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Bounds for in-progress floating-strike Asian options using symmetry

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Abstract This paper studies symmetries between fixed and floating-strike Asian options and exploits this symmetry to derive an upper bound for the price of a floating-strike Asian. This bound only involves fixed-strike Asians and vanillas, and can be computed simply given one of the many efficient methods for pricing fixed-strike Asian options. The bound coincides with the true price until after the averaging has begun and again at maturity. The bound is compared to benchmark prices obtained via Monte Carlo simulation in numerical examples.

Keywords Asian options · Floating strike Asian options · Put call symmetry · Bounds · Change of numéraire

Asian options have a payoff which depends on the average price of the underlying asset during some part of its life. The average is usually arithmetic, and if the asset price is assumed to follow exponential Brownian motion, an explicit option price is not available as the arithmetic average of a set of lognormal distributions is not known explicitly. Instead, pricing of Asian options is usually done numerically.

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There are two types of Asian options: the fixed-strike option, where the average relates to the underlying asset and the strike is fixed; and floating strike options where the average relates to the strike price. The fixed-strike option is in some sense easier, and has received most attention in the literature.

The starting point for this paper is a "symmetry" result in Henderson and Wojakowski (2002) (see also Hoogland and Neumann (2000)) which proves an equivalence between the price of a floating-strike Asian and the price of a related fixed-strike Asian, shown to be valid at the start of the averaging period.

In this paper we extend this symmetry result to "forward starting" Asian options. In particular, for a forward starting floating Asian option, we provide a symmetry with a "starting" fixed-strike Asian option. In the case where the floating option is "starting", we recover the special case given in Henderson and Wojakowski (2002). If the option is "in progress", we show that a floating-strike option can be re-expressed as a generalized "starting" option but not as any type of fixed-strike option. Instead, we derive an approximate method to price "in progress" floating-strike options. This approximation is actually an upper bound, and relates the price of a floating-strike Asian option to the sum of the price of a fixed-strike Asian and the price of a vanilla option.

Pricing of the fixed-strike Asian has been the subject of much research over the last ten years and academic interest in these options has experienced a revival recently. One approach is to re-characterize the average value of the underlying in terms of another stochastic process, and to use this characterization to derive expressions for the price, perhaps in terms of special functions (see Carr and Schröder (2000), Donati-Martin et al. (2001), Schröder (2002) and Yor (2001), continuing the earlier work of Geman and Yor (1993) and Dufresne (1990)).

Alternatively numerical methods may be employed. The current state of the art methods for fixed-strike Asians are numerical inversion of the Laplace transform (Shaw (2000), (2002)), eigenfunction expansions of Linetsky (2004), the stable pde method of Vecer (2001) and the analytical approximation of Vyncke et al. (2003).

The floating-strike Asian option has received far less attention in the literature, perhaps because the problem is more difficult in that the joint law of the stock price and the average stock price is needed. There are closed form approximations for the price based on the replacing the average stock price with a simpler random variable with the same first two (or more) moments (see Ritchken et al. (1993) and Chung et al. (2003)), but these approximations are not very accurate. Ingersoll (1987) and Rogers and Shi (1995) make use of a scaling argument (which is not valid for in-progress options) to develop a pde for the price of a floating-strike Asian, but it is difficult to solve numerically. Marcozzi (2003) solves the general problem (including in-progress options, and American style Asian options) using a finite element method, but this involves solving a pde in two space dimensions plus time. Papers by Vecer (2001), (2002) develop a new pde which reduces the problem to a single space variable, and has better stability properties and hence is more easily solved numerically. However, despite this literature, pricing methods for floating strike options are underdeveloped compared with the more established methods for the fixed-strike option.

Recently, Henderson and Wojakowski (2002) observed that the problem of pricing a starting floating-strike Asian option could be transformed via a symmetry into the problem of pricing a starting fixed-strike option, see also Eberlein and Papapantoleon (2005) where the result is extended to cover exponential Lévy models of the asset price. Hence, given a favorite method of pricing a fixed-strike call, the problem of pricing a starting floating-strike option is essentially solved, or to put it another way, the problems of pricing fixed and floating options are equally difficult, at least as long as the averaging period has not yet started.

Now consider "in progress" options. For fixed-strike options the fact that the option is in progress makes no difference to the level of difficulty of the problem, since the average stock price so far can be incorporated into the strike. For floating-strike options this is not the case; as we show below an in progress option becomes a generalized starting option. Hence the scaling methods of Ingersoll (1987) and Rogers and Shi (1995) are not applicable. In this paper we capitalize on the fact that there are relatively simple methods for the pricing of a fixed strike option to give a simple upper bound on the price of a floating-strike option.

Our upper bound coincides with the true price at times up to and including the time the averaging begins and at maturity. Via symmetry, the bound may be expressed as a combination of fixed-strike puts and vanilla call options, optimized over a weighting parameter. One of the main advantages of the bound is that one can employ existing methods to price the fixed-strike option. As such, the speed and accuracy of the method depend on the chosen algorithm to price the fixed-strike component of the bound. We introduce an approximation to choose the weighting parameter optimally, and demonstrate whilst this has little effect on the accuracy, it reduces the computation time dramatically.

Our pricing bound is derived in the framework of the Black Scholes model and relies on a model dependent symmetry result and a model independent decomposition of the floating-strike Asian into a fixed-strike Asian and a vanilla option. However, given the symmetry result for exponential Lévy models in Eberlein and Papapantoleon (2005), it is clear that our general methodology can be applied for this model also. Given a method for pricing fixed-strike options under a Lévy model, we can derive an upper bound on the price of an in-progress floating-strike option.

Given the relative difficulty of pricing Asian options, various authors have suggested approximations or bounds. One strand of the literature, initiated by Curran (1994), see also Rogers and Shi (1995) and Nielsen and Sandmann (2003), uses conditioning to bound the payoff $\mathbb{E}[(\frac{1}{T}\int_0^T S_t dt - K)^+]$ with

$$\mathbb{E}\left[\left(\mathbb{E}\left[\frac{1}{T}\int_0^T S_t dt \,\middle|\, \Lambda\right] - K\right)^+\right]$$

for a suitable conditioning event Λ. In the fixed-strike Asian problem, good choices for Λ include S_T, and the geometric average of the stock price. Another approach, as described in Vanmaele et al. (2005) and Vyncke et al. (2003) is to observe that the average $\frac{1}{T}\int_0^T S_t dt$ is smaller in convex order sense (and hence gives lower call option prices) than $\frac{1}{T}\int_0^T F_t^{-1}(U)dt$, where F_t is the distribution function of S_t and U is a uniform random variable on [0,1]. In this bound the joint distribution of $(S_t)_{t\le T}$ is replaced by one in which S_t and $S_{t'}$ are co-monotonic. This approximation makes no reference to the fact that the discounted price process is a martingale, which is why it performs relatively poorly, but when combined with conditioning it gives a much improved upper bound on the starting option price.

Vanmaele et al. (2005) apply their techniques to starting floating-strike Asian options, essentially by using the symmetry result of Henderson and Wojakowski (2002) to convert the problem into a fixed-strike option. They also make the claim (Section V) that 'the in progress case can be dealt with in a similar way', but since the symmetry result does not apply in this case it is not clear what they mean.

The main contribution of the paper is to provide an approximation to the price of a in progress floating-strike Asian option which has some desirable properties: it is exact at the time the averaging starts and at maturity; assuming users have an algorithm for pricing fixed-strike Asian options it is very easy to code; and it gives reasonable accuracy in essentially the same time it takes to price a fixed-strike Asian. Further, the paper builds on symmetry relationships for exotic options and provides an interesting application of such symmetries to pricing. Our method is an illustration of a more general philosophy; when pricing new and more sophisticated derivatives one should try to relate them to existing instruments. The prices of these existing derivatives may provide bounds on the price of the new option and may even form a component of a hedging strategy.

As we show this bound is fairly accurate for in progress floating-strike options, with worst case errors of the order of 3% for reasonable parameter values. (Best case errors are zero, at both ends of the time averaging period, when the bound is equal to the true price.) This bound, perhaps based on evaluating special functions, may be preferable to calculating the price of the floating-strike option via Monte-Carlo methods (generally slow, but with the benefit that they give theoretical error bounds) or via a pde (Marcozzi's (2003) implementation requires solving a pde in three variables which will be computationally intensive).

The paper is structured as follows. The next section outlines the model and defines the floating and fixed-strike Asian option. Section 2 gives some general symmetry results for Asian options and recovers the symmetry found in Henderson and Wojakowski (2002) as a special case. The following section derives the upper bound for the price of the floating-strike Asian call. We concentrate in this paper on a bound for the call. Of course, the same method gives a bound for the floating-strike Asian put, this is left to the interested reader. In Section 4 we give an approximate method to reduce the calculation time and report the results of our numerical investigation in Section 5. The final section concludes the paper.

1 The model

We consider the standard Black Scholes economy with a risky asset (stock) and a money market account. We take as given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0\le t\le T_\infty}$, which is right-continuous and such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. Here T_∞ is the termination date of our economy, which is certainly greater than the maturity date of any option we might consider. We also assume the existence of a risk-neutral probability measure $\mathbb{Q}$ (equivalent to $\mathbb{P}$) under which discounted asset prices are martingales, implying no arbitrage. We denote expectation under measure $\mathbb{Q}$ by $\mathbb{E}$, and under $\mathbb{Q}$, the stock price follows

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t \tag{1}$$

where r is the constant continuously compounded interest rate, q is a continuous dividend yield, σ is the instantaneous volatility of asset return and W is a $\mathbb{Q}$-Brownian motion.

We consider an Asian contract which is based on the value A_T where $(A_t)_{t\ge t_0}$ is the arithmetic average

$$A_t = \frac{1}{t-t_0}\int_{t_0}^{t} S_u du \qquad t > t_0,$$

and by continuity, we define $A_{t_0} = S_{t_0}$. The contract is written at time 0 (with $0 \le t_0$) and expires at $T > t_0$. Of interest is to calculate the price of the option at the current time t, where $0 \le t \le T$. The position of t compared to the start of the averaging, t_0 may vary. If $t \le t_0$ the option is "forward starting". We will call the special case $t = t_0$ a "starting" option. If $t > t_0$, the option is termed "in progress" as the averaging has begun. In this paper we are mainly concerned with in progress options.

We consider a *generalized Asian option* with payoff $(aS_T + bA_T + c)^+$ at time T. The important cases in financial options are

- $(a, b, c) = (0, 1, -K)$—the fixed-strike Asian call option,
- $(a, b, c) = (0, -1, K)$—the fixed-strike Asian put,
- $(a, b, c) = (1, -1, 0)$—the floating-strike Asian call, and
- $(a, b, c) = (-1, 1, 0)$—the floating-strike Asian put.

Note that vanilla European puts and calls can also be put in this form:

- $(a, b, c) = (1, 0, -K)$—the European call option,
- $(a, b, c) = (-1, 0, K)$—the European put option.

By standard arbitrage arguments the time-t price of this generalized option is the discounted expected payoff under $\mathbb{Q}$, and we write

$$V_t(a, b, c; r, q; S_t, A_t; t_0, T) = e^{-r(T-t)}\mathbb{E}[(aS_T + bA_T + c)^+|\mathcal{F}_t].$$

We remind the reader of our notation: the averaging begins at t_0, current time is t, and the option expires at T. Note that for forward starting options A_t is not well defined and so we write $V_t(a, b, c; r, q; S_t, \star; t_0, T)$.

2 Symmetry results for Asian options

In this section, we show that the pricing function for the generalized option satisfies certain scaling and symmetry results.

Theorem 1. (i) *V is homogeneous of degree 1 in the parameters a, b, c, so that for $\lambda > 0$,*

$$V_t(\lambda a, \lambda b, \lambda c; r, q; S_t, A_t; t_0, T) = \lambda V_t(a, b, c; r, q; S_t, A_t; t_0, T).$$

(ii) *For an in-progress option ($t > t_0$) we have the identity*

$$V_t(a, b, c; r, q; S_t, A_t; t_0, T) = V_t\left(a, b\frac{T-t}{T-t_0}, c + b\frac{t-t_0}{T-t_0}A_t; r, q; S_t, \star; t, T\right) \tag{2}$$

which allows us to write any generalized in-progress Asian option as a starting Asian option.

(iii) *For a starting option we have the symmetry*

$$V_{t_0}(a, b, c; r, q; S_{t_0}, \star; t_0, T) = V_{t_0}\left(\frac{c}{S_{t_0}}, b, aS_{t_0}; q, r; S_{t_0}, \star; t_0, T\right) \tag{3}$$

This is an extension of the result of Henderson and Wojakowski (2002). Note that the roles of r and q are reversed as well roles of a and c.

(iv) *Combining* (ii) *and* (iii) *we get for in-progress options*

$$V_t(a, b, c; r, q; S_t, A_t; t_0, T) = V_t\left(\frac{c}{S_t} + b\frac{t-t_0}{T-t_0}\frac{A_t}{S_t}, b\frac{T-t}{T-t_0}, aS_t; q, r; S_t, \star; t, T\right) \tag{4}$$

Proof: The linearity of the option pricing function is inherited from the homogeneity of the payoff function: $(\lambda x)^+ = \lambda x^+$ at least for positive λ. The second part is equally trivial and is based on the identity

$$A_T = \frac{t - t_0}{T - t_0} A_t + \frac{T - t}{T - t_0} \frac{1}{T - t} \int_t^T S_u \, du$$

where the first term is $\mathcal{F}_t$ measurable, and the second term is a constant multiplied by the average stock price over the interval $[t, T]$. The final part does indeed follow from earlier parts as indicated, so the main result of this theorem is contained in (iii), the proof of which is relegated to the appendix. This proof is an extension of an argument in Henderson and Wojakowski (2002) and involves a change of measure and an identification of a time-reversal of a Brownian motion. □

Note that it follows from the first part of the theorem that it is sufficient to consider the cases $b = \pm 1$, together with $b = 0$ which corresponds to vanilla European options. If we take a special case of (3), namely a floating-strike option ($c = 0$), we can derive a symmetry between fixed and floating-strike options. This symmetry also holds whilst the option is forward-starting, which is the content of the next theorem.

Theorem 2. *For a forward-starting Asian option, $t \le t_0$ we have*

$$V_t(a, -1, 0; r, q; S_t, \star; t_0, T) = V_t(0, -1, aS_t e^{-q(t_0 - t)}; q, r; S_t e^{-q(t_0 - t)}, \star; t, T + t - t_0)$$

In particular a forward starting floating-strike call has the same price as a starting fixed-strike put with r and q reversing roles and modified maturity. An analogous result holds for $b = 1$, which converts a floating-strike put into a fixed-strike Asian call.

Remark 3. The special case of this result for a "starting" option was proved in Henderson and Wojakowski (2002) and is given as

$$V_{t_0}(a, -1, 0; r, q; S_{t_0}, \star; t_0, T) = V_{t_0}(0, -1, aS_{t_0}; q, r; S_{t_0}, \star; t_0, T)$$

Vanmaele et al. (2005) also obtain this symmetry result when the average is sampled discretely.

Proof of Theorem 2:

$$\begin{aligned} V_t(a, -1, 0, r, q, S_t, \star, t_0, T) &= e^{-r(T-t)} \mathbb{E}_t \left[S_{t_0} \left(\frac{aS_T - A_T}{S_{t_0}} \right)^+ \right] \\ &= e^{-r(T-t)} (\mathbb{E}_t S_{t_0}) \mathbb{E}_{t_0} \left(\frac{aS_T}{S_{t_0}} - \frac{1}{T - t_0} \int_{t_0}^T \frac{S_u}{S_{t_0}} du \right)^+ \end{aligned}$$

where we use the independence of S_{t_0} and increments after t_0. Using a time translation $u \to u - (t_0 - t)$ in the second expectation this becomes

$$e^{-r(T-t_0)} S_t e^{-q(t_0-t)} \mathbb{E}_t \left(\frac{aS_{T+t-t_0}}{S_t} - \frac{1}{T - t_0} \int_t^{T+t-t_0} \frac{S_u}{S_t} du \right)^+$$

which is

$$V_t(a, -1, 0; r, q; S_t e^{-q(t_0-t)}, \star; t, T + t - t_0)$$

a starting floating option (but at time t) with modified maturity. Now applying the symmetry result of Theorem 1 (iii) for a starting option, we can write this as

$$V_t(0, -1, aS_t e^{-q(t_0-t)}; q, r; S_t e^{-q(t_0-t)}, \star; t, T + t - t_0)$$

□

Theorems 1 and 2 are useful as they give relationships between various Asian options. The generalized symmetry of Theorem 1 (iii) can be used to transform starting floating-strike Asians into starting fixed-strike Asians. In addition, a forward starting floating-strike Asian is equivalent to a starting fixed-strike Asian with modified maturity and other parameters, as given in Theorem 2. Any Asian which is in progress may be written as a generalized starting option, as described in (ii).

However, (iv) clarifies that although we can write an in progress Asian (take $a = 1, b = -1, c = 0$ for a call) as a generalized starting Asian, it cannot be reduced to a standard fixed or floating-strike Asian.

Thus, to price a forward starting (and starting) floating-strike call (or put), we can use symmetry and price the equivalent fixed-strike put (or call). If the floating Asian call is in progress however, there is no such symmetry. Instead we derive an upper bound which involves fixed-strike Asian puts and vanilla call options. Similarly we could derive an upper bound for the floating-strike Asian put.

3 An upper bound for the floating-strike Asian option

Since the symmetry in Theorem 2 holds only up to and at the moment the averaging begins, we develop an upper bound for the case when the option is in progress. The payoff of a floating-strike Asian call option

$$(S_T - A_T)^+ = \left(S_T - \frac{1}{T - t_0}\int_{t_0}^{T} S_u\, du\right)^+$$

can be rewritten in terms of pre and post-t parts, for $t_0 < t$

$$\left(S_T - \frac{1}{T - t_0}\int_{t_0}^{t} S_u\, du - \frac{1}{T - t_0}\int_{t}^{T} S_u\, du\right)^+. \tag{5}$$

We can use this representation to obtain the following result.

Theorem 4. *For $t \geq t_0$, an upper bound on the price $V_t(1, -1, 0; r, q; S_t, A_t; t_0, T)$ of an in-progress floating-strike call is given by*

$$\inf_{\alpha}\left\{V_t\left((1-\alpha), 0, -\frac{t - t_0}{T - t_0}A_t; r, q; S_t, \star; t, T\right) + V_t\left(0, -\frac{T - t}{T - t_0}, \alpha S_t; q, r; S_t, \star; t, T\right)\right\} \tag{6}$$

Note that we have bounded the floating in-progress option with the notionally simpler fixed-strike Asian put with modified dynamics together with an ordinary European call option.

Corollary 5. *If $t = t_0$ the infimum is attained at $\alpha = 1$. Conversely, if $t = T$ then the infimum is attained at $\alpha = 0$. Further, the bound in (6) gives the exact price for the floating-strike option for times t at both ends of the averaging interval.*

Proof of Theorem 4. Note $(a + b + c) = [(1 - \alpha)a + b] + (\alpha a + c)$ and $(x + y)^+ \le x^+ + y^+$. Hence for any α, $(a + b + c)^+ \le ((1 - \alpha)a + b)^+ + (\alpha a + c)^+$. Applying these to (5) gives

$$
\begin{aligned}
(S_T - A_T)^+ \le \inf_\alpha \Bigg\{ & \left(S_T (1 - \alpha) - \frac{1}{T - t_0} \int_{t_0}^{t} S_u \, du \right)^+ \\
& + \left(\alpha S_T - \frac{1}{T - t_0} \int_{t}^{T} S_u \, du \right)^+ \Bigg\}
\end{aligned}
$$

Taking discounted expectations will give an upper bound on the price of a floating-strike Asian call

$$
\begin{aligned}
V_t (1, -1, 0; r, q; S_t, A_t; t_0, T) &= e^{-r(T-t)} \mathbb{E} \left[(S_T - A_T)^+ \middle| \mathcal{F}_t \right] \\
&\le \inf_\alpha \Bigg\{ e^{-r(T-t)} \mathbb{E} \left[\left(S_T (1 - \alpha) - \frac{1}{T - t_0} \int_{t_0}^{t} S_u du \right)^+ \middle| \mathcal{F}_t \right] \\
&\quad + e^{-r(T-t)} \mathbb{E} \left[\left(\alpha S_T - \frac{1}{T - t_0} \int_{t}^{T} S_u du \right)^+ \middle| \mathcal{F}_t \right] \Bigg\}. \qquad (7)
\end{aligned}
$$

The first term is a call option and can be rewritten as

$$
V_t \left(1 - \alpha, 0, -\frac{t - t_0}{T - t_0} A_t; r, q; S_t, \star; t, T \right)
$$

Further, by Theorem 1 (iv) the second term can be re-expressed as a fixed-strike Asian put:

$$
V_t \left(\alpha, -\frac{T - t}{T - t_0}, 0; r, q; S_t, \star; t, T \right) = V_t \left(0, -\frac{T - t}{T - t_0}, \alpha S_t; q, r; S_t, \star; t, T \right).
$$

□

We have managed to construct a bound for floating-strike Asians which depends only on vanilla options and fixed-strike Asians. The fixed-strike Asian option has been well studied. Competing methods include integral formulas of Linetsky (2004), inversion of Laplace transform of Geman and Yor (1993) (implemented by Shaw (2000), (2002)), and the stable pde method of Vecer (2001). Each of these methods was shown to give six digit precision by Vecer (2002) and Shaw (2002). Thus, given any of these (or another) method for pricing fixed-strike Asian options, the bound can be calculated without any new algorithms. The bound requires optimizing over the parameter α. This potentially means many calls to a routine to price the fixed-strike put need to be made. It is therefore vital to choose a fast (and accurate) method for pricing the fixed-strike put, as the speed and accuracy of the bound depend on this.

However, if we could avoid this optimization over α by choosing an approximate value which achieved very similar accuracy, this would speed up the computation, as only a single call to the fixed-strike pricing routine would be needed. In the next section we give a method for speeding up the calculation by deriving an approximation to the optimal choice of α for the upper bound.

Note that in this section we have concentrated on the pricing of the "in progress" floating-strike Asian and we have not mentioned hedging at all. However, implicit in the analysis is a simple super-replicating hedge. At time $t \in [t_0, T]$, choose the optimal $\alpha = \alpha(t)$ (or indeed any α) and decompose the floating-strike option into a combination of a vanilla and a fixed-strike Asian option. The vanilla can be replicated in the standard fashion. If the fixed-strike Asian is hedged appropriately, then we have a super-hedge for the floating-strike Asian option.

4 Optimal choice of the parameter α for the upper bound

The purpose of this section is to find an efficient method to choose a value of the parameter α to give a good approximate upper bound in Theorem 4. Recall that if $t = t_0$, so we are in the case of a starting option, then the optimal α is given by $\alpha = 1$. We consider "in-progress" options, so that $t_0 < t < T$, and we will make a series of approximations and assumptions to derive a suitable choice of α. Our linearizing of exponential terms is similar to that used in the pricing approximation in Chung et al. (2003) and Bouaziz et al. (1994).

We begin by recalling

$$(S_T - A_T)^+ \leq \left((1-\alpha)S_T - \frac{t-t_0}{T-t_0}A_t\right)^+ + \left(\alpha S_T - \frac{1}{T-t_0}\int_t^T S_u du\right)^+$$

Note that, for $u \geq v$, $S_u = S_v \exp\{\sigma(W_u - W_v) + (r - q - \sigma^2/2)(u - v)\}$.

Assumption 6. *$r - q$ and σ^2 are small, or more precisely $(r-q)(T-t_0)$ and $\sigma^2(T-t_0)$ are small.*

Under this assumption, for $u \geq t$, we can approximate S_u by $S_t(1 + \sigma(W_u - W_t)) = S_t(1 + \sigma \int_t^u dW_v)$, and then, with $\approx$ denoting approximately equal

$$\begin{aligned}(1-\alpha)S_T - \frac{t-t_0}{T-t_0}A_t &\approx (1-\alpha)S_t - \frac{t-t_0}{T-t_0}A_t + S_t(1-\alpha)\sigma\int_t^T dW_v \\ &\sim N\left((1-\alpha)S_t - \frac{t-t_0}{T-t_0}A_t;\ S_t^2(1-\alpha)^2\sigma^2(T-t)\right). \qquad (8)\end{aligned}$$

Also

$$\frac{1}{T-t_0}\int_t^T S_u du \approx \frac{(T-t)}{T-t_0}S_t + \frac{\sigma S_t}{T-t_0}\int_t^T (W_u - W_t)\,du.$$

and hence

$$\begin{aligned}
&\alpha S_T - \frac{1}{T-t_0}\int_t^T S_u du \\
&\approx \alpha S_t + \alpha S_t \sigma \int_t^T dW_t - \frac{(T-t)}{T-t_0}S_t - \frac{\sigma S_t}{T-t_0}\int_t^T (T-u)dW_u \\
&= \left(\alpha - \frac{T-t}{T-t_0}\right)S_t + S_t\sigma\int_t^T \left(\alpha - \frac{T-u}{T-t_0}\right)dW_u \\
&\sim N\left(\left(\alpha - \frac{T-t}{T-t_0}\right)S_t\ ;\ \frac{T-t_0}{3}S_t^2\sigma^2\left[\alpha^3 - \left(\alpha - \frac{T-t}{T-t_0}\right)^3\right]\right). \qquad (9)
\end{aligned}$$

Note that the covariance of the terms in (8) and (9) is

$$\begin{aligned}
&S_t^2\sigma^2(1-\alpha)\int_t^T \left(\alpha - \frac{T-u}{T-t_0}\right)du \\
&= (T-t_0)S_t^2\sigma^2\frac{(1-\alpha)}{2}\left[\alpha^2 - \left(\alpha - \frac{T-t}{T-t_0}\right)^2\right] \qquad (10)
\end{aligned}$$

Let G_1 and G_2 be normal random variables with distributions

$$G_1 \sim N\left((1-\alpha)S_t - \frac{t-t_0}{T-t_0}A_t;\ S_t^2(1-\alpha)^2\sigma^2(T-t)\right)$$

$$G_2 \sim N\left(\left(\alpha - \frac{T-t}{T-t_0}\right)S_t\ ;\ \frac{T-t_0}{3}S_t^2\sigma^2\left[\alpha^3 - \left(\alpha - \frac{T-t}{T-t_0}\right)^3\right]\right)$$

and covariance as given in (10).

Our goal is to minimize $G_1^+ + G_2^+$. Note that $G_1 + G_2$ has mean $(S_t - A_t)(t-t_0)/(T-t_0)$ which is independent of α. We can imagine choosing α to distribute this mean between the two variables G_i. The proportion of this quantity that we assign to each normally distributed random variable should depend on their respective variances.

In particular we consider α of the form

$$\alpha = \frac{T-t}{T-t_0} + \gamma\left\{\frac{t-t_0}{T-t_0}\left(\frac{S_t - A_t}{S_t}\right)\right\}.$$

Assumption 7. *To leading order $\alpha = (T-t)/(T-t_0)$. Further, when we substitute α in to the expressions for the variances of G_1 and G_2 we can neglect higher order terms.*

For this α, and using the leading order expression for the variances we have

$$G_1 \sim N\left((1-\gamma)\frac{t-t_0}{T-t_0}(S_t - A_t);\ S_t^2\sigma^2(T-t)\frac{(t-t_0)^2}{(T-t_0)^2}\right)$$

$$G_2 \sim N\left(\gamma\frac{t-t_0}{T-t_0}(S_t - A_t)\ ;\ S_t^2\sigma^2\frac{1}{3}\frac{(T-t)^3}{(T-t_0)^2}\right)$$

Note that the ratio of the standard deviations is given by:

$$\sqrt{\mathrm{Var}(G_1)} : \sqrt{\mathrm{Var}(G_2)} = \sqrt{3}(t-t_0) : (T-t)$$

We choose γ such that the ratio of the means is equal to the ratio of the standard deviations, so

$$(1-\gamma) : \gamma = \sqrt{3}(t-t_0) : (T-t)$$

and hence $\gamma = (T-t)/(T-t+\sqrt{3}(t-t_0))$. This choice can be justified rigorously if G_1 and G_2 are uncorrelated (whereas in fact they have correlation $\sqrt{3}/2$), and if the means are

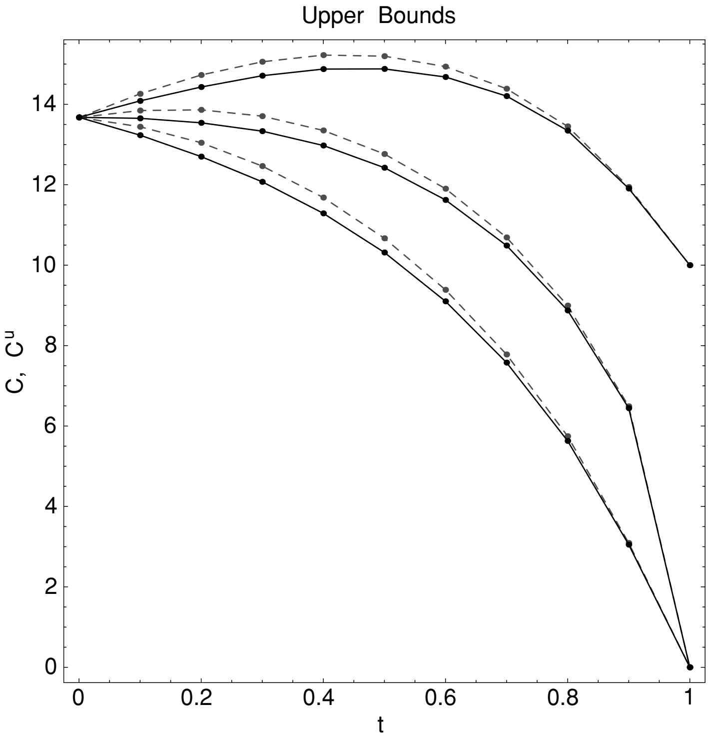

Fig. 1 *Upper Bounds C^u* (dashed lines with dots) vs optimized control variate Monte-Carlo estimates of the arithmetic Asian option price C (solid lines with dots). Parameters are $S_t = 100, \sigma = 0.5, r = 0.1, q = 0, t_0 = 0, T = 1$. The "highest" case is when $A_t = 90$ and either bounds or both prices reach the payoff $(S_t - A_t)^+ = 10$ for $t = T = 1$. The "middle" and "lowest" cases arise when $A_t = 100$ and $A_t = 110$ respectively: bounds and prices reach $(S_t - A_t)^+ = 0$ then

large in comparison with the standard deviations, but remains a plausible choice in many circumstances.

In conclusion, the proposed choice of α is

$$\hat{\alpha} = \frac{T-t}{T-t_0} + \frac{t-t_0}{T-t_0}\left(\frac{T-t}{T-t+\sqrt{3}(t-t_0)}\right)\left(\frac{S_t-A_t}{S_t}\right).$$

5 Implementation and results

We implement the upper bound using Laplace transform inversion methods for the fixed-strike option. Shaw (2000) performs the Laplace transform inversion by direct numerical integration along the truncated Bromwich contour. This contour is a vertical line to the right of any finite singularities, and the truncation can be adjusted to obtain higher accuracy. He has improved upon this implementation in Shaw (2002) by transforming the hypergeometric function into a collection of geometric series using Mellin transforms. This improved the

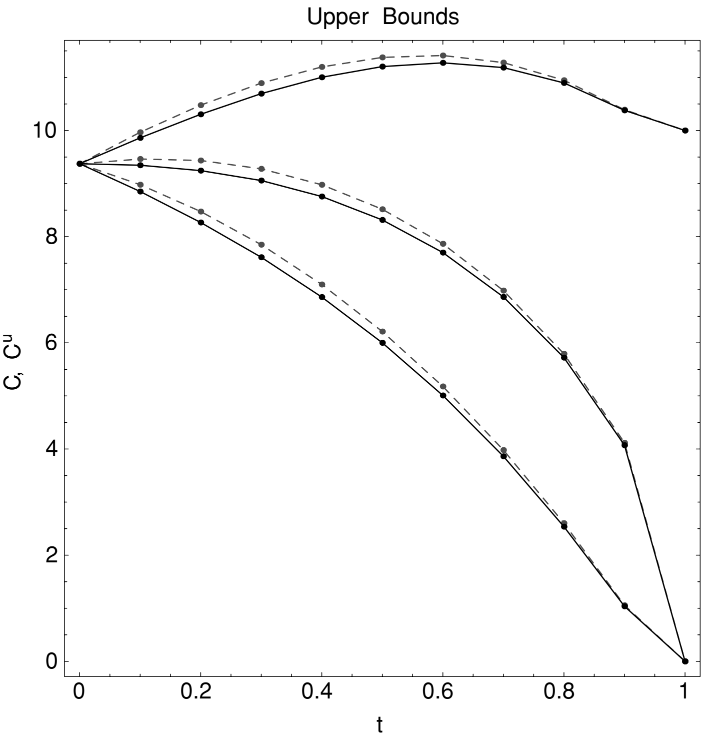

Fig. 2 *Upper Bounds C^u* (dashed lines with dots) vs optimized control variate Monte-Carlo estimates of the arithmetic Asian option price C (solid lines with dots). Parameters are $S_t = 100, \sigma = 0.3, r = 0.1, q = 0, t_0 = 0, T = 1$. The "highest" case is when $A_t = 90$ and either bounds or both prices reach the payoff $(S_t - A_t)^+ = 10$ for $t = T = 1$. The "middle" and "lowest" cases arise when $A_t = 100$ and $A_t = 110$ respectively: bounds and prices reach $(S_t - A_t)^+ = 0$ then

computation time dramatically, especially for low volatility examples. We have employed this improved method in our calculation of the fixed-strike options.

The advantages of this choice are that it is accurate, reasonably fast, and it is relatively easy to code: it can be done in fourteen lines in Mathematica. A further advantage from a coding perspective is that we can then calculate the bound in Mathematica using its in-built optimization routines. In this implementation, we also utilized put-call parity for floating-strike Asians.

Table 1 *Upper Bounds* for $S_t = 100$, $\sigma = 0.5$, $r = 0.1$, $q = 0$, $t_0 = 0$, $T = 1$. In the table, A_t is the arithmetic average realized up to time t, C_g is the price at time t of an otherwise identical geometric average option, C is an optimized control variate Monte-Carlo estimate of the arithmetic Asian price, with $N = 100000$ simulated paths and $m = 3000$ sampling points, δ is the standard deviation of the Monte-Carlo estimate, C^u is the upper bound computed by numerically minimizing over α, $\hat{C}^u$ is the approximate upper bound for approximate $\hat{\alpha}$. Note that the average A_t is unknown if $t = t_0 = 0$ and that at this time point, as well as at $t = T = 1$, both bounds C^u, $\hat{C}^u$ are "exact"

A_t	t	C_g	C	δ	C^u	α	$\hat{C}^u$	$\hat{\alpha}$
–	0	14.8329	13.6756	0.00436	13.6729	–	13.6729	1.
90	0.1	15.3068	14.0854	0.00449	14.2607	0.91211	14.2677	0.90839
	0.2	15.6529	14.4306	0.00441	14.7292	0.81985	14.74	0.81396
	0.3	15.8499	14.7114	0.00393	15.0582	0.724	15.0702	0.71722
	0.4	15.8765	14.8765	0.00334	15.224	0.62525	15.2353	0.61856
	0.5	15.7078	14.8806	0.00264	15.197	0.5242	15.2065	0.5183
	0.6	15.3105	14.6794	0.00189	14.9375	0.42136	14.9447	0.41668
	0.7	14.6338	14.205	0.00122	14.3875	0.31717	14.3922	0.31388
	0.8	13.5896	13.3462	0.00062	13.4512	0.21201	13.4538	0.21009
	0.9	12.	11.9074	0.0002	11.9435	0.1062	11.9445	0.10543
	1	10	10	–	10	–	10	0
–	0	14.8329	13.6756	0.00436	13.6729	–	13.6729	1.
100	0.1	14.7731	13.6521	0.00423	13.8452	0.90388	13.8525	0.9
	0.2	14.5808	13.54	0.00396	13.8625	0.80577	13.8727	0.8
	0.3	14.2363	13.3326	0.00337	13.7055	0.70625	13.7159	0.7
	0.4	13.7166	12.976	0.00276	13.3502	0.60579	13.3591	0.6
	0.5	12.9918	12.4235	0.0021	12.7643	0.50476	12.771	0.5
	0.6	12.019	11.6225	0.00143	11.9019	0.40347	11.9063	0.4
	0.7	10.7285	10.4886	0.00087	10.69	0.30217	10.6924	0.3
	0.8	8.9891	8.8749	0.0004	8.9949	0.20105	8.9959	0.2
	0.9	6.4745	6.4442	0.0001	6.49	0.10028	6.4902	0.1
	1	0	0	–	0	–	0	0
–	0	14.8329	13.6756	0.00436	13.6729	–	13.6729	1.
110	0.1	14.2977	13.2319	0.00402	13.4418	0.89589	13.4501	0.89161
	0.2	13.6413	12.6993	0.00361	13.0442	0.79246	13.0565	0.78604
	0.3	12.8466	12.0728	0.00295	12.4643	0.68984	12.4776	0.68278
	0.4	11.8926	11.2887	0.00233	11.6814	0.58814	11.6937	0.58144
	0.5	10.7517	10.3158	0.00171	10.6685	0.48742	10.6787	0.4817
	0.6	9.3853	9.1014	0.00111	9.3872	0.38774	9.3947	0.38332
	0.7	7.7337	7.577	0.00063	7.7793	0.28913	7.7842	0.28612
	0.8	5.6952	5.6295	0.00027	5.7464	0.19163	5.7491	0.18991
	0.9	3.0649	3.051	0.00006	3.0916	0.09525	3.0926	0.09457
	1	0	0	–	0	–	0	0

We will test our bound against a benchmark price for the floating-strike Asian option, which we calculate by Monte Carlo simulation, with variance reduction techniques. Monte Carlo and Quasi Monte Carlo simulation are used extensively in finance to obtain benchmark prices (see Corwin et al. (1996) and Fu et al. (1999)). Our random numbers were generated using a twisted GFSR (see Matsumoto and Nishimura (1998)). Fu et al. (1999) find that for fixed-strike Asians, the continuous geometric average Asian served as a high quality control variate. We take the same approach and follow Conze and Visvanathan (1991) to derive a formula for the floating-strike geometric average call option and use this as a control variate.

Table 2 *Upper Bounds* for $S_t = 100$, $\sigma = 0.3$, $r = 0.1$, $q = 0$, $t_0 = 0$, $T = 1$. In the table, A_t is the arithmetic average realized up to time t, C_g is the price at time t of an otherwise identical geometric average option, C is an optimized control variate Monte-Carlo estimate of the arithmetic Asian price, with $N = 100000$ simulated paths and $m = 3000$ sampling points, δ is the standard deviation of the Monte-Carlo estimate, C^u is the upper bound computed by numerically minimizing over α, $\hat{C}^u$ is the approximate upper bound for approximate $\hat{\alpha}$. Note that the average A_t is unknown if $t = t_0 = 0$ and that at this time point, as well as at $t = T = 1$, both bounds C^u, $\hat{C}^u$ are "exact"

A_t	t	C_g	C	δ	C^u	α	$\hat{C}^u$	$\hat{\alpha}$
–	0	9.8676	9.3741	0.00159	9.3725	–	9.3725	1.
90	0.1	10.4256	9.8614	0.00174	9.9666	0.91042	9.97	0.90839
	0.2	10.8995	10.3054	0.00177	10.4776	0.81726	10.483	0.81396
	0.3	11.2724	10.6948	0.00164	10.8925	0.72112	10.8988	0.71722
	0.4	11.5293	11.0028	0.00144	11.1978	0.62253	11.2039	0.61856
	0.5	11.655	11.2041	0.00116	11.3775	0.52193	11.383	0.5183
	0.6	11.6326	11.2749	0.00085	11.4128	0.41968	11.4172	0.41668
	0.7	11.4408	11.1851	0.00056	11.2793	0.31611	11.2825	0.31388
	0.8	11.0517	10.8962	0.0003	10.9466	0.2115	10.9485	0.21009
	0.9	10.4468	10.3797	0.0001	10.3915	0.1071	10.3968	0.10543
	1	10	10	–	10	–	10	0
–	0	9.8676	9.3741	0.00159	9.3725	–	9.3725	1.
100	0.1	9.8245	9.3457	0.00156	9.4627	0.90205	9.4659	0.9
	0.2	9.686	9.2441	0.00146	9.4348	0.80304	9.4394	0.8
	0.3	9.4384	9.0576	0.00125	9.2778	0.70329	9.2825	0.7
	0.4	9.0662	8.7564	0.00103	8.9778	0.60304	8.9818	0.6
	0.5	8.5501	8.3147	0.00078	8.5161	0.5025	8.5191	0.5
	0.6	7.8629	7.7003	0.00054	7.8659	0.40182	7.8679	0.4
	0.7	6.9617	6.8647	0.00032	6.9845	0.30114	6.9856	0.3
	0.8	5.7668	5.7215	0.00015	5.793	0.20055	5.7935	0.2
	0.9	4.0824	4.0707	0.00004	4.1165	0.10021	4.1167	0.1
	1	0	0	–	0	–	0	0
–	0	9.8676	9.3741	0.00159	9.3725	–	9.3725	1.
110	0.1	9.2954	8.8508	0.00143	8.979	0.89393	8.983	0.89161
	0.2	8.6481	8.267	0.00124	8.4735	0.78961	8.4798	0.78604
	0.3	7.9171	7.6121	0.00101	7.8495	0.68681	7.8567	0.68278
	0.4	7.0922	6.8623	0.00078	7.0992	0.58537	7.1063	0.58144
	0.5	6.1611	6.0006	0.00056	6.2128	0.48517	6.2191	0.4817
	0.6	5.1084	5.0082	0.00036	5.1782	0.38612	5.1833	0.38332
	0.7	3.9149	3.8619	0.0002	3.9802	0.28813	3.9838	0.28612
	0.8	2.5583	2.537	0.00009	2.6021	0.19115	2.6043	0.18991
	0.9	1.0412	1.0369	0.00002	1.0557	0.09408	1.0564	0.09457
	1	0	0	–	0	–	0	0

For each example following, our calculated simulation prices at time 0 agreed (up to reported accuracy) with those in Chung et al. (2003).

Our first example uses parameters $t = t_0 = 0, T = 1, r = 0.1, q = 0, \sigma = 0.5, S_t = 100$ and A_t is either 90, 100 or 110. The second example uses the same parameters but reduces volatility to 0.3. Both are 'starting' options. All computations were performed on a 1300 Mhz PC running Windows 2000. With more computing power, the times we report at the end of this section should be significantly reduced. Figures 1 and 2 plot the upper bound (C^u calculated using Theorem 4 and optimizing over α by binary search) and the benchmark prices (C) over the 1 year life of the option, for each value of A_t. It can be seen the bound is exact at time 0 and at maturity. If we were to compute the value for a forward starting option, the bound would simply be exact up to and including the time the averaging began. More detail is given in Tables 1 and 2. Simulation benchmark values C with standard deviation δ and the upper bound C^u with optimized value of α are reported. The remaining two columns contain the price bound $\hat{C}^u$ calculated using the approximation $\hat{\alpha}$ given in Section 4.

Our first comparison is between C, the benchmark and C^u the upper bound. For $t = 0$, the option is "starting" and the upper bound should be exact, at least theoretically. We see that 3 or 4 digits of accuracy are obtained between the simulation and the bound, due to errors inherent in the numerical estimation of each. This represents around a 0.02% error. We can use this as a guide for evaluating the errors over the life of the option, since this is a "base" error we are starting with. For volatility 0.5 (Table 1), the accuracy is reduced to 2 digits when $t > 0$ and 1 or 2 if the volatility is 0.3 (Table 2). As a percentage (when volatility is 0.5), errors range from about 0.1% to 3.5%, for the worst out-of-the-money option with $t = 0.4$. These are slightly better for the lower volatility case. If we compare these errors to a 1% mis-specification in volatility, we find the bound in Table 2 is less than the simulated price C with volatility 0.31.

These calculations are time consuming, however. For example, when volatility is 0.5, and $A_t = 90$, it took between 63 and 90 seconds to compute C^u for various points in time, t, with the truncation parameter set to 10000 in Shaw's implementation. These times can be reduced dramatically with little loss of accuracy by using $\hat{C}^u$ and $\hat{\alpha}$, the approximation to α. Under both volatilities, the difference in accuracy between $\hat{C}^u$ and C^u is insignificant, although $\hat{\alpha}$ is closer to α when volatility is 0.3, as expected. Recall $\hat{\alpha}$ does not depend on volatility. Using the approximation for α reduced the computation time for the $A_t = 90, \sigma = 0.5$ example to around a twelfth of the times reported earlier. For example, when $t = 0.2$, the time reduces from 75.6 to 6.3 seconds. Thus the approximation method retains virtually the same accuracy as the full optimized upper bound, but for a fraction of the computation time.

6 Conclusion

This paper has explored symmetries in Asian option pricing and exploited such relationships to derive a new approximation to the price of a floating-strike Asian. This approximation takes the form of a one-sided bound on the true price. The bound depends only on fixed-strike Asians and vanilla options. Given an efficient method for pricing a fixed-strike Asian option, and our approximation for the optimal weights of the fixed-strike and vanilla, the bound can be calculated immediately in one extra line of code. The accuracy and speed of computation of the bound depends on the choice of algorithm for the fixed-strike option. Using Shaw's (2002) implementation, calculations took a few seconds to give prices to within a couple of percent of the Monte Carlo simulation. This approximation may serve as a simple and fast estimate of the price of a floating-strike Asian option. It is particularly accurate near the beginning and end of the averaging period.

Appendix

Proof of Theorem 1 (iii)

This proof is an extension of the argument used in Theorem 1 of Henderson and Wojakowski (2002). Similar ideas have been used repeatedly by Yor (2001). Assume that S has the dynamics given by (1) and that the option is "starting", so $t = t_0$. We begin by rewriting the price of the generalized Asian option as:

$$
\begin{aligned}
V_{t_0}(a, b, c; r, q; S_{t_0}, \star; t_0, T) &= e^{-r(T-t_0)}\mathbb{E}[(aS_T + bA_T + c)^+ | \mathcal{F}_{t_0}] \\
&= S_{t_0} e^{-q(T-t_0)}\mathbb{E}_{t_0}\left[\frac{S_T e^{-(r-q)(T-t_0)}}{S_{t_0}} \frac{(aS_T + bA_T + c)^+}{S_T}\right]
\end{aligned}
$$

Define the measure $\hat{\mathbb{Q}}$ via

$$
\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} = \frac{S_T e^{-(r-q)(T-t_0)}}{S_{t_0}} = \exp\left\{\sigma(W_T - W_{t_0}) - \frac{\sigma^2}{2}(T - t_0)\right\}.
$$

Under $\hat{\mathbb{Q}}$, $\hat{W}_u = W_u - \sigma(u - t_0)$ is a Brownian motion (started at $\hat{W}_{t_0} = W_{t_0}$) and the price becomes

$$
V_{t_0} = S_{t_0} e^{-q(T-t_0)}\hat{\mathbb{E}}_{t_0}\left[\left(a + b\frac{A_T}{S_T} + \frac{c}{S_T}\right)^+\right]. \tag{11}
$$

Again under $\hat{\mathbb{Q}}$ we have $dS_u = S_u\{(r - q + \sigma^2)du + \sigma d\hat{W}\}$ and for $u \geq t_0$,

$$
\frac{S_u}{S_T} = \exp\left\{\left(r - q + \frac{1}{2}\sigma^2\right)(u - T) + \sigma(\hat{W}_u - \hat{W}_T)\right\}.
$$

Now let $(\hat{B}_u)_{t_0 \leq u \leq T}$ be defined via $\hat{B}_u = \hat{B}_{t_0} + \hat{W}_{T+t_0-u} - \hat{W}_T$ for some constant $\hat{B}_{t_0}$. Then $\hat{B}$ is a time-reversal of a Brownian motion and therefore $\hat{B}$ is again a Brownian motion under $\hat{\mathbb{Q}}$. Also

$$
\frac{c}{S_T} = \frac{c}{S_{t_0}}\frac{S_{t_0}}{S_T} = \frac{c}{S_{t_0}} e^{(q-r)(T-t_0)}\exp\left\{\sigma(\hat{B}_T - \hat{B}_{t_0}) - \frac{1}{2}\sigma^2(T - t_0)\right\} = \frac{c}{S_{t_0}}\frac{\hat{S}_T}{\hat{S}_{t_0}},
$$

where $\hat{S}$ solves the stochastic differential equation

$$
\frac{d\hat{S}_u}{\hat{S}_u} = (q - r)du + \sigma d\hat{B}_u \qquad u \geq t_0 \tag{12}
$$

with $\hat{S}_{t_0} \equiv S_{t_0}$. In particular we think of $\hat{S}$ as a stock paying constant rate of dividends r in a market with interest rate δ. Further

$$\begin{aligned}(T-t_0)\frac{A_T}{S_T} &= \int_{t_0}^{T}\frac{S_u}{S_T}du = \int_{t_0}^{T}du\exp\left\{\left(r-q+\frac{1}{2}\sigma^2\right)(u-T)+\sigma(\hat{W}_u-\hat{W}_T)\right\}\\ &= \int_{t_0}^{T}due^{(q-r)(T-u)}\exp\left\{\sigma(\hat{B}_{T+t_0-u}-\hat{B}_{t_0})-\frac{1}{2}\sigma^2(T-u)\right\}\\ &= \int_{t_0}^{T}dve^{(q-r)(v-t_0)}\exp\left\{\sigma(\hat{B}_v-\hat{B}_{t_0})-\frac{1}{2}\sigma^2(v-t_0)\right\}\\ &= \int_{t_0}^{T}dv\frac{\hat{S}_v}{\hat{S}_{t_0}}.\end{aligned}$$

We have that

$$S_{t_0}\frac{(aS_T+bA_T+c)^+}{S_T} = \left(\frac{c}{\hat{S}_{t_0}}\hat{S}_T + b\frac{1}{T-t_0}\int_{t_0}^{T}dv\hat{S}_v + a\hat{S}_{t_0}\right)^+$$

and under $\hat{\mathbb{Q}}$, this last term is the payoff of a generalized Asian option under the dynamics (12). Finally, if we discount this expression at the interest rate δ we get from (11)

$$V_{t_0}(a,b,c;r,q;S_{t_0},\star;t_0,T) = V_{t_0}\left(\frac{c}{S_{t_0}},b,aS_{t_0};q,r;S_{t_0},\star;t_0,T\right)$$

as required. □

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