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A survey of recent developments in multiobjective optimization

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Abstract Multiobjective Optimization (MO) has many applications in such fields as the Internet, finance, biomedicine, management science, game theory and engineering. However, solving MO problems is not an easy task. Searching for all Pareto optimal solutions is expensive and a time consuming process because there are usually exponentially large (or infinite) Pareto optimal solutions. Even for simple problems determining whether a point belongs to the Pareto set is $\mathcal{NP}$-hard. In this paper, we discuss recent developments in MO. These include optimality conditions, applications, global optimization techniques, the new concept of epsilon Pareto optimal solution, and heuristics.

Keywords Multiobjective optimization · Pareto optimality · Duality · Generalized convexity

The general MOP can be written as follows:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{1}$$

where $X \subseteq \mathbb{R}^n$ is a nonempty set, $f(x) = (f_1, f_2, \dots, f_k)^T : X \to \mathbb{R}^k$ is a vector-valued function.

The feasible region X is usually expressed by a number of inequality constraints, that is, $X = \{x \in \mathbb{R}^n \mid g_j(x) \le 0, j = 1, 2, \dots, l\}$. If all the objective functions and the constraint functions are *linear*, then (1) is called a *Multiobjective Linear Programming Problem* (MOLP). If at least one of the objective functions or the constraint functions is *nonlinear*,

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(1) is called a *Nonlinear Multiobjective Optimization Problem* (NMOP). Throughout this paper, we will consider NMOPs. If all the objective functions and the constraint set are *convex*, then (1) becomes a *convex MOP*. When at least one of the objective functions or the constraint set is *nonconvex*, then (1) becomes *nonconvex MOP*.

Here, we generally aim at minimizing all the objective functions at the same time if there is no conflict between the objective functions. However, for general MOP, the objective functions are in contradiction to each other.

The present paper is organized as follows. In Sect. 1, we introduce Pareto optimality which is the main concept of multiobjective optimization. Optimality conditions and duality results are surveyed in Sects. 2 and 3. In Sect. 4, we discuss the optimality conditions and duality for multiobjective fractional programming problems. Multiobjective integer linear programming problems will be discussed briefly in Sect. 5. Section 6 covers multiobjective optimization approaches while some multiobjective combinatorial optimization problems are covered in Sect. 7. Finally, we discuss some applications of MO.

1 Pareto optimality

As we mentioned before there are conflicts between the objective functions of a MOP. Thus, there should be an ordering concept in $\mathbb{R}^k$ in order to solve the problem comparing the objective function values. Pareto optimality (Pareto 1964), named after the Italian economist Vilfredo Pareto (1906), is a measure of efficiency in multiobjective optimization. The concept has a wide range of applications in economics, game theory, multiobjective optimization, and the social sciences generally. According to this concept, we look for objective vectors such that none of the components of each of those vectors can be improved without deterioration to at least one of the other components of the vector. Therefore, Pareto optimality can be defined mathematically as follows:

Definition 1 (Pareto Optimality) A point $x^* \in X$ with $f(x^*)$ is called (globally) Pareto optimal (or efficient or non-dominated, or non-inferior), if and only if there exists no point $x \in X$ such that $f_i(x) \leq f_i(x^*)$ for all $i = 1, 2, \ldots, k$ and $f_j(x) < f_j(x^*)$ for at least one index $j \in \{1, 2, \ldots, k\}$.

We can also define the concept of local Pareto optimality similar to the local optimality concept in scalar valued optimization.

Definition 2 (Local Pareto Optimality) A point $x^* \in X$ with $f(x^*)$ is called locally Pareto optimal, if and only if there exists $\delta > 0$ such that x^* is Pareto optimal in $S \cap B(x^*, \delta)$.

Here, $B(x^*, \delta)$ is the open ball of radius δ centered at point $x^* \in X$, that is, $B(x^*, \delta) = \{x \in \mathbb{R}^n \mid \|x - x^*\| < \delta\}$. Note that every globally Pareto optimal solution is a locally Pareto optimal solution. However, the converse is not always true unless there are some assumptions in the problem.

Theorem 1 *When the MOP is convex then every locally Pareto optimal solution is also globally Pareto optimal.*

Proof Suppose to the contrary that x^* is a locally Pareto optimal solution which is not a globally Pareto optimal. Then there exists $y \in X$ such that $f(y) < f(x^*)$ and $f(y) \neq f(x^*)$.

Consider a scalar $z = \beta x^* + (1-\beta)y$ such that $z \in S \cap B(x^*, \delta)$ and $0 < \beta < 1$. By the convexity of the objective functions, we have

$$f(z) \leq \beta f(x^*) + (1-\beta) f(y) \leq f(x^*)$$

and

$$f(x^*) + (1-\beta) f(y) \neq f(x^*),$$

which show that x^* would not be a locally Pareto optimal solution. □

The following theorem stated without a proof (which is similar to the above theorem's proof) extends the above result. For the proof, see Ruíz-Canales and Rufián-Lizana (1995).

Theorem 2 *Consider the MOP with a convex feasible set and quasiconvex objective functions. If at least one of the objective functions is strictly quasiconvex, then every locally Pareto optimal solution is also globally Pareto optimal.*

Before presenting further results, let us introduce the weakly Pareto optimality.

Definition 3 A point $x^* \in X$ with $f(x^*)$ is called weakly Pareto optimal, if and only if there exists no point $x \in X$ such that $f_i(x) < f_i(x^*)$ for all $i = 1, 2, \ldots, k$.

It is easy to see that every Pareto optimal solution is weakly Pareto optimal.

For multiobjective discrete optimization, the concept of Pareto optimality can be stated in the similar way as we defined in continuous optimization. Finding all Pareto optimal solutions is often computationally problematic since there are usually exponentially (or infinite) large Pareto optimal solutions. Furthermore, for even the simplest problems and even for two objectives, determining whether a point belongs to the Pareto optimal set is $\mathcal{NP}$-hard (Papadimitriou and Yannakakis 2000). One way to handle those problems is to introduce approximate Pareto solutions.

Definition 4 Given a scalar $\epsilon > 0$, an ϵ-approximate Pareto optimal set, denoted by P_ϵ, is a subset of X such that there is no other solution y such that $(1+\epsilon) f_i(y) \leq f_i(x)$ for all $x \in P_\epsilon$ and for some i.

This definition says that every other solution is almost dominated by some solution in P_ϵ, i.e. there is a solution in P_ϵ that is within a factor of ϵ in all objectives.

Next, we introduce results regarding the size of the ϵ-approximate Pareto optimal set by Papadimitriou and Yannakakis (2000). Before presenting the next theorems, we need some assumptions.

Let solutions of a multiobjective optimization problem be polynomially bounded and polynomially recognizable in the size of the problem. We also assume that if $f_i(x) > 0$; then $f_i(x)$ is between $2^{-p(|x|)}$ and $2^{p(|x|)}$ for some polynomial p and for $i = 1, 2, \ldots, k$. Therefore, $f_i(x)$ can be calculated in polynomial time for any feasible point x.

The following theorem was obtained by Papadimitriou and Yannakakis (2000).

Theorem 3 *For any MOP and any $\epsilon > 0$, there is always an ϵ approximate Pareto set consisting of a number of solutions that is polynomial in the input size of the problem and $\frac{1}{\epsilon}$.*

2 Optimality conditions

In this section we discuss optimality conditions for the multiobjective optimization problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \le 0, \\ & x \in X, \end{aligned} \tag{2}$$

where X is an open set in $\mathbb{R}^n$.

We first present necessary conditions which can be derived without any convexity assumptions. We then introduce sufficient optimality conditions under suitable convexity assumptions. In order to define these optimality conditions we need the following notations.

Let

$$I(x) = \{j \in \{1, 2, \ldots, l\} \mid g_j(x) = 0\}$$

be the index set of the active constraints in the inequality constraints at x, and let $D = \{x \in \mathbb{R}^n \mid g(x) \le 0, x \in X\}$.

Theorem 4 (Karush-Kuhn-Tucker 1951) *Let $f, g_j, j = 1, 2, \ldots, l$ be continuously differentiable in an open set containing the feasible set of* (2), *and let x^* be a locally Pareto optimal point. Further, assume that the vectors $\nabla g_j(x^*)$, $j \in I(x^*)$ are linearly independent. Then the following optimality conditions hold*:

(i) $g_j(x^*) \le 0$, $j = 1, 2, \ldots, l$.
(ii) *There exist vectors $\alpha \in \mathbb{R}^k$ and $\lambda \in \mathbb{R}^l$ such that*

$$\begin{aligned} &\sum_{i=1}^{k} \alpha_i \nabla f_i(x^*) + \sum_{j=1}^{l} \lambda_j \nabla g_j(x^*) = 0, \\ &\lambda_j g_j(x^*) = 0, \quad \lambda_j \ge 0, \ j = 1, 2, \ldots, l, \\ &\sum_{i=1}^{k} \alpha_i = 1, \quad \alpha_i \ge 0, \ i = 1, 2, \ldots, k. \end{aligned}$$

Let us introduce the scalar-valued function

$$F(x) = \sum_{i=1}^{k} \alpha_i f_i(x). \tag{3}$$

It is easy to see that the above conditions are equivalent to the claim that x^* is a Karush-Kuhn-Tucker point of the corresponding optimization problem with scalar valued function $F(x)$ and the same constraints. Furthermore, Geoffrion (1968) has shown that, if Problem (2) is convex, then x^* is Pareto optimal in Problem (2) if and only if x^* is global minimum of the corresponding scalar valued function over the same constraint set as those in (2). This argument gives the fact that the above optimality conditions are sufficient for x^* to be (globally) Pareto optimal for convex problems.

In general, the optimality conditions do not provide the complete Pareto optimal set. We also consider second order optimality conditions which are necessary for a point x^* to be a local Pareto optimal solution of (2).

Theorem 5 (Second-order necessary conditions) *Let the objective and the constraint functions of Problem* (2) *be twice continuously differentiable at a feasible point* x^*. *Further, assume that the vectors* $\nabla g_j(x^*)$, $j \in I(x^*)$ *are linearly independent. Then the following optimality conditions hold*:

(i) $g_j(x^*) \le 0$, $j = 1, 2, \dots, l$,
(ii) *There exist vectors* $\alpha \in \mathbb{R}^k$ *and* $\lambda \in \mathbb{R}^l$ *such that*

$$\sum_{i=1}^{k} \alpha_i \nabla f_i(x^*) + \sum_{j=1}^{l} \lambda_j \nabla g_j(x^*) = 0,$$

$$\lambda_j g_j(x^*) = 0, \quad \lambda_j \ge 0, \ j = 1, 2, \dots, l,$$

$$\sum_{i=1}^{k} \alpha_i = 1, \quad \alpha_i \ge 0, \ i = 1, 2, \dots, k.$$

(iii) $d^T(\sum_{i=1}^{k} \alpha_i \nabla^2 f_i(x^*) + \sum_{j=1}^{l} \lambda_j \nabla^2 g_j(x^*))d \ge 0$ *for all* $d \in \{d \in \mathbb{R}^n \setminus \{0\} \mid \nabla f_i(x^*)^T d \le 0, i = 1, 2, \dots, k, \nabla g_j(x^*)^T d = 0, j \in I(x^*)\}$.

For the proof, see Wang (1991).

Further optimality conditions for multiobjective programming can be found in, for example, Wang (1991) and Miettinen (1999).

Several authors have been interested in sufficient optimality conditions for multiobjective optimization problems in connection with generalized convexity. Those generalized convexities include F-convexity in Hanson (1981), (F, ρ)-convexity in Preda (1992), ρ-convexity in Vial (1983), (F, α, ρ, d)-convexity in Liang et al. (2003) and (F, α, ρ, d)-type I functions in Hachimi and Aghezzaf (2004). The latter generalized convexity, (F, α, ρ, d)-type I functions, unifies the previous generalized convexity concepts. Therefore, Hachimi and Aghezzaf (2004) presented sufficient optimality conditions for the special case of the multiobjective optimization problem where all the objective functions and the constraint functions are of (F, α, ρ, d)-type I.

3 Duality in MO problems

In this section we discuss duality theory for the multiobjective optimization problem (2).

Tanina and Sawaragi (1979) introduced a duality theory for multiobjective optimization problems using a vector-valued Lagrangian function and exploring the properties of a primal and dual point to set maps. Bitran (1981) also presented a duality theory for the multiobjective optimization problem based on a vector-valued Lagrangian. In their duality theory, a matrix of dual variables associated with the constraint functions in the original problem. Duality theory of multiobjective optimization can also be found in many other articles and books including Nakayama (1985), Luc (1984), Singh et al. (1996), Das and Nanda (1997) and Sawaragi et al. (1985).

Mond and Weir (1981) have introduced a pair of symmetric dual nonlinear programs under pseudo-convexity (pseudo-concavity) assumptions and derived the corresponding weak and strong duality of these programs. These dual programs were generalized by Egudo (1989) for multiobjective optimization problems. Here we discuss duality results established in Aghezzaf and Hachimi (2000) for the following Mond-Weir dual problem by Egudo (Mond-Weir 1981).

$$\max \quad f(y) \tag{4}$$

$$\text{s.t.} \quad \sum_{i=1}^{k} \lambda_i \nabla f_i(y) + \sum_{j=1}^{l} \mu_j \nabla g_j(y) = 0, \tag{5}$$

$$\mu^T g(y) \geq 0, \tag{6}$$

$$\lambda \geq 0, \quad \mu \geq 0, \quad \lambda \in \mathbb{R}^k, \quad \mu \in \mathbb{R}^l, \tag{7}$$

$$\lambda^T e = 1, \tag{8}$$

where $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^l$.

Before presenting the duality results for the MOP, let us introduce some definitions following Aghezzaf and Hachimi (2000).

Definition 5 (f, g) is said to be (def) with respect to η at $x^* \in X$ if there exists a vector function $\eta(x, x^*)$ defined on $D \times X$ such that, for all $x \in D$, (conds) hold.

(i) def: type I,
conds:

$$f(x) - f(x^*) \geq \nabla f(x^*)\eta(x, x^*),$$
$$-g(x^*) \geq \nabla g(x^*)\eta(x, x^*).$$

(ii) def: weak strictly-pseudoquasi-type I,
conds:

$$f(x) \leq f(x^*), \quad f(x) \neq f(x^*) \quad \Rightarrow \quad \nabla f(x^*)\eta(x, x^*) < 0,$$
$$-g(x^*) \leq 0 \quad \Rightarrow \quad \nabla g(x^*)\eta(x, x^*) \leq 0.$$

(iii) def: strong pseudoquasi-type I,
conds:

$$f(x) \leq f(x^*), \quad f(x) \neq f(x^*) \quad \Rightarrow \quad \nabla f(x^*)\eta(x, x^*) \leq 0, \quad \nabla f(x^*)\eta(x, x^*) \neq 0,$$
$$-g(x^*) \leq 0 \quad \Rightarrow \quad \nabla g(x^*)\eta(x, x^*) \leq 0.$$

(iv) def: weak quasistrictly-pseudo-type I,
conds:

$$f(x) \leq f(x^*), \quad f(x) \neq f(x^*) \quad \Rightarrow \quad \nabla f(x^*)\eta(x, x^*) \leq 0,$$
$$-g(x^*) \leq 0 \quad \Rightarrow \quad \nabla g(x^*)\eta(x, x^*) \leq 0, \quad \nabla g(x^*)\eta(x, x^*) \neq 0.$$

(v) def: weak strictly pseudo-type I,
conds:

$$f(x) \leq f(x^*), \quad f(x) \neq f(x^*) \quad \Rightarrow \quad \nabla f(x^*)\eta(x, x^*) < 0,$$
$$-g(x^*) \leq 0 \quad \Rightarrow \quad \nabla g(x^*)\eta(x, x^*) < 0.$$

We are ready now to introduce the following duality theorems by Aghezzaf and Hachimi (2000).

Theorem 6 (Weak Duality) *Suppose that any of the following holds*:

(a) $(f, \mu g)$ *is strong pseudoquasi-type I at* y *with respect to* η *and* $u > 0$;
(b) $(f, \mu g)$ *is weak strictly pseudoquasi-type I at* y *with respect to* η;
(c) $(f, \mu g)$ *is weak strictly pseudo-type I at* y *with respect to* η

for all feasible x *for* (2) *and all feasible* (y, λ, μ) *for* (4–8). *Then the following cannot hold*:

$$f(x) \leq f(y), \qquad f(x) \neq f(y). \tag{9}$$

Proof Suppose that $f(x) \leq f(y)$ and $f(x) \neq f(y)$.

Since $(f, \mu g)$ is strong pseudoquasi-type I, we have

$$\nabla f(y)\eta(x, y) \leq 0, \qquad \nabla f(y)\eta(x, y) \neq 0,$$
$$\nabla g(y)\eta(x, y) \leq 0.$$

Therefore, counting $\lambda \geq 0$ and $\lambda \neq 0$, we can write

$$\left(\sum_{i=1}^{k} \lambda_i \nabla f_i(y) + \sum_{j=1}^{l} \mu_j \nabla g_j(y)\right)^T \eta(x, y) < 0.$$

This is a contradiction to (5).

For the cases (b) and (c), we can give similar proof to the above. □

Theorem 7 (Strong Duality) *Let* x^* *be a Pareto optimal solution of* (2). *Suppose that* x^* *satisfies a constraint qualification in Marusciac* (1982) *for* (2). *Then there exist* $\lambda^* \in \mathbb{R}^l$ *and* $\mu^* \in \mathbb{R}^m$ *such that* (x^*, λ^*, μ^*) *is feasible for* (4–8). *If also weak duality holds between Problem* (2) *and Problem* (4–8) *then* (x^*, λ^*, μ^*) *is a Pareto optimal solution for* (4–8).

Proof According to the Karush-Kuhn-Tucker conditions by Marusciac (1982) for the multiobjective optimization problem, there exist $\lambda > 0$, $\mu \geq 0$ such that

$$\sum_{i=1}^{k} \lambda_i^* \nabla f_i(x^*) + \sum_{j-1}^{l} \mu_j^* g_j(x^*) = 0,$$
$$\mu_j^* g_j(x^*) = 0, \quad j = 1, 2, \ldots, l.$$

This means that (x^*, λ^*, μ^*) is feasible for Problem (4–8).

We now apply the weak duality theorem and can have the desired result. □

Aghezzaf and Hachimi (2000) also presented the following converse duality theorem. We do not present the proof here because it is quite extensive.

Theorem 8 (Converse Duality) *Let* (x^*, λ^*, μ^*) *be a Pareto point for* (4), *and let the hypotheses of the weak duality theorem hold. If the* $n \times n$ *Hessian matrix* $\nabla^2(\sum_{i=1}^{k} \lambda_i^* f_i(x^*) + \sum_{j=1}^{l} \mu_j^* g_j(x^*))$ *is negative definite and* $\sum_{j-1}^{l} \mu_j^* \nabla g_j(x^*) \neq 0$, *then* x^* *is a Pareto point for* (2).

The above results can be further extended for generalized Mond-Weir duality according to Aghezzaf and Hachimi (2000). Recently, these results were generalized by Hachimi and Aghezzaf (2004) under the (F, α, ρ, d)-type I assumptions.

4 Multiobjective fractional programming problems

Fractional programming problems arise from many applied areas including portfolio selection, stock cutting, game theory and numerous decision problems in management science. Here we discuss some recent efficiency and duality results about multiobjective fractional programming problems.

Efficiency conditions and duality for single-objective fractional programming problem has been studied by many researchers including Craven (1981), Weir (1990), Khan and Hanson (1997), and Reddy and Mukherjee (1999). Hanson (1961) introduced invex functions which are generalized convex functions. Under this invexity assumptions, Khan and Hanson (1997), and Reddy and Mukherjee (1999) have obtained some optimality conditions and duality results for fractional programming problems. Singh and Hanson (1991) have applied invex functions to fractional multiobjective programming problems and derived some duality results. Jeyakumar and Mond (1992) have introduced generalized invex functions, called V-invex functions and extended the results by Singh and Hanson (1991). Recently, Liang et al. (2001) introduced (F, α, ρ, d)-convexity, a unified formulation of the generalized convexity, and derived optimality conditions and duality results for fractional programming problems. Therefore, (F, α, ρ, d)-convexity has been applied to multiobjective fractional programming problems in Liang et al. (2003).

Consider the Multiobjective Fractional Programming Problem (MFP):

$$\begin{aligned} \min \quad & \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_k(x)}{g_k(x)}\right) \\ \text{s.t.} \quad & h(x) \leq 0, \\ & x \in X, \end{aligned} \tag{10}$$

where $f = (f_1, f_2, \ldots, f_k)^T$, $g = (g_1, g_2, \ldots, g_k)^T$ and $h = (h_1, h_2, \ldots, h_l)^T$ are vector valued functions defined on an open set $X \in \mathbb{R}^n$. Suppose that $f(x) \geq 0$, $g(x) > 0$ for all $x \in X$. Further we assume that $f_i,\ g_i,\ i = 1, 2, \ldots, k$ and $h_j,\ j = 1, 2, \ldots, l$ are continuously differentiable over X.

Definition 6 A function $F : \mathbb{R}^n \to \mathbb{R}$ is sublinear if for any $r \in \mathbb{R}_+$, $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}^n$,

$$F(\alpha_1 + \alpha_2) \leq F(\alpha_1) + F(\alpha_2), \qquad F(r\alpha) = rF(\alpha).$$

Definition 7 Let $\rho \in \mathbb{R}$, $\alpha : X \times X \to \mathbb{R}_+ \setminus \{0\}$ and $d : X \times X \to \mathbb{R}_+$. A differentiable function $\phi : X \to \mathbb{R}$ is (F, α, ρ, d)-convex at $x^* \in X$ if for any $x \in X$, $F(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ is sublinear, and $\phi(x)$ satisfies the following conditions:

$$\phi(x) - \phi(x^*) \geq F(x, x^*; \alpha(x, x^*)\nabla\phi(x^*)) + \rho d(x, x^*).$$

In the definition, the function ϕ is (F, ρ)-convex if $\alpha(x, x_0) = 1$ for all $x, x_0 \in X$; ϕ is ρ-invex if $F(x, x_0; \alpha(x, x_0)\nabla\phi(x_0)) = \nabla\phi(x_0)^T\eta(x, x_0)$ for a certain mapping $\eta : X \times X \to \mathbb{R}^n$.

Liang et al. (2003) proved the following property of (F, α, ρ, d)-convex functions.

Theorem 9 *Let $X \subset \mathbb{R}^n$ be an open set, and let p and q be real valued differentiable functions defined on X. If p and $-q$ are (F, α, ρ, d)-convex at $x_0 \in X$ and $p(x) \geq 0$, $q(x) > 0$*

for all $x \in X$, then $\frac{p}{q}$ is (F, α, ρ, d)-convex at x_0, where

$$\bar{\alpha}(x, x_0) = \frac{\alpha(x, x_0)q(x_0)}{q(x)}, \qquad \bar{\rho} = \rho\left(1 + \frac{p(x_0)}{q(x_0)}\right), \quad \text{and} \quad \bar{d}(x, x_0) = \frac{d(x, x_0)}{q(x)}.$$

We will sketch the proof as follows. We can write the following statement for any $x \in X$:

$$\frac{p(x)}{q(x)} - \frac{p(x_0)}{q(x_0)} = \frac{p(x) - p(x_0)}{q(x)} - \frac{p(x_0)(q(x) - q(x_0))}{q(x)q(x_0)}.$$

Therefore, using the (F, α, ρ, d)-convexity assumptions we can rewrite the above as follows:

$$\begin{aligned}\frac{p(x)}{q(x)} - \frac{p(x_0)}{q(x_0)} &\geq \frac{1}{q(x)}(F(x, x_0; \alpha(x, x_0)\nabla p(x_0)) + \rho d(x, x_0)) \\ &\quad + \frac{p(x_0)}{q(x)q(x_0)}(F(x, x_0; -\alpha(x, x_0)\nabla q(x_0)) + \rho d(x, x_0)).\end{aligned}$$

Now, applying sublinearity assumptions to the right hand side of the above inequality, we can have the desired result.

The following efficiency conditions and duality results, stated without proofs, were also obtained by Liang et al. (2003) using the above theorem.

Theorem 10 *Let x^* be a feasible solution of the MFP. Suppose that there exist $\lambda \in \mathbb{R}^k_+$, and $\mu \in \mathbb{R}^l$ such that*

$$\sum_{i=1}^{k} \lambda_i \nabla \frac{f_i(x^*)}{g_i(x^*)} + \sum_{j=1}^{l} \mu_j \nabla h_j(x^*) = 0,$$

$$g(x^*)\mu = 0, \qquad e^T\lambda = 1, \quad \lambda, \mu \geq 0.$$

If f_i and $-g_i$, $i = 1, 2, \ldots, k$ are $(F, \alpha_i, \rho_i, d_i)$-convex at x^*, h_j, $j = 1, 2, \ldots, l$ are $(F, \beta_j, \zeta_j, c_j)$-convex at x^*, and

$$\sum_{i=1}^{k} \lambda_i \bar{\rho}_i \frac{\bar{d}_i(x, x^*)}{\bar{\alpha}_i(x, x^*)} + \sum_{j=1}^{l} \mu_j \bar{\zeta}_j \frac{\bar{c}_j(x, x^*)}{\bar{\beta}_j(x, x^*)} \geq 0,$$

where $\bar{\alpha}_i(x, x^*) = \frac{\alpha_i(x, x^*)g_i(x^*)}{g_i(x)}$, $\bar{\rho}_i = \rho_i(1 + \frac{f_i(x^*)}{g_i(x^*)})$, and $\bar{d}_i(x, x^*) = \frac{d_i(x, x^*)}{g(x)}$, then x^* is a global Pareto optimal solution for the MFP.

The Mond-Weir dual of the problem can be written as follows:

$$\begin{aligned}\max \quad & \frac{f(y)}{g(y)} \\ \text{s.t.} \quad & \sum_{i=1}^{k} \lambda_i \nabla \frac{f_i(x^*)}{g_i(x^*)} + \sum_{j=1}^{l} \mu_j \nabla h_j(x^*) = 0, \\ & g(x^*)\mu \geq 0, \qquad e^T\lambda = 1, \quad \lambda, \mu \geq 0, \\ & y \in X.\end{aligned}$$

The following Duality results have been obtained by Liang et al. (2003).

Theorem 11 (Weak Duality) *Suppose that x^* is a feasible solution of the MOP and (y^*, λ^*, μ^*) is a feasible solution of the corresponding Mond-Weir dual problem. If f_i and g_i, $i = 1, 2, \ldots, k$ are $(F, \alpha_i, \rho_i, d_i)$-convex at y^*, h_j, $j = 1, 2, \ldots, l$ are (F, β, ζ_j, c_j)-convex at y^*, and the inequality*

$$\sum_{i=1}^{k} \lambda_i {\rho^*}_i \frac{{d^*}_i(x^*, y^*)}{{\alpha^*}_i(x^*, y^*)} + \sum_{j=1}^{l} \mu_j {\zeta^*}_j \frac{{c^*}_j(x^*, y^*)}{{\beta^*}_j(x^*, y^*)} \geq 0$$

holds, where ${\alpha^*}_i(x^*, y^*) = \frac{\alpha_i(x^*, y^*) g_i(y^*)}{g_i(x^*)}$, ${\rho^*}_i = \rho_i(1 + \frac{f_i(y^*)}{g_i(y^*)})$, and ${d^*}_i(x^*, y^*) = \frac{d_i(x^*, y^*)}{g(x^*)}$, then the following cannot hold:

$$\frac{f(x^*)}{g(y^*)} \leq \frac{f(y^*)}{g(y^*)} \quad \text{and} \quad \frac{f(x^*)}{g(y^*)} \neq \frac{f(y^*)}{g(y^*)}.$$

Theorem 12 (Strong Duality) *Suppose that x^* is an efficient solution of the MFP and the constraint qualification in Maeda (1994) holds at x^*. Then there exists $\lambda^*, \mu^* \in \mathbb{R}^k_+ \times \mathbb{R}^l_+$ such that (x^*, λ^*, μ^*) is a feasible solution of the corresponding dual problem, and the objective function values of the MFP and its dual at the corresponding points are equal. If the assumptions about the generalized convexity and the inequality in the weak duality theorem are satisfied, then (x^*, λ^*, μ^*) is a Pareto optimal solution of the MFP.*

Further duality results on multiobjective fractional programming problems can be found in Liang et al. (2003).

For a comprehensive review of theory and applications of multiobjective programming, the reader is referred to Hillermeier (2001), Jahn (2004), Pardalos et al. (1995), Steuer (1986) and Miettinen (1999).

5 Multiobjective integer programming problems

Multiobjective problems with discrete variables arise naturally in many practical applications on several different areas, such as transportation and location allocation problems, capital budgeting, and project selection. We consider the following multiobjective integer linear programming problem (MOILP):

$$\begin{aligned} \min \quad & Cx \\ \text{s.t.} \quad & Ax \leq b, \\ & x \geq 0, \qquad x \in \mathbb{Z}^n, \end{aligned} \tag{11}$$

where C is an $k \times n$ matrix, A is an $l \times n$ matrix and b is an l vector.

Bitran (1977, 1979) developed methods, based on enumerative schemes, for solving multiobjective linear programs with binary variables. Klein and Hannan (1982) proposed a sequential procedure for generating all the efficient points of the multiobjective integer programming problem. In this method, one of the objective functions is minimized subject to more constrained feasible sets determined by the other objective functions and previously found efficient solutions. Chalmet et al. (1986) studied the MOILP based on the weighted sum method. Since the weighted sum method can find only the set of *supported* (see Sect. 7.3) nondominated solutions, they introduced an additional constraint to ensure access to the nondominated solutions. Recently, Klamroth et al. (2004) studied multiobjective integer linear programming involving integer programming duality. They proposed a general ϵ-constraint approach, resulting in particular single objective integer programming problems to generate Pareto solutions.

Let $\mathcal{F}$ be the set of all nonincreasing functions $F : \mathbb{R}^{k+l-1} \to \mathbb{R}$, that is,

$$\mathcal{F} = \{F : \mathbb{R}^{k+l-1} \to \mathbb{R} \mid F(a) \geq F(b),\ \forall a, b \in \mathbb{R}^l,\ a \leq b\}.$$

Let $J_i = \{1, 2, \dots, k\} \setminus \{i\}$ and C_{J_i} be the submatrix of C containing of the rows in J_i, i.e., obtained by deleting its ith column.

Klamroth et al. (2004) have shown the following result based on duality of integer programming and the ϵ-constraint method for MO problems.

Theorem 13 *x^* is a Pareto optimal solution of the MOILP, if and only if there exists an index $i \in \{1, 2, \dots, k\}$ and a function $F^* \in \mathcal{F}$ such that x^* is optimal for the following composite integer programming problem*:

$$\begin{aligned} \min \quad & x^T c^i - F^*(C_{J_i} x, b) \\ \text{s.t.} \quad & Ax \leq b, \\ & x \geq 0, \qquad x \in \mathbb{Z}^n, \end{aligned} \tag{12}$$

where c^i is the ith column of matrix C.

They have also introduced a cutting plane and a branch and bound methods to generate optimal functions F.

6 Methods

In this section, we discuss the most popular existing deterministic methods for solving the multiobjective optimization problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{13}$$

where $f(x) = (f_1, f_2, \dots, f_k)^T : X \to \mathbb{R}^k$. In particularly, the weighting method, ϵ-constraint method and weighted L_p-metric method will be considered.

6.1 Weighting method

This is the most widely used traditional approach to MO, where the attainment of each objective is weighted by its importance to the decision maker. These are then used to express a single combined objective function to evaluate the decision. The method was first presented by Zadeh (1963). Let us assign a weight, say $w_i \geq 0$, to each objective function. Those weights are normalized by the following constraint:

$$\sum_{i=1}^{k} w_i = 1.$$

The MOP becomes the following scalar-valued optimization problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^{k} w_i f_i(x) \\ \text{s.t.} \quad & x \in X. \end{aligned} \tag{14}$$

The following results can be found in, for example, Miettinen (1999).

Theorem 14 *A solution of the weighting problem* (14) *is weakly Pareto optimal for* (13).

Theorem 15 *A solution of the weighting problem* (14) *is Pareto optimal for* (13) *when the weighting coefficients are strictly positive, that is,* $w_i > 0$ *for all* $i = 1, 2, \ldots, k$.

Theorem 16 *Suppose that* (13) *is convex. If* x^* *is a Pareto solution for* (13), *then there exist* w_i $(i = 1, 2, \ldots, k)$ *such that* x^* *is optimal for* (14).

Note that the last theorem does not generalize to nonconvex problems. For more theoretical results regarding the method can be found in Miettinen (1999).

6.2 ϵ-constraint method

This method was presented in Haimes et al. (1971). It chooses one individual objective f_j, $j \in \{1, 2, \ldots, k\}$, to be minimized and all the other objective functions are converted into constraints setting upper bounds. The MOP becomes the following scalar-valued optimization problem:

$$\begin{aligned} \min \quad & f_j \\ \text{s.t.} \quad & f_i(x) \le \epsilon_i, \quad \text{for all } i = 1, 2, \ldots, k, \; i \ne j, \\ & x \in X. \end{aligned} \tag{15}$$

Theorem 17 *A solution of the epsilon-constraint problem* (15) *is weakly Pareto optimal for* (13).

Theorem 18 *A feasible point* x^* *is Pareto optimal for* (13) *if and only if it is a solution of* (15) *for every* $j = 1, 2, \ldots, k$, *where* $\epsilon_i = f_i(x^*)$ *for* $i = 1, 2, \ldots, k$ *and* $i \ne j$.

Proofs can be found in any textbooks about MO, for example, Miettinen (1999).

6.3 Weighted L_p-metric method

This method chooses a desired point $y \in \mathbb{R}^k$ and search for a optimal solution which is as close as possible to this point. The L_p metric ($p \in [1, \infty) \cup \{\infty\}$) is used to generate optimal solutions. These metrics can also be weighted in order to produce different Pareto optimal solutions.

$$\begin{aligned} \min \quad & \left(\sum_{i=1}^{k} w_i |f_i(x) - y_i|^p \right)^{1/p} \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{16}$$

where $w_i \ge 0$ for all $i = 1, 2, \ldots, k$.

Theorem 19 *A solution of the weighted L_p-metric problem* (16) (*when* $1 \le p < \infty$) *is Pareto optimal for* (13) *if the solution is unique.*

Theorem 20 *A solution of the weighted L_p-metric problem* (16) (*when* $1 \le p < \infty$) *is Pareto optimal for* (13) *when the coefficients are strictly positive, that is,* $w_i > 0$ *for all* $i = 1, 2, \dots, k$.

For proofs, see, for example, Chankong and Haimes (1983).

7 Multiobjective combinatorial optimization problems

Combinatorial optimization is the process of finding best solutions in a well defined discrete problem space. Combinatorial optimization problems occur in many fields as management, finance, marketing, economics, engineering, biology etc. Combinatorial optimization is extensively studied by many researchers. There are a number of classic textbooks and thousands of articles in this field, however, multiobjective combinatorial optimization has not been studied widely. In Sect. 5, we discussed multiobjective integer linear programming problems briefly. Here we discuss some of the well known multiobjective combinatorial optimization problems. In particularly, multiobjective shortest path problems, the multiobjective minimum spanning tree problem and the multiobjective zero-one knapsack problem are considered. Excellent bibliographical survey of multiobjective combinatorial optimization can be found in Ehrgott and Gandibleux (2000).

7.1 Multiobjective shortest path problems

Multiobjective shortest path problems (MSPP) appear to be the most intensively studied multiobjective combinatorial optimization problems. MSPPs have many applications in different fields including finance, telecommunication and transportation. MSPPs are well known to be $\mathcal{NP}$-hard (Serafini 1986). In MOSPPs, the number of parameters associated with each arc is equal to the number of components of the objective function.

Let $G = (V, E)$ be an undirected and connected graph with the set V of vertices and the set E of edges joining vertices in V. We have multiple cost functions $f_i : E \to \mathbb{R}^+$, $i = 1, 2, \dots, k$ such that a vector $(f_1(e), \dots, f_k(e))$ is the multiple costs associated with an edge $e \in E$. The objective is to find a shortest path in the graph from the source node $s \in V$ to the terminal node $t \in V$. Let $\mathcal{P}$ be the set of all paths between nodes s and t. Then each path $P \in \mathcal{P}$ has multiple costs $F_i(T) = \sum_{p \in P} f_i(e)$, $i = 1, 2, \dots, k$ associated with it. Formally, the general MSPP can be formulated as follows:

$$\begin{aligned} \min \quad & (F_1(T), \dots, F_k(T))^T \\ \text{s.t.} \quad & P \in \mathcal{P}. \end{aligned} \tag{17}$$

Warburton (1987) proposes a method for finding approximate Pareto optimal solution to the problem for any degree of accuracy. The method is polynomial in the problem size and the accuracy.

The biobjective shortest path problem is the most studied problem among MSPPs. Most algorithms for solving the biobjective shortest path problem are applicable to the general MSPP, however, they would add difficulties to the implementation. Huarng et al. (1996) present computational experiments comparing several existing methods for finding Pareto optimal solution of the problem and report that the label correcting algorithm (Brumbaugh-Smith and Shier 1989) is the fastest among those methods. The principle of their algorithm is similar to the one of Dijkstra's shortest path algorithm (Dijkstra 1959), except:

- There is a set of labels at each node,
- The algorithm does not permanently label nodes.

Skriver and Andersen (2000) later improved the label correcting algorithm by imposing some simple domination conditions. They use Dijkstra's shortest path algorithm with each parameter in order to find the upper bounds on the two objectives and set bounds on all labels. Those bounds help to prune some edges from further consideration.

Some other algorithms based on dynamic programming for solving the problem can be found in Kostreva and Wiecek (1993), Sniedovich (1988), Warburton (1987) and Henig (1985).

7.2 The multiobjective minimum spanning tree problem

The minimum spanning tree (MST) problem is to find a least cost tree which spans an edge weighted graph. The cost of a tree is the summation of the weights of all edges in the tree. There are several efficient algorithms for solving the problem (Kruskal 1956, Prim 1957, and Sollin). These algorithms can be found in many textbooks on graph theory and network flows, for example, in Ahuja et al. (1993). Recently, the multiobjective minimum spanning tree problem (MMSTP), an extension of the MST problem, has received great attraction due to some practical demands. The MMSTP can be stated as follows:

Let $G = (V, E)$ be an undirected and connected graph. Multiple cost functions $f_i : E \rightarrow \mathbb{R}^+$, $i = 1, 2, \ldots, k$, and the vector $(f_1(e), \ldots, f_k(e))$ is the multiple cost associated with an edge $e \in E$. Let $ST(G)$ be the set of all spanning trees of G. Each tree $T \in ST(G)$ has multiple costs $F_i(T) = \sum_{e \in T} f_i(e)$, $i = 1, 2, \ldots, k$ associated with it. Then the MMSTP can be formulated as:

$$\begin{aligned} \min \quad & (F_1(T), \ldots, F_k(T))^T \\ \text{s.t.} \quad & T \in ST(G). \end{aligned} \tag{18}$$

Finding the set of Pareto optimal solutions of the MMSTP is $\mathcal{NP}$-hard problem (Camerini et al. 1984). Corley (1985) proposed a method which is a generalization of Prim's algorithm to find efficient trees. Their algorithm is based on the following result.

Theorem 21 *Let $T = (V, E^*)$ be a Pareto optimal spanning tree. Then, $f(v)$ is a Pareto minimum among the edges in the unique cut formed by eliminating v from T.*

We will sketch the proof as follows. Let us assume the contrary, i.e., there exists an edge e in the cut such that $e \leq v$ and $e \neq v$. Then $T' = (V, E^* \setminus \{v\} \cup \{e\})$ is also a spanning tree. Moreover, $F(T') \leq F(T^*)$ and $F(T') \neq F(T^*)$ which contradict our assumption.

Corley's algorithm is very similar to Prim's algorithm for the single objective case except it finds set of trees by adding a new edge to each of the previous trees along a cut at each iteration. Every new edge is selected as a Pareto minimum among the corresponding cut. However, it has been shown that the algorithm may find spanning trees which are not efficient in Hamacher and Ruhe (1994). Hamacher and Ruhe (1994) modified Corley's algorithm as it excludes trees which are not Pareto optimal in each iteration.

Hamacher and Ruhe (1994) also presented an approximative algorithm which finds a subset of the set of biobjective minimum spanning trees. Their algorithm has two phases.

In the first phase, it finds *extremal* efficient spanning trees, which are on the border of the convex hull of the set $\{F_1(T), F_2(T) \mid T \in ST(G)\}$. It is known (Hamacher and Ruhe 1994) that an extremal efficient spanning tree is a solution of the parametric problem

$$\begin{aligned} \min \quad & \lambda_1 F_1(T) + \lambda_2 F_2(T) \\ \text{s.t.} \quad & T \in ST(G), \end{aligned}$$

for some $\lambda_1 + \lambda_2 = 1$ and $\lambda_i > 0$, $i = 1, 2$. In the second phase, they use a neighborhood search to find nonextremal efficient spanning trees based on the extremal efficient spanning trees. However, there is no guarantee that it finds all nonextremal efficient spanning trees.

Ramos et al. (1998) proposed a two phase method for finding the set of efficient spanning trees. First, it finds the set of extremal efficient spanning trees. Second part of the algorithm uses the branch and bound technique to obtain the set of nonextremal efficient spanning trees.

There are many other algorithms such as an algorithm of the Kruskal-type can in Schweigert (1990) and genetic algorithm approach in Zhou and Gen (1999).

7.3 The multiobjective zero one knapsack problem

The zero one knapsack problem is one of the classic $\mathcal{NP}$-complete combinatorial optimization problems. The multiobjective zero one knapsack problem, which is also $\mathcal{NP}$-complete, is an extension of the zero one knapsack problem and has many practical applications including transportation planning, packaging and loading, conservation biology, capital budgeting and financial management. The problem can also be seen as a subproblem of larger problems including the multiobjective assignment problem and multiobjective scheduling problems. The multiobjective zero one knapsack problem can be formulated as follows:

$$\begin{aligned} \min \quad & Cx && (19) \\ \text{s.t.} \quad & a^T x \le b, && (20) \\ & x \in \{0, 1\}^n, && (21) \end{aligned}$$

where C is a $k \times n$ matrix with nonnegative entries, a is an n vector and b is a scalar.

When $k = 1$, the above problem is called the single constraint multiobjective zero one knapsack problem. An excellent survey of algorithmic approaches for solving this problem can be found in Martello and Toth (1990) and Martello et al. (1997).

A solution to the following integer programming problem is well known to be an efficient solution of the multiobjective zero one knapsack problem and called the *supported* efficient solution.

$$\begin{aligned} \min \quad & \sum_{i=1}^{k} \lambda_i c^i x \\ \text{s.t.} \quad & a^T x \le b, \\ & x \in \{0, 1\}^n, \end{aligned}$$

for some $\sum_{i=1}^{k} = 1$ and $\lambda_i \ge 0$, $i = 1, 2, \ldots, k$. Here c^i $(i = 1, 2, \ldots, k)$ is the ith row vector of the matrix C.

Rosenblatt and Sinuany-Stern (1989) presented a branch and bound algorithm to determine the set of supported efficient solutions. However, it is known that there might be nonsupported efficient solutions of the problem. Visée et al. (1996) have shown an example problem where the supported efficient solutions constitute only small percentage of all efficient solutions. The work by Rosenblatt and Sinuany-Stern (1989) was continued in Eben-Chaime (1996). Eben-Chaime employed a network model for the dynamic programming solution of knapsack problems in the case of two objectives. For given weights, a solution to the above knapsack problem is the longest path from the source node to the terminal node in the network.

Recently, Ulungu and Teghem (1994, 1997) and Visée et al. (1996) suggested two-phase methods. Their algorithms construct the set of all supported efficient solutions in the first phase. In the second phase, branch and bound algorithms are applied to find nonsupported efficient solutions based on the supported efficient solutions.

Villarreal and Karwan (1981) proposed dynamic programming approaches for solving the integer multiobjective multiple constraint knapsack problem. In this case, Constraint (21) is changed by $x \in \mathbb{Z}$ and Constraint (20) is changed by $Ax \leq b$, where A is a $k \times n$ matrix and b is a k vector. Recently, Klamroth and Wiecek (2000a, 2000b) also suggested dynamic programming approaches for solving the integer multiobjective knapsack problem. They also discussed how their methods can apply to different models including the zero one multiobjective knapsack problem, multiple constraint knapsack problem, and time dependent knapsack problem. These approaches can be seen as generalizations of works by Garfinkel and Nemhauser (1972) and Ibaraki (1987).

Some other approaches for solving the multiobjective knapsack problem include a genetic algorithmic approach by Zhou and Gen (1999) and a tabu search approach by Gandibleux and Freville (2000).

8 Applications

Most of the optimization problems arising in practice have several objectives which have to be optimized simultaneously. These problems include different problems in engineering design, portfolio selection, game theory, decision problems in management science, web access problems, query optimization in databases etc.

In this section we discuss some of those application problems in multiobjective optimization since it is impossible to mention all of them. In particular, the web access problem, the portfolio selection problem and capital budgeting problem are considered.

8.1 The web access problem

Here, we consider the web access problem first studied in Etzioni et al. (1996). Suppose that one wants to retrieve a list of records from the world-wide web. Then the problem can be formulated as follows.

We are given a collection of n information sources, each of which has a known time delay t_i, cost c_i and probability p_i of providing the needed information, from the world-wide web. We will end up accessing a subset $S \in \{1, 2, \ldots, n\}$ of this sites. Then the total cost is in the form

$$C(s) = \sum_{i \in S} c_i .$$

The total delay can be calculated as follows:

$$T(s) = \max_{i \in S} t_i .$$

Then the failure probability of source i is $1 - p_i$. We also assume that the success of a given source is independent of the success or failure of the other sources. Therefore the overall probability of the result will be:

$$P(s) = 1 - \prod_{i \in S} (1 - p_i).$$

If we assume that the above functions are the objective functions of our problem, of course, the problem will be to minimize the first two objective functions and maximize the third function.

Papadimitriou and Yannakakis (2000) proved the following result regarding the problem.

Theorem 22 *Given an $\epsilon > 0$, there is a polynomial algorithm in the input size of the problem and $1/\epsilon$ for constructing the ϵ approximate set for the web access problem.*

Etzioni et al. (1996) studied two models, cost model and reward model, with respect to the web access problem. In the cost model, we have to seek an order where the expected overall cost is minimized. In the later model, we assume that a constant known reward should be collected if at least one source returns a correct answer. Then we seek a schedule of maximum expected reward. They showed that the second problem is $\mathcal{NP}$-hard and developed approximation algorithms for those problems.

8.2 Portfolio selection problems

In traditional way, we have the portfolio selection problem in the mean-variance formulation by Markowitz (1952) which formulation has been used in finance for the last half century. According to this theory, in the model the risk is measured with variance thus generating a quadratic programming model. Markowitz (1952) model has been frequently criticized as not consistent with axiomatic models of preferences for choice under risk. In Markowitz model, it is usually assumed that at least one of the following hypotheses should be verified: the utility function of the investor is quadratic, the returns of the stocks have normal distributions. However, quadratic utility functions are not consistent in economics. According to the second hypothesis, negative returns are very unlikely. Therefore, Cloquette et al. (1995) have shown that stock returns have asymmetrical or leptokurtic distribution based on empirical tests.

On the other hand, the multidimensional nature of the problem has been emphasized by researchers such as Jacquillat (1972), Bell and Raiffa (1988), Khoury et al. (1993), and Spronk and Hallerbach (1997). During the portfolio evaluation, decision makers face several criteria including return criterion, risk criterion, liquidity criterion and size criteria. Based on these criteria, the portfolio selection problem is usually formulated as an multiobjective optimization problem. Several authors have employed multiobjective methods for the portfolio selection problem including Hurson and Zopounidis (1995), Zopounidis (1999) and Bouri et al. (2002). Further results will be discussed later on.

An excellent bibliographic survey about multiobjective optimization in finance can be found in Steuer and Na (2003).

8.3 Capital budgeting problem

Capital budgeting is the problem of determining whether or not investment projects such as building a new plant or investing in a long-term venture are worthwhile. In other words, the capital budgeting decision a company faces is to select a subset of available projects that gives the highest earnings to the company while it does not exceed a certain budget. Traditionally capital budgeting concerns a single objective function which is usually in the form of a maximization of company's revenue. Capital budgeting process was first formulated as an optimization problem in Weingartner (1963). Ansoff (1968), Thanassoulis (1985), and Lee and Lerro (1974) later studied capital budgeting with multiple objectives. Typical capital budgeting model with multiple criteria is usually expressed as a multiobjective knapsack problem which we discussed in Sect. 7.3.

Klamroth and Wiecek (2000b) presented a time-dependent capital budgeting problem as an multiobjective knapsack type problem.

Let $S = \{1, \ldots, n\}$ be the a set of projects that could be performed and we assume that only one project can be performed at a time. Let $\Delta = \{\sigma \mid \sigma(i) \in S, i = 1, \ldots, p(\sigma);\ \sigma(i) \neq \sigma(j), i \neq j\}$. Every sequence δ of projects represents the order of the projects to be performed. Fixed available budget b, independent of the investment decisions, and the cost $a(i)$ of project i, $i \in S$, are given. There is a vector valued function $c_i(t) = [c_i^1(t), \ldots, c_i^m(t)]^T$ corresponding to the job i, $i = 1, \ldots, n$, related to choosing the project i at time t. These functions might represent different criteria such as the time, revenue, appreciation, and sustainability needed to accomplish the projects. In particularly, let $c_i^1(t)$ be the time needed to accomplish the project i, and let $c_i^j(t)$ be the other criteria such as revenue, appreciation and risk if the project is selected at time t. The objective function of the capital budgeting model can be defined as follows:

$$f(\delta) = (f_1(\delta), \ldots, f_m(\delta))^T,$$

where

$$f_1(\delta) = -\sum_{i=1}^{p(\delta)} c^1_{\delta(i)}(t^i),$$

$$f_2(\delta) = \sum_{i=2}^{p(\delta)} c^2_{\delta(i)}(t^i),$$

$$\ldots$$

$$f_m(\delta) = \sum_{i=1}^{p(\delta)} c^m_{\delta(i)}(t^i).$$

If we assume that the performing the next project is started immediately after the current project finishes, the time constraints for the sequence δ can be written as follows:

$$t^1 = 0,$$

$$t^{i+1} = t^i + c^1_{\delta(i)}(t^i).$$

The aim of the problem is to minimize the time needed to accomplish the projects while maximizing all other criteria such as revenue, appreciation and sustainability.

$$\begin{aligned}
\max \quad & f(\delta) \\
\text{s.t.} \quad & \sum_{i=1}^{p(\delta)} a(\delta(i)) \le b, \\
& t^1 = 0, \\
& t^{i+1} = t^i + c^1_{\delta(i)}(t^i), \\
& \delta \in \Delta.
\end{aligned} \tag{22}$$

The above problem is called the *time-dependent multiobjective knapsack problem* (TDMKP). Klamroth and Wiecek (2000a) have also shown that the dynamic programming approach by Kostreva and Wiecek (1993) can be applied for finding the set of all efficient solutions to the TDMKP when all cost coefficients and the budget are integer. The basic idea of their approach is based on the following theorem:

Theorem 23 *Let us assume that $t_1 + c_i^1(t_1) \le t_2 + c_i^1(t_2)$ for all $0 \le t_1 \le t_2$ and $i \in S$. Then an efficient sequence of projects δ of the k-TDMKP accomplished at time $t^{p(\delta)+1}$ has the property that each subsequence of projects $\{\delta(1), \ldots, \delta(s)\}$, $1 \le s < p(\delta)$ accomplished at time t^{s+1} is an efficient sequence of the $\sum_{i=1}^{s} a_{\delta(i)}$-TDMKP.*

Here, k-TPMKP is the resulting problem after changing the first constraint of Problem (22) by $\sum_{i=1}^{p(\delta)} a(\delta(i)) = k$.

It is not difficult to see that the set of efficient solutions of Problem (22) is the set of efficient solutions among the union of efficient solutions of k-TDMKPs, $k = 1, 2, \ldots, b$.

9 Conclusions

In this paper we have presented a number of selected recent theoretical results for MO. Those results include optimality conditions and duality theory on different MO problems including differentiable multiobjective programming problems, multiobjective fractional programming problems and multiobjective integer linear programming problems. We have also discussed some multiobjective combinatorial optimization problems. There exist several widely used approaches for solving MO problems. These approaches are discussed in Sect. 6. MO has many applications in many different fields. A large number of real world practical problems are usually expressed as MO problems. We presented few applications in the last section of the paper.

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