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A two-stage stochastic integer programming approach as a mixture of Branch-and-Fix Coordination and Benders Decomposition schemes

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Abstract We present an algorithmic approach for solving two-stage stochastic mixed 0–1 problems. The first stage constraints of the Deterministic Equivalent Model have 0–1 variables and continuous variables. The approach uses the *Twin Node Family* (*TNF*) concept within the so-called *Branch-and-Fix Coordination* algorithmic framework to satisfy the *nonanticipativity* constraints, jointly with a Benders Decomposition scheme to solve a given *LP* model at each *TNF integer set*. As a pilot case, the structuring of a portfolio of Mortgage-Backed Securities under uncertainty in the interest rate path on a given time horizon is used. Some computational experience is reported.

Keywords Stochastic programming · Benders Decomposition · Branch-and-Fix Coordination · MBS portfolio structuring

Introduction

Very frequently, mainly in problems with a given time horizon to exploit, some coefficients in the objective function and the right-hand-side (*rhs*, for short) vector and, to a lesser extent,

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the constraint matrix are not known with certainty when decisions are to be made, but some information is available. This type of problem can also have logistic constraints. These circumstances allow Stochastic Integer Programming (*SIP*) to be used to solve 0–1 programs under uncertainty. It has a broad field of application, mainly in production planning, energy generation planning, and finance. See Ziemba and Mulvey (1998), Uryasev and Pardalos (2001) and Wallace and Ziemba (2005), among others.

The main focus and contribution of the paper is on the design and computational assessment of a *Branch-and-Fix Coordination (BFC)* scheme to obtain the optimal mixed 0–1 solution to the two-stage stochastic program, where parameter uncertainty is represented by a set of scenarios. The special structure of the *Deterministic Equivalent Model (DEM)* is exploited. The relaxation of the *nonanticipativity* constraints of the first stage variables allows for the independent solution of the so-called scenario *cluster*-related problems. The constraints related to the 0–1 variables are satisfied by using a scheme that is based on the *Twin Node Family* (*TNF*) concept introduced in Alonso-Ayuso, Escudero, and Ortuño, (2003). The scheme is specifically designed to coordinate node branching selection and pruning, and 0–1 variable branching selection and fixing at each *Branch-and-Fix* (*BF*) tree.

An important feature of our approach with respect to some other approaches for large-scale two-stage *SIP* is that it addresses the problem where 0–1 variables and continuous variables have nonzero elements in their first stage constraints. The difficulty of the algorithmic approach is greatly increased by having continuous variables in the first stage constraints. The approach proposed in this paper introduces the notion of *TNF integer sets* and considers the *compact* representation of the *DEM* at each one of them. By fixing those variables to the nodes' values, the *DEM* has only continuous variables. By exploiting the remaining structure of the model a Benders Decomposition (Benders, 1962) allows the *nonanticipativity* constraints on the first stage continuous variables to be satisfied, thus obtaining the *LP* optimal solution for the given *TNF integer set*. The conditions for pruning a *TNF* are stated. So a mixture of the *BFC* approach and a Benders Decomposition scheme is proposed.

Given a time horizon, a set of available securities and an available budget for investment, the *Mortgage-Backed Securities Portfolio Structuring Problem* (*MBSPSP*) is concerned with determining the subset of securities that will be included in the portfolio as well as determining the fraction of the face value to be considered for each security under uncertainty in the interest rate path over the time horizon. The problem in question can be viewed as the problem considered in Escudero (1995), see also Zenios (1993), but forcing an upper bound on the number of securities in the portfolio and requiring a minimum conditional on the face value for each security, among other constraints for structuring the portfolio. The problem is used as a pilot case for validating the stochastic integer programming approach to be introduced in the paper, since it can be treated as a two-stage stochastic mixed 0–1 model. The first stage constraints in the problem have the 0–1 variables for determining the securities in the portfolio, and the continuous variables for determining the fraction of the related face value to be considered. The second stage constraints determine the net available cash at the so-called dedicated time periods and represent certain types of mismatching related to durations and present values under each scenario. Some computational experience is reported to compare the quality of the solution obtained by our approach and the optimization of the average scenario problem. A comparison is also performed, solving the *DEM* by a plain use of a state-of-the-art optimization engine.

The remainder of the paper is organized as follows. Section 1 sets out the *MBSPSP*. Section 2 presents the mixed 0–1 *DEM*. Section 3 presents the *BFC* algorithmic framework for problem solving. Section 4 reports on the computational results. Section 5 concludes.

1 Statement of the problem

Let a *security* be defined as an asset that entitles the holder to a return over a time horizon. In our case, the asset is a financial right comprising a principal and a yield backed by a mortgage (called *Mortgage-Backed Security*, *MBS* for short), whose principal can be prepaid and even delayed. So each security (e.g. a loan) to be considered for inclusion in the portfolio should have the following features: principal amortization structure up to maturity; (usually adjustable) yield to be paid over time; partial or full potential prepayment, such that the prepayment of a security will affect its duration and the cashflow generated; potential delay of the principal's amortization; and type of risk measured by the interest rate weighting factor, the so-called *Option Adjusted Spread* (*OAS*).

The *OAS* is used to weight the discount rate for obtaining the present value of a given security. It can be interpreted as the implied risk penalty for a particular security, see Hayre and Lauterbach (1991) and Ben-Dov, Hayre and Pica, (1992), among others. Note: The value 0 (respectively, 1) means a neutral factor for an *additive* (respectively, *multiplicative*) scheme, see below.

MBS securitization consists of structuring a portfolio from a set of available securities. The problem in question is how to securitize *MBS* under uncertainty in the interest rate path over a given time horizon, which implies uncertainty in yield, prepayment and payment delay of securities. As said above, uncertainty is represented by a set of scenarios. One characteristic of our problem is the need to resort to an integer formulation (rather than using only continuous variables). That need is motivated by the problem's requirements in terms of the maximum number of securities in the portfolio, the *MBS* face value conditional minimum, etc.

There are three important issues that have not been considered in the paper, namely the recursive contingent claim option Dunn and McConnell, 1981 and (Schwartz and Torous, 1989), the transaction costs on exercising the options, Stanton (1995) and Longstaff (2004) and the heterogeneity among mortgage borrowers for determining the *MBSs* Deng, Quigley, and Van Order (2000). Although important issues, they are not crucial for assessing the performance of the proposed algorithmic approach for optimizing two-stage *SIP* problems.

A feasible structuring of a portfolio requires two types of constraints to be satisfied, namely: (a) first stage constraints that force some types of relationships between securities, e.g. an upper bound on the number of securities in the portfolio, investment budget for their total face value, equilibrium in the total face value of the different types of security, etc.; and (b) second stage constraints for basically analyzing the performance of the securities' portfolio over the time horizon under all the scenarios. Typical constraints are the portfolio's cashflow balance equation including the cash inflow and outflow due to the liability satisfaction for each dedicated time period under any scenario, the lower and upper bounds for the net available cash in those periods under any scenario, the requirement that the present value of the portfolio should not be smaller than the present value of the liabilities under any scenario, the requirement that the absolute mismatchings of the unit durations and the present values of the *MBS* in the portfolio and the set of securities from which it is taken should not be greater than given thresholds, etc.

There are different types of goals. *Scenario tracking* through the minimization of the expected difference between the *MBS* portfolio's and liabilities' duration mismatching and the optimal related mismatching under any scenario is covered in Escudero (1995). However, we consider the minimization of the expected absolute mismatching of the durations of the *MBS* portfolio and the liabilities over the scenarios. This is another approach for hedging the return on the investment against small changes in the interest rate over the time horizon, for given portfolio management fees and others.

The notation used throughout the paper is as follows.

Sets:

$\mathcal{I}$, set of available securities.
$\mathcal{T}$, set of time periods.
Ω, set of scenarios to represent uncertainty.

Deterministic parameters:

b_1, maximum number of securities allowed in the *MBS* portfolio to be structured.
$\vec{b}_2$, right-hand-side vector for the subsystem of constraints for the 0–1 variables δ_i, $i \in \mathcal{I}$.
A_2, constraint matrix for the subsystem of constraints for the 0–1 variables δ_i, $i \in \mathcal{I}$.
b_3, investment budget available at time period 0 to create the *MBS* portfolio.
h, net unit return on investment b_3 (including management fees) as a target to reach for each dedicated time period.
α_t, amortization of investment considered for time period t, for $t \in \mathcal{T}$, such that

$$b_3 = \sum_{t \in \mathcal{T}} \alpha_t. \tag{1}$$

φ_t, liability to be met at (the end of) dedicated time period t, for $t \in \mathcal{T}$. This can be expressed as

$$\varphi_t = \alpha_t + h \sum_{\tau \in \mathcal{T}:\tau > t} \alpha_\tau. \tag{2}$$

ℓ, latest dedicated time period where the cash inflow from the portfolio is committed to satisfy the liabilities, for $\ell \in \mathcal{T}$.
$\underline{\sigma}$, $\overline{\sigma}$, unit lower and upper bounds of the investment face value allowed to be kept in cash at any dedicated time period, respectively.
$\underline{s}_t$, $\overline{s}_t$, lower and upper bounds of cash available at dedicated time period t, respectively, for $t = 1, \ldots, \ell$, such that

$$\underline{s}_t = \underline{\sigma} \sum_{\tau \in \mathcal{T}:\tau > t} \alpha_\tau \tag{3}$$

$$\overline{s}_t = \overline{\sigma} \sum_{\tau \in \mathcal{T}:\tau > t} \alpha_\tau. \tag{4}$$

f_i, principal (face) value of security i, for $i \in \mathcal{I}$.
$\underline{x}_i$, $\overline{x}_i$, conditional lower and upper bounds of the principal (face) value f_i for security i to be included in the *MBS* portfolio, respectively, for $i \in \mathcal{I}$.
t_i, maturity period for security i (i.e., last period where any payment is planned). Note: $t_i \in \mathcal{T}$, $\forall i \in \mathcal{I}$.
a_{it}, unit amortization of principal of security i at (the end of) time period t, for $t = 1, \ldots, t_i$, $i \in \mathcal{I}$.

A_{it}, accumulated unit amortization of principal of security i at time period t, for $t = 1, \ldots, t_i$, $i \in \mathcal{I}$, such that

$$A_{it} = \sum_{\tau=1,\ldots,t} a_{i\tau}, \tag{5}$$

so that $A_{it} = 1$ for $t = t_i$.

c_i^{ξ}, extra interest rate to be charged for each time period with payment delay in security i, for $i \in \mathcal{I}$.

o_i, *OAS* assigned to security i, for $0 \leq o_i$, $i \in \mathcal{I}$.

$\overline{\tau}$, maximum number of time periods where a principal's amortization payment can be delayed for any security. Note: $\overline{\tau} \leq |\mathcal{T}| - t_i$, $i \in \mathcal{I}$.

Uncertain and scenario related parameters:

w^{ω}, weight factor assigned to scenario ω, for $\omega \in \Omega$, such that $\sum_{\omega \in \Omega} w^{\omega} = 1$.

r_t^{ω}, interest rate at time period t under scenario ω, for $t \in \mathcal{T}$, $\omega \in \Omega$. The scenarios for the interest rate path over the time horizon can be generated from the binomial lattice approach given in Black, Derman, and Toy (1990) as in Zenios (1993). See other schemes in Frauendorfer and Schürle (1998) and Mulvey and Thorlacius (1998). See Kleywegt, Shapiro, and Homem-de Mello (2001) and Ahmed and Shapiro (2002) for some approaches to approximating the underlying two-stage stochastic program with integer recourse via sampling, among other approaches for dealing with the size of the scenario set.

c_{it}^{ω}, unit yield of security i at (the end of) time period t under scenario ω. This is a function of the interest rate r_t^{ω} and the security itself under scenario ω, for $t = 1, \ldots, t_i$, $i \in \mathcal{I}$, $\omega \in \Omega$. Notice that $r_1^{\omega} = r_1$, where r_1 is the interest rate at time $t = 1$.

β_{it}^{ω}, (partial or full) prepayment of the accumulated unit amortization of the principal of security i at time period t under scenario ω, for $t = 1, \ldots, t_i$, $i \in \mathcal{I}$, $\omega \in \Omega$. This is a function of the security, the age of the security, the month of the year and the interest rate at the given period. The function is usually obtained by statistical means. However, see Kang and Zenios (1992) for some complete prepayment models.

$\kappa_{it\tau}^{\omega}$, unit payment delay in τ time periods of the amortization of the principal of security i that is due at time period t under scenario ω, for $\sum_{\tau=1,\ldots,\overline{\tau}} \kappa_{it\tau}^{\omega} \leq a_{it}$, $t = 1, \ldots, t_i$, $\tau = 1, \ldots, \overline{\tau}$, $i \in \mathcal{I}$, $\omega \in \Omega$. This is a function of the security, the month of the year, the number of delay periods and the interest rate at the given time period.

e_{it}^{ω}, net unit amortization of principal of security i at time period t plus interest payments due to principal delays. This can be expressed as

$$\begin{aligned} e_{it}^{\omega} = a_{it}\left[1 - \sum_{j=1}^{t-1} \beta_{ij}^{\omega} - \left(1 + c_{it}^{\omega}\right) \sum_{\tau=1}^{\overline{\tau}} \kappa_{it\tau}^{\omega}\right] \\ + \sum_{\tau:1 \leq t-\tau \leq \overline{\tau}} a_{i\tau}\left[1 + (t-\tau)\left(c_{i\tau}^{\omega} + c_i^{\xi}\right)\right]\kappa_{i\tau(t-\tau)}^{\omega} \end{aligned} \tag{6}$$

γ_{it}^{ω}, unit return on security i at time period t under scenario ω, for $t = 1, \ldots, t_i + \overline{\tau}$, $i \in \mathcal{I}$, $\omega \in \Omega$. Under mild assumptions, this can be expressed as

$$\gamma_{it}^{\omega} = e_{it}^{\omega} + \beta_{it}^{\omega} A_{it+1} + c_{it}^{\omega} A_{it} \left(1 - \sum_{j=1}^{t-1} \beta_{ij}^{\omega}\right). \tag{7}$$

Γ_i^{ω}, present value of unit return on security i under scenario ω, for $i \in \mathcal{I}$, $\omega \in \Omega$. This can be expressed as

$$\Gamma_i^{\omega} = \sum_{t=1,\ldots,t_i} \gamma_{it}^{\omega} \prod_{\tau=1,\ldots,t} \left(1 + o_i \cdot r_{\tau}^{\omega}\right)^{-1}. \tag{8}$$

Note that o_i has been used as a *multiplicative* factor of r_{τ}^{ω}, so the zero-value is not allowed. However, it is allowed when the *OAS* is used as an *additive* factor. Notice that the greater the risk penalty *OAS* o_i is, the smaller the present value Γ_i is, $\forall i \in \mathcal{I}$.

d_i^{ω}, change in the present unit value of the return on security i due to a small change in the interest rate over the time horizon under scenario ω, for $i \in \mathcal{I}$, $\omega \in \Omega$. This can be expressed as

$$d_i^{\omega} = -\left(1/\Gamma_i^{\omega}\right) \sum_{t=1,\ldots,t_i} t \cdot \gamma_{it}^{\omega} \cdot o_i \prod_{\tau=1,\ldots,t} \left(1 + o_i \cdot r_{\tau}^{\omega}\right)^{-1}. \tag{9}$$

Note: $|d_i^{\omega}|$ is the so-called *modified Macaulay duration* for a flat interest rate over a time horizon.

Φ^{ω}, present value of the liabilities under scenario ω, for $\omega \in \Omega$. This can be expressed as

$$\Phi^{\omega} = \sum_{t \in \mathcal{T}} \varphi_t \prod_{\tau=1,\ldots,t} \left(1 + r_{\tau}^{\omega}\right)^{-1}. \tag{10}$$

d'^{ω}, change in the present unit value of the liabilities due to a small change in the interest rate over the time horizon under scenario ω, for $\omega \in \Omega$. This can be expressed as

$$d'^{\omega} = -(1/\Phi^{\omega}) \sum_{t \in \mathcal{T}} t \cdot \varphi_t \prod_{\tau=1,\ldots,t} \left(1 + r_{\tau}^{\omega}\right)^{-1}. \tag{11}$$

Additional deterministic parameters:

$\overline{z}$, upper bound on the absolute difference between the unit duration of the *MBS* portfolio to be structured and the unit duration of the set of securities available $\mathcal{I}$.

$\overline{v}$, upper bound on the absolute difference between the present unit value of the *MBS* portfolio to be structured and the present unit value of the set of securities available $\mathcal{I}$.

Note. The parameters $\overline{z}$ and $\overline{v}$ allow some slackness in the representation of the *MBS* portfolio with respect to the set of securities available.

Structuring variables. These are 0–1 variables, such that

$$\delta_i = \begin{cases} 1, & \text{if security } i \text{ is selected for the } MBS \text{ portfolio to be structured} \\ 0, & \text{otherwise.} \end{cases} \qquad \forall i \in \mathcal{I}$$

Face value variables. These are continuous variables, such that

x_i, principal (face) value of f_i for security i that is included in the *MBS* portfolio, where $\underline{x}_i \leq x_i \leq \overline{x}_i$ for $\delta_i = 1$ and, otherwise zero, for $i \in \mathcal{I}$.

Performance variables. These are continuous variables, such that

s_t^ω, cash availability at (the end of) dedicated time period t under scenario ω, for $t = 1, \ldots, \ell$, $\omega \in \Omega$.

y^ω, free variable to take the (positive or negative) difference of the *MBS* portfolio's duration and the liabilities' duration under scenario ω, for $\omega \in \Omega$.

z^ω, free variable to take the (positive or negative) difference of the unit durations of the *MBS* portfolio and the set of available securities $\mathcal{I}$ under scenario ω, for $\omega \in \Omega$.

v^ω, free variable to take the (positive or negative) difference of the present unit values of the *MBS* portfolio and the set of available securities $\mathcal{I}$ under scenario ω, for $\omega \in \Omega$.

2 Mixed 0–1 Deterministic Equivalent Model (*DEM*)

The goal is to structure the *MBS* portfolio (i.e. obtain $x_i, i \in \mathcal{I}$) to dedicate cash availability to satisfy the liabilities for the given set of dedicated time periods, and to protect the present value of investment, such that a set of constraints is satisfied by the portfolio.

The following is a *compact* representation of the mixed 0–1 *DEM* for the two-stage stochastic *MBSPSP* with complete recourse.

Objective: To minimize the expected duration mismatching of the *MBS* portfolio and the liabilities over the scenarios, subject to the constraints (13)–(25).

$$Z_{IP} = \min \sum_{\omega \in \Omega} w^\omega |y^\omega| \tag{12}$$

Constraints:

$$\sum_{i \in \mathcal{I}} \delta_i \leq b_1 \tag{13}$$

$$A_2 \vec{\delta} = \vec{b}_2 \tag{14}$$

$$\delta_i \in \{0, 1\} \quad \forall i \in \mathcal{I} \tag{15}$$

$$\underline{x}_i \delta_i \leq x_i \leq \overline{x}_i \delta_i \quad \forall i \in \mathcal{I} \tag{16}$$

$$\sum_{i \in \mathcal{I}} x_i = b_3 \tag{17}$$

$$\sum_{i \in \mathcal{I}} \Gamma_i^\omega x_i \geq \Phi^\omega \quad \forall \omega \in \Omega \tag{18}$$

$$\left(1 + r_t^\omega\right) s_{t-1}^\omega + \sum_{i \in \mathcal{I}} \gamma_{it}^\omega x_i = \varphi_t + s_t^\omega \quad \forall t = 1, \ldots, \ell, \omega \in \Omega \tag{19}$$

$$\underline{s}_t \leq s_t^\omega \leq \overline{s}_t \quad \forall t = 1, \ldots, \ell, \omega \in \Omega \tag{20}$$

$$\sum_{i\in\mathcal{I}} d_i^{\omega} x_i - d'^{\omega}\Phi^{\omega} = y^{\omega} \quad \forall\omega\in\Omega \tag{21}$$

$$\left(\sum_{i\in\mathcal{I}} d_i^{\omega} x_i\right)\Big/ b_3 - \left(\sum_{i\in\mathcal{I}} d_i^{\omega} f_i\right)\Big/ \sum_{i\in\mathcal{I}} f_i = z^{\omega} \quad \forall\omega\in\Omega \tag{22}$$

$$|z^{\omega}| \le \overline{z} \quad \forall\omega\in\Omega \tag{23}$$

$$\left(\sum_{i\in\mathcal{I}} \Gamma_i^{\omega} x_i\right)\Big/ b_3 - \left(\sum_{i\in\mathcal{I}} \Gamma_i^{\omega} f_i\right)\Big/ \sum_{i\in\mathcal{I}} f_i = v^{\omega} \quad \forall\omega\in\Omega \tag{24}$$

$$|v^{\omega}| \le \overline{v} \quad \forall\omega\in\Omega. \tag{25}$$

The constraint system (13)–(25) has three different subsystems. Subsystem (13)–(17) comprises the first stage constraints to structure the *MBS* portfolio by considering all the scenarios via the other subsystems but without being subordinated to any of them in particular. Subsystem (18)–(20) basically protects the investment and forces some constraints for each dedicated time period under each scenario. Subsystem (22)–(25) forces the representativeness of the portfolio under each scenario.

Constraint (13) bounds above the number of securities in the *MBS* portfolio to be structured. System (14) imposes exclusivity and implicative constraints on the *MBS* portfolio for the 0–1 variables δ_i, for $i\in\mathcal{I}$. Examples of this type of constraint are the exclusion of a particular security if certain others are in the portfolio, and the need to have certain securities in the portfolio if certain others are in it.

Constraints (16) define the semi-continuous character of the x-variables, such that no investment in any security can have more than a given weight in the portfolio, and no security can have less than a given face value, if any.

Constraint (17) forces the total investment in the portfolio to a given budget.

Constraint (18) protects the investment in the sense that the present value of the *MBS* portfolio cannot be smaller than the liabilities' present value under any scenario.

Constraints (19)–(20) give the balance equations for the cashflow in the dedicated time periods, such that the return on the amortization and yield of the investment as well as the management fees are guaranteed under any scenario. It is assumed that the available cash is short-time invested in a risk free environment and, in any case, is bounded below and above by given values.

Constraint (21) gives the duration balance equations of the *MBS* portfolio and the liabilities under each scenario. The goal is precisely to minimize the expected absolute difference in the durations.

The constraint system (22)–(25) forces the representativeness of the *MBS* portfolio with respect to the set of available securities $\mathcal{I}$, as measured by the unit duration and the unit present value under any scenario. It allows some upper bounds in the related differences.

Consider the *compact* representation of the mixed 0–1 *DEM* (12)–(25).

$$Z_{IP} = \min \sum_{\omega\in\Omega} w^{\omega} |y^{\omega}|$$

$$
\begin{array}{llllllll}
\text{s.t.} & \vec{e}\,\vec{\delta} & & & & & \le & b_1 \\
 & A_2\,\vec{\delta} & & & & & = & \vec{b}_2 \\
 & \vec{\delta} \in \{0,1\}^n & & & & & & \\
 & -I_{\overline{x}}\vec{\delta} & +I_x\vec{x} & & & & \le & \vec{0} \\
 & -I_{\underline{x}}\vec{\delta} & +I_x\vec{x} & & & & \ge & \vec{0} \\
 & & \vec{e}\,\vec{x} & & & & = & b_3 \\
 & & \vec{a}_4^{\omega}\,\vec{x} & & & & \ge & b_4^{\omega} \quad \forall\omega\in\Omega \\
 & & A_5{}^{\omega}\vec{x} & +B^{\omega}\vec{s}^{\omega} & & & = & \vec{b}_5 \quad \forall\omega\in\Omega \\
\underline{\vec{s}} \le & & & I_s\,\vec{s}^{\omega} & & & \le & \vec{\overline{s}} \quad \forall\omega\in\Omega \\
 & & \vec{a}_6^{\omega}\vec{x} & & +y^{\omega} & & = & b_6^{\omega} \quad \forall\omega\in\Omega \\
 & & \vec{a}_7^{\omega}\vec{x} & & & +z^{\omega} & = & b_7^{\omega} \quad \forall\omega\in\Omega \\
 & & & & & |z^{\omega}| & \le & \overline{z} \quad \forall\omega\in\Omega \\
 & & \vec{a}_8^{\omega}\vec{x} & & & \quad +v^{\omega} & = & b_8^{\omega} \quad \forall\omega\in\Omega \\
 & & & & & \quad |v^{\omega}| & \le & \overline{v} \quad \forall\omega\in\Omega,
\end{array}
\tag{26}
$$

where the additional notation is as follows: $n = |\mathcal{I}|$, b_4^{ω}, b_6^{ω}, b_7^{ω} and b_8^{ω} are the right-hand-side (*rhs* for short) parameters for the second stage constraints under scenario ω; $\vec{b}_5$ is the *rhs* vector of the parameters for the cashflow balance equations; $\vec{e}$ is the unit row vector; $I_{\underline{x}}$ and $I_{\overline{x}}$ are the diagonal matrices whose diagonal vectors are the conditional lower and upper bounds of the x-variables, respectively; I_x and I_s are the unit diagonal matrices for the x-and s^{ω}-variables, respectively, $\vec{a}_4^{\omega}$, $\vec{a}_6^{\omega}$, $\vec{a}_7^{\omega}$ and $\vec{a}_8^{\omega}$ are the constraint row vectors related to the x-variables for the second stage constraints; A_5^{ω} and B^{ω} are the constraint matrices related to the x-and s^{ω}-variables for the second stage constraints under scenario ω, respectively, for $\omega \in \Omega$; and the pair $(\underline{\vec{s}}, \vec{\overline{s}})$ gives the vectors of the lower and upper bounds for the s^{ω}-variables.

The *compact* representation (26) can be transformed into a *splitting variable* representation, such that the variables δ_i and x_i are replaced with δ_i^{ω} and x_i^{ω}, respectively, $\forall\omega \in \Omega$, $i \in \mathcal{I}$. So there is a model for each scenario $\omega \in \Omega$, but they are linked by the so-called *nonanticipativity* constraints

$$\delta_i^{\omega} - \delta_i^{\omega'} = 0 \tag{27}$$

$$x_i^{\omega} - x_i^{\omega'} = 0, \tag{28}$$

$\forall i \in \mathcal{I}, \omega, \omega' \in \Omega : \omega \neq \omega'$. Then the *splitting variable* representation is as follows,

$$Z_{IP} = \min \sum_{\omega\in\Omega} w^{\omega}|y^{\omega}|$$

$$
\begin{array}{llllllll}
\text{s.t.} & \vec{e}\,\vec{\delta}^{\omega} & & & & & \le b_1 & \forall \omega \in \Omega \\
 & A_2\,\vec{\delta}^{\omega} & & & & & = \vec{b}_2 & \forall \omega \in \Omega \\
 & \vec{\delta}^{\omega} \in \{0,1\}^n & & & & & & \forall \omega \in \Omega \\
 & -I_{\overline{x}}\vec{\delta}^{\omega} & +I_x\vec{x}^{\omega} & & & & \le \vec{0} & \forall \omega \in \Omega \\
 & -I_{\underline{x}}\vec{\delta}^{\omega} & +I_x\vec{x}^{\omega} & & & & \ge \vec{0} & \forall \omega \in \Omega \\
 & & \vec{e}\,\vec{x}^{\omega} & & & & = b_3 & \forall \omega \in \Omega \\
 & & \vec{a}_4^{\omega}\,\vec{x}^{\omega} & & & & \ge b_4^{\omega} & \forall \omega \in \Omega \\
 & & A_5{}^{\omega}\vec{x}^{\omega} & +B^{\omega}\vec{s}^{\omega} & & & = \vec{b}_5 & \forall \omega \in \Omega \\
\underline{\vec{s}} \le & & & I_s\,\vec{s}^{\omega} & & & \le \overline{\vec{s}} & \forall \omega \in \Omega \\
 & & \vec{a}_6^{\omega}\vec{x}^{\omega} & & +y^{\omega} & & = b_6^{\omega} & \forall \omega \in \Omega \\
 & & \vec{a}_7^{\omega}\vec{x}^{\omega} & & +z^{\omega} & & = b_7^{\omega} & \forall \omega \in \Omega \\
 & & & & |z^{\omega}| & & \le \overline{z} & \forall \omega \in \Omega \\
 & & \vec{a}_8^{\omega}\vec{x}^{\omega} & & & +v^{\omega} & = b_8^{\omega} & \forall \omega \in \Omega \\
 & & & & & |v^{\omega}| & \le \overline{v} & \forall \omega \in \Omega \\
 & \vec{\delta}^{\omega} - \vec{\delta}^{\omega'} & & & & & = \vec{0} & \forall \omega, \omega' \in \Omega : \omega \ne \omega' \\
 & \vec{x}^{\omega} - \vec{x}^{\omega'} & & & & & = \vec{0} & \forall \omega, \omega' \in \Omega : \omega \ne \omega'.
\end{array}
\qquad (29)
$$

Notice that the dualization (or, as the case may be, the relaxation) of the constraints (27) and (28) from model (29) results in $|\Omega|$ independent mixed 0–1 models. To solve the original model (29), we propose to execute a *Branch-and-Fix Coordination* (*BFC*) scheme for each of the scenario-related models to ensure the integrality condition on the δ-variables, such that the *nonanticipativity* constraints (27) are satisfied while selecting the branching nodes and the branching variables. For this purpose the *Twin Node Family* (*TNF*) concept introduced in Alonso-Ayuso, Escudero, and Ortuño (2003) is used. Additionally, the approach proposed optimizes the *LP* model that results from model (26) at each *TNF integer set*, so that the *nonanticipativity* constraints (28) are also satisfied, see below.

3 Branch-and-Fix Coordination algorithmic framework

3.1 *BFC* methodology

The scenario-related model for $\omega \in \Omega$ results from the relaxation of the *nonanticipativity* constraints (27) and (28) in model (29).

Instead of obtaining the optimal solution of the resulting programs independently , we propose a specialization of the *BFC* approach, see Alonso-Ayuso, Escudero and Ortuño. (2003). It is especially designed to coordinate the selection of the branching node and branching variable for each scenario-related *Branch-and-Fix* (*BF*) tree, such that the relaxed constraints (27) are satisfied when fixing the appropriate variables at either one or zero. The approach also coordinates and reinforces the scenario-related *BF* node pruning, the variable fixing and the objective function bounding of the subproblems attached to the nodes. See similar decomposition approaches in Carøe and Schultz (1999), Takriti and Birge (2000), Hemmecke and Schultz (2001), Klein Haneveld and van der Vlerk (2001), Römisch and Schultz (2001), Nowak, Schultz, and Westphalen (2002) and Schultz (2003), among others. However, those

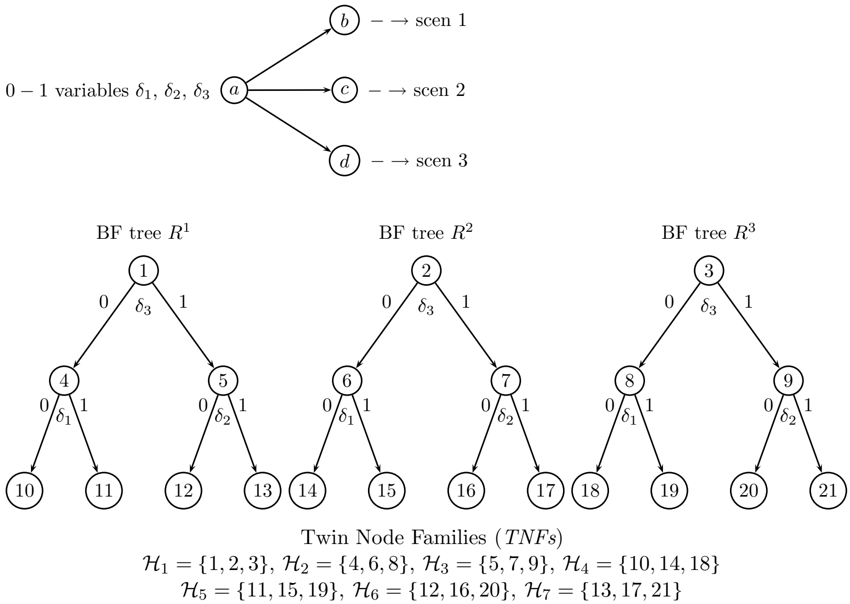

Fig. 1 Branch-and-Fix Coordination scheme

approaches focus more on using a Lagrangian relaxation of the constraints (27) to obtain good lower bounds, and less on branching and variable fixing. In any case, Lagrangian relaxation schemes can be added on top.

For the specialization of the *BFC* approach to solve problem (29), let $\mathcal{R}^{\omega}$ denote the *BF* tree associated with scenario ω, and $\mathcal{G}^{\omega}$ the set of active nodes in $\mathcal{R}^{\omega}$, $\omega \in \Omega$. Any two active nodes, say, $g \in \mathcal{G}^{\omega}$ and $g' \in \mathcal{G}^{\omega'}$ are said to be *twin* nodes if they are either the *root* nodes or the paths from the *root* nodes to each of them in their own *BF* trees $\mathcal{R}^{\omega}$ and $\mathcal{R}^{\omega'}$, respectively, have branched on or are fixed at the same 0–1 values for the same variables δ_i^{ω} and $\delta_i^{\omega'}$, for $\omega, \omega' \in \Omega$, $i \in \mathcal{I}$. A *Twin Node Family* (*TNF*), say $\mathcal{H}_f$, is a set of nodes, such that any one is a *twin* node to all the other members of the family, for $f \in \mathcal{F}$, where $\mathcal{F}$ is the set of *TNFs*. Note that $g, g' \in \mathcal{H}_f$ for any family $f \in \mathcal{F}$ implies that $\omega \neq \omega'$ for $g \in \mathcal{G}^{\omega}$ and $g' \in \mathcal{G}^{\omega'}$, $\omega, \omega' \in \Omega$. A *TNF integer set* is a a set of integer *BF* nodes, one per tree, where the *nonanticipativity* constraints (27) of the 0–1 variables are satisfied.

Let us consider the scenario tree and the *BF* trees shown in Fig. 1, where δ_i gives the generic notation for the variables δ_i^{ω}, $\forall \omega \in \Omega$. Notice that the first *TNF* to be used is $\mathcal{H}_1$. Based on the *LP* optimal solution of the scenario related models attached to the nodes in $\mathcal{H}_1$, let us assume that the selected branching variable is δ_3 and, so, the nodes 4 and 5, 6 and 7, and 8 and 9 are created. The new *TNFs* are $\mathcal{H}_2 = (4, 6, 8)$ and $\mathcal{H}_3 = (5, 7, 9)$, and so forth.

It is clear that the relaxation of the *nonanticipativity* constraints (27) is not required for all pairs of scenarios in order to obtain computational efficiency. So the number of scenarios to be considered in a given model basically depends on the dimensions of the scenario related model (i.e, the parameters $|\mathcal{I}|$ and t_i, $\forall i \in \mathcal{I}$). The criterion for scenario clustering in the sets, say, $\Omega_1, \ldots, \Omega_q$, where q is the number of *clusters* to consider, could alternatively be based on the smallest internal deviation of the uncertain parameter (i.e., the interest rate r_t^{ω}, $\forall t \in \mathcal{T}$), the greatest deviation, etc. The determination of the most efficient criterion is instance dependent. In any case, notice that $\Omega_p \cap \Omega_{p'} = \emptyset,\ p, p' = 1, \dots, q : p \neq p'$ and $\Omega = \cup_{p=1}^{q} \Omega_p$. The specific measure for quantifying the deviation of the interest rate path for any two scenarios is another instance dependent element. In any case, by slightly abusing the previous notation, the problem to be considered for the scenario *cluster* $p = 1, \dots, q$ can be expressed as follows:

$$Z_{IP}^{p} = \min \sum_{\omega \in \Omega_p} w^{\omega} |y^{\omega}|$$

$$\begin{array}{llllllllll}
\text{s.t.} & \vec{e}\,\vec{\delta}^p & & & & & & \leq & b_1 & \\
& A_2\,\vec{\delta}^p & & & & & & = & \vec{b}_2 & \\
& \vec{\delta}^p \in \{0,1\}^n & & & & & & & & \\
& -I_{\overline{x}}\vec{\delta}^p & +I_x\vec{x}^p & & & & & \leq & \vec{0} & \\
& -I_{\underline{x}}\vec{\delta}^p & +I_x\vec{x}^p & & & & & \geq & \vec{0} & \\
& & \vec{e}\,\vec{x}^p & & & & & = & b_3 & \\
& & \vec{a}_4^{\omega}\,\vec{x}^p & & & & & \geq & b_4^{\omega} & \forall \omega \in \Omega_p \\
& & A_5{}^{\omega}\vec{x}^p & +B^{\omega}\vec{s}^{\omega} & & & & = & \vec{b}_5^{\omega} & \forall \omega \in \Omega_p \\
\underline{\vec{s}} \leq & & & I_s\,\vec{s}^{\omega} & & & & \leq & \overline{\vec{s}} & \forall \omega \in \Omega_p \\
& & \vec{a}_6^{\omega}\vec{x}^p & & +y^{\omega} & & & = & b_6^{\omega} & \forall \omega \in \Omega_p \\
& & \vec{a}_7^{\omega}\vec{x}^p & & & +z^{\omega} & & = & b_7^{\omega} & \forall \omega \in \Omega_p \\
& & & & & |z^{\omega}| & & \leq & \overline{z} & \forall \omega \in \Omega_p \\
& & \vec{a}_8^{\omega}\vec{x}^p & & & & +v^{\omega} & = & b_8^{\omega} & \forall \omega \in \Omega_p \\
& & & & & & |v^{\omega}| & \leq & \overline{v} & \forall \omega \in \Omega_p.
\end{array} \tag{30}$$

The q problems (30) are linked by the *nonanticipativity* constraints

$$\delta_i^p - \delta_i^{p'} = 0 \tag{31}$$

$$x_i^p - x_i^{p'} = 0, \tag{32}$$

$\forall i \in \mathcal{I},\ p, p' = 1, \dots, q : p \neq p'$.

3.2 All x-variables alone. Benders Decomposition scheme

By slightly abusing the notation, let the following represent the *LP* model after fixing the δ-variables at the 0–1 values related to a given *TNF integer set* in model (26). In the new model, $\vec{x}^1$ will denote the vector of the x-variables whose related δ-variables have taken the value 1, and the pair $(\underline{\vec{x}}^1, \overline{\vec{x}}^1)$ gives the related lower and upper bounds.

$$Z_{LP}^{TNF} = \min \sum_{\omega \in \Omega} w^{\omega} |y^{\omega}|$$

$$
\begin{array}{llllllll}
\text{s.t.} & \vec{e}\,\vec{x}^1 & & & & = & b_3 & \\
 & \vec{a}_4^{\omega}\,\vec{x}^1 & & & & \geq & b_4^{\omega} & \forall\omega\in\Omega \\
\underline{\vec{x}}^1 \leq & \vec{x}^1 & & & & \leq & \overline{\vec{x}}^1 & \\
 & A_5{}^{\omega}\vec{x}^1 & +B^{\omega}\vec{s}^{\omega} & & & = & \vec{b}_5 & \forall\omega\in\Omega \\
\underline{\vec{s}} \leq & & I_s\,\vec{s}^{\omega} & & & \leq & \overline{\vec{s}} & \forall\omega\in\Omega \\
 & \vec{a}_6^{\omega}\vec{x}^1 & & +y^{\omega} & & = & b_6^{\omega} & \forall\omega\in\Omega \\
 & \vec{a}_7^{\omega}\vec{x}^1 & & & +z^{\omega} & = & b_7^{\omega} & \forall\omega\in\Omega \\
 & & & & |z^{\omega}| & \leq & \overline{z} & \forall\omega\in\Omega \\
 & \vec{a}_8^{\omega}\vec{x}^1 & & & +v^{\omega} & = & b_8^{\omega} & \forall\omega\in\Omega \\
 & & & & |v^{\omega}| & \leq & \overline{v} & \forall\omega\in\Omega.
\end{array}
\tag{33}
$$

By assuming that the x^1–variables are the *complicating* ones and replacing the free variables y^{ω}, z^{ω} and v^{ω} with $y_1^{\omega}- y_2^{\omega}$, $z_1^{\omega}- z_2^{\omega}$ and $v_1^{\omega}- v_2^{\omega}$, respectively, for $y_1^{\omega}, y_2^{\omega}, z_1^{\omega}, z_2^{\omega}$ $v_1^{\omega}, v_2^{\omega} \geq 0$, the original program (33) can be expressed

$$\min_x F_x$$

$$
\begin{aligned}
\text{s.t.}\quad & \vec{e}\,\vec{x}^1 = b_3 \\
& \vec{a}_4^{\omega}\vec{x}^1 \geq b_4^{\omega} \quad \forall\omega\in\Omega \\
& \underline{\vec{x}}^1 \leq \vec{x}^1 \leq \overline{\vec{x}}^1,
\end{aligned}
\tag{34}
$$

where

$$F_x = \sum_{\omega\in\Omega} w^{\omega} F_x^{\omega} \tag{35}$$

and

$$
\begin{array}{llllll}
 & F_x^{\omega} = \min\ \ y_1^{\omega}+y_2^{\omega} & & & & \\
\text{s.t.} & B^{\omega}\vec{s}^{\omega} & & & = & \vec{b}_5 - A_5^{\omega}\vec{x}^1 \\
 & & y_1^{\omega}-y_2^{\omega} & & = & b_6^{\omega}-\vec{a}_6^{\omega}\vec{x}^1 \\
 & & & z_1^{\omega}-z_2^{\omega} & = & b_7^{\omega}-\vec{a}_7^{\omega}\vec{x}^1 \\
 & & & z_1^{\omega}+z_2^{\omega} & \leq & \overline{z} \\
 & & & v_1^{\omega}-v_2^{\omega} & = & b_8^{\omega}-\vec{a}_8^{\omega}\vec{x}^1 \\
 & & & v_1^{\omega}+v_2^{\omega} & \leq & \overline{v} \\
\underline{\vec{s}} \leq & I_s\vec{s}^{\omega} & & & \leq & \overline{\vec{s}} \\
 & & y_1^{\omega}, y_2^{\omega}, & z_1^{\omega}, z_2^{\omega},\ \ v_1^{\omega}, v_2^{\omega} & \geq & 0.
\end{array}
\tag{36}
$$

The dual of the primal *LP* problem (36) can be expressed

$$\begin{array}{llllllll}
 & F_x^\omega = \max\ \left(\vec{b}_5 - A_5^\omega \vec{x}^1\right)^T \vec{\mu}_5^\omega + \left(b_6^\omega - \vec{a}_6^\omega \vec{x}^1\right)\mu_6^\omega + \left(b_7^\omega - \vec{a}_7^\omega \vec{x}^1\right)\mu_7^\omega - \overline{z}\lambda^\omega \\
 & \quad + \left(b_8^\omega - \vec{a}_8^\omega \vec{x}^1\right)\mu_8^\omega - \overline{v}\beta^\omega + \vec{\underline{s}}^T \vec{\alpha}_1^\omega - \vec{\overline{s}}^T \vec{\alpha}_2^\omega \\
\text{s.t.} & B^{\omega T}\vec{\mu}_5^\omega \quad + I_s\vec{\alpha}_1^\omega \quad - I_s\vec{\alpha}_2^\omega & \le \vec{0} \\
-1 \le & \mu_6^\omega & \le 1 \\
 & \mu_7^\omega \quad -\lambda^\omega & \le 0 \\
 & \mu_7^\omega \quad +\lambda^\omega & \ge 0 \\
 & \mu_8^\omega \quad -\beta^\omega & \le 0 \\
 & \mu_8^\omega \quad +\beta^\omega & \ge 0 \\
 & \vec{\alpha}_1^\omega, \vec{\alpha}_2^\omega, \lambda^\omega, \beta^\omega \ge 0 \\
 & \vec{\mu}_5^\omega, \mu_7^\omega, \mu_8^\omega \text{ unrestricted.}
\end{array} \tag{37}$$

Given the structure of the constraint matrix that defines the feasible region in problem (37), it can be decomposed into a series of independent subproblems, such that

$$F_x^\omega = F_x^\omega\left(\vec{\mu}_5^\omega, \vec{\alpha}_1^\omega, \vec{\alpha}_2^\omega\right) + F_x^\omega\left(\mu_6^\omega\right) + F_x^\omega\left(\mu_7^\omega, \lambda^\omega\right) + F_x^\omega\left(\mu_8^\omega, \beta^\omega\right) \quad \forall\omega \in \Omega, \tag{38}$$

where

$$\begin{array}{ll}
F_x^\omega\left(\vec{\mu}_5^\omega, \vec{\alpha}_1^\omega, \vec{\alpha}_2^\omega\right) = \max & \left(\vec{b}_5 - A_5^\omega \vec{x}^1\right)^T \vec{\mu}_5^\omega + \vec{\underline{s}}^T \vec{\alpha}_1^\omega - \vec{\overline{s}}^T \vec{\alpha}_2^\omega \\
\text{s.t.} & B^{\omega T}\vec{\mu}_5^\omega + I_s\vec{\alpha}_1^\omega - I_s\vec{\alpha}_2^\omega \le \vec{0} \\
 & \vec{\alpha}_1^\omega, \vec{\alpha}_2^\omega \ge 0 \\
 & \vec{\mu}_5^\omega \text{ unrestricted,}
\end{array} \tag{39}$$

$$\begin{array}{ll}
F_x^\omega\left(\mu_6^\omega\right) = \max & \left(b_6^\omega - \vec{a}_6^\omega \vec{x}^1\right)\mu_6^\omega \\
\text{s.t.} & -1 \le \mu_6^\omega \le 1,
\end{array} \tag{40}$$

$$\begin{array}{ll}
F_x^\omega\left(\mu_7^\omega, \lambda^\omega\right) = \max & \left(b_7^\omega - \vec{a}_7^\omega \vec{x}^1\right)\mu_7^\omega - \overline{z}\lambda^\omega \\
\text{s.t} & \mu_7^\omega - \lambda^\omega \le 0 \\
 & \mu_7^\omega + \lambda^\omega \ge 0 \\
 & \lambda^\omega \ge 0 \\
 & \mu_7^\omega \text{ unrestricted,}
\end{array} \tag{41}$$

and

$$\begin{array}{ll}
F_x^\omega\left(\mu_8^\omega, \beta^\omega\right) = & \max\ \left(b_8^\omega - \vec{a}_8^\omega \vec{x}^1\right)\mu_8^\omega - \overline{v}\beta^\omega \\
\text{s.t} & \mu_8^\omega - \beta^\omega \le 0 \\
 & \mu_8^\omega + \beta^\omega \ge 0 \\
 & \beta^\omega \ge 0 \\
 & \mu_8^\omega \text{ unrestricted.}
\end{array} \tag{42}$$

The assumption of feasibility in the original model (33) requires the feasibility of the primal problems (36) $\forall\omega \in \Omega$ for all feasible values of the vector $\vec{x}^1$ in the model (33). So, by the Duality Theorem, F_x^ω in the model (37), and therefore F_x (35), also have finite values.

Let $\mathcal{J}^p$ and $\mathcal{J}^r$ denote the sets of the extreme points and extreme rays of the feasible region in each problem (37), respectively. And let an extreme point from $\mathcal{J}^p$ and an extreme ray from $\mathcal{J}^r$ be denoted as follows,

$$\vec{v}_j^{\omega} \equiv \left(\vec{\mu}_5^{\omega}, \mu_6^{\omega}, \mu_7^{\omega}, \mu_8^{\omega}, \vec{\alpha}_1^{\omega}, \vec{\alpha}_2^{\omega}, \lambda^{\omega}, \beta^{\omega}\right)_j \qquad \omega \in \Omega, j \in \mathcal{J}^p \cup \mathcal{J}^r. \tag{43}$$

The problem (37) for $\omega \in \Omega$ is finite if and only if

$$-\vec{c}_j^{\,\omega}\vec{x}^1 + k_j^{\omega} \le 0 \qquad j \in \mathcal{J}^r, \tag{44}$$

where

$$\begin{aligned} k_j^{\omega} &= \left[\vec{\mu}_5^{\omega}\right]_j^t \vec{b}_5 + \underline{\vec{s}}^t\left[\vec{\alpha}_1^{\omega}\right]_j - \overline{\vec{s}}^t\left[\vec{\alpha}_2^{\omega}\right]_j + b_6^{\omega}\left[\mu_6^{\omega}\right]_j + b_7^{\omega}\left[\mu_7^{\omega}\right]_j - \overline{z}[\lambda^{\omega}]_j + b_8^{\omega}\left[\mu_8^{\omega}\right]_j - \overline{v}[\beta^{\omega}]_j \\ c_j^{\,\omega} &= \left[\vec{\mu}_5^{\omega}\right]_j^t A_5^{\omega} + \left[\mu_6^{\omega}\right]_j \vec{a}_6^{\omega} + \left[\mu_7^{\omega}\right]_j \vec{a}_7^{\omega} + \left[\mu_8^{\omega}\right]_j \vec{a}_8^{\omega}. \end{aligned} \tag{45}$$

We can outer linearize the infimal value function in (37), such that it can be expressed as

$$\max_{j \in \mathcal{J}^p} \sum_{\omega \in \Omega} w^{\omega}\left(-\vec{c}_j^{\,\omega}\vec{x}^1 + k_j^{\omega}\right). \tag{46}$$

By expressing the infimal value function by the outer linearized dual functions (37) and letting Z denote the smallest upper bound, the original problem (33) for the given *Twin Node Family* (*TNF*) can be represented as follows,

$$Z_{LP}^{TNF} = \min Z \tag{47}$$

$$\text{s.t.} \quad \vec{e}\,\vec{x}^1 = b_3 \tag{48}$$

$$\vec{a}_4^{\omega}\vec{x}^1 \ge b_4^{\omega}, \quad \forall \omega \in \Omega \tag{49}$$

$$\underline{\vec{x}}^1 \le \vec{x}^1 \le \overline{\vec{x}}^1 \tag{50}$$

$$Z \ge \sum_{\omega \in \Omega} w^{\omega}\left(-\vec{c}_j^{\,\omega}\vec{x}^1 + k_j^{\omega}\right), \quad \forall j \in \mathcal{J}^p \tag{51}$$

$$-\vec{c}_j^{\,\omega}\vec{x}^1 + k_j^{\omega} \le 0 \;\; \forall \omega \in \Omega, \quad j \in \mathcal{J}^r. \tag{52}$$

Problem (47)–(52) is known as the Benders *Master Program*, see Benders (1962). It is not efficient to compute all its extreme points and rays (if any) (43), and on the other hand very few induced cuts (51)–(52) are frequently active at its optimal solution. A necessary condition for the implementation of this procedure is that the feasible region defined by (48)–(50) be finite. So the solution can be iteratively obtained by identifying extreme points and rays–based cuts from the optimization of the so-called *Auxiliary Program* (*AP*), and appending them to the so-called *Relaxed Master Program* (*RMP*) for its optimization. The *RMP* can be expressed as

$$\begin{aligned} \underline{Z} &= \min Z \\ \text{s.t.} \quad & \vec{e}\,\vec{x}^1 = b_3 \\ & \vec{a}_4^{\omega}\vec{x}^1 \ge b_4^{\omega} \qquad \forall \omega \in \Omega \end{aligned}$$

$$\underline{\vec{x}}^1 \le \vec{x}^1 \le \overline{\vec{x}}^1$$
$$Z \ge \sum_{\omega\in\Omega} w^\omega\big(-\vec{c}_j^{\,\omega}\vec{x}^1 + k_j^\omega\big) \quad \forall j \in \overline{\mathcal{J}}^p \tag{53}$$
$$-\vec{c}_j^{\,\omega}\vec{x}^1 + k_j^\omega \le 0 \;\; \forall \omega \in \Omega, \quad j \in \overline{\mathcal{J}}^r,$$

where $\overline{\mathcal{J}}^p \subseteq \mathcal{J}^p$ and $\overline{\mathcal{J}}^r \subseteq \mathcal{J}^r$ are the subsets of the extreme points and extreme rays already identified, respectively.

At the first iteration, *RMP* is only included by the submodel (47)–(50). The *AP* is given by the model (37), whose value (38) is obtained by solving independently the models (39)–(42) for a given value, say $\hat{\vec{x}}^1$, of the vector of the $\vec{x}^1$-variables. This value is the optimal solution in the *RMP* solved in the previous iteration, its solution value being $\underline{Z}$.

Notice that the primal infeasibility (i.e., dual unboundness) of model (36) is detected for the vector $\hat{\vec{x}}^1$ if there is a scenario whose model (39)–(42) is unbounded for that vector. In this case, by Farkas' lemma, there exists an extreme ray $\vec{\nu}_j^\omega$ (43) such that $\vec{\nu}_j^\omega W \le 0$ and $-\vec{c}_j^\omega\vec{x}^1 + k_j^\omega > 0$, where W is the matrix of the feasible region for the dual problem (37). Then, at least one *feasible cut* from set (54) should be appended to the *RMP*.

$$-\vec{c}_j^\omega\vec{x}^1 + k_j^\omega \le 0 \quad \forall \omega \in \Omega^0, \tag{54}$$

where Ω^0 gives the set of scenarios from Ω whose related models (39)–(42) are unbounded, and (43) gives the corresponding extreme ray.

On the other hand, if all dual models (39)–(42), $\forall \omega \in \Omega$ are bounded for the vector $\hat{\vec{x}}^1$, let $\overline{Z} = F_{\hat{x}}$ denote the optimal value of the objective function (38) and (55) be the *optimality cut* to be appended to the *RMP* if $\overline{Z}$ (38) $>$ $\underline{Z}$ (53).

$$Z \ge \sum_{\omega\in\Omega} w^\omega\big(-\vec{c}_j^\omega\vec{x}^1 + k_j^\omega\big), \tag{55}$$

where (43) gives the corresponding extreme point as the *AP* optimal solution for the point $\hat{\vec{x}}^1$.

Notice that if $\overline{Z} = \underline{Z}$ then $\hat{\vec{x}}^1$ is the optimal solution of model (33), with $Z_{LP}^{TNF} = \underline{Z}$.

3.3 All x-variables with fractional δ-variables. Benders Decomposition scheme

By again abusing the notation let $\vec{\delta}^f$ denote the vector of the δ-variables to be allowed to take fractional values, $\vec{\delta}^1$ the vector of the δ-variables that have been branched or fixed at one, $\vec{x}^{1f}$ the vector of the x-variables whose related δ-variables have not branched or fixed to zero in model (33), and $\vec{e}^f$ and A_2^f (res., $\vec{e}^1$ and A_2^1) the unit row vector and constraint matrix for the variables' vector $\vec{\delta}^f$ (res., $\vec{\delta}^1$). The model can be expressed as follows,

$$Z_{LP}^f = \min \sum_{\omega\in\Omega} w^\omega |y^\omega|$$

$$\begin{array}{llllllll}
\text{s.t.} & \vec{e}\,\vec{\delta}^f & & & & \leq b_1 - \vec{e}^1\vec{\delta}^1 & & \\
& A_2^f\,\vec{\delta}^f & & & & = \vec{b}_2 - A_2^1\vec{\delta}^1 & & \\
& \vec{\delta}^f \in [0,1]^n & & & & & & \\
& -I_{\overline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & & & \leq \vec{0} & & \\
& -I_{\underline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & & & \geq \vec{0} & & \\
& & \vec{e}\,\vec{x}^{1f} & & & = b_3 & & \\
& & \vec{a}_4^\omega\,\vec{x}^{1f} & & & \geq b_4^\omega & \forall\omega\in\Omega & \\
& & A_5{}^\omega\vec{x}^{1f} & +B^\omega\vec{s}^\omega & & = \vec{b}_5 & \forall\omega\in\Omega & \quad (56)\\
\underline{\vec{s}} \leq & & & I_s\,\vec{s}^\omega & & \leq \overline{\vec{s}} & \forall\omega\in\Omega & \\
& & \vec{a}_6^\omega\vec{x}^{1f} & +y^\omega & & = b_6^\omega & \forall\omega\in\Omega & \\
& & \vec{a}_7^\omega\vec{x}^{1f} & & +z^\omega & = b_7^\omega & \forall\omega\in\Omega & \\
& & & & |z^\omega| & \leq \overline{z} & \forall\omega\in\Omega & \\
& & \vec{a}_8^\omega\vec{x}^{1f} & & +v^\omega & = b_8^\omega & \forall\omega\in\Omega & \\
& & & & |v^\omega| & \leq \overline{v} & \forall\omega\in\Omega. &
\end{array}$$

By assuming that the δ^f- and x^{1f}-variables are the *complicating* ones and replacing the free variables y^ω, z^ω and v^ω with $y_1^\omega - y_2^\omega$, $z_1^\omega - z_2^\omega$ and $v_1^\omega - v_2^\omega$, respectively, for y_1^ω, y_2^ω, z_1^ω, z_2^ω v_1^ω, $v_2^\omega \geq 0$ as above, the program (56) can be expressed as

$$\min_x F_x$$

$$\begin{array}{lllll}
\text{s.t.} & \vec{e}\,\vec{\delta}^f & & \leq & b_1 - \vec{e}^1\vec{\delta}^1 \\
& A_2^f\,\vec{\delta}^f & & = & \vec{b}_2 - A_2^1\vec{\delta}^1 \\
& \vec{\delta}^f \in [0,1]^n & & & \\
& -I_{\overline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & \leq & \vec{0} \qquad\qquad (57)\\
& -I_{\underline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & \geq & \vec{0} \\
& & \vec{e}\,\vec{x}^{1f} & = & b_3 \\
& & \vec{a}_4^\omega\vec{x}^{1f} & \geq & b_4^\omega \quad \forall\omega\in\Omega,
\end{array}$$

where

$$F_x = \sum_{\omega\in\Omega} w^\omega F_x^\omega \tag{58}$$

and F_x^ω can be expressed following the same rationale as in (36)–(46), but replacing $\vec{x}^1$ with $\vec{x}^{1f}$. From which it results that Z_{LP}^f can be expressed as

$$Z_{LP}^f = \min Z$$

$$
\begin{array}{llcll}
\text{s.t.} \quad \vec{e}\,\vec{\delta}^f & & \leq & b_1 - \vec{e}^1\vec{\delta}^1 & \\
A_2^f\,\vec{\delta}^f & & = & \vec{b}_2 - A_2^1\vec{\delta}^1 & \\
\quad \vec{\delta}^f \in [0,1]^n & & & & \\
-I_{\overline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & \leq & \vec{0} & \\
-I_{\underline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & \geq & \vec{0} & \\
 & \vec{e}\,\vec{x}^{1f} & = & b_3 & \\
 & \vec{a}_4^{\omega}\vec{x}^{1f} & \geq & b_4^{\omega} & \forall \omega \in \Omega \\
 & Z & \geq & \sum_{\omega\in\Omega} w^{\omega}\big(-\vec{c}_j^{\,\omega}\vec{x}^{1f} + k_j^{\omega}\big) & \forall j \in \mathcal{J}^p \\
 & -\vec{c}_j^{\,\omega}\vec{x}^{1f} + k_j^{\omega} & \leq & 0 & \forall \omega \in \Omega,\ j \in \mathcal{J}^r.
\end{array}
\tag{59}
$$

Problem (59) is the Benders *Master Program*. The *Relaxed Master Program* (*RMP*) can be expressed as the same problem (59), where the sets $\mathcal{J}^p$ and $\mathcal{J}^r$ are replaced with the subsets $\overline{\mathcal{J}}^p \subseteq \mathcal{J}^p$ and $\overline{\mathcal{J}}^r \subseteq \mathcal{J}^r$. Again, the feasible region of the initial relaxed master program must be finite.

The *Auxiliary Problem* (*AP*) is given by model (37) whose value (38) is obtained by solving models (39)–(42) independently, but now replacing the vector $\vec{\hat{x}}^1$ with the vector $\vec{\hat{x}}^{1f}$.

The *feasibility* and *optimality* cuts from *AP* to be appended to *RMP* are given by the constraints (54) and (55), respectively, where again $\vec{\hat{x}}^1$ is replaced with $\vec{\hat{x}}^{1f}$.

3.4 *BFC* implementation

Different *BFC* implementations can be considered. We present the version that has been implemented to perform the computational experimentation reported in Section 4.

Notice that the δ-and x-variables have zero coefficients in the objective function (12). In fact the y-variables are the only variables in the objective function. These variables give the residual values of the duration balance equation (21) of the *MBS* portfolio and liabilities under each scenario. So there is no clear criterion for assigning branching priorities to the δ-variables. We have chosen the model's input order (i.e. a random order) as the branching priority.

Based on the same reason, the value of the objective function is not a good indication for node branching selection. So we have chosen the *depth first* strategy for *TNF* branching selection, having first ”branching on the zeros” and then ”branching on the ones” for the chosen δ–variable to satisfy the *nonanticipativity* constraints (31) for the selected *TNF* to be branched.

Notice that a *TNF* can be pruned due to any of the following reasons: (a) the *LP* relaxation of the scenario-*cluster* model (30) attached to a given node member is infeasible; (b) there is no guarantee that a better solution than the *incumbent* one can be obtained from the best descendant *TNF integer set* (in our current implementation, it is based on its objective function value, also called solution value); (c) the *LP* model (33) attached to the *TNF integer set* is infeasible or its solution value is not better than the solution value of the *incumbent* solution if all δ-variables have already been branched on or fixed for the family; and (d) see below when there is a δ-variable in the *TNF integer set* that has not yet been branched on or fixed.

Once a *TNF* has been pruned, the same branching criterion allows one to perform either a “branching on the ones” (if case “branched on the zeros” has already taken place) or a *backtracking* to the previous branched *TNF*.

The solution to be obtained by solving the *LP* model (33) attached to a *TNF integer set* could be the *incumbent* solution. However, this does not necessarily mean that it should be pruned, except if all δ-variables have been branched on or fixed for the family, as said above. Otherwise, a better solution can still be obtained by branching on the not-yet branched on or fixed δ-variables. Let Z_{LP}^{TNF} be the solution value in (33) that satisfies the *nonanticipativity* constraints (28) by fixing the δ-variables at their 0–1 values (where constraints (27) are already satisfied). The family can be pruned if $Z_{LP}^{TNF} = Z_{LP}^{f}$, where Z_{LP}^{f} is the solution value of model (56), where both constraint types are satisfied, but the not-yet branched on or fixed δ-variables are allowed to take fractional values. Notice that the solution space defined by model (33) is included in the space defined by model (56). In this case, there is no better solution than Z_{LP}^{TNF} to be obtained from the descendant *TNF integer sets*.

To present the *BFC* algorithm to solve model (29), the following additional notation is adopted:

$\mathcal{R}^p$, *BF* tree for the scenario *cluster p*, for $p = 1, \ldots, q$.

LP^p, *LP* relaxation of the scenario *cluster*-related model (30) attached to a given node member of the *BF* tree $\mathcal{R}^p$ in the given *TNF*, for $p = 1, \ldots, q$.

Z_{LP}^p, solution value of the *LP* model LP^p, for $p = 1, \ldots, q$. By convention, let $Z_{LP}^p = +\infty$ in case of infeasibility. Note: Z_{LP}^p is the expected duration mismatching of the *MBS* portfolio and the liabilities over the scenarios in *cluster p*, for the *LP* relaxation case.

$\underline{Z}_{IP}$, lower bound of the solution value of the original model (29) to be obtained from the best descendant *TNF integer set* for a given family. This will be computed as $\underline{Z}_{IP} = \sum_{p=1,\ldots,q} Z_{LP}^p$ for any family, except the one included by the root nodes of the *BF* trees. For the latter family, $\underline{Z}_{IP}$ is given by the LP relaxation of the original problem (26); the value is reported as Z_{LP} in the computational experience shown in Section 4 when computed in Step 1 below, and it is obtained by solving problem (56), via Benders Decomposition, without fixing *a priori* any δ-variable.

By convention, $Z_{LP}^{TNF} = +\infty$, for the infeasible problem (33) related to a given *TNF integer set*, and $Z_{LP}^{f} = +\infty$, for the infeasible problem (56).

BFC Algorithm

Step 0: Initialize $\overline{Z}_{IP} := +\infty$.

Step 1: Solve the *LP* relaxation of the original problem (26) and compute $\underline{Z}_{IP}$. If there is any δ-variable that takes a fractional value then go to Step 2. Otherwise, the optimal solution to the original problem has been found and, so, $\overline{Z}_{IP} := \underline{Z}_{IP}$ and stop.

Step 2: Initialize $i := 1$ and go to Step 4.

Step 3: Reset $i := i + 1$. If $i = |\mathcal{I}| + 1$ then go to Step 8.

Step 4: Branch $\delta_i^p := 0$ and, so, fix $x_i^p := 0$, $\forall p = 1, \ldots, q$.

Step 5: Solve the linear problems LP^p, $\forall p = 1, \ldots, q$ and compute $\underline{Z}_{IP}$.
If $\underline{Z}_{IP} \geq \overline{Z}_{IP}$ then go to Step 7. If there is any δ-variable that either takes fractional values or takes different values for some of the q scenario *clusters* then go to Step 3.
If all the x-variables take the same value for all scenario *clusters* $p = 1, \ldots, q$ then update $\overline{Z}_{IP} := \underline{Z}_{IP}$ and go to Step 7.

Step 6: Solve the *LP* model (33) to satisfy the constraints (32) for the x^1-variables in the given *TNF integer set*. Notice that the solution value is denoted by Z_{LP}^{TNF}.
Update $\overline{Z}_{IP} := \min\{Z_{LP}^{TNF}, \overline{Z}_{IP}\}$. If $i = |\mathcal{I}|$ then go to Step 7.
Solve the *LP* model (56), where the fractional δ-variables are those not-yet branched on or fixed in the current *TNF*. Notice that the solution value is denoted by Z_{LP}^{f}. If $Z_{LP}^{TNF} = Z_{LP}^{f}$ or $Z_{LP}^{8} \geq \overline{Z}_{IP}$ then go to Step 7, otherwise go to Step 3.

Table 1 Test bed dimensions

Case	$\|\mathcal{I}\|$	l	$\|\mathcal{T}\|$	b_1	$\|\Omega\|$
P1	10	5	10	4	10
P2	20	8	12	7	20
P3	20	5	10	6	50
P4	20	5	10	4	50
P5	20	5	10	12	50
P6	20	5	10	4	1000
P7	20	5	10	8	1000
P8	40	10	12	20	1000
P9	100	5	10	30	1000
P10	100	5	10	50	2000
P11	200	5	10	50	2000
P12	300	5	10	200	2000
P13	500	5	10	300	1500
P14	700	5	10	400	1000
P15	1000	5	10	600	1000

Step 7: Prune the branch.
If $\delta_i^p = 0$, $\forall p = 1, \ldots, q$ then go to Step 10.
Step 8: Reset $i := i - 1$.
If $i = 0$ then stop, since the optimal solution $\overline{Z}_{IP}$ has been found.
Step 9: If $\delta_i^p = 1$, $\forall p = 1, \ldots, q$ then go to Step 8.
Step 10: Branch $\delta_i^p := 1$ and, therefore, $\underline{x}_i \le x_i^p \le \overline{x}_i$, $\forall p = 1, \ldots, q$.
Go to Step 5.

4 Computational results

We report the results of the computational experiment obtained while optimizing the model for structuring the *MBS* portfolio for a set of instances by using the *BFC* approach presented in the previous section.

The scenarios are generated as follows:

1. The scenarios for the interest rate path $r_t^\omega, \forall t \in \mathcal{T}, \omega \in \Omega$ have been generated by using the binomial lattice approach given in Black, Derman, and Toy (1990).
2. The unit returns on the securities in the dedicated time periods for the scenarios have been randomly generated as a function of the interest rate.
3. The *Option Adjusted Spread* o_i has been obtained for each security i by solving the nonlinear function

$$\Gamma_i^0 = \sum_{\omega \in \Omega} w^\omega \left(\sum_{t=1}^{t_i} \gamma_{it}^\omega \prod_{\tau=1}^{t} \left(1 + o_i \cdot r_\tau^\omega\right)^{-1} \right),$$

where Γ_i^0 is the current unit return value of security i, for $i \in \mathcal{I}$.

We should mention again that *MBSPSP* has been used as a pilot case to test out the stochastic integer approach proposed in the paper: it is not intended to serve as a basis for drawing economic conclusions for decision making in managing portfolios of Mortgage Backed Securities.

Table 1 gives the dimensions of the cases. They can be split into three categories. The first includes those cases with a maximum of $|\Omega| = 50$ scenarios, the second includes cases with $|\Omega| = 1000$ and 2000 scenarios and $|\mathcal{I}| \leq 100$ securities, and the third includes cases with $|\Omega| = 1000$, 1500 and 2000 scenarios and $200 \leq |\mathcal{I}| \leq 1000$ securities.

Our algorithmic approach has been implemented in a FORTRAN 90 experimental code, which uses the optimization engine IBM OSL v2.0 to solve the *LP* models. The computational experiments were conducted in a WS Sun Park under the Solaris 2.5 operating system.

Table 2 gives the dimensions of the *DEM* (12)–(25), compact representation (26). It also gives the dimensions of the scenario cluster related deterministic model (30). The new headings are as follows: m, number of constraints; $n\delta$, number of (0–1) δ-variables (and also number of x–variables); $n2$, number of (continuous) second-stage variables; nc, total number of continuous variables; nel, number of nonzero elements in the constraint matrix; $dens$, constraint matrix density (in %).

Table 3 shows the main results of our computational experimentation for given values of the number of scenario *clusters*. The headings are as follows: Z_{LP}, solution value of the *LP* relaxation of the original problem (12)–(25); Z_{IP}, solution value of the original problem; GAP, optimality gap defined as $(Z_{IP} - Z_{LP})/Z_{LP}\%$; nn, number of *TNF* branches for the set of *BF* trees; T_{LP} and T_{LP}^{B}, the elapsed time (sec.) to obtain the LP solution without using Benders Decomposition (*BD*) and using it, respectively; T, T^{B} and T^{OSL}, the total elapsed time (sec.) to obtain the optimal solution to the original problem by using the *BFC* procedure without *BD*, by using *BFC* jointly with *BD* and by plain use of the optimization engine for solving the *DEM*, respectively. Notice that the *LP* relaxation of the original problem (12)–(25) is optimized in Step 1 of the *BFC* algorithm, the *LP* relaxation of the scenario *cluster* model (30) is optimized in Step 5, and the linear programs (33) and (56) are optimized in Step 6 by using Benders Decomposition for the *TNF integer sets*.

The first conclusion that can be drawn from the results shown in Table 3 is that our approach obtains the optimal solution in all the cases we have experimented with. Generally speaking, it seems that the optimization engine requires less computational effort than the approach proposed when the cases have small dimensions. In other words, it seems that the greater the dimensions of the cases (particularly the number of scenarios and securities), the better the performance of the proposed approach is, especially considering that our testing has been done with an experimental code. Note that our algorithm when using the *BD* scheme (besides the *BFC* approach) reduces by one order of magnitude the elapsed time required by the plain use of the optimization engine for the second category of cases.

Additionally, we can observe the good performance of the *BD* scheme in Table 3 by comparing the elapsed times T_{LP} and T_{LP}^{B} to obtain the LP solution value without using *BD* and using it, respectively. In any case, the time spent by our approach not counting these times (e.g. Step 1 of the algorithm) is relatively small. Notice that Step 1 is only used for computing the lower bound of the solution.

The computational results for the third category of cases are also very interesting. Notice in Table 3 that the optimization engine cannot find any solution within the time limit allowed (6 hours) except in cases P11 and P12. On the other hand, the mixture *BFC–BD* obtains the optimal solution in relatively small elapsed time, for a rather large number of scenario *clusters* and securities in all cases. Moreover, the performance of steps 2 to 10 of the algorithm is much better when using BD than when not using it, in all cases.

Another interesting observation in Table 3 is that the GAP is zero in 12 of the 15 test cases. This result is entirely different from the result obtained when the LP relaxation of the original problem is also included by the relaxation of the *nonanticipativity* constraints (i.e. the solution value of the LP models $LP^{p}, \forall p = 1, \ldots, q$). We have not reported the

Table 2 Model dimensions. Compact representation

Case	Deterministic equivalent model							Scenario cluster model					
	$\|\Omega\|$	m	$n\delta$	$n2$	nc	nel	dens	m	$n\delta$	$n2$	nc	nel	dens
P1	10	142	10	110	120	1170	6.33	43	10	11	21	189	14.18
P2	20	342	20	280	300	5460	4.98	76	20	14	34	425	10.35
P3	50	612	20	550	570	10110	2.79	73	20	11	31	359	9.64
P4	50	612	20	550	570	10110	2.79	73	20	11	31	359	9.64
P5	50	612	20	550	570	10110	2.79	73	20	11	31	359	9.64
P6	1000	11062	20	11000	11020	199160	0.16	1162	20	1100	1120	20060	1.51
P7	1000	11062	20	11000	11020	199160	0.16	1162	20	1100	1120	20060	1.51
P8	1000	16122	40	16000	16040	589320	0.22	1722	40	1600	1640	59220	2.05
P9	1000	11302	100	11000	11100	919800	0.72	1402	100	1100	1200	92700	5.09
P10	2000	22302	100	22000	22100	1838800	0.37	1402	100	1100	1200	92700	5.09
P11	2000	22602	200	22000	22200	3639600	0.78	1702	200	1100	1300	183500	7.19
P12	2000	22902	300	22000	22300	5440400	1.05	1012	300	110	410	29590	4.12
P13	1500	18002	500	16500	17000	6782500	2.15	1612	500	110	610	49190	2.75
P14	1000	13102	700	11000	11700	6324600	3.84	2157	700	55	755	37195	1.19
P15	1000	14002	1000	11000	12000	9027000	4.96	3057	1000	55	1055	53095	0.85

Table 3 Stochastic solution

Case	q	Z_{LP}	Z_{IP}	GAP	nn	T_{LP}	T	T^B_{LP}	T^B	T^{OSL}
P1	10	2583.62	2583.62	0.00	16	0.04	0.53	0.47	0.70	0.13
P2	20	23693.57	23693.57	0.00	29	0.23	2.36	0.42	1.65	0.31
P3	50	1225.11	1225.11	0.00	34	0.78	5.48	0.71	5.16	0.94
P4	50	2853.19	4907.18	71.99	22	0.83	2.14	0.54	1.33	2.79
P5	50	1225.11	1225.11	0.00	28	0.74	4.95	0.69	5.13	0.97
Total time for the 1st category of cases						2.62	15.46	2.83	13.97	5.14
P6	10	2447.11	4825.39	97.19	22	159.22	173.38	2.93	13.58	437.61
P7	10	5163.87	5163.87	0.00	31	283.32	394.91	4.98	56.14	393.23
P8	10	57179.60	57179.60	0.00	73	1226.64	1982.61	13.88	160.63	2182.69
P9	10	13.74	13.74	0.00	108	803.60	1060.61	14.24	266.44	1188.78
P10	20	13341.88	13341.88	0.00	221	3696.32	5959.14	30.17	379.83	5713.93
Total time for the 2nd category of cases						6169.10	9570.65	66.20	876.62	9916.24
P11	20	26255.09	26255.09	0.00	256	7362.13	8927.12	60.81	850.58	12184.14
P12	200	38736.99	38736.99	0.00	422	10326.12	16951.55	100.02	2323.06	20257.54
P13	150	86086.38	87808.01	2.00	584	11951.17	17231.34	147.65	4267.04	–
P14	200	183384.04	183384.04	0.00	742	8586.08	16721.12	275.66	7995.64	–
P15	200	260870.26	260870.26	0.00	1030	12551.44	–	295.71	14123.76	–
Total time for the 3rd category of cases						50776.94	–	879.85	29560.08	–

Elapsed time greater than time limit (6 hours).

Table 4 Performance of the BFC approach

q	nn	$T - T_{LP}$	$T^B - T^B_{LP}$
		(a) Case P6	
2	22	255.39	85.57
5	22	84.92	25.51
10	22	14.16	10.65
50	22	85.98	4.77
100	22	81.75	5.07
1000	22	94.99	16.71
		(b) Case P9	
2	106	712.85	703.33
5	108	415.45	371.98
10	108	257.01	252.20
50	107	385.12	184.12
100	106	397.25	172.69
1000	106	457.14	194.17

Table 5 The value of the stochastic solution

Case	*EV*	*WS*	Z_{IP}	*EEV*	*VSS*
P1	0.00	964.19	2583.62	2731.96	148.34
P2	23696.07	23622.99	23693.57	*(76.18)	*
P3	0.00	263.52	1225.11	2224.75	999.64
P4	3412.16	4749.08	4907.18	4907.18	0.00
P5	0.00	431.45	1225.11	2223.40	998.29
P6	2447.11	4754.12	4825.39	4825.39	0.00
P7	0.00	1115.81	5163.87	5476.30	312.43
P8	57023.95	56782.23	57179.60	*(50.01)	*
P9	0.00	7.43	13.74	26.74	13.00
P10	11000.72	12691.41	13341.88	14893.33	1551.45
P11	21628.06	24973.57	26255.09	29306.11	3051.02
P12	31922.18	36845.90	38736.99	43238.46	4501.47
P13	76879.07	83518.02	87808.01	93177.51	5369.50
P14	168948.95	174405.56	183384.04	184490.58	1106.54
P15	240386.53	248095.70	260870.26	262444.67	1574.41

*Infeasible solution. (.)Weighted percentage of infeasible scenarios

related GAP that is obtained by using this other approach, but it is very frequently greater than 100%.

Tables 4a and 4b show the performance of the *BFC* approach for different sizes of the scenario *clusters* and then different dimensions of model (30) for cases P6 and P9. We can observe how sensitive the elapsed time for the solution to the problem is to the number of scenario *clusters* (all of which have the same dimensions for each q value).

Table 5 shows some parameters for analyzing the goodness of the stochastic approach, see e.g. Birge and Louveaux (1997) for more details. The headings are as follows: WS (*Wait-and-See*), which can be expressed as $WS = \sum_{\omega\in\Omega} w^\omega Z^\omega_{IP}$, where Z^ω_{IP} is the solution value of the model for scenario ω; EV is the solution value of the model for the average scenario (i.e., the *Expected Value* of the interest rate over the time horizon); EEV is the *Expected* result of the *Expected Value*, which can be expressed as $EEV = \sum_{\omega\in\Omega} w^\omega Z^\omega$, where Z^ω is the solution value of the model for scenario ω, whose solution for the first stage variables has been fixed at the optimal solution for the average scenario model; and VSS is the *Value* of the *Stochastic Solution*, which can be expressed as $VSS = EEV - Z_{IP}$.

We can observe in Table 5 that *VSS* is strictly positive in 13 out of the 15 test cases. There are two cases, namely P2 and P8, where the *EV* solution is infeasible; they have 15 and 500 infeasible scenario related models, respectively. The results demonstrate that the use of stochastic programming is worthwhile, as opposed to using average scenario approaches, even though there are two cases where $VSS = 0$.

5 Conclusions

In this paper a new scheme for assessing the performance of the standard Benders Decomposition in two-stage stochastic integer programming is presented for cases where the first stage includes 0–1 variables plus continuous variables, and the second stage has only continuous variables. The approach is based on a mixture of *Branch-and-Fix Coordination* and Benders Decomposition schemes. The first scheme coordinates the execution of the branch-and-bound phases to satisfy the *nonanticipativity* constraints for the 0–1 variables among the scenario *cluster*-related sub-problems. The second scheme is designed to satisfy the *nonanticipativity* constraints for the first stage continuous variables at each *TNF integer set*. We have used the *Mortgage-Backed Securities (MBS)* structuring portfolio problem as an illustrative case to test our approach. The goal is to minimize the expected absolute mismatching of the durations of the *MBS* portfolio and the liabilities over the scenarios. The results have been obtained using an experimental code. They are very interesting when compared them with the non-stochastic strategy based on the average scenario approach. They also show a remarkable reduction in the elapsed time when comparing the new approach with the plain use of a state-of-the-art optimization engine. In any case, further experimentation with the hybrid decomposition approach that we have presented seems worthwhile.

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