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Higher-order optimality conditions for strict local minima

Bienvenido Jiménez · Vicente Novo

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Abstract In this work, we study a nonsmooth optimization problem with generalized inequality constraints and an arbitrary set constraint. We present necessary conditions for a point to be a strict local minimizer of order k in terms of higher-order (upper and lower) Studniarski derivatives and the contingent cone to the constraint set. In the same line, when the initial space is finite dimensional, we develop sufficient optimality conditions. We also provide sufficient conditions for minimizers of order k using the lower Studniarski derivative of the Lagrangian function. Particular interest is put for minimizers of order two, using now a special second order derivative which leads to the Fréchet derivative in the differentiable case.

Keywords Optimality conditions · Strict minimizer of higher order

The notion of strict minimizer of order k has turned out to be very fruitful in optimization theory. Let us recall that given a normed vector space X, $f : X \to \mathbb{R}$ and $M \subset X$, the point $x_0 \in M$ is said to be a strict local minimizer of order k ($k \geq 1$ an integer), denoted $x_0 \in \mathrm{Strl}(k, f, M)$, for the optimization problem

$$\mathrm{Min}\{f(x) : x \in M\}$$

if there exist $\alpha > 0$ and a neighborhood U of x_0 such that

$$f(x) > f(x_0) + \alpha\|x - x_0\|^k \quad \forall x \in M \cap U \setminus \{x_0\}.$$

The notion (without a specific name) was handled by Hestenes (1966, 1975) for the values $k = 1$ and $k = 2$ in order to prove sufficient optimality conditions.

B. Jiménez · V. Novo (✉)
Departamento de Matemática Aplicada, E.T.S.I. Industriales, Universidad Nacional de Educación a Distancia, Calle Juan del Rosal 12, 28040 Madrid, Spain
e-mail: vnovo@ind.uned.es

B. Jiménez
e-mail: bjimenez@ind.uned.es

Cromme (1978) used this notion in the context of the convergence of iterative numerical procedures. Auslender (1984) studied the strict minimality to develop stability conditions in nonsmooth optimization.

Studniarski (1986) extending some Auslender's results provided necessary and sufficient conditions for a strict local minimizer of order k for any function f and an arbitrary subset M of a finite dimensional space X. For this aim, he used directional derivatives that are generalizations of the lower and upper Hadamard derivatives (see Definition 1.2). Ward (1994) follows the line of Studniarski using other derivatives and tangent cones. Studniarski (1997) considers locally Lipschitz functions and the Clarke subdifferential. This notion (under the name k-unique) was also studied in linear semi-infinite programs in $\mathbb{R}^n$ by Goberna et al. (1995).

Jiménez (2002) extends the notion of strict minimizer to vector optimization problems, this is, to a function from X to another normed space Y partially ordered by a convex cone. In several papers, Jiménez (2003) and Jiménez and Novo (2003a, 2003b, 2004) develop a theory on strict minimizers of order k considering different frameworks.

In this paper, we discuss a particular (scalar) mathematical programming problem (see (1)) and provide necessary and sufficient optimality conditions formulated through the Studniarski derivatives. Moreover, we present some sufficient conditions for strict minimality of order k using the Lagrangian function. Further, a second order sufficient condition is established using a particular second order derivative which becomes the second order Fréchet derivative when the involved functions are twice Fréchet differentiable. Some examples are also given.

1 Notations and preliminaries

Let M be a subset of X. We denote by $\operatorname{int} M$, $\operatorname{cl} M$ and $\operatorname{cone} M$ the interior, closure and cone generated by the set M, respectively, and $B(x_0, \varepsilon)$ represents the open ball of center x_0 and radius $\varepsilon > 0$.

The following tangent cones will be used in this paper.

Definition 1.1 (a) The tangent (contingent) cone to M at $x_0 \in M$ is

$$T(M, x_0) = \{v \in X : \exists t_n \to 0^+,\ x_n \in M,\ x_n \to x_0 \text{ such that } (x_n - x_0)/t_n \to v\}.$$

(b) The interior tangent cone is

$$IT(M, x_0) = \{v \in X : \exists \varepsilon > 0 \text{ such that } x_0 + tu \in M,\ \forall t \in [0, \varepsilon],\ \forall u \in B(v, \varepsilon)\}.$$

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we are interested in the following optimization problem

$$\operatorname{Min} f(x) \quad \text{subject to} \quad g(x) \in -K, \quad x \in Q, \tag{1}$$

where $g : X \to Y$, Y is a normed space, Q is an arbitrary subset of X, and K is a convex cone of Y with nonempty interior. We denote $G = \{x \in X : g(x) \in -K\}$, thus the feasible set of problem (1) is $M = G \cap Q$. Problem (1) includes the usual problem defined by inequality constraints $g(x) \leq 0$ and equality constraints $h(x) = 0$ taking $Y = \mathbb{R}^m$, $K = \mathbb{R}^m_+$ (the nonnegative orthant of $\mathbb{R}^m$) and $Q = h^{-1}(0)$.

The topological dual of Y is denoted by Y^*. The positive polar cone to K is $K^+ = \{\mu \in Y^* : \langle \mu, y\rangle \geq 0 \quad \forall y \in K\}$ and the negative polar cone to K is $K^- = -K^+$.

The Hadamard derivative of g at x_0 in the direction $v \in X$ is

$$dg(x_0, v) = \lim_{(t,u)\to(0^+,v)} \frac{g(x_0 + tu) - g(x_0)}{t}.$$

Throughout the paper, we will assume that g is Hadamard differentiable at x_0, that is, $dg(x_0, v)$ exists for all $v \in X$.

Moreover, the next directional derivatives introduced by Studniarski (1986) will be used.

Definition 1.2 Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be finite at $x_0 \in X$ and $k \geq 1$ an integer number. The lower (resp. upper) Studniarski derivative of order k of f at x_0 in the direction $v \in X$ is

$$\underline{d}^k f(x_0, v) = \liminf_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0)}{t^k}$$

(resp. $\bar{d}^k f(x_0, v) = \limsup_{(t,u)\to(0^+,v)} (f(x_0 + tu) - f(x_0))/t^k$).

If $k = 1$, $\underline{d}^1 f(x_0, v)$ and $\bar{d}^1 f(x_0, v)$ are called the lower and upper Hadamard derivatives and are denoted by $\underline{d} f(x_0, v)$ and $\bar{d} f(x_0, v)$, respectively.

We shall also use the following second order derivative:

$$\underline{d}_r^2 f(x_0, v) = \liminf_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0) - t\underline{d} f(x_0, u)}{t^2/2}.$$

The function $\underline{d}^k f(x_0, \cdot)$ is lower semicontinuous (see Proposition 2.1.1 in Demyanov and Rubinov 1995), positively homogeneous of degree k and can take the values $-\infty$ and $+\infty$.

Let us recall that the function I_M defined as $I_M(x) = 0$ if $x \in M$ and $I_M(x) = +\infty$ if $x \notin M$ is called the indicator function of the set $M \subset X$.

In the next lemma two basic properties are collected.

Lemma 1.3 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be finite at $x_0 \in X$ and $v \in X$.*

(i) *If $k > 1$ and $\underline{d} f(x_0, v) > 0$, then $\underline{d}^k f(x_0, v) = +\infty$.*
(ii) *If $v \notin T(Q, x_0)$ then $\underline{d}^k (f + I_Q)(x_0, v) = +\infty$ $\forall k \geq 1$.*

For part (i) see Lemma 3.4 in Jiménez (2003), and for part (ii) it suffices to take into account that $T(Q, x_0)^c = IT(Q^c, x_0)$ and to apply the definition (here $(\cdot)^c$ means algebraic complement).

2 Necessary conditions

In this section, we provide necessary optimality conditions for problem (1). In Theorem 2.1 the conditions are formulated in terms of the upper Studniarski derivative, whereas in Theorem 2.2 we use the lower derivative, however the latter includes the indicator function of the set Q, which is a drawback.

Theorem 2.1 *Let $k \geq 1$. If $x_0 \in \text{Strl}(k, f, G \cap Q)$ then*

$$\bar{d}^k f(x_0, v) > 0 \quad \forall v \in C_0(G, x_0) \cap T(Q, x_0) \setminus \{0\},$$

where $C_0(G, x_0) = \{v \in X : dg(x_0, v) \in \operatorname{int}\operatorname{cone}(-K - g(x_0))\}$.

Proof Assume that $\bar{d}^k f(x_0, v) \leq 0$ for some $v \in C_0(G, x_0) \cap T(Q, x_0) \setminus \{0\}$. Then there exist $t_n \to 0^+$ and $x_n \in Q$ such that $v_n := (x_n - x_0)/t_n \to v$. Without loss of generality (taking a subsequence if necessary) we can assume that $\lim_{n\to\infty}(f(x_0 + t_n v_n) - f(x_0))/t_n^k \leq 0$ since $\bar{d}^k f(x_0, v) \leq 0$. Hence, for each $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ such that

$$\frac{f(x_0 + t_{n_j} v_{n_j}) - f(x_0)}{t_{n_j}^k} < \frac{1}{j}. \tag{2}$$

Since (as it will be proved latter)

$$C_0(G, x_0) \subset IT(G, x_0) \tag{3}$$

and $v \in C_0(G, x_0)$, it follows that $v \in IT(G, x_0)$, and consequently, $x_0 + t_{n_j} v_{n_j} \in G$ for all j large enough. So $x_0 + t_{n_j} v_{n_j} \in G \cap Q$. On the other hand, by assumption, there exist $\alpha > 0$ and $\delta > 0$ such that

$$f(x) > f(x_0) + \alpha\|x - x_0\|^k \quad \forall x \in G \cap Q \cap B(x_0, \delta) \setminus \{x_0\}.$$

In particular, for $x = x_0 + t_{n_j} v_{n_j}$, for all j large enough, taking into account (2),

$$\frac{1}{j} > \frac{f(x_0 + t_{n_j} v_{n_j}) - f(x_0)}{t_{n_j}^k} > \alpha\|v_{n_j}\|^k.$$

Taking the limit when $j \to \infty$ we deduce that $\|v\| = \lim_{j\to\infty} \|v_{n_j}\| = 0$, which is a contradiction to the fact that $v \neq 0$.

Now let us prove the inclusion (3). Choose $v \in C_0(G, x_0)$, then $dg(x_0, v) \in \operatorname{int}\operatorname{cone}(-K - g(x_0))$. By Proposition 2.3(ii) in Jiménez and Novo (2003b), $\operatorname{int}\operatorname{cone}(-K - g(x_0)) = IT(-K, g(x_0))$. Hence, $dg(x_0, v) \in IT(-K, g(x_0))$ and by the definition of $IT(-K, g(x_0))$, there exists $\varepsilon > 0$ such that

$$g(x_0) + \beta w \in -K \quad \forall \beta \in (0, \varepsilon),\ \forall w \in B(dg(x_0, v), \varepsilon).$$

By the definition of $dg(x_0, v)$, for this $\varepsilon > 0$ there exists $\delta_0 > 0$ such that

$$\frac{g(x_0 + tu) - g(x_0)}{t} \in B(dg(x_0, v), \varepsilon) \quad \forall t \in (0, \delta_0),\ \forall u \in B(v, \delta_0).$$

Taking $\delta = \min\{\delta_0, \varepsilon\}$ we have

$$g(x_0 + tu) = g(x_0) + t\frac{g(x_0 + tu) - g(x_0)}{t} \in -K \quad \forall t \in (0, \delta),\ \forall u \in B(v, \delta).$$

This implies that $v \in IT(G, x_0)$ and the proof is concluded. □

Theorem 2.2 *Let $k \geq 1$. If $x_0 \in \text{Strl}(k, f, G \cap Q)$ then*

$$\underline{d}^k (f + I_Q)(x_0, v) > 0 \quad \forall v \in C_0(G, x_0) \cap T(Q, x_0) \setminus \{0\}.$$

Proof Suppose that the conclusion is false, this is, that there exists $v \in C_0(G, x_0)$, $v \neq 0$, such that $\underline{d}^k(f + I_Q)(x_0, v) \leq 0$. Then as

$$\underline{d}^k(f + I_Q)(x_0, v) = \sup_{\delta>0} \inf_{t\in(0,\delta),u\in B(v,\delta)} \frac{(f + I_Q)(x_0 + tu) - f(x_0)}{t^k} \leq 0$$

we deduce that

$$\inf_{t\in(0,1/n),u\in B(v,1/n)} \frac{f(x_0 + tu) + I_Q(x_0 + tu) - f(x_0)}{t^k} \leq 0 \quad \forall n \in \mathbb{N}.$$

By the property of infimum, for each $n \in \mathbb{N}$, there exist $t_n \in (0, 1/n)$ and $v_n \in B(v, 1/n)$ such that

$$\frac{f(x_0 + t_n v_n) + I_Q(x_0 + t_n v_n) - f(x_0)}{t_n^k} < \frac{1}{n}.$$

From these expressions we deduce that $x_n := x_0 + t_n v_n \in Q$, $t_n \to 0^+$ and $v_n \to v$. As $v \in C_0(G, x_0) \subset IT(G, x_0)$ (see (3)) we have that $x_0 + t_n v_n \in G$ for all n large enough. Now we conclude as in the proof of Theorem 2.1. □

Remark 2.3 (1) Notice that we can change, in the hypothesis, $C_0(G, x_0)$ by $IT(G, x_0)$ and the proof is valid.

(2) If $Q = X$ we obtain Corollary 2.3(a) in Ward (1994).

(3) If $G = X$ we obtain Theorem 2.1(i) in Studniarski (1986).

(4) Theorem 2.2 is not better if we put in the conclusion $\forall v \in C_0(G, x_0) \setminus \{0\}$ (instead of $C_0(G, x_0) \cap T(Q, x_0) \setminus \{0\}$), because when $v \notin T(Q, x_0)$ one has $\underline{d}^k(f + I_Q)(x_0, v) = +\infty$ (Lemma 1.3(ii)).

(5) Theorem 2.2 is specially simple to apply when the set Q is defined by parametric equations: $Q = h(S)$, since $(f + I_Q)(x) = f(h(y))$ with $h(y) = x$ and $y \in S$. Outside Q, $(f + I_Q)(x) = +\infty$.

(6) For $k > 1$, Theorem 2.2 is only meaningful for those $v \in X$ satisfying $\underline{d}f(x_0, v) \leq 0$. Indeed, if $\underline{d}f(x_0, v) > 0$ then $\underline{d}(f + I_Q)(x_0, v) > 0$, and from Lemma 1.3(i) it follows that $\underline{d}^k(f + I_Q)(x_0, v) = +\infty$.

3 Sufficient conditions

In this section, we develop sufficient conditions for a point to be a strict local minimizer of order k for problem (1). First, this is made with the Studniarski derivative of $f + I_Q$ distinguishing the cases $k > 1$ and $k = 1$; second, we use the derivative of the Lagrangian function.

We will assume that the space X is finite dimensional from now on and x_0 is a feasible point for problem (1), i.e., $x_0 \in G \cap Q$. In this section, it is not necessary to assume that K has nonempty interior. We denote

$$C(G, x_0) = \{v \in X : dg(x_0, v) \in \operatorname{cl}\operatorname{cone}(-K - g(x_0))\},$$
$$C(f, x_0) = \{v \in X : \underline{d}f(x_0, v) \leq 0\}.$$

These sets are closed cones.

Theorem 3.1 *Let $k > 1$. If $\forall v \in C(G, x_0) \cap T(Q, x_0) \cap C(f, x_0) \setminus \{0\}$ we have*

$$\underline{d}^k(f + I_Q)(x_0, v) > 0,$$

then $x_0 \in \mathrm{Strl}(k, f, G \cap Q)$.

Proof We are going to apply Theorem 2.1(i) in Studniarski (1986). For this aim, we need $\underline{d}^k(f + I_M)(x_0, v) > 0\ \forall v \in T(M, x_0) \cap C(f, x_0) \setminus \{0\}$, where $M = G \cap Q$. Now if $v \in T(M, x_0)$ it is enough to take into account the hypothesis, that

$$T(M, x_0) = T(G \cap Q, x_0) \subset C(G, x_0) \cap T(Q, x_0)$$

(Lemma 4.2 in Jiménez and Novo 2003a) and that $\underline{d}^k(f + I_Q)(x_0, v) \leq \underline{d}^k(f + I_M)(x_0, v)$. □

The case $k = 1$ is considered in the following theorem. Its proof is very similar to that the previous one, only we now apply Theorem 2.1(ii) in Studniarski (1986) instead of Theorem 2.1(i) in Studniarski (1986).

Theorem 3.2 *If $\forall v \in C(G, x_0) \cap T(Q, x_0) \setminus \{0\}$ we have*

$$\underline{d}(f + I_Q)(x_0, v) > 0,$$

then $x_0 \in \mathrm{Strl}(1, f, G \cap Q)$.

Notice that the theorem is not more restrictive if in the hypothesis we put $\forall v \in C(G, x_0) \setminus \{0\}$, because if $v \notin T(Q, x_0)$ we have $\underline{d}(f + I_Q)(x_0, v) = +\infty$. Something similar can be said about Theorem 3.1. In this way, both theorems are very close to the necessary conditions expressed in Theorem 2.2. In this last theorem we cannot substitute $C_0(G, x_0)$ with $C(G, x_0)$ as the following example shows.

Example 3.3 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x) = x_2 + x_2^2$, where $x = (x_1, x_2) \in \mathbb{R}^2$, $G = \{x : x_2 \geq x_1^2\}$, $Q = \{x : x_2 \geq 0\}$ and $x_0 = (0, 0)$. We have $\underline{d}^2(f + I_Q)(x_0, v) = 0$ for $v = (1, 0) \in C(G, x_0) \cap T(Q, x_0)$ and $x_0 \in \mathrm{Strl}(2, f, G \cap Q)$.

The necessary condition of Theorem 2.2 is not sufficient. It is enough to consider in $\mathbb{R}^2$, $Q = \{x : x_2 \geq x_1^4\}$, $G = \{x : g(x) := -x_2 \leq 0\}$, $f(x) = x_2 + x_2^2$ and $x_0 = (0, 0)$, where $x = (x_1, x_2) \in \mathbb{R}^2$. Then $\underline{d}^2(f + I_Q)(x_0, v) > 0\ \forall v \in C_0(G, x_0)$, but $x_0 \notin \mathrm{Strl}(2, f, G \cap Q)$.

Now we are going to discuss sufficient conditions with the Lagrangian function:

$$L_\mu(x) = f(x) + \langle \mu, g(x) \rangle, \tag{4}$$

where $\mu \in Y^*$.

Theorem 3.4 *Assume the following*:

(a) $C(G, x_0) \cap T(Q, x_0) \cap \ker \underline{d} f(x_0, \cdot) = \{0\}$,
(b) $\forall v \in C(G, x_0) \cap T(Q, x_0) \setminus \{0\}$ *there exists* $\mu \in K^+$ *such that* $\langle \mu, g(x_0) \rangle = 0$ *and* $\underline{d}L_\mu(x_0, v) \geq 0$.

Then $x_0 \in \mathrm{Strl}(1, f, G \cap Q)$.

Proof Suppose that $x_0 \notin \mathrm{Strl}(1, f, G \cap Q)$. Then there exist sequences $\alpha_n > 0$ and $x_n \in G \cap Q \cap B(x_0, \alpha_n) \setminus \{x_0\}$ such that

$$\alpha_n \to 0^+ \quad \text{and} \quad \frac{f(x_n) - f(x_0)}{t_n} \le \alpha_n, \tag{5}$$

where $t_n = \|x_n - x_0\|$. Choosing a subsequence if were necessary we can assume that

$$v_n := \frac{x_n - x_0}{t_n} \to v \in T(Q, x_0), \quad \text{with } \|v\| = 1.$$

As g is Hadamard differentiable at x_0, and $x_n = x_0 + t_n v_n$ with $t_n \to 0^+$ and $v_n \to v$ we have

$$\lim_{n\to\infty} \frac{g(x_n) - g(x_0)}{t_n} = dg(x_0, v);$$

and as $x_n \in G$, i.e., $g(x_n) \in -K$, it follows that $dg(x_0, v) \in \mathrm{cl\,cone}(-K - g(x_0))$. Therefore $v \in C(G, x_0)$. From (5) we deduce that

$$\underline{d} f(x_0, v) \le \liminf_{n\to\infty} \frac{f(x_n) - f(x_0)}{t_n} \le 0,$$

and as $v \in C(G, x_0) \cap T(Q, x_0) \setminus \{0\}$ it follows from the hypothesis (a) that $\underline{d} f(x_0, v) < 0$. Using hypothesis (b) there exists $\mu \in K^+$ such that $\langle \mu, g(x_0)\rangle = 0$ and $\underline{d} L_\mu(x_0, v) \ge 0$. From the definition of L_μ (4) it follows that

$$\underline{d} L_\mu(x_0, u) = \underline{d} f(x_0, u) + \langle \mu, dg(x_0, u)\rangle \quad \forall u \in X. \tag{6}$$

Since $dg(x_0, v) \in \mathrm{cl\,cone}(-K - g(x_0))$ and $\mu \in (-K - g(x_0))^- = [\mathrm{cl\,cone}(-K - g(x_0))]^- = \{\mu \in K^+ : \langle \mu, g(x_0)\rangle = 0\}$ we deduce that $\langle \mu, dg(x_0, v)\rangle \le 0$. Moreover, as $\underline{d} f(x_0, v) < 0$, from (6) (for $u = v$) it follows that $\underline{d} L_\mu(x_0, v) < 0$ and we obtain a contradiction. □

Theorem 3.5 *Let* $k > 1$. *If* $\forall v \in C(G, x_0) \cap T(Q, x_0) \cap C(f, x_0) \setminus \{0\}$ *there exists* $\mu \in K^+$ *such that* $\langle \mu, g(x_0)\rangle = 0$ *and* $\underline{d}^k L_\mu(x_0, v) > 0$, *then* $x_0 \in \mathrm{Strl}(k, f, G \cap Q)$.

Proof Suppose that $x_0 \notin \mathrm{Strl}(k, f, G \cap Q)$. Then there exist sequences $\alpha_n > 0$ and $x_n \in G \cap Q \cap B(x_0, \alpha_n) \setminus \{x_0\}$ such that

$$\alpha_n \to 0^+ \quad \text{and} \quad \frac{f(x_n) - f(x_0)}{t_n^k} \le \alpha_n, \tag{7}$$

where $t_n = \|x_n - x_0\|$. Proceeding as in the proof of Theorem 3.4 we can assume that

$$v_n := \frac{x_n - x_0}{t_n} \to v \in T(Q, x_0) \cap C(G, x_0) \cap C(f, x_0), \quad \text{with } \|v\| = 1.$$

By hypothesis, there exists a Lagrangian function L_μ such that $\underline{d}^k L_\mu(x_0, v) > 0$. Now $f(x) \ge L_\mu(x)$ for all $x \in M = G \cap Q$ and therefore

$$\underline{d}^k (f + I_M)(x_0, v) \ge \underline{d}^k (L_\mu + I_M)(x_0, v) \ge \underline{d}^k L_\mu(x_0, v) > 0.$$

In view of (7) it follows that

$$\underline{d}^k (f + I_M)(x_0, v) \le \liminf_{n\to\infty} \frac{f(x_n) - f(x_0)}{t_n^k} \le \liminf_{n\to\infty} \alpha_n = 0,$$

which is a contradiction. □

We finish with a second order sufficient condition.

Theorem 3.6 *If Q is a convex set and $\forall v \in C(G,x_0)\cap T(Q,x_0)\cap C(f,x_0)\setminus\{0\}$ there exists $\mu \in K^+$ such that $\langle \mu, g(x_0)\rangle = 0$,*

$$\underline{d}L_\mu(x_0,u) \geq 0 \quad \forall u \in T(Q,x_0), \tag{8}$$

$$\underline{d}_r^2 L_\mu(x_0,v) > 0, \tag{9}$$

then $x_0 \in$ Strl$(2, f, G\cap Q)$.

Proof Assume that $x_0 \notin \mathrm{Strl}(2,f,G\cap Q)$. Then there exist sequences $\alpha_n \to 0^+$ and $x_n \in G\cap Q\cap B(x_0,\alpha_n)\setminus\{x_0\}$ such that $f(x_n) \leq f(x_0) + \alpha_n\|x_n - x_0\|^2$, this is,

$$\frac{f(x_n)-f(x_0)}{t_n^2} \leq \alpha_n, \tag{10}$$

where $t_n = \|x_n - x_0\|$. As in the proof of Theorem 3.5 we can assume that

$$v_n := \frac{x_n - x_0}{t_n} \to v \in T(Q,x_0)\cap C(G,x_0)\cap C(f,x_0)\setminus\{0\},$$

and so $x_n = x_0 + t_n v_n$. By hypothesis, there exists a Lagrangian function L_μ which satisfies (8), (9) and $\langle \mu, g(x_0)\rangle = 0$. Now, by the definition of L_μ and as $\underline{d}L_\mu(x_0,\cdot)$ is positively homogeneous, we have:

$$L_\mu(x_n) - L_\mu(x_0) - t_n\underline{d}L_\mu(x_0,v_n) = f(x_n) - f(x_0) + \langle \mu, g(x_n)\rangle - \underline{d}L_\mu(x_0, x_n - x_0). \tag{11}$$

As $x_n \in G$ we have $g(x_n) \in -K$ and so $\langle \mu, g(x_n)\rangle \leq 0$, and as $T(Q,x_0) = \mathrm{cl\,cone}(Q - x_0)$ and $x_n \in Q$ it follows that $x_n - x_0 \in T(Q,x_0)$ and by (8), $\underline{d}L_\mu(x_0, x_n - x_0) \geq 0$. From (11) we deduce that

$$\underline{d}_r^2 L_\mu(x_0,v) \leq \liminf_{n\to\infty} \frac{L_\mu(x_n) - L_\mu(x_0) - t_n\underline{d}L_\mu(x_0,v_n)}{t_n^2/2} \leq \liminf_{n\to\infty}\frac{f(x_n)-f(x_0)}{t_n^2/2}.$$

Using (10) it results that $\underline{d}_r^2 L_\mu(x_0,v) \leq \liminf_{n\to\infty} 2\alpha_n = 0$ which contradicts (9). □

Let us observe in connection with the convexity of Q, we only need that $(Q - x_0)\cap B(0,\delta) \subset T(Q,x_0)$ for some $\delta > 0$.

Notice that the first order conditions of previous theorem are very close to the necessary conditions for local minimum given by Bender (1978).

Theorem 3.6 generalizes Theorem 3.1 in Ward (1994) taking $X = Q = \mathbb{R}^n$, $Y = \mathbb{R}^m \times \mathbb{R}^p$ and $K = \mathbb{R}^m_+ \times \{0_p\}$. Indeed, if f and g are Fréchet differentiable at x_0 then $\underline{d}L_\mu(x_0,u) = \nabla L_\mu(x_0)u$ $\forall u \in \mathbb{R}^n$, where $\nabla L_\mu(x_0)$ is the Fréchet derivative of L_μ at x_0. Moreover, if $\nabla L_\mu(x_0) = 0$ then $\underline{d}_r^2 L_\mu(x_0,u) = \underline{d}^2 L_\mu(x_0,u)$ $\forall u \in \mathbb{R}^n$ (this follows from the definitions) and the set $D(x_0)$ of Ward coincides with the cone $C(G,x_0)\cap T(Q,x_0)\cap C(f,x_0)$.

When the involved functions in the problem are twice Fréchet differentiable we immediately obtain the following corollary taking into account that $\underline{d}_r^2 L_\mu(x_0,v) = \nabla^2 L_\mu(x_0)(v,v)$ $\forall v \in X$, where $\nabla^2 L_\mu(x_0)$ is the second order Fréchet derivative of L_μ at x_0.

Corollary 3.7 *Assume that Q is convex and f and g are twice Fréchet differentiable at x_0. If $\forall v \in C(G, x_0) \cap T(Q, x_0) \cap C(f, x_0) \setminus \{0\}$ there exists $\mu \in K^+$ such that $\langle \mu, g(x_0)\rangle = 0$,*

$$\nabla L_\mu(x_0)u \geq 0 \quad \forall u \in T(Q, x_0), \tag{12}$$

$$\nabla^2 L_\mu(x_0)(v, v) > 0, \tag{13}$$

then $x_0 \in \mathrm{Strl}(2, f, G \cap Q)$.

To illustrate these results we give two examples.

Example 3.8 Let $X = Y = \mathbb{R}^2$, $K = \mathbb{R}_+ \times \{0\}$, $x = (x_1, x_2) \in \mathbb{R}^2$, $Q = \{x : x_2 \leq 0\}$, $x_0 = (0, 0)$,

$$f(x) = \begin{cases} -4x_1 - x_2 & \text{if } x_2 \geq 0, \\ -4x_1 - x_2 \sin^2(1/x_2) & \text{if } x_2 < 0, \end{cases}$$

and $g(x) = (x_2 - 4x_1^2, x_1 + x_1^2 + x_2^2)$. It is not difficult to obtain:

$$\underline{d} f(x_0, v) = \begin{cases} -4v_1 - v_2 & \text{if } v_2 \geq 0, \\ -4v_1 & \text{if } v_2 < 0, \end{cases}$$

$C(G, x_0) = \{v : v_1 = 0, v_2 \leq 0\}$, $T(Q, x_0) = Q$ and $C(f, x_0) = \{v : -4v_1 - v_2 \leq 0$ or $v_1 \geq 0\}$, where $v = (v_1, v_2)$. Then, condition (8) becomes

$$\underline{d} L_\mu(x_0, u) = (\mu_2 - 4)u_1 + \mu_1 u_2 \geq 0 \quad \forall u = (u_1, u_2) \in T(Q, x_0).$$

This condition is satisfied if we choose $\mu_1 = 0$ and $\mu_2 = 4$. For

$$v = (0, v_2) \in C(G, x_0) \cap T(Q, x_0) \cap C(f, x_0) \setminus \{0\} = \{(0, v_2) : \ v_2 < 0\},$$

condition (9) is satisfied with $\mu = (0, 4)$ because

$$\underline{d}_r^2 L_\mu(x_0, v) = \underline{d}_r^2 f(x_0, v) + \mu_1 \nabla^2 g_1(x_0)(v, v) + \mu_2 \nabla^2 g_2(x_0)(v, v) = 8v_2^2 > 0$$

since $\underline{d}_r^2 f(x_0, v) = 0$. Therefore, applying Theorem 3.6, we conclude that x_0 is a strict local minimizer of order 2.

Example 3.9 Let $X = \mathbb{R}^3$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $f(x) = 2x_2 - x_3^2$, $g(x) = x_3 - x_2 + x_1^2$, $Q = \{x : x_3 \geq |x_1|^{3/2}\}$ and $x_0 = (0, 0, 0)$. It is easy to verify the following facts:

$$C(f, x_0) = \{v = (v_1, v_2, v_3) : v_2 \leq 0\},$$

$$C(G, x_0) = \{v : -v_2 + v_3 \leq 0\}, \qquad T(Q, x_0) = \{v : v_3 \geq 0\},$$

$$C(G, x_0) \cap T(Q, x_0) \cap C(f, x_0) = \{v : v_2 = 0, \ v_3 = 0\},$$

$$L_\mu(x) = (2 - \mu)x_2 + \mu x_3 + \mu x_1^2 - x_3^2 \quad \text{with } \mu \geq 0,$$

$$\nabla L_\mu(x_0)u \geq 0 \quad \forall u \in T(Q, x_0) \quad \text{if } \mu = 2, \quad \text{and}$$

$$\nabla^2 L_\mu(x_0)(v, v) = 2(\mu v_1^2 - v_3^2) > 0$$

$$\forall v \in C(G, x_0) \cap T(Q, x_0) \cap C(f, x_0) \setminus \{0\} \quad \text{if } \mu > 0.$$

Applying Corollary 3.7 we obtain that x_0 is a strict local minimizer of order 2.

Conclusions

We have attained both necessary and sufficient conditions for a point to be a strict local minimizer of order k for a nonsmooth problem with generalized inequality constraints and a set constraint. The inequality constraints are assumed to be Hadamard differentiable while the objective is an arbitrary functional. These conditions are expressed in terms of the Studniarski derivatives in certain directions belonging to the contingent cone to the constraint set. These results are important in stability analysis and in the study of the convergence of numerical procedures. A difficulty of these results, from a practical point of view, is to determine the contingent cone (to the constraint set) and to calculate some directional derivatives.

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