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# Modeling mixed push and pull promotion flows in Manpower Planning

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**Abstract** Manpower Planning is a useful tool for human resource management in large organizations. Classical Manpower Planning models are analytical time-discrete push and pull models. Push models are characterized by the same promotion and wastage probabilities for people within the same group. This assumption is suitable in organizations where for instance promotions are used for reasons of personnel motivation or employees are promoted after succeeding in an exam. In many organizations, people are only promoted when there are vacancies at other levels. In those cases, pull models can be used. Pull models only assume known wastage probabilities. In practice, both assumptions may occur simultaneously. In this paper, a mixed push-pull model is developed for organizations in which both types of flows are considered.

**Keywords** Manpower planning · Stochastic modeling · Push models · Pull models

## 1 Introduction

Operational research techniques are used in several domains of personnel management. In personnel scheduling for example, mathematical methods are developed to assign the available workforce to the different tasks that should be performed by the company. This type of workforce planning is located at the short-term tactical planning level of the organization. Another important domain of workforce planning is long-term strategic Manpower Planning. Operational Research techniques in this area have been extensively developed since the sixties (Smith and Bartholomew 1988). While in personnel scheduling the available personnel is more or less fixed, the long-term supply of employees in the company can be adapted to the forecasted needs by recruitments, layoffs or retraining the current workforce. To be able to take such decisions, the personnel manager should have a notion of the available personnel in the future. This depends on the evolution of the current workforce. The current employees might leave the organization voluntarily or might develop a broader range

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of skills, such that they can handle a broader range of tasks. In the last case, those people can be used to fulfill the needs for another type of employees within the company. Because strategic Manpower Planning is approached within a long-term view, the evolution of new recruits should also be considered to estimate the future available workforce.

Several techniques were developed to investigate the evolution of the current workforce. Two common approaches are the push and the pull models. While the push models are based on the principles of Markov, the pull models use the renewal theory. Since the behavior of employees within a population is very heterogeneous, the population is firstly divided in several homogeneous groups with respect to the probabilities to leave the organization and to move to other homogeneous groups. In this way, every employee in a homogeneous group is assumed to have the same transition probability. Various personnel characteristics can be used to divide the total group of employees in different homogeneous subgroups, for example gender, job description, age and length of service (De Feyter 2006; Bartholomew et al. 1991). In most cases, the different grades in the hierarchy of the enterprise are used in this disaggregation. Transitions among homogeneous groups are therefore usually denoted in Manpower Planning literature by the term 'promotions', although they might not always be movements towards a higher level in the organization's hierarchy (e.g. Bartholomew 1977).

Pull models are developed for organizations where a promotion only takes place when there is a vacancy in another group. This transition creates a new vacancy in the group from which the individual is promoted. A vacancy also arises when someone leaves the organization. The evolution of the employees through the different groups in the system, are determined by probabilities, denoted by $s_{ij}$. This is the probability for a vacancy in group $i$ to be filled by a promotion from group $j$. $s_{ii}$ refers to the probability for a vacancy in group $i$ to remain in group $i$. The latter event occurs in case the vacancy is filled by an employee from within group $i$ itself. While in pull models employees are 'pulled up' once there is a vacancy at another level, employees in push models have a certain probability to leave or to be promoted independently from vacancies at other levels. The probability to be promoted from group $i$ to group $j$ is denoted by $p_{ij}$. The probability $p_{ii}$ refers to the probability for an employee in group $i$ to stay in the same group (Bartholomew et al. 1991).

Several approaches were used to develop push models for prediction, control or optimization. The simplest way is to consider constant promotion probabilities over time (see e.g. Parker and Caine 1996; Bartholomew et al. 1991). Non-homogeneous Markov systems are used to model systems in which the transition probabilities are not constant over time (see e.g. Yadavalli et al. 2002; Georgiou and Vassiliou 1997). Finally, semi-Markov models use conditional transition probabilities, depending on the duration in the grade (see e.g. Yadavalli and Natarajan 2001; McClean and Montgomery 2000). The transition and wastage probabilities in the push models can be estimated based on a historical dataset of the personnel movements in the past (De Feyter 2003; McClean 1991).

In this work, we will develop a mixed push-pull model, in the sense that the model uses the assumption that push as well as pull promotions are possible to occur in the same system at the same time. There exist a number of reasons why such an approach is reasonable:

- Firstly, there is an important reason from a *modeling point of view*: most often the different levels of a company are used for disaggregation. Enterprises which only promote if vacancies exist need to use a pull approach to model those flows. The models however highly depend on the assumption of homogeneity with respect to all transition probabilities. Therefore most often supplementary division is necessary to get reliable forecasting results (De Feyter 2006), such that there might arise groups between which transitions should be modeled by a push approach. An example of a personnel system requiring a model in which both push and pull transitions occur, is an organization in which vacancies are filled by promotions from groups of employees that succeeded in an exam. A transition between the group of people that not yet passed an exam and the group of people that succeeded in the exam happens with a certain probability. This is a typical push movement. Meanwhile, the actual promotion (only if there is a vacancy at another level) has to be considered as a pull transition.

– Secondly, a mixed push-pull model has an advantage from the *practitioner's point of view*. Often, organizations promote employees because of several reasons: Obviously, vacancies at higher levels can be filled by promotions from lower levels. In a Manpower Planning model, this should be considered as a pull flow. Human Resource Management (HRM) literature however frequently reports on other reasons for managers to promote employees to motivate them or on promotion as a retention management technique (Ferris et al. 1992). This should be modeled as a push flow, since it is independent from the vacancies at higher levels. The mixed push-pull model allows considering several reasons for promotion at the same time.

van Veen et al. (2001) and Geerlings et al. (2001) describe a simulation model that incorporates mixed push-pull transitions. Their model however only allows either push or pull transitions between two groups of the system. This doesn't meet the motivation for a mixed push-pull model from the practitioner's point of view, since therefore both push and pull transitions should be possible between two groups of the system. Moreover, they explicitly choose to use a simulation approach instead of an analytical approach like in traditional forecasting models. Their model is very useful for predicting the future evolution of the personnel structure. However, they abandon the classical analytical approach, which has the disadvantage that the model can't be used anymore for optimization or for identifying interesting properties of the personnel system that influence the future personnel structure (as in e.g. Tsantas 2001; Tsantas and Georgiou 1998; Bartholomew 1977). Therefore, we will develop a mixed push-pull forecasting model from an analytical point of view.

In Sect. 2 we will provide the mixed push-pull model and indicate how the separate push and pull models are special cases of the mixed push-pull model. Various researchers in Manpower Planning have done a great deal of effort on the investigation of the asymptotic behavior of the organizations' personnel structure, under several different model assumptions. In Sect. 3, we will give some results on the investigation of the asymptotic behavior of the personnel structure in the mixed pull-push model. Section 4 illustrates how the mixed push-pull model contributes to the decision support of an organization. A numerical application is given in Sect. 5. Finally, Sect. 6 contains some topics for further research on this approach of strategic Manpower Planning.

## 2 The mixed push-pull model

This section starts with a short review of the classical push and pull model, followed by the development of the mixed push-pull model framework. We consider an organization in which the total population of employees is divided into $k$ homogeneous groups. These homogeneous groups form a partition of the total population. The number of people in group $i$ at time $t$ is denoted by $n_i(t)$. We use a discrete time scale. The length of one time interval is chosen in such a way that it can be assumed that one employee can make at most one transition during the time interval. This is analogue to the approach in classical pull models: the length of one time interval is the time that it takes to fill a vacancy by a promotion or recruitment. This implicates the assumption that vacancies are not filled instantaneous. This is a realistic hypothesis since it takes time to take and to perform a promotion or recruitment decision. This assumption also implicates that a vacancy does not disappear in the company when it is filled by a promotion. When an employee promotes from group $i$ to group $j$, the initial vacancy in group $j$ creates a new vacancy in group $i$. In fact, the initial vacancy moves in the opposite direction of the employee. This means that $n_i(t)$ is possibly smaller than the desired number of employees $n_i^*(t)$ at time $t$.

Classical pull models most often assume that vacancies which need to be filled in the next time interval are determined at the end of the time period in which they turned up. This way, the model is given by:

$$\begin{aligned} V(t) &= V(t-1)S(t-1) + [n^*(t-2) - V(t-2)]W(t-1) + \Delta n^*(t-1), \\ \Delta n^*(t) &= n^*(t) - n^*(t-1), \end{aligned} \tag{1}$$

where $V(t)$ being a $(1 \times k)$ row vector formed by the vacancies in every group to be filled in time interval $[t-1, t)$; $S(t)$ is a $(k \times k)$ matrix with elements $s_{ij}(t)$; $W(t)$ being a $(k \times k)$ diagonal matrix formed by the voluntary wastage probabilities $w_i(t)$; $w_i(t)$ is the probability that an employee in group $i$ will leave the organization in time period $[t-1, t)$; $n^*(t)$ denoting the $(1 \times k)$ row vector $\{n_i^*(t)\}$ and $n(t)$ denoting the $(1 \times k)$ row stock vector $\{n_i(t)\}$.

This classical model assumes that vacancies will only be filled in the first time period following the one in which they arose. Meanwhile, the company will suffer a personnel shortage. Therefore, we choose to make the mixed push-pull model an anticipating model. We presume that at the start of a time interval, the enterprise will think ahead and try to prevent problems caused by an employees' shortage. This shortage would appear if it is not foreseen that people leaving in the new time interval, have to be replaced. The vacancies to be filled in the next time interval are based on the expected leavers in that time period. We have adapted the classical pull framework to the anticipating approach:

$$V(t) = V(t-1)S(t-1) + [n^*(t-1) - V(t-1)S(t-1)]W(t) + \Delta n^*(t). \tag{2}$$

Although this anticipating assumption is in harmony with the philosophy of HRM which arguments that management of human resources should rather be proactive than reactive (Brockbank 1999), managers might feel a lot more comfortable with opening a vacancy for an experienced shortage rather than an expected shortage. Anyway, the mixed push-pull model is developed according to the proactive HRM philosophy.

Although the pull model is developed to compute the number of promotions and recruitments needed in the future (Bartholomew et al. 1991), the known framework also allows computing the expected number of employees in every group using a system of difference equations:

$$n(t) = n^*(t) - V(t)S(t). \tag{3}$$

In contrast to the classic pull model, the main objective of the push models is estimating the future number of employees in every group. The model uses the assumption that the total recruitments in the time interval $[t-1, t)$ $R(t)$ are divided over the $k$ homogeneous groups according to a certain distribution. The proportion of the recruits assigned to group $i$ is denoted by $r_i(t)$. The expected number of employees in every group can be computed by a system of difference equations:

$$n(t) = n(t-1)P(t) + R(t)r(t), \tag{4}$$

where $P(t)$ is a $(k \times k)$ matrix with elements $p_{ij}(t)$ and the $(1 \times k)$ vector $r(t)$ being the $(1 \times k)$ row vector $\{r_i(t)\}$ (Bartholomew et al. 1991).

Georgiou and Tsantas (2002) use this push model to build an extended model. They include a trainee class into the model, next to (what they call) the active classes. Employees in the trainee class will only promote to the other groups in case there are vacancies in those groups. In that sense, their model is a mixed push-pull model. They use the assumption that personnel in the trainee class is 'pulled' up to the other classes, once there are vacancies at the active level. This 'pull' effect however is restricted to transitions from the trainee class to the active classes, while the mixed push-pull model is in this paper allows pull as well as push promotions between all classes in the model. Moreover, the vacancies in the model of Georgiou and Tsantas (2002) are not calculated and filled as common in the classical pull approach. Because it is convenient from a mathematical point of view, like most research in Manpower Planning, their discussion is restricted to an embedded Markov chain model, implying that the total number of individuals in the system is fixed or at least known (see also e.g. Tsantas 2001; Guerry 1999). This means that there does not, as in the pull models, exist a desired number of employees in every group, but a desired number of employees in the whole system. The model of Georgiou and Tsantas (2002) assumes that the needed size of the active classes as a whole is known. Once the size at the active level of the system is smaller than the desired size, vacancies arise. For most companies however, it is a lot more interesting to have a model available in which the needed employees in every group are used to determine the vacancies rather than only the total size of the personnel system. Such enterprises need personnel to execute corporate plans, but allocated to various well-defined jobs. It is most likely necessary to have the right personnel size in every employee category, defined by the jobs to be fulfilled, rather than having a certain number of employees available in the company as a whole.

The mixed push-pull model is capable incorporating this additional constraint. This model is based on the assumption of the classical pull models, in which vacancies arise in case that the number of employees in a specific group is less than the desired one. It allows the organization to choose a policy to fulfil those vacancies. According to the pull strategy, the vacancies are filled by promotions or by external recruitments.[1] Besides, in the mixed push-pull model, push promotions are possible in case not enough people had the opportunity to promote after all vacancies at higher levels were filled.

Since the internal dynamics of the personnel system is regulated by a mixed push-pull approach, we create a framework which allows computing the expected number of personnel in every homogeneous group by a system of difference equations. This system is given by:

$$n(t) = V(t) + \{n(t-1)[I - W(t)] - V(t)S(t)\}Q(t) + R(t)r(t). \tag{5}$$

This expression consists of three terms: the first term regulates the internal dynamics defined by the pull promotions; the second term represents the push promotions; the last one defines the external push-recruitments. We will now explain those three terms in the model in more detail.

*Pull promotions:* The first term $V(t)$ in expression (5) defines all the people who will get a pull-promotion together with all pull-recruitments in time interval $[t-1, t)$. This is nothing

[1]The assumptions of the mixed pull-push approach assure that the right personnel size in every employee category is reached by recruitments and/or by promotions. Georgiou and Tsantas (2002) on the contrary investigate the attainability or maintainability of a certain personnel distribution over the different grades. The organisation can only adapt his recruitment policy to get the right personnel size in every employee category.

else but the vacancies one expects to arise in that period. Those initial vacancies are created by the voluntary wastage that is expected in the next time interval. They will be filled either by a promotion from another group, either by an external recruitment. We need to add an extra set of equations to (5) to compute the vacancies:[2]

$$V(t) = \text{Max}\{\bar{0}; n^*(t) - n(t-1)[I - W(t)]\}. \tag{6}$$

The mixed push-pull model allows additional promotions after achieving the desired number of employees by pull transitions. That is the reason why it is possible that in a following time interval, the number of employees in a certain group is larger than the desired one, even after the voluntary wastage in that following time interval. This possibility is reflected in the model equations by the computation of the expected vacancies in the estimation of $n(t)$. If the number of employees in time period $[t-1, t)$ left after the expected voluntary wastage is larger than the desired personnel size in the specific group, the number of vacancies in that group is zero.

*Push promotions:* The employees in every group that did not make any pull transition and did not leave the organization are submitted to the push transition assumption. We call them the stayers. In expression (5) the expected number of stayers is computed by $n(t-1)[I - W(t)] - V(t)S(t)$. While $n(t-1)[I - W(t)]$ results in the expected number of people who keep being employed in the organization, $V(t)S(t)$ represents the number of vacancies created by filling the initial vacancies. We expect the stayers moving through the system determined by the row stochastic promotion matrix $Q(t)$. This matrix is different from the transition matrix $P(t)$ used in the classical push model. While $Q(t)$ is row stochastic, $P(t)$ is not in case there is voluntary wastage in at least one of the groups. There exists a relationship between the transition probabilities $p_{ij}(t)$ and the voluntary wastage probabilities:

$$w_i(t) = 1 - \sum_{j=1}^{k} p_{ij}(t). \tag{7}$$

In this way, the push model considers the voluntary wastage in forecasting the future stock $n(t)$. In the mixed push-pull model, we already considered the leavers while computing the stayers. Therefore, we introduce a row stochastic matrix $Q(t)$ which contains the transition probabilities of the stayers.

*External recruitment:* The term $R(t)r(t)$ in expression (5) represents the external recruitments. Such as in the classical push models, we leave the possibility open to recruit employees besides recruitments done to fill the initial vacancies. This is useful for groups like for example in a situation analogue to the model of Georgiou and Tsantas (2002). For the trainee class, there should only be enough people to fill the vacancies in the other grades in the future. Therefore, external recruitments are necessary. Moreover, as will be shown in Sect. 4, external recruitment is an important tool in the decision support provided by the mixed push-pull model.

The mixed push-pull model is in fact a generalization of the classical push and pull models. If we restrict the mixed model to a push model, we set the pull-parameters of the mixed model equal to zero. This means that $n^*(t)$ becomes a null vector. There is no explicit

[2] To take the maximum of vector $A$ and $B$, we take $\text{Max}\{A_i, B_i\}\ \forall i$.

desired personnel distribution given. The model (5) is reduced to the classical push model of (4):

$$\begin{aligned} n(t) &= n(t-1)[I - W(t)]Q(t) + R(t)r(t) = n(t-1)P(t) + R(t)r(t), \\ V(t) &= \text{Max}\{\bar{0}; n^*(t) - n(t-1)[I - W(t)]\} = \bar{0}. \end{aligned} \tag{8}$$

In the same way, we show that the pull model is a special case of the mixed push-pull model. $Q(t)$ becomes the identity matrix because there will be no push promotions. The push recruitments $R(t)$ also becomes zero. The model (5) becomes:

$$\begin{aligned} n(t) &= V(t) + n(t-1)[I - W(t)] - V(t)S(t) \\ &= \text{Max}\{\bar{0}; n^*(t) - n(t-1)[I - W(t)]\} + n(t-1)[I - W(t)] - V(t)S(t) \\ &= n^*(t) - n(t-1)[I - W(t)] + n(t-1)[I - W(t)] - V(t)S(t) \\ &= n^*(t) - V(t)S(t). \end{aligned} \tag{9}$$

The third equality will hold if $n^*(t)$ is larger than or equal to $n(t-1)[I - W(t)]$.[3] This is assumed in the classical pull models. In case $n^*(t)$ is constant over time, this assumption is valid in the mixed push-pull model since it's impossible to have other transitions than the pull movements. We can conclude that the model equation (9) is the same as (3).

## 3 Asymptotic behaviour

Under certain conditions, the push model evolves towards a limiting distribution. For example, the already mentioned fixed size model with time homogeneous transition probabilities evolves like a Markov chain towards an equilibrium. It is also known that the classical push model under the condition of time homogeneous transition probabilities and fixed recruitment policy $R(t) = R$ and $r(t) = r$ has a limiting stock vector (Bartholomew et al. 1991). We investigate the asymptotic behavior of the mixed push-pull model under analogue assumptions, namely time homogeneous transition probabilities $Q(t) = Q$, $W(t) = W$ and $S(t) = S$ and a fixed recruitment policy.

The evolution of the personnel structure in the mixed push-pull model depends on the evolution of $n^*(t)$, since this variable will determine the vacancies that need to be filled in the future. However, in the present uncertain and ever changing economic environment, it will hardly be possible to exactly know the number of employees needed in the far future. However, precisely because of this uncertain future, the organizations experience an increased need for planning. In that sense, Manpower Planning models can be used to investigate where the personnel structure is heading for under certain personnel strategies. Considering the uncertain future needs for employees, it is not unreasonable to presume that the Manpower Planner will leave the future model parameters as they are. We introduce the additional assumption that the desired personnel distribution $n^*(t)$ is fixed over time. The fixed desired employees in group $i$ and the fixed desired personnel distribution are denoted respectively by $n_i^*$ and by the $(1 \times k)$ row vector $n^* = \{n_i^*\}$.

Besides on the parameters of the model, the long-term evolution highly depends on the initial personnel distribution. We need to consider several cases:

[3] $A \geq B \Leftrightarrow \forall i : A_i \geq B_i$.

– Consider a situation where $n_i(0) > n_i^*$ $(\forall i)$. In that case, the pull aspect of the mixed model can be ignored, since the number of vacancies is zero. $n(0)$ will start his evolution towards the known fixpoint of the push model $n_e^p = Rr(I - P)^{-1}$ (Bartholomew et al. 1991). If this equilibrium is larger than $n^*$ and $|n(0)[I - W] - n_e^p| \leq |n_e^p - n^*|$[4] and all eigenvalues of $P$ are real, we know for sure that the system will evolve towards this equilibrium, without ever reaching a stock vector that would introduce a pull-effect in the model. Indeed, the difference between the stock and the equilibrium will only reduce in time. If the push equilibrium is smaller than $n^*$, the stock will follow the push path towards this fixpoint until it reaches a stock smaller than the desired distribution. This will induce pull transitions in the system.
– Consider another case in which the initial distribution and all the following distributions are below the desired one. This way, we can simplify (6):

$$V(t) = n^* - n(t-1)[I - W]. \tag{10}$$

For computational reasons, we put $A = I - SQ$ and $M = [I - W]\{[I + S]Q - I\}$. Expression (5) becomes:

$$n(t) = n(t-1)M + n^*A + Rr. \tag{11}$$

It is now possible to state the following theorem.

**Theorem** *If in every time period both pull and push transitions occur, the row sums of $M$ are strictly less than one and $M$ is such that*:

(i) *$M$ is a power-nonnegative matrix of arbitrary degree $k$,*[5] *or*
(ii) *$M + I$ is a totally nonnegative matrix*[6]

*then the mixed push-pull model will evolve towards the limiting distribution $n_e = (n^*A + Rr)(I - M)^{-1}$.*

*Proof* If the model evolves towards a limiting distribution $n_e$, in the limit, $n(t) = n(t-1) = n_e$. It follows that the limiting vector equals:

$$n_e = n_eM + n^*A + Rr \Leftrightarrow n_e = (n^*A + Rr)(I - M)^{-1}. \tag{12}$$

Since $0 \leq s_{ij} \leq 1$ $\forall i, j$ and $Q$ is a row stochastic matrix, the elements on the diagonal of $\{[I + S]Q - I\}$ (and therefore $M$) are between $-1$ and 1; the other elements are greater than or equal to 0 and smaller than or equal to 2. Since the row sum of $M$ are assumed to be strictly smaller than one, the row sum of the nonnegative matrix $[M + I]$ is strictly smaller than two. According to the Perron–Frobenius theorem, the largest eigenvalue of a nonnegative matrix is positive (MacCluer 2000). An important corollary of this theorem states that the largest eigenvalue is smaller than the largest row sum and larger than the smallest row sum of the nonnegative matrix (Seneta 1973). This means that the largest eigenvalue of $[M + I]$ is strictly smaller than two. If $\lambda$ is an eigenvalue of matrix $M$ then matrix $[M + I]$

[4] If $A$ is a vector with elements $a_i$, $|A|$ is the vector with the elements $|a_i|$.

[5] A matrix $A$ is called power-nonnegative of degree $k$ (with $k$ a positive integer) if $\exists k : A^k > 0$ and $k$ is the smallest integer for which this condition holds.

[6] A matrix is called totally nonnegative if all his minors are nonnegative.

has an eigenvalue $\tilde{\lambda} = \lambda + 1$. This way the largest eigenvalue of $M$ is strictly smaller than one. We can conclude that $[I - M]^{-1}$ exists.

Let us now study the difference $v(t) = n(t) - n_e$. Using (12), we know that:

$$\begin{aligned} v(t) &= v(t-1)M + n_e M + n^* A + Rr - n_e \\ &= v(t-1)M. \end{aligned} \tag{13}$$

We use the *Jordan normal form theorem* (Gantmacher 1964), which allows us to rewrite $M$ as:

$$M = A^{-1} D A \tag{14}$$

with $A$ a $(k \times k)$ non-singular matrix and $D$ a block diagonal matrix with $m$ the number of eigenvalues of $M$ and:

$$D = \operatorname{diag}(J_1(\lambda_1), J_2(\lambda_2), \ldots, J_m(\lambda_m)) \tag{15}$$

with $J_i(\lambda_i)$ a $(p_i \times p_i)$ matrix with eigenvalue $\lambda_i$ (with algebraic multiplicity $p_i$):

$$J_i(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ & & \ddots & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}.$$

This allows us to rewrite (13) as:

$$v(t) = v(0) A^{-1} D^t A. \tag{16}$$

The evolution of $v(t)$ can now be studied by considering the eigenvalues of $M$:

(i) Kemp and Kimura (1978) have proved that if the row sums of a power-nonnegative matrix are smaller than one, the eigenvalue of that matrix with the greatest magnitude is less than one. Since we know that the row sums of $M$ are strictly smaller than one, we know that if $M$ is power-nonnegative, the magnitude of every eigenvalue of $M$ is smaller than one.

(ii) It is well known that if a matrix is totally nonnegative, its eigenvalues are all real and positive (Koev 2005). If $M + I$ is totally nonnegative, we can conclude that all the eigenvalues of $M$ are greater than $-1$ and smaller than 1.

Since in case (i) as well as in case (ii) the eigenvalues of $M$ are between $-1$ and 1, using the Jordan decomposition, we conclude that in both cases the difference between the stock and the desired stock would evolve towards zero as $t \to +\infty$. Consequently, the stock $n(t)$ evolves towards the limiting stock $n_e$. □

The assumption of the theorem that the row sums of $M$ are strictly less than one, is very likely to hold in practice. Since the row sums of $S$ are smaller than or equal to one and $Q$ is a row stochastic matrix, this assumption can only be violated in case the voluntary wastage probability in at least one of the groups is zero. This is highly unlikely to happen, because this would mean that wastage in a certain grade is never possible. Moreover, even then, this would only create problems if in that specific group all the vacancies would be filled by pull-promotions and none by pull-recruitments.

## 4 Decision support

### 4.1 Maintainability and attainability under control by recruitment

A lot of previous research on push models is dedicated to the maintainability and attainability problem under control by recruitment. The push models allow studying the evolution of the future number of people in every class under a certain recruitment strategy. This enables organizations to investigate the impact of a recruitment policy on the future personnel structure and if necessary to adapt it in order to reach or approach the desired personnel distribution in the future (see e.g. Guerry 1999; Tsantas and Georgiou 1998). Remark that in order to reach a certain desired personnel structure, it may be required that $R(t)r(t)$ contains some negative elements, implying that redundancies are necessary (Bartholomew 1977).

The mixed push-pull model as stated in (5) and (6) allows a comparable analysis. Although the model explicitly takes into account the desired personnel structure $n^*(t)$, it was pointed out that because of the model assumptions (stated for reasons of compatibility with situations in practice), the organization might not always reach this desired personnel structure. Nevertheless, analogue to the classical push models, it is possible to identify the optimal push recruitment strategy, such that the desired personnel structure is reached anyway. This recruitment policy is given by:

$$R(t)r(t) = n^*(t) - V(t) - \{n(t-1)[I - W(t)] - V(t)S(t)\}Q(t). \tag{17}$$

In case that $n^*(t)$ is larger than or equal to $n(t-1)[I - W(t)]$, (17) is simplified by:

$$R(t)r(t) = n^*(t)S(t)Q(t) - n(t-1)[I - W(t)]\{[I + S(t)]Q(t) - I\}. \tag{18}$$

Formula (18) is used to give the optimal recruitment policy for maintaining a desired stock vector, since in the maintainability problem $n^*(t-1) = n^*(t)$ is always larger than or equal to $n^*(t-1)[I - W(t)]$:

$$R(t)r(t) = n^*(t)[S(t)Q(t) - [I - W(t)]\{[I + S(t)]Q(t) - I\}]. \tag{19}$$

Although those optimal recruitment strategies assure the organization to reach its desired personnel structure at time $t$, it might not be a preferable policy since it influences the evolution of the future personnel structure. The model equations however can be used to do a 'What-if' analysis to get an insight in the influence of recruitment policies on the future personnel structure. It is obvious that the Manpower Planner should take into account the effects of its recruitment strategy on other aspects like future opportunities, financial consequences, ethical or psychological problems. The organization should be very careful, especially in case one or more elements of its recruitments policy $R(t)r(t)$ are negative. This will not only influence the future personnel structure, but might also have an effect on the motivation or commitment of the 'survivors' or on the industrial relations.

What-if analysis based on the mixed push-pull model also allows the Manpower Planner identifying bottlenecks in the personnel system. It might become clear that the transition pattern systematically causes deviations of the preferable behavior of the system. The organization might be able to influence some of the transition probabilities: HRM academics put a lot of effort in for example retention management and practices creating High Performance or High Commitment Work Systems. The mixed push-pull model can be used in a deterministic way to investigate the effect of changes in the transition pattern.

### 4.2 Anticipating all vacancies

Although the desired personnel structure is explicitly taken into account in the mixed push-pull model, the discussion in this paper shows that it will not always be reached after the pull flows were carried out. This is a common problem in pull models under the assumption that vacancies can not be filled instantaneous, because the pull model assumptions bring along the fact that filling a vacancy by a promotion creates a vacancy at another level. By implementing the optimal recruitment strategy discussed above, it can be assured that the desired personnel structure is reached anyway. Nevertheless, it might not always be desirable to complete the personnel size by external recruitments. An organization might prefer to control the system more and to fill (at least a part of the) vacancies by promotions rather than by recruitments. This preferred control mechanism is explicitly expressed by the $S$ matrix in the mixed push-pull model. In Sect. 2, we expressed the objective to create an anticipating model. We succeeded to a certain extent, because the mixed push-pull model assumes that the user will incorporate the expected future voluntary wastage in his promotion and recruitment decisions. However, the model can be adapted by not only anticipating the initial vacancies created by the future voluntary wastage, but also the vacancies that will be created by all pull promotions. Remark that those extra vacancies will again be filled by pull promotions resulting in more vacancies at other levels. To incorporate the vacancies arising out of all pull promotions in the coming time period, (6) in the mixed push-pull model needs to be replaced by:

$$V(t) = \mathrm{Max}\{\bar{0}; n^*(t) - n(t-1)[I - W(t)]\} \cdot [I - S(t)]^{-1}. \tag{20}$$

Since the vacancies within one time period evolve as a chain satisfying the Markov properties with $S$ acting as a transition matrix, the initial vacancies as estimated by (6) need to be multiplied by the fundamental matrix. Indeed, it is very well known that the fundamental matrix gives the expected number of visits to each state before absorption occurs (Bartholomew et al. 1991).

This full anticipating model requires the decision maker to adjust his way of planning. Moreover, analogue to the remark made in Sect. 2, the anticipating approach brings along the difficulty that the Manpower Planner has to be able to convince its superiors to provide more promotions than obviously needed to replace the employees that actually will leave the company. This might not always be easy, since nowadays personnel planning is subject to strong budget constraints.

## 5 Numerical illustration

We present two examples in which time homogeneous transition probabilities are assumed. The total personnel system is divided into three homogeneous groups with respect to the transition probabilities. The desired personnel distribution is fixed over time and given by:

$$n^* = (800 \quad 1000 \quad 900).$$

The transition probabilities of the mixed push-pull models are estimated based on a historical dataset of the movements in the past:

$$Q = \begin{pmatrix} 0.85 & 0.10 & 0.05 \\ 0.05 & 0.85 & 0.10 \\ 0.05 & 0.05 & 0.90 \end{pmatrix}, \qquad S = \begin{pmatrix} 0.00 & 0.15 & 0.05 \\ 0.05 & 0.00 & 0.45 \\ 0.20 & 0.40 & 0.00 \end{pmatrix}, \qquad w = \begin{pmatrix} 0.10 \\ 0.05 \\ 0.02 \end{pmatrix}.$$

**Table 1** Extrapolation Example 1

| | $n(t-1)[I-W]$ | | | $n(t)$ | | |
|---|---|---|---|---|---|---|
| $t=0$ | | | | 1200 | 1100 | 950 |
| $t=1$ | 1080 | 1045 | 931 | 1157 | 1153 | 996 |
| $t=2$ | 1041 | 1095 | 976 | 1129 | 1194 | 1040 |
| $t=3$ | 1016 | 1134 | 1020 | 1111 | 1227 | 1082 |
| $t=4$ | 1000 | 1165 | 1060 | 1101 | 1253 | 1121 |
| | | ... | | | ... | |
| $t=154$ | 1179 | 1566 | 1790 | 1310 | 1649 | 1826 |
| $t=155$ | 1179 | 1566 | 1790 | 1310 | 1649 | 1827 |
| $t=156$ | 1179 | 1566 | 1790 | 1310 | 1649 | 1827 |
| $t=157$ | 1179 | 1566 | 1790 | 1310 | 1649 | 1827 |
| $t=158$ | 1179 | 1566 | 1790 | 1310 | 1649 | 1827 |
| $t=159$ | 1179 | 1566 | 1790 | 1310 | 1649 | 1827 |

*Example 1* Assume that the current personnel size in every group is higher than the desired one and that the initial personnel stock is given by:

$$n(0) = (1200 \quad 1100 \quad 950).$$

The recruitment policy for the vacancies is implicitly given by $S$ because vacancies that are not filled by promotions will be compensated by recruitments. There are also recruitments independent from the vacancies created by the desired personnel stock. Every year, the enterprise recruits $R = 250$ people in this way, of which 140 in group 1 and 110 in group 2:

$$r = (0.56 \quad 0.44 \quad 0).$$

We use formula (5) to do a what-if analysis. The results are given in Table 1.

Since the eigenvalues of $P = [I - W]Q$ are all real and the (absolute value of the) difference between $n(0)[I - W]$ and $n_e$ is smaller than the (absolute value of the) difference between $n_e$ and $n^*$, we know that there will never be pull promotions or recruitments. Table 1 confirms that $n(t-1)[I - W]$ is always larger than $n^*$; there will be never vacancies in the model. The mixed model will be reduced to an ordinary push model and the model evolves towards the classical limiting distribution:

$$n_e = Rr(I - P)^{-1} = (1310 \quad 1649 \quad 1827).$$

It is interesting to remark that, although the mixed push-pull model is applied here in conformity with the mixed push-pull promotion strategy determined by the organisation, the evolution of the personnel structure is fully dominated by push transitions. Indeed, because of the personnel surplus in the system, (6) sets vacancies to zero and no pull promotions occur. The company should reconsider its (push) promotion and/or recruitment policy to reduce its personnel size to the desired personnel size.

*Example 2* Consider another example with the same transition probabilities and desired stock. The current personnel size in every group is smaller than the desired one. Conse-

**Table 2** Extrapolation Example 2

| | $n(t)$ | | | $n(t-1)(I-W)$ | | | $V(t)$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $t=0$ | 750 | 800 | 850 | | | | | | |
| $t=1$ | 749 | 948 | 818 | 675 | 760 | 833 | 125 | 240 | 67 |
| $t=2$ | 758 | 918 | 891 | 674 | 901 | 801 | 126 | 99 | 99 |
| $t=3$ | 770 | 952 | 873 | 682 | 872 | 873 | 118 | 128 | 27 |
| $t=4$ | 768 | 944 | 892 | 693 | 905 | 856 | 107 | 95 | 44 |
| $t=5$ | 772 | 952 | 887 | 692 | 897 | 874 | 108 | 103 | 26 |
| $t=6$ | 771 | 950 | 892 | 695 | 905 | 869 | 105 | 95 | 31 |
| $t=7$ | 772 | 952 | 890 | 694 | 902 | 874 | 106 | 98 | 26 |
| $t=8$ | 772 | 951 | 891 | 695 | 904 | 872 | 105 | 96 | 28 |
| $t=9$ | 772 | 952 | 891 | 695 | 904 | 873 | 105 | 96 | 27 |
| $t=10$ | 772 | 952 | 891 | 695 | 904 | 873 | 105 | 96 | 27 |
| $t=11$ | 772 | 952 | 891 | 695 | 904 | 873 | 105 | 96 | 27 |

quently, both aspects of the mixed push-pull model have an influence on the stock evolution.

$$n(0) = (750 \quad 800 \quad 850).$$

There are no push recruitments.

The extrapolations, computed by (5) and (6), are summarized in Table 2. Since in every year, $n^*(t)$ is larger than to $n(t-1)[I - W(t)]$ both push and pull transitions take place.

Remark that, although matrix $M + I$ is not totally nonnegative and matrix $M$ is not power-nonnegative, the system evolves towards the limiting distribution anyway:

$$n_e = (n^* A + Rr)(I - M)^{-1} = (772 \quad 952 \quad 891).$$

Under this recruitment policy $R(t)r(t) = \bar{0}$ , this company never reaches the desired personnel structure $n^*$. The decision maker might consider changing its policy. From Table 2 we know that the optimal recruitment policy is $R(1)r(1) = (51\ 52\ 82)$. It is obvious that formula (18) gives the same result. Once at $t = 1$ this desirable personnel structure is reached, the maintainability formula (19) gives the optimal recruitment policy:

$$R(t)r(t) = n^*(t)[S(t)Q(t) - M] = (24 \quad 45 \quad -17), \qquad t > 1.$$

According to this optimal recruitment policy, there should be 17 redundancies is group 3 every year, while at the same time people are recruited at the other groups. It is clear that there exists a structural problem in this organization: the promotion system is not compatible with the desired personnel structure. As mentioned in Sect. 4, the organisation should consider trying to influence and change its promotion system.

## 6 Conclusion and further research

In this paper, we have developed a time-discrete mixed push-pull Manpower Planning model. This model allows taking into account push and pull transitions of employees through an organization at the same time. This is useful because the need for homogeneous groups in manpower modeling might create classes among which both push as pull transitions are possible. This extension will increase the accuracy of the forecasting results. Furthermore, there are motivations from the user's point of view. From the HRM literature we know that promotions are used to fulfill the firm's needs and for reasons of personnel motivation. Those reasons will provoke respectively pull and push promotions at the same time. Besides, Georgiou and Tsantas (2002) discussed the potential use of the introduction of a trainee class from which promotions are only possible to fill vacancies at the other so-called active classes. The promotions among those active classes are modeled as push flows. While Georgiou and Tsantas (2002) only use the constraint of fixed needs for employees over all active classes together, we introduced two constraints, namely a desired need in all the groups separately and a certain pull promotion and recruitment policy to fill the vacancies. van Veen et al. (2001) and Geerlings et al. (2001) on the other hand discussed a Manpower Planning simulation model including pull and push transitions. While simulation models only enable forecasting, classical analytical Manpower Planning models (like the mixed push-pull model developed in this paper) allow optimization, control and determining special properties of the personnel system. For example, this paper discussed some topics on decision support based on the mixed push-pull model. Moreover, we investigated the asymptotic behavior of the mixed push-pull model under specific conditions. It is shown that under specific conditions, the system evolves towards stability. In this stage, it is only possible to formulate those conditions in mathematical terms; it is not clear whether those conditions can be interpreted intuitively. Finally we mention that Example 2 shows that also in another case the personnel distribution evolves towards $n_e$. Those situations need further research.

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