined by taking into account the system's ability to meet the quoted delivery dates. In order to avoid earliness-tardiness penalties, including the possibility of losing customers, companies are under increasing pressure to quote attainable delivery dates. At the same time, promising delivery dates too far into the future may not be acceptable to the customer or may force a company to offer price discounts in order to retain the business. Thus there is an important tradeoff between assigning relatively short due dates to customer orders and avoiding tardiness penalties. This is why an increasingly large number of recent studies have viewed due date assignment as part of the scheduling process, and showed how the ability to control due dates can be a major factor in improving system performance. In many scheduling problems, the job processing times can also be controlled by changing the allocation of resources to the jobs, which may result in further efficiencies. In this paper, we study a single-machine scheduling problem with an integrated objective function which includes earliness-tardiness penalties, resource consumption costs and the costs of due date assignment and schedule duration.

Sequencing problems with controllable processing times have been studied extensively by researchers since 1980 (e.g., Vickson 1980; Van Wassenhove and Baker 1982; Janiak 1987; Daniels 1990; Alidaee and Ahmadian 1993; Hoogeveen and Woeginger 2002; Ng et al. 2003). A survey of results up to 1990 can be found in Nowicki and Zdrzalka (1990). A more recent survey was given by Shabtay and Steiner (2007). In most of the above-mentioned studies of scheduling with controllable processing times, researchers assumed that the job processing time is a bounded linear function of the amount of resource allocated to the processing of the job, i.e., the *resource consumption function* is of the form

$$p_j(u_j) = \overline{p}_j - a_j u_j, \quad j = 1, \ldots, n, \quad 0 \le u_j \le \overline{u}_j < \overline{p}_j / a_j, \tag{1}$$

where n is the number of non-preemptive jobs, p_j is the processing time of job j, u_j is the amount of resource allocated to job j, $\overline{p}_j$ is the non-compressed processing time for job j, $\overline{u}_j$ is the upper bound on the amount of resource that can be allocated to job j and a_j is the positive compression rate of job j. For many resource allocation problems in physical or economic systems, however, they do not use a linear resource consumption function, since it fails to reflect the *law of diminishing marginal returns*. This law states that productivity increases at a decreasing rate with the amount of resource employed. In order to model this, other studies on scheduling with resource allocation assumed that the job processing time is a convex decreasing function of the amount of resource allocated to the processing of the job (e.g., Monma et al. 1990; Shabtay 2004; Shabtay and Kaspi 2004). For a convex resource consumption function, we assume the following relationship between the job processing time and the resource allocated to the job:

$$p_j(u_j) = \left(\frac{w_j}{u_j} \right)^k, \tag{2}$$

where w_j is a positive parameter, which represents the *workload* of the processing operation for job j and k is a positive constant. This resource consumption function has been used extensively in continuous resource allocation theory (e.g., Monma et al. 1990; Scott and Jefferson 1995; Armstrong et al. 1995, 1997; Shabtay 2004 and Shabtay and Kaspi 2004). In fact, Monma et al. (1990) pointed out that $k = 1$ corresponds to many actual government and industrial operations and the $k = 0.5$ case arises from VLSI (very large scale integration) circuit design, where the product of the silicon area (resource) and the square of time spent equals a constant value (the workload) for an individual job.

The objective in the problem we study is to determine the job sequence $\pi^* \in \Pi$, the set of due dates $\mathbf{d}^* = (d_1^*, d_2^*, \ldots, d_n^*)$ and the resource allocation $\mathbf{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)$ which minimize a cost function that includes the costs of earliness, tardiness, due date assignment, makespan and resource consumption as given by the following equation:

$$Z(\pi, \mathbf{d}, \mathbf{u}) = \alpha \sum_{j=1}^{n} E_j + \beta \sum_{j=1}^{n} T_j + \gamma \sum_{j=1}^{n} d_j + \delta C_{\max} + \sum_{j=1}^{n} v_j u_j, \tag{3}$$

where Π is the set of all $n!$ permutations of the n jobs, C_j is the completion time of job j in π, $E_j = \max(0, d_j - C_j)$ is the earliness of job j, $T_j = \max(0, C_j - d_j)$ is the tardiness of job j, $C_{\max} = \max_{j=1,\ldots,n} C_j$ is the maximal completion time (makespan), α, β, γ and δ are nonnegative parameters representing the cost of one unit of earliness, tardiness, due date and operation time, respectively, and v_j is the cost of one unit of resource allocated to job j.

A large variety of due date assignment methods have been studied in the literature. Recent surveys of these were given by Gordon et al. (2002a, 2002b). In this paper, we study our problem with the three most frequent due date assignment methods:

- The *common* due date assignment method (usually referred to as CON) where all jobs are assigned the same due date, that is $d_j = d$ for $j = 1, \ldots, n$.
- The *slack* due date assignment method (usually referred to as SLK) where jobs are given an equal flow allowance that reflects equal waiting time (i.e., equal slacks), that is, $d_j = p_j + slk$ for $j = 1, \ldots, n$, where $slk \geq 0$ is a decision variable.
- The *unrestricted* due date assignment method where each job can be assigned a different due date with no restrictions. (We will refer to this method as DIF in short.)

We study optimal schedules under these three due date assignment methods with both types of resource consumption function (1, 2). For a given feasible resource allocation, which fixes the processing times and the makespan and resource consumption costs, our problem is reduced to finding a job sequence π and a set of due dates $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ which minimize $Z(\pi, \mathbf{d}) = \alpha \sum_{j=1}^{n} E_j + \beta \sum_{j=1}^{n} T_j + \gamma \sum_{j=1}^{n} d_j$. Seidmann et al. (1981) presented an $O(n \log n)$ optimization algorithm to solve this problem for the DIF due date assignment method. Panwalkar et al. (1982) and Adamopoulos and Pappis (1996) showed that this problem can also be solved in $O(n \log n)$ time for the CON and the SLK due date assignment methods, respectively.

There are several previous papers which combined due date assignment and continuous resource allocation decisions to minimize an objective which includes earliness and tardiness penalties (Panwalkar and Rajagopalan 1992; Alidaee and Ahmadian 1993; Cheng et al. 1996; Biskup and Jahnke 2001 and Ng et al. 2003). All of these papers assumed a linear resource consumption function as given by (1) and most of them dealt only with the CON method. These results are summarized in the first five rows of Table 1, and we briefly review them next. For the case when $a_j = 1$, $j = 1, \ldots, n$, Panwalkar and Rajagopalan (1992) presented an algorithm to find the job sequence, resource allocation and due date which minimize a cost function containing earliness, tardiness and resource consumption costs for the CON method. They proved that each job will be processed either with its non-compressed (maximal) or its most compressed (minimal) processing time, and reduced the problem to an assignment problem solvable in polynomial time ($O(n^3)$). Alidaee and Ahmadian (1993) extended Panwalkar and Rajagopalan's results to the case of identical parallel machines and solved the problem by reducing it to a transportation problem. Cheng et al. (1996) also extended Panwalkar and Rajagopalan's research by adding the due date cost to the objective and by also solving the problem for the case of slack due date assignments (SLK). For the

Table 1 Summary of results

Objective	Res. Cons. Fun.	Due Date Ass. St.	Complexity	Reference
(3) with $\gamma=\delta=0$	(1) with $a_j=1$	CON	$O(n^3)$	(Alidaee and Ahmadian 1993; Panwalkar and Rajagopalan 1992)
(3) with $\delta=0$	(1) with $a_j=1$	CON	$O(n^3)$	(Cheng et al. 1996)
(3) with $\delta=0$	(1) with $a_j=1$	SLK	$O(n^3)$	(Cheng et al. 1996)
(3) with $\delta=0$ and $u_j=u$	$p_j(u)=\overline{p}_j(1-u)$	CON	$O(n\log n)$	(Biskup and Jahnke 2001)
(3) with $\delta=0$ and $u_j=u$	(1) with $u_j=u$	CON	$O(n^2\log n)$	(Ng et al. 2003)
(3)	(1)	CON/SLK/DIF	$O(n^3)$	Theorem 1
(3)	(2)	CON/SLK/DIF	$O(n\log n)$	Theorem 2

CON method, Biskup and Jahnke (2001) studied the special case where the job processing times are jointly reducible by the *same proportional* amount, i.e., the case where $a_j=\overline{p}_j$ and $u_j=u$ for $j=1,\dots,n$. They presented $O(n\log n)$ optimization algorithms to minimize a cost function containing earliness, tardiness, resource consumption and due date assignment costs. Ng et al. (2003) extended Biskup and Jahnke's results to the case where the job processing times are jointly reducible by the *same amount* of the resource, i.e., for the case where $u_j=u$ for $j=1,\dots,n$, and presented $O(n^2\log n)$ optimization algorithms for the same objective.

In most classical single-machine due date assignment problems where job processing times are fixed, one of the main properties is that an optimal schedule does not have any idle time. As a result, the makespan is identical for all job permutations. An important issue when dealing with controllable processing times, however, is that the makespan is no longer a constant value, it is a function of the resource allocation strategy. Therefore, it is quite surprising that previous studies that combined resource allocation with due date assignment did not include the makespan cost in the objective function.

Our paper extends the existing research on due-date assignment with controllable job processing times in several aspects: Since minimizing the makespan is one of the most important scheduling objectives, we include the makespan cost in our cost function; For the linear resource consumption function, we study the problems with the general function given in (1), while previous studies dealt only with special cases of this function; We include a complete analysis of the three due date assignment methods with the convex resource consumption function given in (2); Our algorithms for both resource consumption functions represent a *unified* solution approach for the different due date assignment methods. In the last two rows of Table 1, we include a summary of our main results.

The rest of the paper is organized as follows. In Sect. 2, we present some relevant preliminary results for the reduced problems with fixed processing times. Section 3 presents an $O(n^3)$ optimization algorithm to solve the problem for each of the three due date assignment methods assuming a linear resource consumption function. For the convex resource consumption function, we show in Sect. 4 that the optimal schedule can be determined in $O(n\log n)$ time for all three due date assignment methods. Section 5 includes a numerical example, while the last section contains our concluding remarks.

1 Summary of preliminary results for related problems with fixed processing times

Any given feasible resource allocation **u** fixes the job processing times, the makespan and the resource consumption cost. Thus in this case, our problem is reduced to finding a job sequence π and a set of due dates $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ which minimize $Z(\pi, \mathbf{d}) = \alpha \sum_{j=1}^{n} E_j + \beta \sum_{j=1}^{n} T_j + \gamma \sum_{j=1}^{n} d_j$ for the three different due date assignment methods. In the following we present some earlier results obtained by Panwalkar et al. (1982), Seidmann et al. (1981) and Adamopoulos and Pappis (1996) for the CON, DIF and SLK due date assignment methods, respectively. The following lemma is applicable for all three due date assignment methods.

Lemma 1 *The optimal schedule does not include idle times with any of the three due date assignment strategies.*

As a result, we can conclude from Lemma 1 that

$$C_{[j]} = \sum_{i=1}^{j} p_{[i]}, \tag{4}$$

where $[j]$ represents the job that is in the jth position in schedule π for $j = 1, 2, \ldots, n$.

1.1 Preliminary results for the CON due date assignment method

The following results are given by Panwalkar et al. (1982) for the CON due date assignment method.

Lemma 2 *For the CON due date assignment method, there exists an optimal due-date equal to* $C_{[l^*]}$, *where*

$$l^* = \min\left(\max\left(\left\lceil \frac{n \times (\beta - \gamma)}{\alpha + \beta} \right\rceil, 0\right), n\right), \tag{5}$$

and $C_0 = 0$ *by definition.*

Note that the value of l^*, given by (5), is independent of the job processing times and the job sequence. Therefore, it is optimal for *any* job sequence and processing times. Thus for any **u** and π, the following holds:

$$d^*_{[j]} = d^* = C_{[l^*]} = \sum_{j=1}^{l^*} p_{[j]} \quad \text{for } j = 1, \ldots, n; \tag{6}$$

$$E_{[j]} = \sum_{i=j+1}^{l^*} p_{[i]}, \quad \text{for } j = 1, \ldots, l^* - 1 \quad \text{and} \quad E_{l^*} = 0; \tag{7}$$

$$T_{[j]} = \sum_{i=l^*+1}^{j} p_{[i]}, \quad \text{for } j = l^* + 1, \ldots, n. \tag{8}$$

By substituting (6–8) into (3), we get a new expression for our objective function under an optimal due date assignment strategy, denoted by $\mathbf{d}^*(\pi, \mathbf{u})$, for the CON due date assignment method:

$$Z(\pi, \mathbf{u}, \mathbf{d}^*(\pi, \mathbf{u})) = \sum_{j=1}^{l^*} (\alpha(j-1) + \gamma n + \delta) \times p_{[j]} + \sum_{j=l^*+1}^{n} (\beta(n-j+1) + \delta) \times p_{[j]} + \sum_{j=1}^{n} v_{[j]} u_{[j]}. \quad (9)$$

1.2 Preliminary results for the SLK due date assignment method

Adamopoulos and Pappis (1996) showed that the CON and the SLK methods have similar properties and presented the following result for the SLK due date assignment method.

Lemma 3 *For the SLK due date assignment method, there exists an optimal slack allowance, slk^*, equal to $C_{[l^*-1]}$, where l^* is given by* (5).

As a result, the following holds for any $\mathbf{u}$ and π:

$$slk^* = C_{[l^*-1]} = \sum_{i=1}^{l^*-1} p_{[i]}; \quad (10)$$

$$d^*_{[j]} = p_{[j]} + slk^* = p_{[j]} + \sum_{i=1}^{l^*-1} p_{[i]} \quad \text{for } j = 1, \dots, n; \quad (11)$$

$$E_{[j]} = \sum_{i=j}^{l^*-1} p_{[i]}, \quad \text{for } j = 1, \dots, l^* - 1 \quad \text{and} \quad E_{[l^*]} = 0; \quad (12)$$

$$T_{[j]} = \sum_{i=l^*}^{j-1} p_{[i]}, \quad \text{for } j = l^* + 1, \dots, n. \quad (13)$$

By substituting (10–13) into (3) we get a new expression for our objective function under an optimal due date assignment strategy for the SLK due date assignment method:

$$Z(\pi, \mathbf{u}, \mathbf{d}^*(\pi, \mathbf{u})) = \sum_{j=1}^{l^*-1} (\alpha j + \gamma(n+1) + \delta) \times p_{[j]} + \sum_{j=l^*}^{n} (\beta(n-j) + \gamma + \delta) \times p_{[j]} + \sum_{j=1}^{n} v_{[j]} u_{[j]}. \quad (14)$$

As we can see, there is a lot of similarity between the CON and SLK methods.

1.3 Preliminary results for the DIF due date assignment method

The DIF due date assignment method to minimize earliness, tardiness and due date assignment costs was studied by Seidmann et al. (1981) and they presented the following lemma which defines the optimal due date assignment strategy for a given π. This lemma is also applicable for our problem with a given π and fixed $\mathbf{u}$ and processing times.

Lemma 4 *For a given π and fixed processing times, the optimal due date assignment strategy for the DIF due date assignment method is defined as follows: if $\gamma \geq \beta$ then set $d_j = 0$, otherwise set $d_j = C_j$ for $j = 1, \ldots, n$.*

From Lemma 4, we can conclude that $E_j = 0$ for $j = 1, \ldots, n$. Therefore, with an optimal due date assignment strategy, our objective function becomes

$$Z(\pi, \mathbf{u}, \mathbf{d}^*(\pi, \mathbf{u})) = \begin{cases} \beta \sum_{j=1}^n C_j + \delta C_{\max} + \sum_{j=1}^n v_j u_j & \text{if } \gamma \geq \beta, \\ \gamma \sum_{j=1}^n C_j + \delta C_{\max} + \sum_{j=1}^n v_j u_j & \text{if } \gamma < \beta. \end{cases}$$

By using (4) and $C_{\max} = \sum_{j=1}^n p_{[j]}$, this can be further written as

$$Z(\pi, \mathbf{u}, \mathbf{d}^*(\pi, \mathbf{u})) = \sum_{j=1}^{n} (\epsilon(n - j + 1) + \delta) \times p_{[j]} + \sum_{j=1}^{n} v_{[j]} u_{[j]}, \tag{15}$$

where $\epsilon = \min(\beta, \gamma)$.

2 Solution with a linear resource consumption function

In this section, we will show that with a linear resource consumption function the optimal schedule for all three due date assignment methods can be obtained in $O(n^3)$ time. By solving (1) for each $u_{[j]}$ and substituting these into (9), (14) and (15), we obtain the following objective functions under an optimal due date assignment for the CON, SLK and DIF due date assignment methods, respectively.

$$\begin{aligned} Z(\pi, \mathbf{u}, \mathbf{d}^*(\pi, \mathbf{u})) &= \sum_{j=1}^{l^*} (\alpha(j-1) + \gamma n + \delta - v_{[j]}/a_{[j]}) \times p_{[j]} \\ &\quad + \sum_{j=l^*+1}^{n} (\beta(n - j + 1) + \delta - v_{[j]}/a_{[j]}) \times p_{[j]} + \sum_{j=1}^{n} \frac{v_{[j]}}{a_{[j]}} \overline{p}_{[j]}; \end{aligned} \tag{16}$$

$$\begin{aligned} Z(\pi, \mathbf{u}, \mathbf{d}^*(\pi, \mathbf{u})) &= \sum_{j=1}^{l^*-1} (\alpha j + \gamma(n+1) + \delta - v_{[j]}/a_{[j]}) \times p_{[j]} \\ &\quad + \sum_{j=l^*}^{n} (\beta(n - j) + \gamma + \delta - v_{[j]}/a_{[j]}) \times p_{[j]} + \sum_{j=1}^{n} \frac{v_{[j]}}{a_{[j]}} \overline{p}_{[j]}; \end{aligned} \tag{17}$$

$$Z(\pi, \mathbf{u}, \mathbf{d}^*(\pi, \mathbf{u})) = \sum_{j=1}^{n} (\epsilon(n - j + 1) + \delta - v_{[j]}/a_{[j]}) \times p_{[j]} + \sum_{j=1}^{n} \frac{v_{[j]}}{a_{[j]}} \overline{p}_{[j]}. \tag{18}$$

Note that the value $\sum_{j=1}^n \frac{v_{[j]}}{a_{[j]}} \overline{p}_{[j]}$ is a constant for any sequence of the jobs.

Let us define $\varpi_{[j]}$ for the three due date assignment methods as follows.

$$\varpi_{[j]} = \begin{cases} \alpha \times (j - 1) + \gamma n + \delta - v_{[j]}/a_{[j]} & \text{for } j = 1, \ldots, l^*; \\ \beta \times (n - j + 1) + \delta - v_{[j]}/a_{[j]} & \text{for } j = l^* + 1, \ldots, n, \end{cases} \tag{19}$$

for the CON method;

$$\varpi_{[j]} = \begin{cases} \alpha j + \gamma(n+1) + \delta - v_{[j]}/a_{[j]} & \text{for } j = 1, \ldots, l^* - 1; \\ \beta \times (n-j) + \gamma + \delta - v_{[j]}/a_{[j]} & \text{for } j = l^*, \ldots, n, \end{cases} \tag{20}$$

for the SLK method; and

$$\varpi_{[j]} = \epsilon(n - j + 1) + \delta - v_{[j]}/a_{[j]} \quad \text{for } j = 1, \ldots, n, \tag{21}$$

for the DIF method; i.e., $\varpi_{[j]}$, $j = 1, \ldots, n$, represents the cost of job $[j]$ in the objective function. In other words, $\varpi_{[j]}$ could be viewed as the *positional* penalty for the job in the jth position. In the following lemma, we present the optimal resource allocation as a function of the job sequence for all three due date assignment methods.

Lemma 5 *Expressed as a function of the job sequence, the optimal resource allocation,* $\mathbf{u}^*(\pi)$, *for all three due date assignment methods can be determined as follows*:

$$u^*_{[j]} = \begin{cases} 0 & \textit{if } \varpi_{[j]} < 0; \\ u_{[j]} \in [0, \overline{u}_{[j]}] & \textit{if } \varpi_{[j]} = 0; \\ \overline{u}_{[j]} & \textit{if } \varpi_{[j]} > 0 \end{cases} \quad \textit{for } j = 1, \ldots, n. \tag{22}$$

Proof Substituting (1) for $p_{[j]}$ into (16–18) for the three due date assignment methods and taking the derivative by $u_{[j]}$, we see that $\frac{dZ(\pi,\mathbf{u},\mathbf{d}^*(\pi,\mathbf{u}))}{du_{[j]}} = -a_{[j]} \times \varpi_{[j]}$ for $j = 1, \ldots, n$. Since $a_{[j]}$ is a positive parameter, we can conclude that the sign of $\frac{dZ(\pi,\mathbf{u},\mathbf{d}^*(\pi,\mathbf{u}))}{du_{[j]}}$ must be the opposite of the sign of $\varpi_{[j]}$ in order to minimize Z. Therefore, if $\varpi_{[j]} < 0$, we should not allocate any resource to job $[j]$; if $\varpi_{[j]} > 0$, we will allocate the maximal feasible amount of resource to job $[j]$; and if $\varpi_{[j]} = 0$, any feasible resource allocation can be optimal. □

Lemma 6 *The optimal sequence, for all three due date assignment methods, can be determined by solving a linear assignment problem requiring* $O(n^3)$ *time.*

Proof For $1 \leq i, j \leq n$, let us define

$$\varpi_{ij} = \begin{cases} \alpha \times (i-1) + \gamma n + \delta - v_j/a_j & \text{for } i = 1, \ldots, l^*; \\ \beta \times (n-i+1) + \delta - v_j/a_j & \text{for } i = l^* + 1, \ldots, n, \end{cases} \tag{23}$$

for the CON method;

$$\varpi_{ij} = \begin{cases} \alpha i + \gamma(n+1) + \delta - v_j/a_j & \text{for } j = 1, \ldots, l^* - 1; \\ \beta \times (n-i) + \gamma + \delta - v_j/a_j & \text{for } j = l^*, \ldots, n, \end{cases} \tag{24}$$

for the SLK method;

$$\varpi_{ij} = \epsilon(n - i + 1) + \delta - v_j/a_j, \tag{25}$$

for the DIF method. It is clear that ϖ_{ij} represents the positional penalty if job j is assigned to position i in the schedule.

Using (22), we obtain that the optimal length of job j if it is scheduled in position i is as follows:

$$p_{ij} = \begin{cases} \overline{p}_j & \text{if } \varpi_{ij} < 0; \\ p_j \in [\overline{p}_j - a_j \times \overline{u}_j, \overline{p}_j] & \text{if } \varpi_{ij} = 0; \\ \overline{p}_j - a_j \times \overline{u}_j & \text{if } \varpi_{ij} > 0. \end{cases} \tag{26}$$

Therefore, if we define the value c_{ij} by

$$c_{ij} = \varpi_{ij} \times p_{ij} = \begin{cases} \varpi_{ij}\overline{p}_j & \text{if } \varpi_{ij} < 0; \\ 0 & \text{if } \varpi_{ij} = 0; \\ \varpi_{ij}(\overline{p}_j - a_j \times \overline{u}_j) & \text{if } \varpi_{ij} > 0, \end{cases} \tag{27}$$

it represents the minimum possible cost resulting from assigning job j to position i in the sequence. Let us also define $x_{ij} = 1$ if job j is assigned to position i and $x_{ij} = 0$ otherwise. Our sequencing problem then can be formulated as the following linear assignment problem:

$$\begin{aligned} \text{(P1)} \quad & \min \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \\ & \text{s.t.} \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, \ldots, n, \\ & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, n, \\ & x_{ij} = 0 \text{ or } 1, \quad i, j = 1, \ldots, n. \end{aligned}$$

The first set of constraints in the formulation assures that each job will be assigned only to one position, the second set assures that each position will be assigned only once, and the penalty for each assignment under an optimal resource allocation appears in the objective. It is well known that a linear assignment problem can be solved in $O(n^3)$ time (see Papadimitriou and Steiglitz 1982). □

The results of our analysis are summarized in the following optimization algorithm that solves our problem, for all three due date assignment methods, with a linear resource consumption.

Algorithm 1 (*The optimization algorithm for all three due date assignment methods, assuming a linear resource consumption function*).

Step 1. (*Apply this step only for the CON and the SLK methods*) Calculate l^* by (5).
Step 2. Calculate the c_{ij} values for each of the three due date assignment methods by (23–27).
Step 3. For each due date assignment method, solve the assignment problem (P1) to determine the optimal job sequence, and for each method denote the resulting optimal sequence by $\pi^* = [1], [2], \ldots, [n]$.
Step 4. For each due date assignment method, allocate the resources according to (22) and determine the optimal job processing times, $p_j^*(u_j^*)$, by (1).
Step 5. For the CON method, assign the due date according to (6), and for the SLK method, assign the due dates according to (10, 11) with $p_j = p_j^*(u_j^*)$. For the DIF method, assign

the due dates according to Lemma 4 with $\pi = \pi^*$ and the job completion times calculated according to (4) with $p_j = p_j^*(u_j^*)$.

Theorem 1 *Algorithm 1 solves the earliness-tardiness scheduling problem for each of the three due date assignment methods and linear resource consumption function in $O(n^3)$ time.*

Proof The correctness of the algorithm follows from Lemmas 1–6. Step 1 takes constant time; Steps 4 and 5 can be performed in linear time; Step 2 requires $O(n^2)$ and Step 3 $O(n^3)$ time. Thus the overall computational complexity of the algorithm is $O(n^3)$, which is equal to the computational complexity of Step 3. □

Remark 1 The single-machine *weighted* earliness-tardiness due date assignment problem with non-controllable processing times, i.e., the problem of minimizing $\sum_{j=1}^{n} \alpha_j E_j + \sum_{j=1}^{n} \beta_j T_j + \sum_{j=1}^{n} \gamma_j d_j$ is known to be $\mathcal{NP}$-hard for the CON and the SLK due date assignment method (see Hall and Posner 1991 and Gupta et al. 1990) even if $\alpha_j = \beta_j$ and $\gamma_j = 0$ for $j = 1, \ldots, n$. Therefore, the problem is also $\mathcal{NP}$-hard for the case when the processing times are controllable by a linear resource consumption function. For the DIF due date assignment method, Shabtay and Steiner (2006) showed that minimizing $\sum_{j=1}^{n} \alpha_j E_j + \sum_{j=1}^{n} \beta_j T_j + \sum_{j=1}^{n} \gamma_j d_j$ is equivalent to a weighted completion time problem on a single machine, which is known to be $\mathcal{NP}$-hard with a linear resource consumption function (Wan et al. 2001). Thus the weighted versions of our problem with a linear resource consumption function are $\mathcal{NP}$-hard for all three due date assignment methods.

3 The solution with a convex resource consumption function

By substituting (2) into (9), (14) and (15), we obtain the following expressions for the objective functions under an optimal due date assignment strategy for the three due date assignment methods:

$$Z(\pi, \mathbf{u}, \mathbf{d}^*(\pi, \mathbf{u})) = \sum_{j=1}^{l^*} (\alpha(j-1) + \gamma n + \delta) \times \left(\frac{w_{[j]}}{u_{[j]}}\right)^k + \sum_{j=l^*+1}^{n} (\beta(n-j+1) + \delta) \times \left(\frac{w_{[j]}}{u_{[j]}}\right)^k + \sum_{j=1}^{n} v_{[j]} u_{[j]} \tag{28}$$

for the CON method;

$$Z(\pi, \mathbf{u}, \mathbf{d}^*(\pi, \mathbf{u})) = \sum_{j=1}^{l^*-1} (\alpha j + \gamma(n+1) + \delta) \times \left(\frac{w_{[j]}}{u_{[j]}}\right)^k + \sum_{j=l^*}^{n} (\beta(n-j) + \gamma + \delta) \times \left(\frac{w_{[j]}}{u_{[j]}}\right)^k + \sum_{j=1}^{n} v_{[j]} u_{[j]} \tag{29}$$

for the SLK method; and

$$Z(\pi, \mathbf{u}, \mathbf{d}^*(\pi, \mathbf{u})) = \sum_{j=1}^{n} (\epsilon(n-j+1) + \delta) \times \left(\frac{w_{[j]}}{u_{[j]}}\right)^k + \sum_{j=1}^{n} v_{[j]} u_{[j]} \tag{30}$$

for the DIF method.

In the following lemma, we determine the optimal resource allocation, denoted by $\mathbf{u}^*(\pi)$, as a function of the job sequence for each of the three due date assignment methods.

Lemma 7 *The optimal resource allocation as a function of the job sequence*, $\mathbf{u}^*(\pi)$, *is*:

$$u^*_{[j]} = \begin{cases} \left(\frac{k\times(\alpha(j-1)+\gamma n+\delta)}{v_{[j]}}\right)^{\frac{1}{k+1}} \times w_{[j]}^{\frac{k}{k+1}}, & \textit{for } j = 1, \dots, l^*; \\ \left(\frac{k\times(\beta(n-j+1)+\delta)}{v_{[j]}}\right)^{\frac{1}{k+1}} \times w_{[j]}^{\frac{k}{k+1}}, & \textit{for } j = l^*+1, \dots, n, \end{cases} \tag{31}$$

for the CON method;

$$u^*_{[j]} = \begin{cases} \left(\frac{k\times(\alpha j+\gamma(n+1)+\delta)}{v_{[j]}}\right)^{\frac{1}{k+1}} \times w_{[j]}^{\frac{k}{k+1}}, & \textit{for } j = 1, \dots, l^*-1; \\ \left(\frac{k\times(\beta(n-j)+\gamma+\delta)}{v_{[j]}}\right)^{\frac{1}{k+1}} \times w_{[j]}^{\frac{k}{k+1}}, & \textit{for } j = l^*, \dots, n, \end{cases} \tag{32}$$

for the SLK method; *and*

$$u^*_{[j]} = \left(\frac{k\times(\epsilon(n-j+1)+\delta)}{v_{[j]}}\right)^{\frac{1}{k+1}} \times w_{[j]}^{\frac{k}{k+1}}, \quad \textit{for } j = 1, \dots, n, \tag{33}$$

for the DIF method.

Proof By taking the derivative of each of the objectives given by (28–30) with respect to $u_{[j]}$, $j = 1, \dots, n$, equating it to zero and solving it for $u^*_{[j]}$, we obtain (31–33). Since each of the objectives is a convex function, (31–33) provide necessary and sufficient conditions for optimality. □

By substituting (31) into (28), (32) into (29), and (33) into (30), we obtain a new *unified* expression for the cost function for the three due date assignment methods under an optimal resource allocation and due date assignment and as a function of the job sequence:

$$Z(\pi, \mathbf{u}^*(\pi), \mathbf{d}^*(\pi, \mathbf{u})) = \left(k^{\frac{-k}{k+1}} + k^{\frac{1}{k+1}}\right) \times \sum_{j=1}^{n} \theta_{[j]} \times \eta_j, \tag{34}$$

where

$$\theta_j = (w_j \times v_j)^{\frac{k}{k+1}}, \quad j = 1, \dots, n, \tag{35}$$

and

$$\eta_j = \begin{cases} (\alpha(j-1)+\gamma n+\delta)^{\frac{1}{k+1}}, & \text{for } j = 1, \dots, l^*; \\ (\beta(n-j+1)+\delta)^{\frac{1}{k+1}}, & \text{for } j = l^*+1, \dots, n, \end{cases} \tag{36}$$

for the CON method;

$$\eta_j = \begin{cases} (\alpha j+\gamma(n+1)+\delta)^{\frac{1}{k+1}}, & \text{for } j = 1, \dots, l^*-1; \\ (\beta(n-j)+\gamma+\delta)^{\frac{1}{k+1}}, & \text{for } j = l^*, \dots, n, \end{cases} \tag{37}$$

for the SLK method; and

$$\eta_j = (\epsilon(n-j+1)+\delta)^{\frac{1}{k+1}}, \quad \text{for } j = 1, \dots, n, \tag{38}$$

for the DIF method.

In order to find the job sequence that minimizes $Z(\pi, \mathbf{u}^*(\pi), \mathbf{d}^*(\pi, \mathbf{u}))$, we have to optimally match the *positional* penalties η_j with the *job-dependent* costs θ_j. The optimal matching is obtained by applying the following lemma.

Lemma 8 *The optimal job sequence is obtained by matching the smallest η_j value to the job with the largest θ_j value, the second smallest η_j value to the job with the second largest θ_j value, and so on. The index of the η matched with θ_j specifies the position of job j in the optimal sequence for $j = 1, \dots, n$.*

Proof This follows from a well-known result in linear algebra about the minimization of a scalar product of two vectors (see Hardy et al. 1934). □

The results of our analysis are summarized in the following optimization algorithm that solves our problem for all three due date assignment methods with the convex resource consumption given in (2).

Algorithm 2 (*The optimization algorithm for all three due date assignment methods with a convex resource consumption function*).

Step 1. (*Apply this step only for the CON and the SLK methods*) Calculate l^* by (5).
Step 2. For each due date assignment method, calculate θ_j and η_j for $j = 1, \dots, n$ by (35–38).
Step 3. Sequence the jobs according to Lemma 8, and for each method denote the resulting optimal sequence by $\pi^* = [1], [2], \dots, [n]$.
Step 4. Allocate the resources according to (31) for the CON method, according to (32) for the SLK method and according to (33) to the DIF method. For each method, determine the optimal job processing times, $p^*_{[j]}(u^*_{[j]})$ by (2).
Step 5. For the CON method, assign the due date according to (6) and for the SLK method assign the due dates according to (10, 11), where $p_{[j]} = p^*_{[j]}(u^*_{[j]})$. For the DIF method, assign the due dates according to Lemma 4 using $\pi = \pi^*$ and calculate the job completion times by (4) with $p_{[j]} = p^*_{[j]}(u^*_{[j]})$.

Theorem 2 *Algorithm 2 solves the earliness-tardiness scheduling problem for each of the three due date assignment methods and a convex resource consumption function in $O(n \log n)$ time.*

Proof The correctness of the algorithm follows from Lemmas 1–4, and Lemmas 7–8. Step 1 takes constant time; Steps 2, 4 and 5 can be performed in linear time and Step 3 requires $O(n \log n)$ time. Thus, the overall computational complexity of the algorithm is $O(n \log n)$, which is the computational complexity of Step 3. □

Table 2 The per unit resource consumption penalties

j	1	2	3	4	5
v_j	20	14	27	8	24

Table 3 The linear resource consumption parameters

j	1	2	3	4	5
a_j	2	1	3	1	4
$\overline{p}_j$	12	10	14	7	20
$\overline{u}_j$	4	6	3	5	3

Table 4 The ϖ_{ij} values

$\varpi_{ij} =$

$i\backslash j$	1	2	3	4	5
1	−2	−6	−1	0	2
2	−1	−5	0	1	3
3	0	−4	1	2	4
4	−1	−5	0	1	3
5	−3	−7	−2	−1	1

Table 5 The p_{ij} values

$p_{ij} =$

$i\backslash j$	1	2	3	4	5
1	12	10	14	2	8
2	12	10	5	2	8
3	4	10	5	2	8
4	12	10	5	2	8
5	12	10	14	7	8

4 Numerical example

The following numerical example with $n = 5$, $\alpha = 1$, $\beta = 2$, $\gamma = 0.6$, $\delta = 5$ and per unit resource cost as given in Table 2 is used to illustrate the optimization algorithms.

Since the algorithm is similar for the three due date assignment methods, in the numerical example we will apply the algorithms only for the CON due date assignment method.

4.1 The linear resource consumption function

In this subsection, we assume that the processing time of each job follows the linear resource consumption function given in (1). In Table 3, we specify the resource consumption function parameters for each job.

We apply Algorithm 1 to solve the problem for the CON due date assignment strategy.

Step 1. $l^* = \min(\max(\lceil \frac{5\times(2-0.6)}{1+2} \rceil, 0), 5) = 3$.

Step 2. The ϖ_{ij} values were calculated according to (23) and are given in Table 4. The p_{ij} values were calculated according to (26) and are given in Table 5 (when $\varpi_{ij} = 0$, we arbitrarily set $p_{ij} = \overline{p}_j - a_j\overline{u}_j$). The resulting c_{ij} values, which were calculated according to (27), are given in Table 6.

Table 6 Table c_{ij} values

$c_{ij} =$

$i\backslash j$	1	2	3	4	5
1	**−24**	−60	−14	0	16
2	−12	**−50**	0	2	24
3	0	−40	5	**4**	32
4	−12	−50	0	2	**24**
5	−36	−70	**−28**	−7	8

Table 7 The optimal resource allocations and job processing times

j	1	2	3	4	5
$[j]$	1	2	4	5	3
u_j^*	0	0	0	5	3
p_j^*	12	10	14	2	8

Table 8 The job workloads

j	1	2	3	4	5
w_j	20	30	50	60	70

Table 9 The θ_j and η_j values

j	1	2	3	4	5
θ_j	20	20.494	36.742	21.909	40.988
η_j	2.828	3	3.162	3	2.645

Step 3. The costs for the solution of the assignment problem are highlighted in bold in Table 6 and the optimal job sequence is $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 3$.

Step 4. The optimal resource allocations (obtained by (22)) and the compressed processing times appear in Table 7.

Step 5. $d^* = C_{[l^*]} = C_4 = 24$.

4.2 The convex resource consumption function

In this subsection, we assume that the processing time of each job follows the convex resource consumption function given in (2). In Table 8, we specify the *workload* of each job and we also assume that $k = 1$.

We apply Algorithm 2 to solve the problem for the CON due date assignment strategy.

Step 1. $l^* = \min(\max(\lceil \frac{5\times(2-0.6)}{1+2} \rceil, 0), 5) = 3$.

Step 2. The θ_j and η_j values obtained by (35, 36) are given in Table 9.

Step 3. The optimal job sequence is $3 \rightarrow 2 \rightarrow 1 \rightarrow 4 \rightarrow 5$.

Step 4. The optimal resource allocations (obtained by (31)) and processing times (calculated by (2)) appear in Table 10.

Step 5. $d^* = C_{[l^*]} = C_1 = 26.144$.

Table 10 The optimal resource allocations and job processing times

j	1	2	3	4	5
$[j]$	3	2	1	4	5
u_j^*	3.162	4.392	3.849	8.216	4.518
p_j^*	13.000	6.826	6.328	7.302	15.496

5 Concluding remarks and future research

We have studied the single-machine earliness-tardiness scheduling problem where both the job processing times and the due dates are decision variables to be determined by the scheduler. We have presented unified polynomial-time algorithms under three of the most frequently used due date assignment methods for linear and convex resource consumption functions.

Future extensions of this research may include different objective functions (such as studying the weighted version of our problem with a convex resource consumption function), additional due date assignment strategies and different resource consumption functions.

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