Ann Oper Res (2006) 144:201–234
DOI 10.1007/s10479-006-0016-9

Combinatorial aspects of the sensor location problem

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Published online: 24 May 2006

Abstract In this paper we address the Sensor Location Problem, that is the location of the minimum number of counting sensors, on the nodes of a network, in order to determine the arc flow volume of all the network. Despite the relevance of the problem from a practical point of view, there are very few contributions in the literature and no combinatorial analysis is performed to take into account particular structure of the network. We prove the problem is $\mathcal{NP}$-complete in different cases. We analyze special classes of graphs that are particularly interesting from an application point of view, for which we give low order polynomial solution algorithms.

Keywords Traffic problems · Combinatorial optimization · Complexity analysis · Graph theoretical approach

1. Introduction

The continuous growth in the demand for private transportation in large urban areas is the cause of severe congestion, pollution, time loss in traffic jams and a deterioration in the quality of life.

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Monitoring flows on the network allows traffic managers to control and manage these problematic situations. Even though communication technologies for monitoring traffic networks in real time, via sensors and video cameras, are currently available, in most cases we have a very large network which is monitored only in small part. In the case of large traffic volumes, traffic flows tend to be spread out from monitored major boulevards to non-monitored minor roads due to route diversions and, hence, new flow patterns are created. Thus, a relevant problem for traffic manager and planners is how to monitor and predict flow propagation in these cases. Indeed, since the installation of traffic sensors on every arc of the net is not affordable, it is necessary to design optimal strategies to determine sensor locations, to better predict traffic flows on the whole network at minimum cost.

Sensor can be located either on the arcs or on the nodes of a network, and, following the differentiation introduced in Gentili and Mirchandani (2005) and Gentili (2002), we can distinguish between *passive* and *active* sensors. *Active* sensors is the class of sensors that decode active transmission provided by vehicles, for example, freight information from trucks or path/schedule information from buses. *Counting* and *video image* sensors belong to the class of passive sensors since the vehicle is being detected and observed without it providing any other signal. In particular, counting sensors are the most commonly used (see Chandnani and Mirchandani (2002) for a detailed description). A counting sensor located on an arc (a node) of the network counts the number of vehicles on the arc (on the arcs incident to the node) in the unit time interval.

Despite the relevance of the problem from an application point of view, there are few contributions in the literature addressing the problem of locating sensors on traffic network. In particular, in Lam and Lo (1990), some heuristic procedures to decide where to locate counting sensors on the arc in order to obtain a better estimate of the O/D matrix are proposed. Yang, Iida, and Sasaki (1991) defined the O/D covering rule to locate counting sensors on arcs and obtain the O/D estimation error to be bounded. Yang and Zhou (1998) defined three additional rules that an optimal location of counting sensors on arcs should respect to obtain a better estimate of the O/D matrix and proposed some heuristic procedures. Gentili and Mirchandani (2005) addressed the problem of locating active sensors (path-ID sensors) on the arcs of a network to determine route flow volumes. The same problem has been addressed in Gentili and Mirchandani (2006) when *video image* sensors are located on the nodes of a network.

In this paper, we address the specific problem of locating the minimum number of counting sensors on the nodes of a network in order to determine the arc flow volume on all the network. This problem has been addressed in Bianco, Confessore, and Reverberi (2001). A combinatorial optimization problem (the Sensor Location Problem) was defined and two heuristics giving lower and upper bounds to the solution value were presented. Finally, Bianco, Confessore, and Reverberi (2001) defined a combinatorial optimization problem (the Sensor Location Problem (*SLP*)) to locate counting sensors on nodes and get information about the arc flow volumes on the non-monitored portion of the network. Despite the strict dependence of the solution to the underlying graph structure, the complexity of the problem was not assessed in the general case and no investigation on specific graph classes was developed. In this paper we (i) formally assess the complexity of the problem in different cases, and (ii) find graph classes for which is polynomially solvable. Therefore, we define a new optimization problem on graphs, the *Dominating Paths Problem (DPP)*, that is proved to be equivalent to *SLP* and that better captures the graph theoretic aspect of the problem. We formally prove *DPP* is NP-complete on general graph and we analyze some subclasses of triangle free graphs. We defined linear time exact algorithms for paths, cycles and combs.

The paper is organized as follows. Section 2, introduces the problem and recalls the existing results. Section 3 defines the *Dominating Paths Problem* and proves the equivalence with *SLP*. NP-completeness results are presented in Section 4. The analysis of *DPP* for the class of paths and combs is developed in Sections 5 and 6. Conclusions and further research are the object of Section 7.

2. Location of counting sensors on nodes

Conventional inductive loop sensors installed on a lane(s) of a road count vehicles, thereby measuring flow volume (vehicle per unit time) on that lane(s). When flow on all the outgoing lanes from a node is monitored then, through the knowledge of *split ratios* at the node, we are able to calculate the flows on incoming arcs to the node. The split ratios (Nobe 2002), which specify the fraction $0 < p_{v,w} \leq 1$ of the incoming flow $F(v)$ that leaves the node v on each of the outgoing arcs (v, w), are assumed known, either from historical data or from a calibrated network loading model that has assigned traffic demand from origins to destinations (see for example Berman, Krass, and Xu (1995)). Therefore, in general, when we state that a *counting sensor is located on a node* it will be assumed that a configuration of loop detectors (or any detectors that count vehicles, based on technologies such as video, sonar, microwave, etc.) are located at the node which give flow volumes on all arcs incident to the node.

If counting sensors are located on all the nodes of the network, then all the arc flow volumes of the network can be directly known. Obviously, it is unrealistic to locate sensors on all the nodes of a traffic network. In this problem, we seek the smallest subset of nodes that, if monitored, would allow us to obtain the flow on all the arcs of the network.

2.1. Split ratios and counting sensors on nodes

Given a network $\Gamma = (N, A)$, flow on arcs contains subflows that are generated and/or absorbed from different origin/destination pairs. In the sequel we say a node to be a *transfer node*, if no flow is generated or absorbed by it, otherwise it is a *centroid*. Let $T \subseteq N$ be the set of transfer nodes of the network and $B \subseteq N$ the set of centroids of the network, obviously, we have $T \cup B = N$ and $T \cap B = \emptyset$.

For each $v \in T$, the flow conservation constraints hold:

$$\sum_{(v,w)\in A_v^-} f_{v,w} - \sum_{(w,v)\in A_v^+} f_{w,v} = 0 \tag{1}$$

where A_v^- and A_v^+ are the outgoing and incoming arcs of node v, respectively.

For each centroid $v \in B$, we have the following flow conservation constraint:

$$\sum_{(v,w)\in A_v^-} f_{v,w} - \sum_{(w,v)\in A_v^+} f_{w,v} = S_v \tag{2}$$

where S_v is the *balancing flow* at v, that is, a source or a sink flow so that (2) holds. For example, see the network in Fig. 1, where the label on each arc represents the unit of flow in

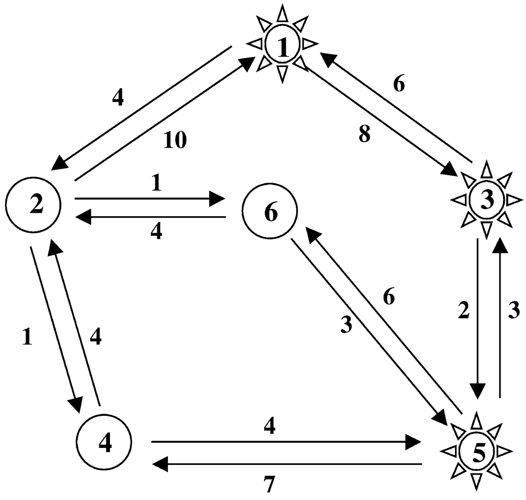


Fig. 1 A network with three centroids 1, 3 and 5. The balancing flows are $S_1 = -4$, $S_3 = -3$ and $S_5 = 7$

the time interval unit. Nodes 1, 3, 5 are centroids with balancing flows $S_1 = -4$, $S_3 = -3$ and $S_5 = 7$.

Let the split ratios associated with node 2 be: $p_{2,1} = \frac{10}{12}$, $p_{2,4} = \frac{1}{12}$, $p_{2,6} = \frac{1}{12}$. Formally, for each $v \in N$ and each outgoing arc (v, w), by using these split ratios, we have:

$$f_{v,w} = F(v) \cdot p_{v,w} \tag{3}$$

From (3) and considering any other outgoing arc of v, say (v, z), we obtain:

$$f_{v,z} = \frac{f_{v,w}}{p_{v,w}} \cdot p_{v,z} \tag{4}$$

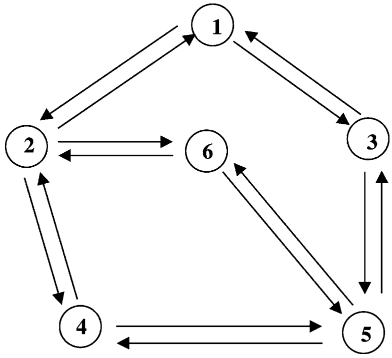
Indeed, by using split ratios, we can express the total incoming flow $F(v)$ as a function of the flow volume of any outgoing arc.

Therefore, assuming known split ratios at nodes, if at least one outgoing flow of node v is known, it is possible to determine the total incoming flow $F(v)$ (by Eq. (3)), and, consequently, flow volumes on all the outgoing arcs of v (by Eq. (4)). Conversely, knowing the flow volume of an incoming arc of v does not imply we can determine the other incoming flows, because split ratios relate to the relationship among the outgoing arcs. However, if we know, somehow, the total ingoing flow $F(v)$, we can determine all the outgoing flows of v. Note that, if $v \in T$, then knowing $F(v)$ means we know all the incoming flows, while if $v \in B$, it means we know also the balancing flow S_v.

Consider again the network in Fig. 1 and we locate a counting sensor on node 1. We obtain the following flow volumes directly monitored:

$$\begin{aligned} f_{1,2} &= 4 \\ f_{2,1} &= 10 \\ f_{1,3} &= 8 \\ f_{3,1} &= 6 \end{aligned}$$

Fig. 2 A transportation network



and, also the balancing flow $S_1 = -4$. Moreover, when we know split ratios, we can determine the following non-monitored flow volumes:

$$\begin{aligned} f_{2,4} &= \frac{f_{2,1}}{p_{2,1}} \cdot p_{2,4} \\ f_{2,6} &= \frac{f_{2,1}}{p_{2,1}} \cdot p_{2,6} \\ f_{3,5} &= \frac{f_{3,1}}{p_{3,1}} \cdot p_{3,5} \end{aligned}$$

Summarizing, assuming split ratios to be known, by locating a counting sensor on a node v we determine (i) the flow volume of all the arcs incident to v and (ii) the flow volumes of all the outgoing arcs of the nodes adjacent to v; and, if node $v \in B$, we know also the value S_v of the balancing flow.

Considering also flow conservation constraints at nodes, we can derive additional arc flow volumes, as explained next. For the network of Fig. 2 suppose that nodes 4 and 5 are centroids and that a counting sensor is located on each of them. We directly monitor the following flows: $f_{4,2}$, $f_{2,4}$, $f_{4,5}$, $f_{5,4}$, $f_{5,6}$, $f_{6,5}$, $f_{5,3}$ and $f_{3,5}$. Moreover, by using the split ratios we derive flows: $f_{2,6}$, $f_{2,1}$, $f_{6,2}$ and $f_{3,1}$. Suppose, nodes 1, 2, 3 are transfer nodes. Then, since flow conservation constraints (1) hold for each of them and, in particular for node 1, we have $F(1) = f_{2,1} + f_{3,1}$ and we can derive also flows $f_{1,2}$ and $f_{1,3}$ by using (3).

On the other hand, if node 1 is a centroid and node 2 and 3 are transfer nodes, by using flow conservation constraints, we can obtain $f_{1,2}$, $f_{1,3}$ and S_1 as follows:

$$\begin{aligned} f_{1,2} &= f_{2,1} + f_{2,6} + f_{2,4} - f_{4,2} - f_{6,2} \\ f_{1,3} &= f_{3,1} + f_{3,5} - f_{5,3} \\ S_1 &= f_{1,2} + f_{1,3} - f_{2,1} - f_{3,1} \end{aligned}$$

In general, with a set of measured nodes we can associate a system of linear equations (in the sequel referred to as *Flow Conservation System*) where the unknown variables are arc flow volumes and centroid balancing flows. As an example, suppose on the network of Fig. 2, nodes 4 and 5 are centroids and we locate a sensor on node 1. It is possible to define a linear system where (i) a single unknown variable, representing an outgoing arc flow, is associated with every node, except the monitored node and its adjacent nodes, and, (ii) equations correspond to flow conservation constraints relative to every node of the network. Additional variables S_v, denoting the centroid balancing flows, are also included in the linear system. Note that, the flow conservation equation corresponding to the measured node 1, does not contain any unknown flow, and thus can be omitted from the system. We obtain the following system corresponding to the flow conservation constraints of the network when node 1 is monitored:

$$
\begin{array}{llcll}
\textit{node } 2 & (f_{2,1} + f_{2,6} + f_{2,4}) & - & \left(f_{1,2} + x_{6,5} \cdot \dfrac{p_{6,2}}{p_{6,5}} + x_{4,2}\right) & = 0 \\
\textit{node } 3 & (f_{3,1} + f_{3,5}) & - & \left(f_{1,3} + x_{5,4} \cdot \dfrac{p_{5,3}}{p_{5,4}}\right) & = 0 \\
\textit{node } 4 & \left(x_{4,2} + x_{4,2} \cdot \dfrac{p_{4,5}}{p_{4,2}}\right) & - & (x_{5,4} + f_{2,4}) & = S_4 \\
\textit{node } 5 & \left(x_{5,4} + x_{5,4} \cdot \dfrac{p_{5,3}}{p_{5,4}} + x_{5,4} \cdot \dfrac{p_{5,6}}{p_{5,4}}\right) & - & \left(f_{3,5} + x_{4,2} \cdot \dfrac{p_{4,5}}{p_{4,2}} + x_{6,5}\right) & = S_5 \\
\textit{node } 6 & \left(x_{6,5} \cdot \dfrac{p_{6,2}}{p_{6,5}} + x_{6,5}\right) & - & \left(f_{2,6} + x_{5,4} \cdot \dfrac{p_{5,6}}{p_{5,4}}\right) & = 0
\end{array}
\tag{5}
$$

where: the unknown variables are $x_{4,2}$, $x_{5,4}$, $x_{6,5}$, S_4 and S_5, flows $f_{1,2}$, $f_{2,1}$, $f_{1,3}$, $f_{3,1}$ are directly monitored, and flows $f_{2,4}$, $f_{2,6}$ and $f_{3,5}$ are obtained using split ratios. The remaining unknown flows are obtained using equation (4). Note that, in this system, the number of variables is equal to the number of equations and thus, we obtain a unique solution.

Let us assume now, node 2 and 6 are centroid too. Because the number of variables is increased, the system does not have a unique solution anymore. Hence, additional monitored nodes are needed. Note that, a trivial solution can be obtained by locating a counting sensor on all the centroids. However, it is enough to locate a single counting sensor on node 5 to obtain all flow volumes. Indeed, the corresponding system would be:

$$
\begin{array}{llcll}
\textit{node } 1 & \left(x_{1,3} \cdot \dfrac{p_{1,2}}{p_{1,3}} + x_{1,3}\right) & - & (x_{2,1} + f_{3,1}) & = 0 \\
\textit{node } 2 & \left(x_{2,1} + x_{2,1} \cdot \dfrac{p_{2,4}}{p_{2,1}} + x_{2,1} \cdot \dfrac{p_{2,6}}{p_{2,1}}\right) & - & \left(x_{1,3} \cdot \dfrac{p_{1,2}}{p_{1,3}} + f_{6,2} + f_{4,2}\right) & = S_2 \\
\textit{node } 3 & (f_{3,1} + f_{3,5}) & - & (x_{1,3} + f_{5,3}) & = 0 \\
\textit{node } 4 & (f_{4,2} + f_{4,5}) & - & \left(x_{2,1} \cdot \dfrac{p_{2,4}}{p_{2,1}} + f_{5,4}\right) & = S_4 \\
\textit{node } 6 & (f_{6,2} + f_{6,5}) & - & \left(x_{2,1} \cdot \dfrac{p_{2,6}}{p_{2,1}} + f_{5,6}\right) & = S_6
\end{array}
\tag{6}
$$

with variables $x_{1,3}$, $x_{2,1}$, S_2, S_4 and S_6.
It should be clear now, the question we want to answer is:

Question 1 What is the minimum number of counting sensors to locate on the nodes and where to place them such that the corresponding Flow Conservation System has a unique solution?

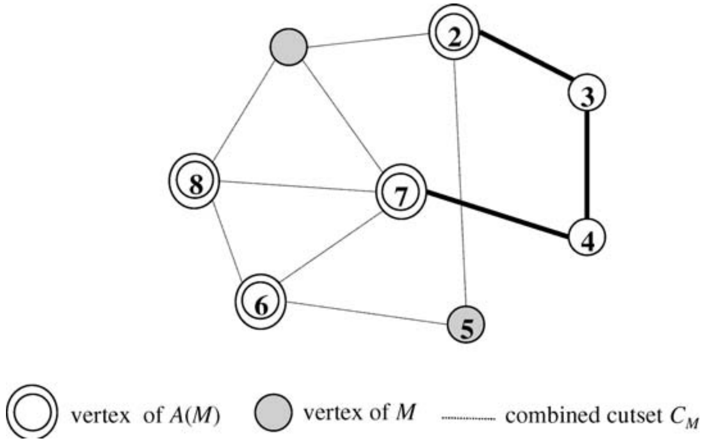


Fig. 3 The combined cut C_M associated with the subset of vertices $M = \{1, 5\}$

This problem has been addressed in Bianco, Confessore, and Reverberi (2001) on symmetric transportation networks. The main result is recalled in the next section.

2.2. The sensor location problem

To answer Question 1, we need to explore the conditions under which the Flow Conservation System, associated with a set M of measured nodes, admits a unique solution. In order to do that, let us introduce the concept of *combined cut* associated with a subset M. For clarity of presentation, the traffic network will be referred in terms of nodes and arcs, whereas the graph representing the network in terms of vertices and edges. We recall that a cut in a graph G is a set of edges joining vertices in the subset S and in the complementary set $V \backslash S$. A *combined cut* in G is a cut with some additional edges among vertices in the same subset. More formally, given an undirected graph $G = (V, E)$ and $M \subseteq V$, the *combined cut* C_M is the set of edges of the subgraph $G_{M \cup A(M)}$, induced by both M and its adjacent vertices $A(M) = \{v \in V \backslash M: \exists (v, w) \in E \text{ with } w \in M\}$, that is the set of vertices that are adjacent to at least one element of M. Let q_M be the number of connected components of $G^M = (V \backslash M, E \backslash C_M)$.

In Fig. 3, the set $M = \{1, 5\}$ and the set $A(M) = \{2, 6, 7, 8\}$. Dotted lines are the combined cut C_M. Let us denote by $G_i = (V_i, E_i), i = 1, 2, \ldots q_M$ the connected components induced by C_M. There are $q_M = 3$ connected components induced by the combined cut C_M: $G_1 = (\{6\}; \cdot)$, $G_2 = (\{8\}, \cdot)$, $G_3 = (\{2, 3, 4, 7\}; (2, 3), (3, 4), (4, 7))$. The following theorem holds.

Theorem 1. *(Bianco, Confessore, and Reverberi (2001)) The Flow Conservation System associated with M admits a unique solution if and only if for each connected component G_i the following relation is satisfied:*

$$|A(M) \cap V_i| \geq |(B \backslash M) \cap V_i|$$

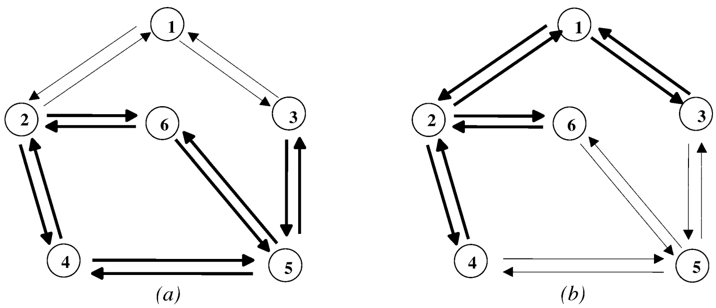


Fig. 4 The connected components associated with (a) $M = \{1\}$ and (b) with $M = \{5\}$ for the graph of Fig. 2

That is, locating sensors on M, allows us to determine all the flow volumes if and only if the number of centroids, in each connected component induced by C_M, in not greater that the number of nodes, adjacent to M, and contained in the component.

Figure 4(a) shows the connected components induced by $M = \{1\}$ and Fig. 4(b) the components induced by $M = \{5\}$. If nodes 4 and 5 are centroids then all the flows are determined by locating a sensor on node 1 (this corresponds to system (5)). Indeed, the single connected component $G_1 = \{2, 3, 4, 5, 6\}$ contains the two adjacent nodes 2 and 3, and the two centroids 4 and 5. Observe that, in this case, by locating a sensor on node 5 we can also determine all the flow volumes: the connected component $G_1 = \{1, 2, 3, 4, 6\}$ contains the three adjacent nodes 3, 4 and 6 and the single centroid 4.

If we assume nodes 2, 4, 5, 6 are centroids, then, locating a sensor on node 1 is not enough to determine all the flow volumes, while by locating a sensor on node 5 the condition of Theorem 1 is satisfied and we determine all arc flow volumes (see the corresponding flow system (6)).

Therefore, by Theorem 1, the answer to Question 1 is obtained by solving the following optimization problem stated in its decision version:

Sensor Location Problem (*SLP*)

INSTANCE: An undirected graph $G = (V, E)$, a subset $B \subseteq V$, a positive integer $K \leq |B|$.
QUESTION: Is there a sensor set $M \subseteq V$ with $|M| \leq K$, such that each of the q_M connected components $G_i = (V_i, E_i), i = 1, 2, \ldots, q_M$ induced by C_M has a number of vertices belonging to $B \backslash M$ not greater than the number of vertices belonging to $A(M)$, i.e. $|A(M) \cap V_i| \geq |(B \backslash M) \cap V_i|, i = 1, 2, \ldots, q_M$?.

In the rest of the paper, we refer to such a set M as a *sensor set* for *SLP*. Moreover, we refer to the set B as the set of *bound vertices*, (that is, the set of vertices corresponding to the centroids of the network).

Before concluding this section, we recall a simple lower bound to the solution value of *SLP*, that will be useful in the next sections. Let $\delta(v)$ be the number of edges incident to v (the degree of v), and, $\Delta(G)$ be the maximum degree among vertices of G (the degree of G).

Property 1. *(Bianco, Confessore, and Reverberi (2001)) Let m be the optimal solution value of the SLP on G and b = |B|, we have that:* $\lceil \frac{b}{\Delta(G)+1} \rceil \leq m$.

From the definition of the problem, the strict dependence of the feasible solutions on the structure of the graph can be noticed. Indeed, the combined cut C_M associated with a given set M, for particular graph classes may define a set of connected components having simple structures that may be known in advance. In Bianco, Confessore, and Reverberi (2001), the problem has been studied from an algorithmic point of view on general graphs and two heuristics giving lower and upper bounds to the solution value are given. The authors do not take into account the structure of the graph.

However, from an application point of view, the network that needs to be monitored may range from a single intersection, to a major arterial road, to a major arterial road with its main entrances and exits, and, eventually, to the whole city network. Practically, the deep knowledge of subnetworks of the entire traffic network is of great interest for traffic planners and managers. For example, for the city of Rome's network, actually, most of the city managers' interest is mainly concentrated on one of the main arterial (i.e. Muro Torto road) and on the Grande Raccordo Anulare that is a road that makes a ring all around the city. These portions of the network can be modelled using graph structures as paths, cycles or combs.

In order to analyze the *SLP* on particular graph structures, we introduce, in the next section, an equivalent problem allowing us to derive some properties which will be useful in designing solution algorithms.

3. Dominating paths problem

Before introducing the new problem, we resume the notation that will be used in the sequel and introduce some definitions.

Let $G = (V, E)$ be an undirected graph (representing a symmetric transportation network), where V is the set of vertices and E is the set of edges. Given $M \subseteq V$, we denote by $A(M)$ the set of all vertices of G not in M that are adjacent to at least one vertex of M, i.e. $A(M) = \{v \in V \backslash M : \exists (v, w) \in E \text{ with } w \in M\}$. The degree $\delta(v)$ of a vertex v is the number of edges incident to it, and the degree $\Delta(G)$ of the graph G is the maximum degree of its vertices. The subgraph $G_U = (U, E_U)$ is the graph induced by a set $U \subseteq V$ where $E_U = \{(v, w) \in E\colon v, w \in U\}$. We denote $P_{v_1,v_h} = (v_1, v_2, \ldots, v_h)$ as a path from vertex v_1 to vertex v_h of length h where all the vertices are distinct, and we call it a Ph path. We define a $P1$ path as a path $P_{v,v}$ consisting of the single vertex v. Given a path P_{v_1,v_h}, the distance between two vertices v_i and v_j, $h \geq j \geq i \geq 1$ of the path is the length of the subpath P_{v_i,v_j}.

The *Dominating Paths Problem* is based on the notion of MB-feasible paths.

Definition 1. Let $M \subseteq V$, and $B \subseteq V$, be two subsets of vertices of G not necessarily disjoint. A path $P_{v_1,v_h} = \{v_1, v_2, \ldots, v_h\}$ is **MB-feasible** for M and B if it satisfies the following conditions:

Condition 1 $v_1 \in B \backslash M$ and $v_h \in A(M)$;
Condition 2 $P_{v_1,v_h} \cap M = \emptyset$;
Condition 3 If $h \geq 2$, $\forall \{v_i, v_{i+1}\} \in P_{v_1,v_h}$, $i = 1, 2, \ldots, h-1$, $|\{v_i, v_{i+1}\} \cap A(M)| \leq 1$.

These conditions define paths that avoid vertices of the set M and that assure a connection between a vertex of B and a vertex of $A(M)$. Indeed, Condition 1 states that an MB-feasible path has to begin from a vertex of B that is not in M and to end "near" a vertex of M. Condition 2 assures that the path does not contain vertices of M. The last condition is to

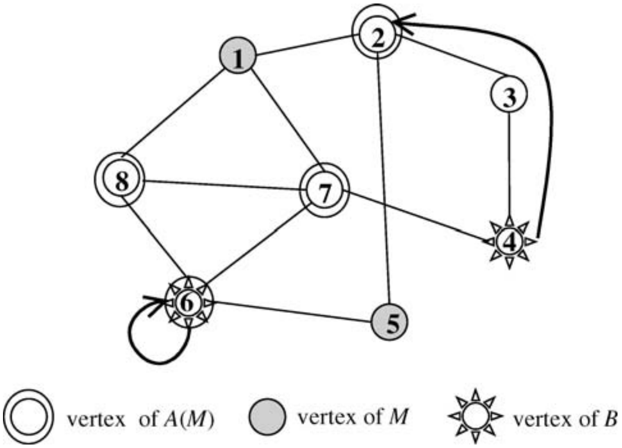


Fig. 5 The MB-feasible paths (shown as arrows) associated with the subsets $M = \{1, 5\}$ and $B = \{4, 6\}$

avoid two consecutive vertices of $A(M)$ in a path. Given the MB-feasible path P_{v_i,v_h} we say the adjacent vertex v_h covers (*dominates*) the bound vertex v_i.

Observation 1. Observe that if $h = 1$ then $P_{v,v}$ is MB-feasible if $v \in B \cap A(M)$, that is when v is a bound vertex and is adjacent to an element of M.

Definition 2. Let $B \subseteq V$ be a subset of bound vertices of G. A set $M \subseteq V$ **dominates by paths** the bound vertices in B if there exists a set $\mathcal{P}_M$ of $|B\backslash M|$ MB-feasible paths that differ in the origin and the destination vertex, i.e., $\{P_{u_1,v_1}, \ldots, P_{u_{|B\backslash M|},v_{|B\backslash M|}}\}$ with $u_i \neq u_j$, $v_i \neq v_j$ for each $i, j = 1, 2, \ldots, |B\backslash M|$, $i \neq j$.

In the sequel, we will refer to the subset M satisfying Definition 2 as an *MB-set* for (G, B) and the pair $(M, \mathcal{P}_M)$ as an *MB-couple* for (G, B).

We can see an example of an MB-couple in Fig. 5. The bound vertices are $B = \{4, 6\}$. The set $M = \{1, 5\}$ dominates by paths the set B. Indeed, we may consider the two MB-feasible paths $P_{4,2} = \{4, 3, 2\}$ and $P_{6,6} = \{6\}$ (in bold line in the figure). Note that there exists for vertex 4 also the MB-feasible path $P_{4,7} = \{4, 7\}$, but for vertex 6 the paths $P_{6,8} = \{6, 8\}$ and $P_{6,7} = \{6, 7\}$ are not MB-feasible because they do not satisfy Condition 3 in Definition 1.

Now we are ready to define the decision version of *DPP* and subsequently prove the equivalence between it and the *SLP*.

Dominating Paths Problem (*DPP*)

INSTANCE: An undirected graph $G = (V, E)$, a subset $B \subseteq V$, a positive integer $K \leq |B|$.
QUESTION: Is there an MB-couple $(M, \mathcal{P}_M)$, $M \subseteq V$ with $|M| \leq K$ for (G, B), i.e. a subset $|M| \leq K$ such that there exist $|B\backslash M|$ MB-feasible paths that are distinct for the origin vertex and the end vertex?

An MB-couple is said to be an optimal solution for *DPP* on (G, B) if the MB-set M has minimum size. We note, as for *SLP*, that a feasible solution to the *DPP* on (G, B) is given by $M = B$ where $|B \backslash M| = 0$ and, thus, there is no MB-feasible path to find, i.e. $\mathcal{P}_M = \emptyset$. Therefore, an implicit upper bound to the optimal solution value is $|B|$.

Let us prove the equivalence between the *SLP* and *DPP* by the following theorem.

Theorem 2. *The Sensor Location Problem and the Dominating Paths Problem are polynomially equivalent.*

Proof: Given a graph $G = (V, E)$, a subset $B \subseteq V$, and an integer $K \leq |B|$ we have to show that G has a sensor set if and only if G has an MB-couple.

First suppose that M, with $|M| \leq K$ is a sensor set for G, now we prove that we can define a set of MB-feasible paths $\mathcal{P}_M$ such that $(M, \mathcal{P}_M)$ is an MB-couple for (G, B). Let consider the set of connected components of G associated with M: $\{G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_q = (V_q, E_q)\}$. Denote by $b_1, b_2, \ldots b_q$, with $\sum_{i=1}^{q} b_i = |B \backslash M|$, the number of bound vertices that do not belong to M contained in each connected component, and by $a_1, a_2, \ldots a_q$, with $\sum_{i=1}^{q} a_i = |A(M)|$, the number of adjacent vertices to M contained in each connected component. Since the sensor set M is feasible, we have $b_i \leq a_i$, $\forall i = 1, 2, \ldots q$. Without loss of generality, we can analyze the existence of the MB-feasible paths within a single component, say $G_1 = (V_1, E_1)$. For the connectivity, there exists in G_1 a path connecting any couple of vertices, in particular there exists a path for each couple (v, w) with $v \in B \cap V_1$ and $w \in A(M) \cap V_1$. Consider all the b_1 vertices $v \in B \cap V_1$ and an equal number of vertices $w \in A(M) \cap V_1$. Let $\mathcal{P}_1$ be the set of the b_1 paths connecting each distinct couple (v, w). These paths are MB-feasible. Indeed, the connected component, by the definition of combined cut, does not contain vertices of M and does not contain edges connecting two vertices of $A(M)$. Denoting $\mathcal{P}_M = \bigcup_{i=1}^{q} \mathcal{P}_i$, we have that the couple $(M, \mathcal{P}_M)$ is a feasible solution for *DPP* on (G, B).

Conversely, suppose that on G there is an MB-couple $(M, \mathcal{P}_M)$, with $|M| \leq K$. We have to show that M is a sensor set for G. Let us consider the combined cut associated with M. This defines a set of q connected components. Each path $P \in \mathcal{P}_M$, by definition of MB-feasible path, belongs entirely to exactly one connected component. Thus, in each component, the number of vertices belonging to the set B is not greater than the number of vertices belonging to the set $A(M)$. □

4. Complexity

The aim of this section is to analyze the complexity of *DPP*. For the *SLP* (and thus *DPP*) no proof of NP-hardness exists in the literature.

Clearly, *DPP* is in NP, a non-deterministic algorithm needs to guess an MB-couple $(M, \mathcal{P}_M)$ and verifies whether (i) $|\mathcal{P}_M| = |B \backslash M|$, (ii) all the paths in $\mathcal{P}_M$ are MB-feasible and (iii) have different origin and destination vertices. We prove that the decision version of the problem is NP-complete, when we assume $B = V$, on general graphs by reduction from the *Dominating Set Problem* (Theorem 3).

Even though this proof could suffice from a theoretical point of view to assess the complexity of the problem, the instances that we can have in practice are much different from this particular case. Indeed, very often the bound vertices are not adjacent, and, a more interesting case is when they form a set of mutually non adjacent nodes (stable set).

We give a second proof of the NP-completeness of the problem when the set of bound vertices B is a stable set of the graph (Theorem 4).

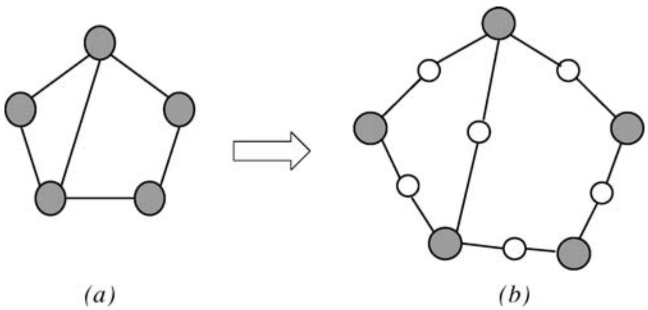


Fig. 6 The reduction from *DSP*

For the sake of completeness, we recall the decision version of the *Dominating Set Problem*.

Dominating Set problem (*DSP*)

INSTANCE: An undirected graph $G = (V, E)$, a positive integer $K \leq |V|$.
QUESTION: Is there a Dominating set D of size K or less for G, i.e. a subset $D \subseteq V$ with $|D| \leq K$ such that for all $u \in V \backslash D$ there is a $v \in D$ for which $\{u, v\} \in E$?
Now, we prove *DPP* is NP-complete.

Theorem 3. *DPP is polynomially equivalent to DSP when $B = V$.*

Proof: First, suppose M, with $|M| \leq K$, is a dominating set for G; we prove there exists a set $\mathcal{P}_M$ of size $|B\backslash M|$ of MB-feasible paths that have all the origin and the end vertices distinct. By definition of a dominating set, each vertex $v \in B\backslash M$ is in $A(M)$, thus we can consider the set of MB-feasible paths of type $P1$ consisting of such vertices.

Conversely, suppose there is in G an MB-couple $(M, \mathcal{P}_M)$ for (G, B) with $|M| \leq K$. We have to show that M is a dominating set for G. The set $\mathcal{P}_M$ has size $|B\backslash M|$. Since $B = V$, then there are $|V\backslash M|$ distinct MB-feasible paths which differ for the initial and the end vertices. Each path contains a bound vertex and a vertex adjacent to an element of M, thus, there are $|V\backslash M|$ bound vertices and at least $|V\backslash M|$ vertices adjacent to element of M. Therefore, all the paths are of type $P1$, and M is a dominating set of G. □

Theorem 4. *The DPP is NP-complete also when the bound vertices form a stable set.*

Proof: Let $G = (V, E)$ and $K \leq |V|$ be any instance of the DSP. We must construct a graph $G' = (V', E')$ associated with G with a stable set $B \subseteq V'$ and a positive integer $K' \leq |B|$ such that G has a dominating set D with $|D| \leq K$ if and only if G' has an MB-couple $(M, \mathcal{P}_M)$ with $|M| \leq K'$.

For each edge $e = (v_i, v_j)$ in G, add in G' a vertex a_e and the two edges (v_i, a_e) and (a_e, v_j). The set $B = \{v_1, v_2, \ldots, v_n\} = V$ is a stable set and each a-vertex has exactly two adjacent vertices in B (Fig. 6). This construction is made in polynomial time. We have to show that G has a dominating set D of size K or less if and only if G' has an MB-couple $(M, \mathcal{P}_M)$ with $|M| \leq K'$. First suppose that $D \subseteq V$ is a dominating set for G with $|D| \leq K$. We set in G': $M = D$ and $K = K'$. In this way, the set $A(M)$ is entirely composed of a-vertices. Keep in mind that $B = V$. By construction, we know that a vertex v of G' may be such that: either $v \in M \cap B$ or $v \in B$. Each path from a vertex $v \in B \backslash M$ to a vertex $w \in A(M)$ satisfies conditions 1 and 3 of MB-feasibility by construction of G'. Moreover, for each vertex $v \in B$ there exists at least one vertex $a_e \in A(M)$, otherwise v would not be dominated by D. Thus, we may consider the MB-feasible path $P_{v,a_e} = \{v, a_e\}$. It remains to be shown that each $a_e \in A(M)$ belongs to exactly one MB-feasible path. Indeed, by construction, for each a_e there exist just the two edges (v_i, a_e) and (a_e, v_j) in G'. If $a_e \in A(M)$ then at most one vertex among v_i and v_j does not belong to M, and so at most one between the two edges incident to a_e may be chosen as an MB-feasible path. Hence, defining $\mathcal{P}_M = \bigcup P_{v,a_e}$, the couple $(M, \mathcal{P}_M)$ is a feasible solution for *DPP* on (G', B).

Conversely, let $M \subseteq V'$ be an MB-set for G' with $|M| \leq K'$. We obtain a dominating set D for G by the following procedure:

FOR EACH $v \in M$ DO:

Step 1. IF $v \in B$ THEN insert v in D;
Step 2. IF $v \notin B$, let w and z be its adjacent bound vertices, THEN

IF $w \notin M$ THEN insert w in D;
ELSE IF $z \notin M$ THEN insert z in D.

Setting $K = K'$, we have to show that all the vertices in $V \backslash D$ are covered by vertices in D and that $|D| \leq K$. Indeed, the size of the so-obtained set D is at most K'. The procedure adds vertices to the set $D = M \cap V$. Let us consider this set. It does not cover all the vertices of $V \backslash D$, then, Step 2 adds to D for each vertex a-vertex $a_e \in M$ one of its adjacent vertex when necessary. Let a_e be such a vertex and let w and z be its adjacent vertices, Step 2, if both w and z do not belong to M (and thus are not inserted in D by Step 1), adds one of them into D. In this way, since w and z are adjacent in G, the so-obtained set D covers all the vertices in $V \backslash D$. □

In the next section we explore the *DPP* on particular graph classes.

First, we notice that the *DPP* remains NP-complete for the graph classes where the *DSP* also is. For an extensive survey of domination problems and a comprehensive bibliography, see for example Hedetniemi and Laskar (1991). We would point out that in Cockaine, Goodman, and Hedetniemi (1975) a polynomial algorithm solving the DSP on trees is presented. Actually, that algorithm solves a slightly more general problem (the *Mixed Dominating Set*) that is more similar to *DPP*. Let the vertices V of a graph G be partitioned into three subsets V_1, V_2, V_3, where V_1 consists of *free* vertices, V_2 consists of *bound* vertices and V_3 consists of *required* vertices. A *mixed dominating set* in G is a set M of vertices which contains all the required vertices, i.e. $V_3 \subseteq M$, and which dominates (by adjacency) all the bound vertices, i.e. every vertex $v \in V_2$ is either in M or is adjacent to at least one vertex in M. Free vertices need not be dominated by M, but may be included in M in order to dominate bound vertices. A mixed dominating set of minimum size solves the *Mixed Dominating Set (MDS)*. *MDS* was defined in Cockaine, Goodman, and Hedetniemi (1975) as a generalization of the *DSP*. Indeed, when $V_2 = V$ and $V_1 = V_3 = \emptyset$ *MDS* is the Dominating Set Problem. The similarity between the *MDS* and *DPP* is evident. However, the dominance relation for the *DPP* appears more complex. Indeed, while for the *MDS* each vertex $v \in V$, if selected, dominates only its adjacent bound vertices, for the *DPP* it can dominate bound vertices connected to v by a path. This makes more difficult to capture the combined dominance of selecting two vertices v, w simultaneously (or a set of vertices): for the *MDS* we have to look just at the adjacent bound vertices of both v and w, for the *DPP* we have to look at all the possible MB-feasible paths. Hence, despite the *MDS* is polynomially solvable on trees, the *DPP* seems to be more difficult on these structures. Therefore, we analyzed the *DPP* on simpler structures first. More in particular, we show in this section two polynomial algorithms solving *DPP* on paths and combs where the MB-feasible paths can be handled, as can be seen in the following paragraphs.

5. A linear algorithm on paths and cycles

Given a path $P = (V, E) = P_{v_1,v_n} = \{v_1, v_2, \ldots, v_n\}$ and a subset $B \subseteq \{v_1, v_2, \ldots, v_n\}$ of bound vertices with $|B| = b$, let $(M, \mathcal{P}_M)$ be an optimal solution for *DPP* on (P_{v_1,v_n}, B) with $|M| = m$.

The first property we prove is that on paths there exists an optimal solution of *DPP* entirely composed of bound vertices.

Theorem 5. *There exists an optimal solution* $(M, \mathcal{P}_M)$ *of DPP such that* $M \subseteq B$.

Proof: Let M contain, by contradiction, a vertex $v_i \notin B$. We want to show that there exists another MB-set M' such that $|M'| = |M|$ and $M' \subseteq B$ with which we can associate a set $\mathcal{P}_{M'}$ of MB-feasible paths.

Since M is optimal, there exist at most two MB-feasible paths $P_{v_h,v_{i-1}}$, $P_{v_k,v_{i+1}} \in \mathcal{P}_M$, $h \leq i-1$, $i+1 \leq k$, connecting the bound vertices v_h and v_k respectively to the adjacent vertices v_{i-1} and v_{i+1}. We can assume v_h (v_k) is the nearest bound vertex to v_{i-1} (v_{i+1}) among $\{v_1, \ldots v_{i-1}\}$ ($\{v_{i-1}, \ldots v_n\}$). Otherwise, let v_s be the bound vertex in the analyzed MB-feasible path and let $P_{v_s,v_{i-1}} \in \mathcal{P}_M$ and $P_{v_h,v_l} \in \mathcal{P}_M$ be two MB-feasible paths with $l \leq s < h \leq i-1$, such that v_h is the nearest bound vertex to v_{i-1} among $\{v_1, \ldots v_{i-1}\}$. Instead of $\mathcal{P}_M$ we could associate with M the set $\mathcal{P}'_M = \mathcal{P}_M \backslash \{P_{v_s,v_{i-1}}, P_{v_h,v_l}\}$ $\cup \{P_{v_h,v_{i-1}}, P_{v_s,v_l}\}$. The same reasoning holds for $P_{v_k,v_{i+1}}$. Now, let us consider the subpath $P_{v_h,v_k} = \{v_h, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k\}$, such that $P_{v_h,v_k} \cap B = \{v_h, v_k\}$ and $P_{v_h,v_k} \cap M = \{v_i\}$. Finally, examine the set $M' = M \backslash \{v_i\} \cup \{v_{i+k}\}$, that has the same size of M and is such that $M' \subseteq B$. M' is also optimum because we can associate with it the set $\mathcal{P}_{M'} = \mathcal{P}_M \backslash \{P_{v_h,v_{i-1}}, P_{v_k,v_{i+1}}\} \cup \{P_{v_h,v_{k-1}}\}$ of MB-feasible paths for M' and B. □

We show next how we can find such a solution.

By the proof of Theorem 5 we can observe that if $v_i \in M$ we may simply define two MB-feasible paths connecting v_{i-1} with the nearest bound vertex among $\{v_1, \ldots v_{i-1}\}$ and v_{i+1} with the nearest bound vertex among $\{v_{i+1}, \ldots v_n\}$.

Moreover, since we want to construct $|B \backslash M|$ MB-feasible paths distinct in the origin and destination, it is also more desirable, when possible, to select vertices at a distance of at least 4.

The above mentioned observation may be stated in the following two rules:

1. Select, if possible, vertices $v \in B$ which are between two not selected bound vertices;
2. Select, if possible, vertices at a distance on the path of at least 4.

As we show next, the first rule assures that a subpath between an adjacent vertex and the nearest bound vertex is MB-feasible; the second rule assures that the total number of these paths is exactly $|B \backslash M|$.

We present now the *Path-algorithm* that finds the optimal solution of *DPP* on paths, then we prove it is exact.

Path-Algorithm
Input: a path P_{v_1,v_n} of n vertices; a set $B \subseteq \{v_1, v_2, \ldots, v_n\}$.
Output: an optimum MB-set M for *DPP*.

Step 1. (INITIALIZATION)
    $M = \emptyset$; count:=0; position:=0;
Step 2. FOR EACH vertex v_i, $i = 1, 2, \ldots, n$ DO
    IF v_i is a bound vertex THEN
        begin
        count:= count +1;
        IF count = 1 THEN
            position := i;
        ELSE IF count = 2 THEN
            $M = M \cup \{v_i\}$;
        ELSE IF count = 3 THEN
            count:= 0;
        end;
Step 3. IF count = 1 THEN
    $M = M \cup \{v_{\text{position}}\}$.

Example. In the example (Fig. 7), there are five bound vertices $B = \{2, 3, 5, 7, 8\}$. The double rounded vertices are the adjacent vertices $A(M)$, the MB-set is $M = \{3, 8\}$ and the MB-feasible paths are: $P_{2,2}$, $P_{5,4} = \{5, 4\}$ and $P_{7,7}$.

In order to prove the exactness of the algorithm we restate for paths the lower bound stated in Property 1 by the following relation:

$$\left\lceil \frac{b}{3} \right\rceil \leq m.$$

We prove the *Path-Algorithm* is exact showing that the number of selected vertices defines a feasible MB-set whose size is equal to the lower bound.

Theorem 6. *The Path-Algorithm finds an optimal solution of DPP on* (P_{v_1,v_n}, B).

Proof: Without loss of generality, let us start to consider $b = 3$. We can order the set $B = \{v_i, v_j, v_k\}$ such that $i < j < k$. Then the algorithm finds the optimum solution $|M| = 1$ selecting the vertex v_j for which the MB-feasible paths are the subpaths $P_{v_i,v_{j-1}}$ and $P_{v_k,v_{j+1}}$. Now let $b > 3$. First, suppose that b *mod* $3 = 0$. In this case the algorithm enters in the IF statement of Step 2 (count = 2) exactly $\frac{b}{3}$ times, then $|M| = \frac{b}{3}$ and for each selected vertex $v_i \in M$ we have two MB-feasible paths $P_{v_{i-h},v_{i-1}}$, $P_{v_{i+k},v_{i+1}}$ with $P_{v_{i-h},v_{i-1}} \cap B = \{v_{i-h}\}$ and $P_{v_{i+k},v_{i+1}} \cap B = \{v_{i+k}\}$.

On the other hand, if b *mod* $3 = 1$, we have that for $\lceil \frac{b}{3} \rceil - 1$ times the algorithm enters in the IF statement of Step 2 (count = 2) and 1 time in the IF statement of Step 3 (count = 1). In this case we have $\lceil \frac{b}{3} \rceil$ selected vertices with the same MB-feasible paths as before, since the last selected vertex is the last non-dominated bound vertex.

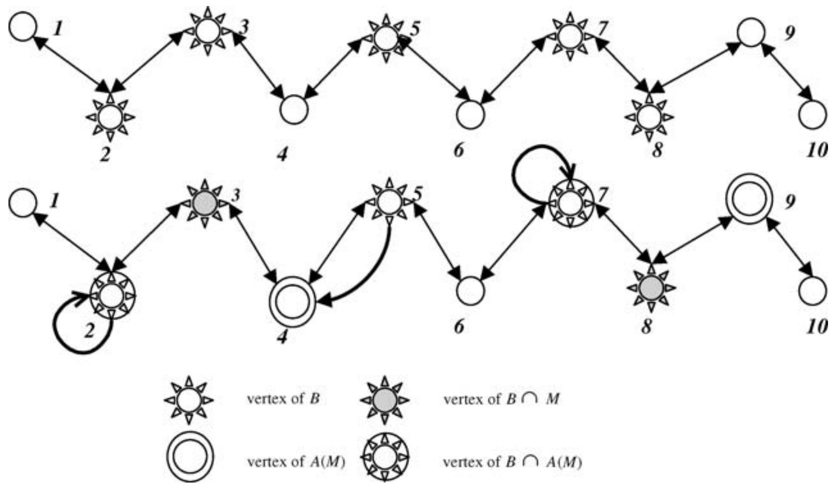


Fig. 7 Application of the Path-Algorithm

Finally, if b *mod* $3 = 2$, we have that the algorithm enters for $\lceil \frac{b}{3} \rceil$ times in the IF statement of Step 2 (count = 2), where the last selected bound vertex v_i induce only one MB-feasible path $P_{v_{i-h}, v_{i-1}}$, with $P_{v_{i-h}, v_{i-1}} \cap B = \{v_{i-h}\}$. □

Remark 1. The Path-Algorithm solves *DPP* also when applied on cycles, starting from any vertex.

As for time complexity, note that the *Path-Algorithm* operates in linear time with respect to the number of vertices V of the path P_{v_1, v_n}.

6. A linear algorithm on combs

In this section the *Comb-Algorithm* solving *DPP* on combs is presented. Despite the simplicity of solving *DPP* on paths, proving the exactness of *Comb-Algorithm* came out to be more complex. Therefore, to make this section readable, we give here the proofs of the main properties that are useful to design the algorithm and in the Appendix the additional proofs we refer to.

We start by giving the needed notations (see, for example, Gionfriddo, Harary, and Tuza (1997)). A comb $C = (V, E)$ is a tree obtained by adding at most one pending edge to each vertex of a path. We call the vertices of the path spinal vertices of C, and the remaining pending vertices the leaves of C. Hereafter, we consider complete combs, where each spinal vertex has exactly one pending vertex. Formally, $C = (V, E)$, with $V = \{v_1, v_1', v_2, v_2', \ldots, v_n, v_n'\}$, where, $V' = \{v_1', v_2', \ldots, v_n'\}$ are the leaves and $V \backslash V'$ are the spinal vertices. We define *brick* as the couple of vertices (v_i, v_i'). $C_{i,h}$, where $i < h$, denotes the subcomb $C_{\{v_i, v_i', \ldots, v_h, v_h'\}}$ induced by the subset of vertices $V_{i,h} = \{v_i, v_i', \ldots, v_h, v_h'\} \subseteq V$.

6.1. Basic properties

Let $C = (V, E)$ be a complete comb, $B \subseteq V$ be the set of bound vertices and $(M, \mathcal{P}_M)$ be the optimal solution of *DPP* on (C, B). We denote by $B_{i,h}$ the set of bound vertices in $C_{i,h}$.

Next three properties state that:

1. an optimum MB-set does not contains both the vertices of a brick (Proposition 1);
2. the set $\mathcal{P}_M$ of MB-feasible paths contains $P1$ paths when they exist (Proposition 2);
3. we can focus on combs where the vertices of a brick are either (i) both bound vertices, or, (ii) there is a single bound vertex that is the spinal one (Proposition 3).

Proposition 1. *M does not contain both vertices v_i, v_i' of a brick (v_i, v_i').*

Proof: Let us suppose $v_i, v_i' \in M$. First observe that there is no MB-feasible paths induced by v_i'. Now consider the set $\hat{M} = M \backslash \{v_i'\}$. If $v_i' \notin B$, then we can associate with $\hat{M}$ the set $\mathcal{P}_{\hat{M}} = \mathcal{P}_M$ of MB-feasible paths. If $v_i' \in B$, then we can associate with $\hat{M}$ the set $\mathcal{P}_{\hat{M}} = \mathcal{P}_M \cup \{P_{v_i', v_i'}\}$. Thus $\hat{M}$ is feasible for *DPP* on (C, B) and is $|\hat{M}| < |M|$, which is a contradiction because M is minimum. □

Proposition 2. *If $u \in B$ is such that $u \in A(M)\backslash M$, then it is possible to associate with M a set $\mathcal{P}_M$ such that $P_{u,u} \in \mathcal{P}_M$.*

Proof: If $u \in V'$ then the only MB-feasible path covering u is the P1 path. Suppose u is spinal and $P_{u,u} \notin \mathcal{P}_M$. For the optimality of M, there exists an MB-feasible path covering u, say $P_{u,v}$, $v \neq u$, $v \in A(M)\backslash M$. Let $P_{z,u}$ be the MB-feasible path connecting the bound vertex $z \in B\backslash M$ and the adjacent vertex u (Fig. 8). The new set $\mathcal{P} = (\mathcal{P}_M \backslash \{P_{u,v}, P_{z,u}\}) \cup \{P_{u,u}, P_{z,v}\}$ is also a set of MB-feasible paths that can be associated with the optimal set M and contains the $P1$ path $P_{u,u}$. □

Proposition 3. *Given a set $B \subseteq V$ of bound vertices such that there exists a brick (v_i, v_i'), $v_i \in B$ and $v_i' \notin B$. The optimal MB-couple $(M, \mathcal{P}_M)$ for (C, B) is optimal also for the set $B^* = (B\backslash\{v_i\}) \cup \{v_i'\}$ on C.*

Proof. By Proposition 1, v_i, v_i' cannot both belong to M. If $v_i \in M$ then setting $\mathcal{P}_M^* = \mathcal{P}_M \cup \{P_{v_i', v_i'}\}$ the thesis follows. If $v_i' \in M$ then, by Proposition 2, $P_{v_i, v_i} \in \mathcal{P}_M$ and setting $\mathcal{P}_M^* = \mathcal{P}_M \backslash \{P_{v_i, v_i}\}$ the thesis follows. Finally, if both $v_i, v_i' \notin M$ then let $P_{v_i,z}$ the MB-feasible path covering vertex v_i. The new path $P_{v_i',z} = \{v_i'\} \cup P_{v_i,z}$ is MB-feasible, and the couple $(M, \mathcal{P}_M^*)$ where $\mathcal{P}_M^* = (\mathcal{P}_M \backslash \{P_{v_i,z}\}) \cup \{P_{v_i',z}\}$ is an optimum MB-couple for (C, B^*).

6.2. The main result

The following Lemma 1 states that an optimal MB-couple for a comb can be decomposed (under certain conditions) into two optimal solutions for two sub-combs. Theorem 7, to follow, states that there exists an optimum MB-set entirely composed of spinal vertices. Both the proofs are given in the Appendix.

Lemma 1. *Let $(M, \mathcal{P}_M)$ be an optimal solution of DPP on (C, B). Let $C^1 = C_{1,h}$ and $C^2 = C_{h+1,n}$ be a partition of C into two components. If it is possible to define the set $\mathcal{P}_M$ such that all the bound vertices in C^i are covered by adjacent vertices of the set $M \cap V^i$, $i = 1, 2$,*

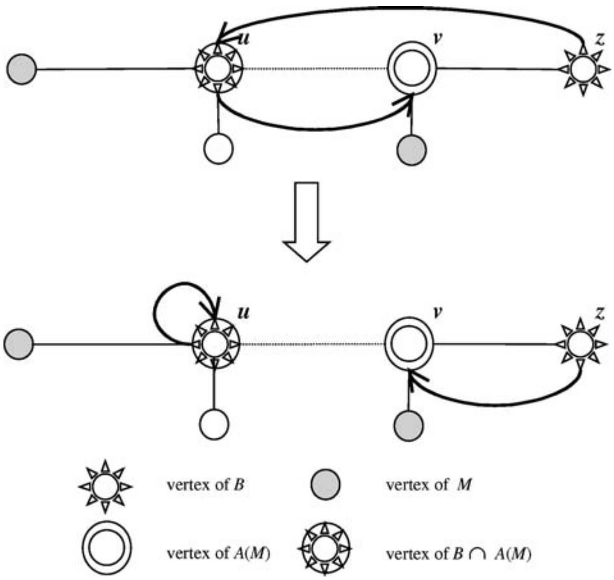


Fig. 8 Example of Proposition 2. Given an optimum MB-set M it is possible to define a set $\mathcal{P}_M$ of MB-feasible paths containing $P1$ paths if they exist

then there exists a partition of M *and of* $\mathcal{P}_M$ *into two subsets such that:* $M = M^1 \cup M^2$, $\mathcal{P}_M = \mathcal{P}_{M^1} \cup \mathcal{P}_{M^2}$ *where* $(M^i, \mathcal{P}_{M^i})$ *is optimal for* $(C^i, V^i \cap B)$, $i = 1, 2$.

Theorem 7. *There exists an optimal solution* $(M, \mathcal{P}_M)$ *of DPP on* (C, B) *such that* $M \subseteq V \backslash V'$.

If all the vertices of the optimal MB-set are spinal vertices of the comb C then simple properties follow immediately. Such properties are useful to design *Comb-Algorithm* and are given in the sequel. Let $M \subseteq V \backslash V'$ be an optimal spinal MB-set.

Corollary 1. *Let* $x_i = v_h \in M$ *and* $x_{i+1} = v_k \in M$, $h < k$, *be two consecutive vertices of* M. *Then* $|B_{h+1,k-1}| \leq 2$.

Proof: The number of adjacent vertices in $C_{h+1,k-1}$ is at most two. Since v_h and v_k are spinal vertices, then all the bound vertices in $C_{h+1,k-1}$ must be covered by adjacent vertices in $C_{h+1,k-1}$. Since M is optimum, the bound vertices in $C_{h+1,k-1}$ cannot be more than the number of adjacent vertices, and thus we have the thesis. □

With the same reasoning we derive:

Corollary 2. *Let* $x_1 = v_h \in M$ *be the first element of* M, *then* $|B_{1,h-1}| \leq 1$.

Remark 2. Note that the position of the unique bound vertex in $C_{1,h-1}$ does not influence the optimality of the MB-set M.

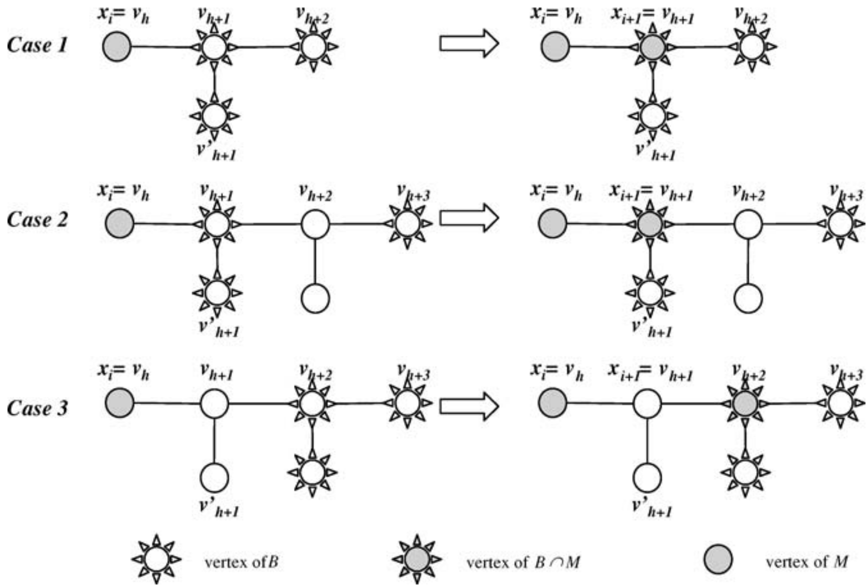


Fig. 9 Illustration to prove Corollary 3. If one of these cases occurs then the MB-feasible path with the adjacent vertex v_{h+1} does not exist

Remark 3. Considering the first brick, the first spinal element of M must be chosen in the following way:

1. If $v_1 \in B$ and $v'_1 \in B$ then $v_1 \in M$;
2. If $v_1 \in B$ and $v'_1 \notin B$, let $k = min\{i : B_{i,n} \neq \emptyset\}$, if $|B| > 2$, then $v_k \in M$ otherwise either $v_1 \in M$ or $v_k \in M$.

Corollary 3. *Let the i-th element of M be $x_i = v_h \in M$. If one of the following cases occurs:*

1. $v_{h+1} \in B$ *and* $v'_{h+1} \in B$ *and* $v_{h+2} \in B$;
2. $v_{h+1} \in B$ *and* $v'_{h+1} \in B$ *and* $v_{h+2} \notin B$ *and* $v_{h+3} \in B$;
3. $v_{h+1} \notin B$, $v_{h+2} \in B$ *and* $v'_{h+2} \in B$ *and* $v_{h+3} \in B$;

then, for cases 1 and 2, the vertex v_{h+1} must belong to M, and, for case 3, v_{h+2} must belong to M.

Proof: See Fig. 9. For case 1, if vertex $v_{h+1} \notin M$ then , by Corollary 1, the vertex v_{h+2} should be in M. In this way, the bound vertex v'_{h+1} is not covered by an MB-feasible path and then M would not be optimal. For case 2 (case 3) if vertex $v_{h+1} \notin M$ ($v_{h+2} \notin M$), then by Corollary 1 the vertex v_{h+3} (v_{h+2}) should be in M. However, the only path covering the vertex v'_{h+1} (v'_{h+2}) is not MB-feasible because contains the two consecutive adjacent vertices v_{h+1} and v_{h+2} and then M would not be optimal. □

Remark 4. Observe that if one of the cases of Corollary 3 occurs then v_{h+1} does not dominate any bound vertex u, that is there is no MB-feasible paths $P_{u,v_{h+1}}$ for each $u \in B$. Moreover, they are the only cases for which such a path does not exist.

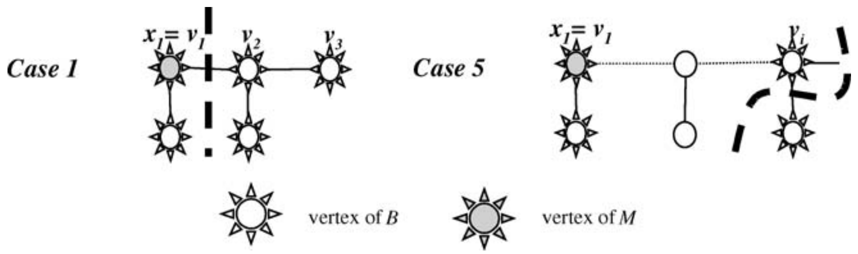


Fig. 10 Illustration for Proposition 4 (Case 1) and Proposition 5 (Case 5)

Lemma 1 above states an optimal MB-couple for a comb can be decomposed (under certain conditions) into two optimal solutions for two sub-combs. In the propositions to follow, we state the conditions for the converse of this lemma when we look for a spinal MB-set. These conditions are useful to build iteratively an optimum MB-set. In particular, they state, after a vertex of an optimum MB-set is chosen, how to choose optimally the next one. Indeed, after choosing the first spinal vertex x_1 of M (see Remark 3), the comb is divided into two subcombs C^1 and C^2 according to the ten possibilities that are summarized in Fig. 11. The second spinal vertex $x_2 \in M$ is chosen according to Remark 3 on the subcomb C^2 which is divided again into two parts and so on. We give the proof for only two of the ten cases (Case 1 and Case 5) since the others are obtained easily by following the same reasoning and by applying different cases of Remark 3 and Corollary 3.

Proposition 4. *Given a comb C and a set B of bound vertices such that $\{v_1, v_1', v_2, v_2', v_3\} \subseteq B$, an optimum spinal MB-set M for DPP on (C, B) is given by $M = M_1 \cup M_2$, $M_1 \cap M_2 = \emptyset$, where M_1 in an optimum spinal MB-set for $(C_{1,1}, B_{1,1})$ and M_2 in an optimum spinal MB-set for $(C_{2,n}, B_{2,n})$.*

Proof: Refer to Fig. 10. Let M_1 and M_2 such optimum MB-sets. We have that, by Remark 3 (Case 1), $M_1 = \{v_1\}$ and $\mathcal{P}_{M_1} = P_{v_1',v_1'}$. Moreover, by Remark 3 (Case 1) and by Corollary 3 (Case 1), $v_2 \in M_2$ and $\mathcal{P}_{M_2} \supseteq \{P_{v_2',v_2'}, P_{v_3,v_3}\}$. The couple $M = M_1 \cup M_2$ and $\mathcal{P}_M = \mathcal{P}_{M_1} \cup \mathcal{P}_{M_2}$ is a feasible MB-couple. It is also optimum. Otherwise, let $(M^*, \mathcal{P}_{M^*})$ be an optimum MB-couple for (C, B). By Remark 3 (Case 1) and by Corollary 3 (Case 1), $v_1, v_2 \in M^*$. Thus, it is possible to apply Lemma 1 to $(C_{1,1}, B_{1,1})$ and $(C_{2,n}, B_{2,n})$ obtaining two new optimum MB-sets, $|M_1^*|$ and $|M_2^*|$, for these two subcombs; this leads to a contradiction because $|M_1^* \cup M_2^*| = |M^*| < |M| \Leftrightarrow |M_2^*| < |M_2|$, and M_2 is optimum. □

Proposition 5. *Let C be a comb and B a set of bound vertices such that $v_1, v_1 \in B$ $v_i, v_i' \in B$, $i \geq 4$ and there is no bound vertex in $C_{2,i-1}$. An optimum spinal MB-set M on (C, B) is given by $M = M_1 \cup M_2$, $M_1 \cap M_2 = \emptyset$, where M_1 in an optimum spinal MB-set for $(C_{1,i}, B_{1,i} \backslash \{v_i'\})$ and M_2 in an optimum spinal MB-set for $(C_{i,n}, B_{i,n} \backslash \{v_i\})$.*

Proof: Refer to Fig. 10. Let M_1 and M_2 be optimum spinal MB-sets as stated by hypothesis. By Remark 3 (Case 1), $v_1 \in M_1$ and $\mathcal{P}_{M_1} = \{P_{v_1',v_1'}, P_{v_i,v_2}\}$. Moreover, by Remark 3 (Case 2), we can assume $v_i \notin M_2$ and there exists $P_{v_i',z} \in \mathcal{P}_{M_2}$ for some vertex z.

We want to show $M = M_1 \cup M_2$ and $\mathcal{P}_M = \mathcal{P}_{M_1} \cup \mathcal{P}_{M_2}$ is an optimum couple for (C, B). First observe that $(M, \mathcal{P}_M)$ is feasible for DPP on (C, B) and M is spinal. Let us suppose

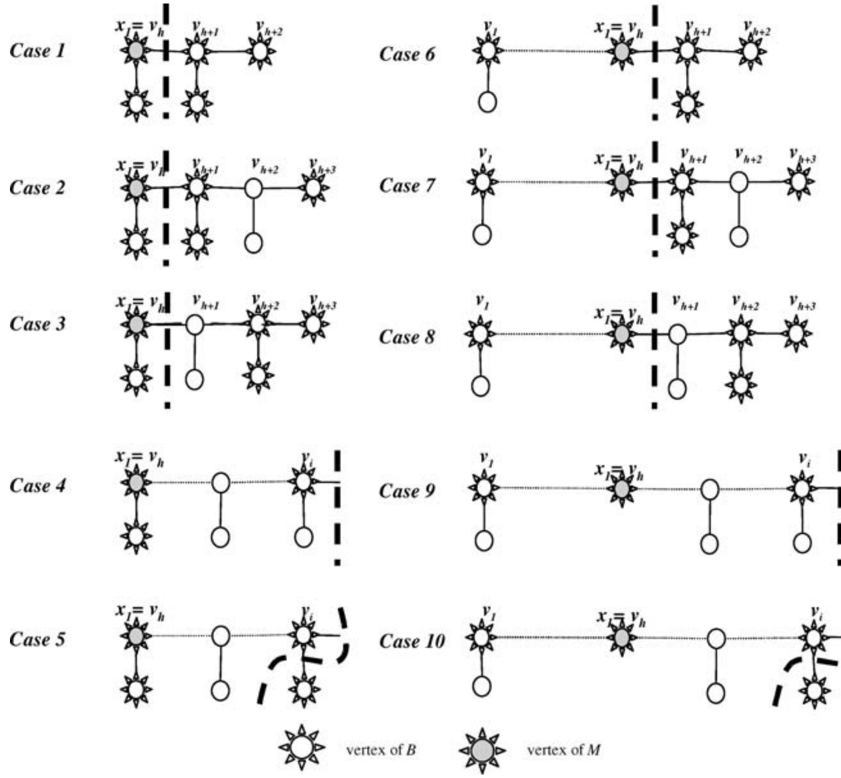


Fig. 11 Rules to choose the second element of an optimum spinal MB-set. An optimum spinal MB-set M for a comb can be obtained by the union of optimum MB-sets M_1 and M_2 on sub-combs. The colored vertex is the set M_1, the dotted line divides the original comb into two sub-combs

there exists another feasible MB-couple $(M^*, \mathcal{P}_{M^*})$, M^* spinal and $|M^*| < |M|$. By Remark 3, $v_1 \in M^*$. Note also that:

- $v_i \notin M^*$
  If $v_i \in M^*$, then, by Lemma 1, there exists M_1^* optimum and spinal for $(C_{1,i-1}, B_{1,i-1})$ and M_2^* optimum and spinal for $(C_{i,n}, B_{i,n})$. However, $|M_1^*| = |M_1|$ and M_2^* is feasible also for $(C_{i,n}, B_{i,n} \backslash \{v_i\})$. Since $|M_1^* \cup M_2^*| = |M^*| < |M|$, then we would have $|M_2^*| < |M_2|$ that is a contradiction because M_2 is optimum.
- $v_j \notin M^*, 2 \leq j \leq i$
  If such a vertex exists then the new set $M^* \backslash \{v_j\} \cup \{v_i\}$ would be optimum too, and this is not possible.

Hence, since $v_i \notin M^*$ then we can have the following two cases :

- $P_{v_i,v_2} \in \mathcal{P}_{M^*}$ and $P_{v_i',v_2} \notin \mathcal{P}_{M^*}$;
- $P_{v_i,v_2} \notin \mathcal{P}_{M^*}$ and $P_{v_i',v_2} \in \mathcal{P}_{M^*}$;

both the cases lead to a contradiction. We prove it for the first one, the second follows the same reasoning. If $P_{v_i,v_2} \in \mathcal{P}_{M^*}$ and $P_{v_i',v_2} \notin \mathcal{P}_{M^*}$, then there exists $P_{v_i,v_k} \in \mathcal{P}_M^*$, $k > i$. We can partition the set $M^* = M_1^* \cup M_2^*$, $M_1^* \cap M_2^* = \emptyset$, and the set $\mathcal{P}_{M^*} = \mathcal{P}_{M_1^*} \cup \mathcal{P}_{M_2^*}$ such that:

- $(M_1^*, \mathcal{P}_{M_1^*})$ is feasible for $(C_{1,i}, B_{1,i} \setminus \{v_i'\})$;
- $(M_2^*, \mathcal{P}_{M_2^*})$ is feasible for $(C_{i,n}, B_{i,n} \setminus \{v_i\})$.

Since $|M_1^*| = |M_1|$, then we have $|M_2^*| < |M_2|$ that is a contradiction and then the thesis holds. □

6.3. The comb-algorithm

From the analysis of the above cases, now the algorithm to solve the *DPP* on combs follows directly. Given a comb C and a set B of bound vertices, choose the first spinal vertex to put in M (according to Remark 3) and consider a sub-comb C^2 as the new instance according to the cases in Fig. 11. Choose the second spinal vertex $x_2 \in M$, by applying Remark 3 to C^2 and define a new subcomb C^3 as the new instance and so on. The main steps of the *Comb-Algorithm* are summarized in the following Remark.

Remark 5. Given a comb $C = (V, E)$ and a set of bound vertices $B \subseteq V$. Let $x_1 = v_h$ be the first spinal vertex in M. If we can build an MB-feasible path $P_{u,v_{h+1}}$ (Fig. 11, Cases 4, 5, 9, 10) then, two cases may occur:

- ***Case A***
  (Fig. 11, Cases 5 and 10)
  $u = v_i \in B$ and $v_i' \in B$.
  Then, we can define M^1, M^2, $\mathcal{P}_{M^1}$ and $\mathcal{P}_{M^2}$ such that:
  $M = M^1 \cup M^2$;
  $\mathcal{P}_M = \mathcal{P}_{M^1} \cup \mathcal{P}_{M^2}$;
  $(M^1, \mathcal{P}_{M^1})$ is optimal for *DPP* on $(C_{1,i}, B_{1,i} \setminus \{v_i'\})$;
  $(M^2, \mathcal{P}_{M^2})$ is optimal for *DPP* on $(C_{i,n}, B_{i,n} \setminus \{v_i\})$.
- ***Case B***
  (Fig. 11, Cases 4 and 9)
  $u = v_i \in B$ and $v_i' \notin B$.
  Then, we can define M^1, M^2, $\mathcal{P}_{M^1}$ and $\mathcal{P}_{M^2}$ such that:
  $M = M^1 \cup M^2$;
  $\mathcal{P}_M = \mathcal{P}_{M^1} \cup \mathcal{P}_{M^2}$;
  $(M^1, \mathcal{P}_{M^1})$ is optimal for *DPP* on $(C_{1,i}, B_{1,i})$;
  $(M^2, \mathcal{P}_{M^2})$ is optimal for *DPP* on $(C_{i+1,n}, B_{i+1,n})$.

On the other hand, if the MB-feasible path $P_{u,v_{h+1}} \notin \mathcal{P}_M$, for any $u \in B_{h+1,n}$ then:

- ***Case C***
  (Fig. 11, Cases 1, 2, 3, 6, 7, 8)
  we can define M^1, M^2, $\mathcal{P}_{M^1}$ and $\mathcal{P}_{M^2}$ such that:
  $M = M^1 \cup M^2$;
  $\mathcal{P}_M = \mathcal{P}_{M^1} \cup \mathcal{P}_{M^2}$;
  $(M^1, \mathcal{P}_{M^1})$ is optimal for *DPP* on $(C_{1,h}, B \cap V_{1,h})$;
  $(M^2, \mathcal{P}_{M^2})$ is optimal for *DPP* on $(C_{h+1,n}, B \cap V_{h+1,n})$.

The *Comb-Algorithm* is given in the sequel.
Comb-Algorithm
Input: a comb $C = (V, E)$, a subset $B \subseteq V$;

Output: an optimal spinal MB-set M for *DPP* on (C, B).

Step 1 {*Initialization*}

1.1 $M = \emptyset$, $j = 0$, $i = 1$;
1.2 $C^{(j)} = C$, $B^{(j)} = B$;
1.3 {*Proposition 3*}
`For each` brick (v_h, v'_h) such that $v_h \notin B^{(j)}$ and $v'_h \in B^{(j)}$ `do` $B^{(j)} = (B^{(j)} \backslash \{v'_h\}) \cup \{v_h\}$;

Step 2 {*First element*}

`If` $B^{(j)} = \emptyset$
`then` return M and `STOP`
`else`

2.1 $k = \min\{i : C^{(j)}_{i,n} \cap B^{(j)} \neq \emptyset\}$
2.2 `If` $k < n$ `then` Select Case:

Case 1: {*Remark 3: Case 1*}
`If` $v_k \in B^{(j)}$ and $v'_k \in B^{(j)}$ `then`
`begin`
$M = M \cup \{v_k\}$;
$B^{(j)} = B^{(j)} \backslash \{v_k, v'_k\}$;
$i = k$;
`end`
Case 2: {*Remark 3: Case 2*}
`If` $v_k \in B^{(j)}$ and $v'_k \notin B^{(j)}$ and $v_{k+1} \in B^{(j)}$ `then`
`begin`
$M = M \cup \{v_{k+1}\}$;
$B^{(j)} = B^{(j)} \backslash \{v_k, v_{k+1}\}$;
$i = k + 1$;
`end`
Case 3: {*Remark 2*}
`If` $v_k \in B^{(j)}$ and $v'_k \notin B^{(j)}$ and $v_{k+1} \notin B^{(j)}$ `then`
`begin`
$B^{(j)} = (B^{(j)} \backslash \{v_k\}) \cup \{v_{k+1}\}$;
$k = k + 1$;
Goto Step 2.2;
`end`
`else` $M = M \cup \{v_n\}$, return M and `STOP`

Step 3 { *Building MB-feasible path* }
`If` $B^{(j)} = \emptyset$
`then` return M and `STOP`
`else`

3.1 $k = \min\{s > i : C^{(j)}_{s,n} \cap B^{(j)} \neq \emptyset\}$
`If` $k < n$ `then`
{

3.1.1 `If` the bound vertex v_k is such that the MB-feasible path $P_{v_k, v_{i+1}}$ can be built (see Fig. 11, Cases 4, 5, 9, 10) `then` select case:

Case 4: {*Remark 6: Case A*}
`If` $v'_k \in B^{(j)}$ `then` begin
$C^{(j+1)} = C^{(j)}_{k,n}$;
$B^{(j+1)} = B^{(j)} \backslash \{v'_k\}$;
$j = j + 1$;
Goto Step 2;
`end`
Case 5: {*Remark 6: Case B*}
`If` $v'_k \notin B^{(j)}$ `then` begin
$C^{(j+1)} = C^{(j)}_{k+1,n}$;
$B^{(j+1)} = B^{(j)} \backslash \{v_k\}$;
$j = j + 1$;
Goto Step 2;
`end`

3.1.2 {*Corollary 3*}
`If` the bound vertex v_k is such that the MB-feasible path $P_{v_k, v_{i+1}}$ cannot be built (see Fig. 11, Cases 1, 2, 3, 6, 7, 8) `then`
{*Remark 6: Case C*}
`begin`
$C^{(j+1)} = C^{(j)}_{k,n}$;
$B^{(j+1)} = B^{(j)}$;
$j = j + 1$;
Goto Step 2;
`end`
}
`else` ($k = n$)
{
`If` $v'_n \in B^{(j)}$ `then` $M = M \cup \{v_n\}$
return M and STOP
}

As for time complexity, note that the *Comb-Algorithm* operates in linear time with respect to the number of vertices V of C. Note that, *Comb-Algorithm* finds an optimal solution of *DPP* on C even if C is a path.

Example. Let us consider the complete comb of 10 spinal vertices, shown in Figs. 12 and 13. The set of bound vertices is $B = \{v_1, v'_2, v_4, v'_4, v_5, v'_5, v_7, v'_{10}\}$. At first the algorithm changes (Step 1.3) the set of bound vertices in a new set $B^{(0)}$ (Proposition 3). The new set of bound vertices is: $B^{(0)} = \{v_1, v_2, v_4, v'_4, v_5, v'_5, v_7, v_{10}\}$. The first bound vertex is v_1, since $v'_1 \notin B$

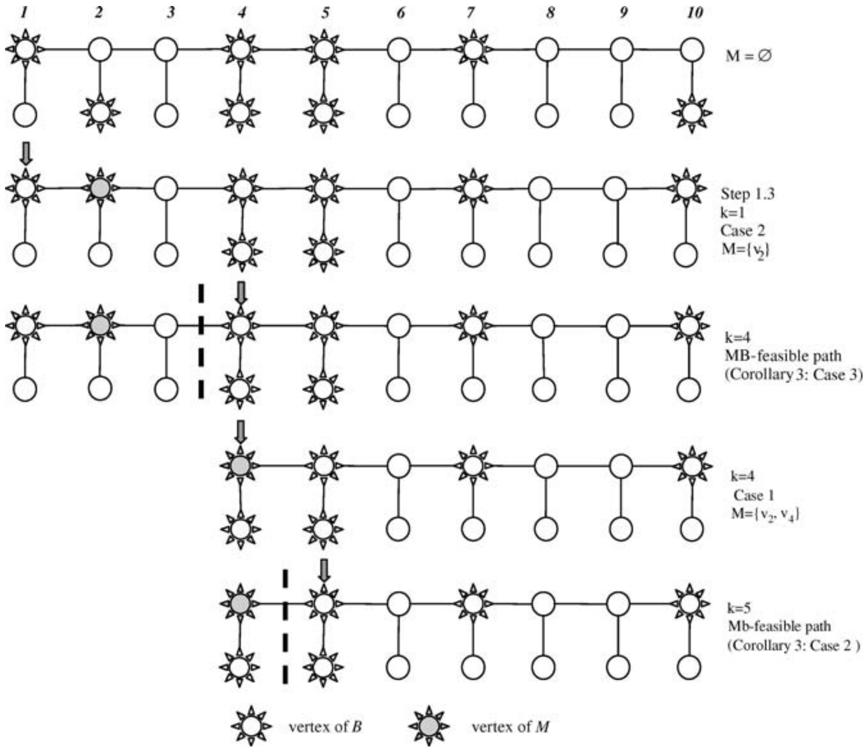


Fig. 12 The Comb-Algorithm (1/2)

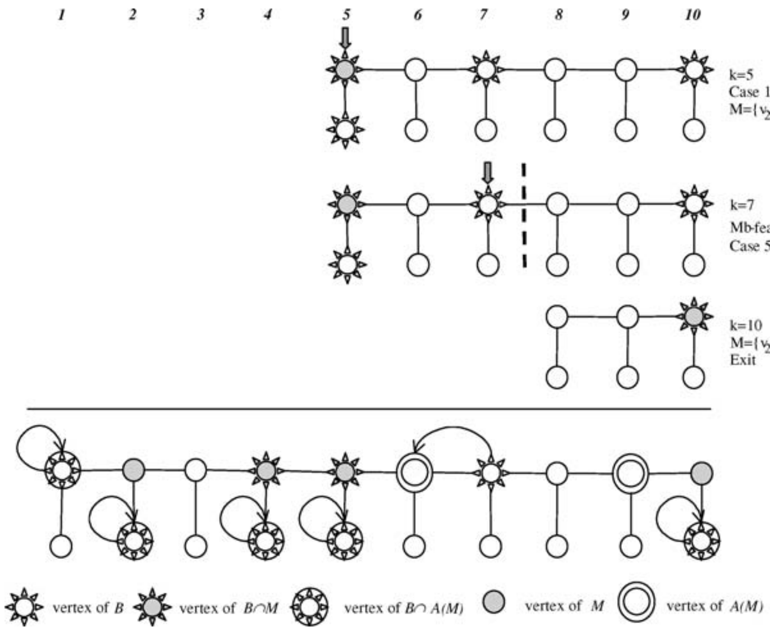


Fig. 13 The Comb-Algorithm (2/2)

and $v_2 \in B$ then ***Case 2*** of Step 2 holds and v_2 is selected. Note now, that the MB-feasible path covering vertex v_1 is the P1 path P_{v_1,v_1}. However, it is not possible to build an MB-feasible path with the other adjacent vertex v_3. Indeed, Case 3 of Corollary 3 holds and the new instance of the problem is the subcomb $C^{(1)} = C_{4,10}$ with $B^{(1)} = \{v_4, v'_4, v_5, v'_5, v_7, v_{10}\}$. Since ***Case 1*** of Step 2 occurs, then vertex v_4 is selected and the MB-feasible path $P_{v'_4,v'_4}$ covers the bound vertex v'_4. Now, it is not possible, again, to build an MB-feasible path with the adjacent vertex v_5 (Case 2 of Corollary 3) and then a new instance is defined: $C^{(2)} = C_{5,10}$ with $B^{(2)} = \{v_5, v'_5, v_7, v_{10}\}$. The first bound vertex of $C^{(2)}$ is vertex v_5 that is selected. It is possible to build the two MB-feasible paths $P_{v'_5,v'_5}$ and P_{v_7,v_6}, covering respectively the bound vertices v'_5 and v_7. The new instance is: $C^{(3)} = C_{8,10}$ and $B^{(3)} = \{v_{10}\}$, since vertex v_{10} is the only bound vertex, then, it is selected and the algorithm stops. The optimum couple for $(C, B^{(0)})$ is the MB-set $M = \{v_2, v_4, v_5, v_{10}\}$ and the set $\mathcal{P}_M = \{P_{v_1,v_1}, P_{v'_4,v'_4}, P_{v'_5,v'_5}, P_{v_7,v_6}\}$. The optimum couple for (C, B) is $(M, \mathcal{P}_M \cup \{P_{v_2,v_2}, P_{v_{10},v_{10}}\})$.

7. Conclusions

In this paper we considered the problem of locating the minimum number of counting sensors on the nodes of a network in order to determine arc flow volumes of the entire network. To characterize the problem from a graph theoretical point of view, we defined the *Dominating Paths Problem* (*DPP*). We proved it is NP-complete on general graphs by reduction from the Dominating Set Problem (DSP). We analyzed special structure of graphs that are particularly interesting from an application point of view, and for which we gave low order polynomial solution algorithms. In particular, we proved that the *DPP* is polynomially solvable on paths, cycles and combs, and we designed algorithms that find the solution in linear time for these structures.

It is also interesting to analyze *DPP* on other graph classes relevant from an application point of view (such as caterpillars, grids, etc.) and to better understand the complexity of the problem in relation with different distributions on the graph of the bound vertices. Such analysis and an approximation algorithm for the general case are object of a future work.

8. Appendix

In this section we give the additional properties to formally prove there exists an optimum MB-set on combs entirely composed of spinal vertices. The proofs of some propositions are already given in the paper, here, for completeness of exposition, we recall their statements.

Basic properties

Let $C = (V, E)$ be a complete comb, $B \subseteq V$ be the set of bound vertices and $(M, \mathcal{P}_M)$ be the optimal solution of *DPP* on (C, B). We denote by $B_{i,h}$ the set of bound vertices in $C_{i,h}$.

Proposition 6. *M does not contain both vertices v_i, v'_i of a brick (v_i, v'_i).*

Proposition 7. *If the brick (v_1, v'_1) is such that $v_1, v'_1 \notin B$, then the size of the optimal MB-set M for DPP on (C, B) and the size of the optimal MB-set $\hat{M}$ of DPP on $(C_{V\setminus\{v_1,v'_1\}}, B)$ are equal.*

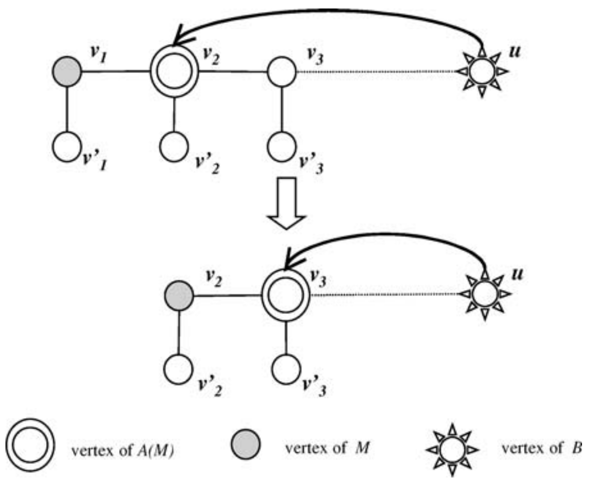


Fig. 14 Example of Proposition 6. If the first brick (v_1, v_1) of a comb C is composed of non-bound vertices, then the optimal solution value of *DPP* on C is the same of the optimal solution value of *DPP* on $C_{V\backslash\{v_1,v_1\}}$

Proof: First suppose $v_1, v_1' \notin M$. In this case, since $v_1, v_1' \notin B$, no MB-feasible path exists in $\mathcal{P}_M$ that contains v_1 or v_1' and thus we have the thesis. Now suppose that at least one vertex between $v_1, v_1' \in M$ (by Proposition 6, both v_1, v_1' cannot belong to M). We show that there exists an optimal solution $(M, \mathcal{P}_M)$ of *DPP* on (C, B) such that M contains neither v_1 nor v_1'. If $v_1 \in M$, then there is the MB-feasible path $P_{u,v_2,} \in \mathcal{P}_M$, $u \neq v_1, v_1'$, covering a bound node u. The set $\hat{M} = (M\backslash\{v_1\}) \cup \{v_2\}$ has the same size of M and we can associate with it the following set of MB-feasible paths:

- if $u = v_2$ then $\mathcal{P}_{\hat{M}} = \mathcal{P}_M \backslash\{P_{v_2,v_2}\}$;
- if $u = v_2'$ then $\mathcal{P}_{\hat{M}} =(\mathcal{P}_M \backslash\{P_{v_2',v_2}\}) \cup \{P_{v_2',v_2'}\}$.
- if $u \neq v_2, v_2'$ then $\mathcal{P}_{\hat{M}} = (\mathcal{P}_M \backslash\{P_{u,v_2}\}) \cup \{P_{u,v_3}\}$;

A similar reasoning holds if $v_1' \in M$ considering $\hat{M} = (M\backslash\{v_1'\}) \cup \{v_2\}$.
Figure 14 shows the case when $u \neq v_2, v_2'$. □

Remark 6. By Proposition 7 we can restrict the analysis to comb $C = (V, E)$ with $(\{v_1, v_1'\} \cap B) \neq \emptyset$.

The following two propositions state that the set $\mathcal{P}_M$ of MB-feasible paths associated with the MB-set M is not unique. Moreover, it is shown that, given two MB-feasible paths, having nonempty intersection, it is possible to obtain from them other two MB-feasible paths. Illustration of Proposition 8 is shown in Fig. 15.

Proposition 8. *Given a solution $(M, \mathcal{P}_M)$ of DPP on (C, B) and two MB-feasible paths $P_{b,a} \in \mathcal{P}_M$, $P_{b^*,a^*} \in \mathcal{P}_M$ such that $P_{b,a} \cap P_{b^*,a^*} \neq \emptyset$, $b, b^* \in B$ and $a, a^* \in A(M)$ not adjacent, $b \neq a$, $b^* \neq a^*$. The new paths P_{b,a^*} and $P_{b^*,a}$ are MB-feasible for both M and B.*

Proof: Note that the adjacent vertices a, a^* must be spinal vertices for the MB-feasibility of the paths, that are not of type P1. First suppose that b, b^* are spinal vertices (Fig. 15). By the MB-feasibility of the paths and since a and a^* are not adjacent, the path $P = P_{b,a} \cup P_{b^*,a^*}$ is such that (i) does not contain vertices of M and (ii) does not contain consecutive vertices of $A(M)$. This is true for any subpath of P, in particular for P_{b,a^*} and $P_{b^*,a}$ that are MB-feasible, since $b, b^* \in B$ and $a, a^* \in A(M)$. Let us suppose now that b^* is a leaf of C, and consider the path $\tilde{P}$ composed of the spinal vertices of the subcomb $\hat{C} = P_{b,a} \cup P_{b^*,a^*}$. $\tilde{P}$ satisfies

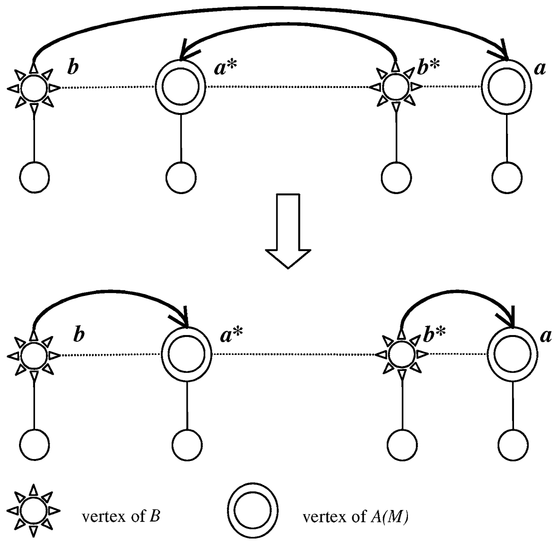


Fig. 15 Example of Proposition 8. From two MB-feasible paths having non-empty intersection, it is possible to obtain two new MB-feasible paths

conditions (i) and (ii), and thus any subpath of $\tilde{P}$. Let us consider now the subpath $P_{A(\{b^*\}),a}$. This path satisfies the two conditions (i) and (ii) and since, by hypothesis $b^* \notin A(M)$, the path $P_{b^*,a} = \{b^*\} \cup P_{A(\{b^*\}),a}$ is MB-feasible. The same reasoning holds for P_{b,a^*}. $\square$

Proposition 9. *It is possible to associate with M a set $\mathcal{P}_M$ such that $P_{u,u} \in \mathcal{P}_M$.*

Proposition 10. *Given a set $B \subseteq V$ of bound vertices such that there exists a brick (v_i, v_i'), $v_i \in B$ and $v_i' \notin B$. The optimal MB-couple $(M, \mathcal{P}_M)$ for (C, B) is optimal also for the set $B^* = (B\backslash\{v_i\}) \cup \{v_i'\}$ on C.*

The main result

Remember that we denote by $C_{i,h} = C_{\{v_i, v_i', \ldots, v_h, v_h'\}}$ the subcomb induced by the subset of vertices $\{v_i, v_i' \ldots, v_h, v_h'\}$, $i < h$ and by $B_{i,h}$ the bound vertices in $C_{i,h}$.

The following Lemma 2 states that there exists an optimum MB-set with the first element that is a spinal vertex. Theorem 8 to follow states that there exists an optimum MB-set entirely composed of spinal vertices.

Lemma 2. *There exists an optimal solution $(M,\mathcal{P}_M)$, $M = \{x_1, \ldots, x_m\}$, of DPP on (C, B) such that x_1 is a spinal vertex.*

Proof: Let us suppose $x_1 = v_i' \in M$ is not a spinal vertex. We want to show that it is possible to obtain an optimal solution $(\hat{M},\mathcal{P}_{\hat{M}})$ where $\hat{M} = \{\hat{x}_1, \ldots, \hat{x}_m\}$, of *DPP* on (C, B) such that $\hat{x}_1$ is spinal.

By Remark 6, at least one vertex between v_1 and v_1' belongs to B.

First consider the case $x_1 = v_1' \in M$. By Proposition 6, $v_1 \notin M$, and, by Propositions 10 and 2, we can suppose $v_1 \in B$ and $P_{v_1,v_1} \in \mathcal{P}_M$. Let us consider the set $\hat{M} = (M\backslash\{v_1'\}) \cup \{v_1\}$. If $x_2 \neq v_3'$ then we can associate with $\hat{M}$ the set $\mathcal{P}_{\hat{M}} = (\mathcal{P}_M\backslash\{P_{v_1,v_1}\}) \cup \{P_{v_1',v_1'}\}$, where the path $P_{v_1',v_1'}$ is added only if $v_1' \in B$. Since $|\hat{M}| = |M|$, the couple $(\hat{M}, \mathcal{P}_{\hat{M}})$ is also

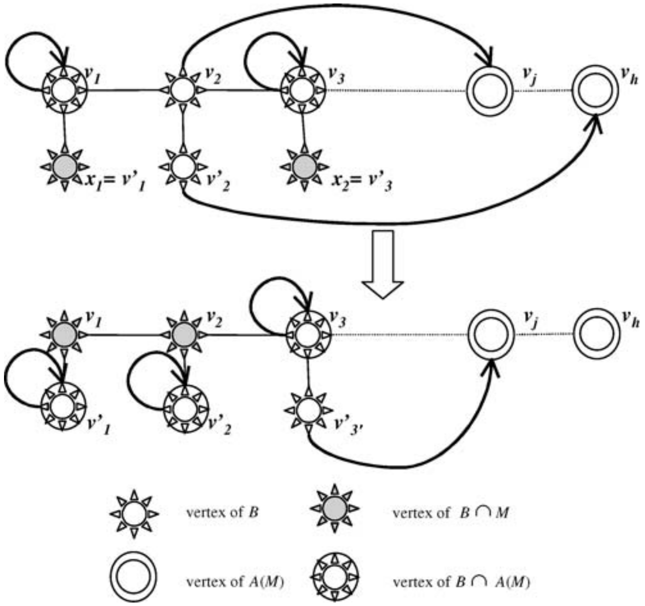


Fig. 16 Example of Lemma 2 when when $x_1 = v_1'$ and $x_2 = v_3'$. There exists an optimum MB-set M such that the first element is a spinal vertex

optimal for *DPP* on (C, B) and has the first element x_1 that is spinal. On the other side, if $x_2 = v_3'$ the above defined set $\hat{M}$ is such that the vertices v_2 and v_3 are adjacent and then any MB-feasible path in $\mathcal{P}_M$ covering a bound vertex $z = v_2$ or $z = v_2'$ is no more feasible for $\hat{M}$. However, since M is optimum, there exist two MB-feasible paths covering the bound vertices v_2 and v_2', say P_{v_2,v_j} and P_{v_2',v_h}, $j, h \geq 3$, $v_j, v_h \in A(M)$ (Fig. 16). We define in this case, $\hat{M} = (M \backslash \{v_1', v_3'\}) \cup \{v_1, v_2\}$. Associating with $\hat{M}$ the set $\mathcal{P}_{\hat{M}} = (\mathcal{P}_M \backslash \{P_{v_1,v_1}, P_{v_2,v_j}, P_{v_2',v_h}\}) \cup \{P_{v_1',v_1'}, P_{v_2',v_2'}, P_{v_3',v_j}\}$, where the paths $P_{v_1',v_1'}$ and P_{v_3',v_j} are added only if $v_1', v_3' \in B$ respectively, then the couple $(\hat{M}, \mathcal{P}_{\hat{M}})$ is also optimal for *DPP* on (C, B). Figure 16 shows the case when all the vertices $\{v_1', v_1', v_2, v_2', v_3, v_3'\} \subseteq B$.

Suppose now, $x_1 = v_i' \in M$, $i > 1$. Consider the subcomb $C_{1,i-1}$. We have, by Remark 6, that $|B_{1,i-1}| \geq 1$.

We prove the thesis for the case $|B_{1,i-1}| = 2$ without loss of generality. By Proposition 10, we can assume $v_1 \in B_{1,i-1}$ and let $u \in V_{1,i-1}$ be the second bound vertex, $u \neq v_1$.

The set $\hat{M} = (M \backslash \{v_i'\}) \cup \{v_1\}$ has the same size of M. We will show that it is possible to associate with it a set $\mathcal{P}_{\hat{M}}$ of MB-feasible paths covering all the bound vertices not in $\hat{M}$, that is, $(\hat{M},\mathcal{P}_{\hat{M}})$ is an optimal solution of *DPP* on (C, B) with the first vertex being spinal. Since M is optimum for (C, B), there exist two MB-feasible paths in $\mathcal{P}_M$ covering the bound vertices v_1 and u, say P_{v_1,v_z} and P_{u,v_h}, $i \leq z < h$, $v_z, v_h \in A(M)$. We can have the following cases:

Case 1: $v_i \notin B$ and $v_i' \notin B$;
Case 2: $v_i \in B$ and $v_i' \notin B$;
Case 3: $v_i \notin B$ and $v_i' \in B$;
Case 4: $v_i \in B$ and $v_i' \in B$.

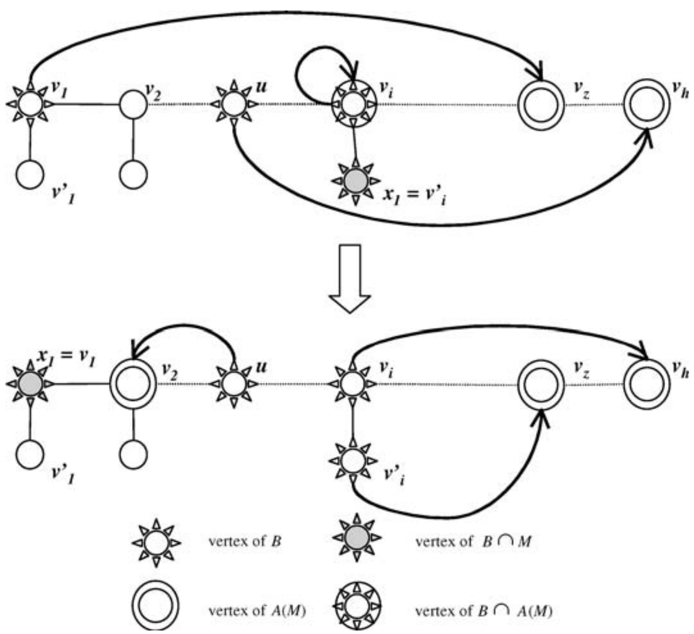


Fig. 17 Example of Lemma 2 when $x_1 = v_i' \neq v_1'$. There exists an optimum MB-set M such that the first element is a spinal vertex

We define the set $\mathcal{P}_{\hat{M}}$ to be associated with $\hat{M}$ for *Case 4*, as the other cases are trivially derived from it. Let $v_i \in B$ and $v_i' \in B$. By Proposition 2 we can suppose the MB-feasible path $P_{v_i,v_i} \in \mathcal{P}_M$ (see Fig. 17). We can associate with $\hat{M}$ the set $\mathcal{P}_{\hat{M}} = (\mathcal{P}_M \setminus \{P_{v_i,v_i}, P_{v_1,v_z}, P_{u,v_h}\}) \cup \{P_{u,v_2}, P_{v_i,v_h}, P_{v_i',v_z}\}$ (if $u = v_1'$ we can consider instead of P_{u,v_2} the path $P_{u,u}$ that is MB-feasible for $\hat{M}$). All the paths in $\mathcal{P}_M \setminus \{P_{v_i,v_i}, P_{v_1,v_z}, P_{u,v_h}\}$ are MB-feasible also for $\hat{M}$. It remains to be shown that P_{u,v_2}, P_{v_i,v_h} and P_{v_i',v_z} are MB-feasible. Recall that a path $P_{b,a}$, $b \in B$, $a \in A(\hat{M})$, is MB-feasible for $\hat{M}$ if: (i) it does not contain vertices of $\hat{M}$ and (ii) it does not contain consecutive vertices of $A(\hat{M})$. First observe that all the subpaths of $P_{v_1,v_z} \in \mathcal{P}_M$ satisfy these two conditions for the set $\hat{M}$. Then the path $P_{u,v_2} \subset P_{v_1,v_z}$ is MB-feasible. The same consideration for the path $P_{v_i,v_h} \subset P_{u,v_h}$. Let us consider now the path $P_{v_i',v_z} = \{v_i', v_i, v_{i+1}, \ldots v_z\}$. The subpath $\{v_{i+1}, \ldots v_z\} \subset P_{v_1,v_z}$ satisfies conditions (i) and (ii). It remains to be shown that also the subpath $\{v_i', v_i, v_{i+1}\}$ satisfies these two conditions. Since $v_i' \in M$, then, by Proposition 6, $v_i \notin M$ and thus $v_i \notin \hat{M}$ by construction. It follows that $v_i' \notin A(\hat{M})$. Moreover, $v_i \notin A(\hat{M})$ because $v_i' \notin \hat{M}$ by construction, $v_{i-1} \notin \hat{M}$ by hypothesis and $v_{i+1} \notin \hat{M}$ by the MB-feasibility of path P_{v_1,v_z} for M. Then, it follows that the entire path P_{v_i',v_z} is MB-feasible for $\hat{M}$ and B, and thus the couple $(\hat{M}, \mathcal{P}_{\hat{M}})$ is optimal for *DPP* on (C, B). □

Lemma 3. *Let $(M,\mathcal{P}_M)$ be an optimal solution of DPP on (C, B). Let $C^1 = C_{1,h}$ and $C^2 = C_{h+1,n}$ be a partition of C into two components. If it is possible to define the set $\mathcal{P}_M$ such that all the bound vertices in C^i are covered by adjacent vertices of the set $M \cap V^i$, $i = 1, 2$, then there exists a partition of M and of $\mathcal{P}_M$ into two subsets such that: $M = M^1 \cup M^2$, $\mathcal{P}_M = \mathcal{P}_{M^1} \cup \mathcal{P}_{M^2}$ where $(M^i,\mathcal{P}_{M^i})$ is optimal for $(C^i, V^i \cap B)$, $i = 1, 2$.*

Proof: Let $C^1 = C_{1,h}$ and $C^2 = C_{h+1,n}$, $h \geq 1$. By hypothesis all the bound vertices in $B^1 = B \cap V_{1,h}$ are covered by adjacent vertices of $C_{1,h}$. That is, the set $M^1 = M \cap V_{1,h}$ covers by MB-feasible paths all the bound vertices in $V_{1,h} \backslash M$. Let us denote by $\mathcal{P}_{M^1}$ the set of these MB-feasible paths. The couple $(M^1, \mathcal{P}_{M^1})$ is thus a feasible solution for *DPP* on $B^1 \backslash M^1$. In the same way the couple $(M^2, \mathcal{P}_{M^2})$ is feasible for *DPP* on (C^2, B^2), where $M^2 = M \backslash M^1$, $B^2 = B \backslash B^1$ and the set $\mathcal{P}_{M^2}$ is a set of MB-feasible paths covering all the bound vertices of B^2 by the adjacent vertices of M^2. We show that these couples are also optimum, respectively, for (C^1, B^1) and (C^2, B^2). Indeed, let $(M^3, \mathcal{P}_{M^3})$, $M^3 \subseteq V_{1,h}$ and $|M^3| < |M^1|$ be a feasible solution for *DPP* on $(C_{1,h}, B^1)$. Thus, we could define the feasible solution $(M^*, \mathcal{P}_{M^*})$ of *DPP* on (C, B) where $M^* = M^3 \cup M^2$, $\mathcal{P}_{M^*} = \mathcal{P}_{M^3} \cup \mathcal{P}_{M^2}$ such that $|M^*| = |M^3| + |M^2| < |M^1| + |M^2| = |M|$, which is a contradiction because M is minimum. □

Finally, the following theorem states that there exists an optimum solution $(M, \mathcal{P}_M)$ of *DPP* on (C, B) such that the MB-set M is entirely composed of spinal vertices.

Theorem 8. *There exists an optimal solution $(M, \mathcal{P}_M)$ of DPP on (C, B) such that $M \subseteq V \backslash V'$.*

Proof: The proof is by induction on the vertices of $M = \{x_1, \ldots, x_m\}$.
By Lemma 2 the thesis is true for $k = 1$, that is x_1 is spinal.
Let us suppose now that $x_1, \ldots, x_{k-1}$ are spinal vertices of the optimal set M and $x_k \in M$ is not spinal.

Without loss of generality we can assume $x_1 = v_{i_1}, x_2 = v_{i_2}, \ldots, x_{k-1} = v_{i_{k-1}}$ and $x_k = v'_{i_k}$, $i_1 < i_2 < \cdots < i_{k-1} < i_k$.

We show that we can define an optimal set $\hat{M} = \{x_1, \ldots, x_{k-1}, \hat{x}_k, x_{k+1}, \ldots, x_m\}$ such that $x_1, \ldots, x_{k-1}, \hat{x}_k$ are spinal.

Let $x_{k-1} = v_h \in M$ be the $k-1$th spinal vertex of M, and, let $x_k = v'_j \in M$, $j > h$, be the kth vertex of M which is not spinal.

Since $v_h \in M$, then $v_{h+1} \in A(M)$. Since $v_h \in M$ is spinal, thus, by definition of MB-feasibility of a path, there are not any MB-feasible paths connecting vertices of $C_{1,h}$ with vertices of $C_{h+1,n}$.

If $j = h + 1$, let $P_{u,v_{h+1}}$ be the MB-feasible path containing vertex v_{h+1} (if it exists). The set $M^* = (M \backslash \{v'_{h+1}\}) \cup v_{h+1}$ has the same size of M. The set $\mathcal{P}_{M^*} = (\mathcal{P}_M \backslash \{P_{u,v_{h+1}}\}) \cup \{P_{u,v_{h+2}}\} \cup \{P_{v'_{h+1},v'_{h+1}}\}$ covers all the bound vertices by MB-feasible paths (note that $P_{u,v_{h+2}}$ and $P_{v'_{h+1},v'_{h+1}}$ are added only if $u \neq v_{h+1}$ and $v'_{h+1} \in B$ respectively), therefore we have the thesis.

Let $j > h + 1$ and $\mathcal{P}_M$ be the set of MB-feasible paths associated with M and covering all the bound vertices in $B \backslash M$.

We can have the following two cases:

- for each bound vertex $u \in B_{h+1,n}$, no MB-feasible path exists $P_{u,v_{h+1}} \in \mathcal{P}_M$;
- there exists a bound vertex $u \in B_{h+1,n}$ such that $P_{u,v_{h+1}} \in \mathcal{P}_M$.

Case A: *no MB-feasible path exists $P_{u,v_{h+1}} \in \mathcal{P}_M$, where $u \in B_{h+1,n}$.*
See Fig. 18. We can assume $v_{h+1} \notin B$, because otherwise, by Proposition 9, we could associate with M a set $\mathcal{P}_M$ such that $P_{v_{h+1},v_{h+1}} \in \mathcal{P}_M$ and thus we would have *Case B*.

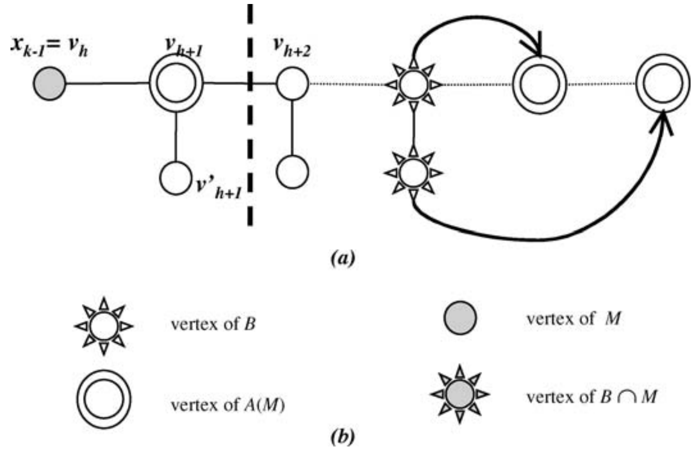


Fig. 18 Example of Case A of Theorem 8

We can assume $v'_{h+1} \notin B$, because otherwise, the MB-feasible path $P_{v'_{h+1},w}$, $w \in A(M)$ and $w \neq v_{h+1}$ connecting the bound vertex v'_{h+1} to the adjacent vertex w could be substituted, by Proposition 8, with the MB-feasible path $P_{v'_{h+1},v_{h+1}}$ and then we would have *Case B*. Therefore, there are no MB-feasible paths connecting vertices of $C_{1,h+1}$ with vertices of $C_{h+2,n}$. More in detail, all the bound vertices of $C_{1,h+1}$ (respectively $C_{h+2,n}$) are covered by adjacent vertices of $M \cap V_{1,h+1}$ (respectively $M \cap V_{h+2,n}$). By Lemma 3 we can define the sets $M^1 = \{x_1, \ldots, x_{k-1}\}$ and $M^2 = \{x_k, \ldots, x_m\}$ such that $(M^1, \mathcal{P}_{M^1})$ is optimal for *DPP* on $(C_{1,h+1}, B_{1,h+1})$ and $(M^2, \mathcal{P}_{M^2})$ is optimal for *DPP* on $(C_{h+2,n}, B_{h+2,n})$. Applying Lemma 2 to $(M^2, \mathcal{P}_{M^2})$ we can define the new optimal solution $(M^{2*}, \mathcal{P}_{M^{2*}})$ on $(C_{h+2,n}, B_{h+2,n})$, $M^{2*} = \{\hat{x}_k, \ldots, x_m\}$, where $\hat{x}_k$ is spinal. Thus, the couple $(\hat{M}, \mathcal{P}_{\hat{M}})$, $\hat{M} = M^1 \cup M^{2*}$ and $\mathcal{P}_{\hat{M}} = \mathcal{P}_{M^1} \cup \mathcal{P}_{M^{2*}}$ is an optimal solution of *DPP* on (C, B) and it is such that the set $\hat{M}$ contains k spinal vertices.

Case B: *there exists an MB-feasible path $P_{u,v_{h+1}} \in \mathcal{P}_M$ for a bound vertex $u \in V_{h+1,n}, \cap B$.* By Proposition 8, we can suppose that $|P_{u,v_{h+1}} \cap B| = 1$, that is, we suppose the adjacent vertex v_{h+1} covers the nearest bound vertex in $\{v_{h+1}, \ldots, v_n\}$. Let $u \in \{v_i, v'_i\}$, $i \geq h+1$. The partition $\mathcal{P}_M = \mathcal{P}_{M^1} \cup \{P_{u,v_{h+1}}\} \cup \mathcal{P}_{M^2}$, is such that $\mathcal{P}_{M^1}$ is the set of MB-feasible paths connecting bound vertices and adjacent vertices of the set $V_{1,h}$, and $\mathcal{P}_{M^2}$ is the set of MB-feasible paths connecting bound vertices and adjacent vertices of the set $V_{h+1,n} \backslash \{u, v_{h+1}\}$.

If $h+1 \leq i \leq j$ (Fig. 19), then defining $M^2 = \{x_k, \ldots, x_m\}$ the couple $(M^2, \mathcal{P}_{M^2})$ is optimal for *DPP* on $(C_{i,n}, B_{i,n} \backslash \{u\})$. Otherwise, there exist a set $|M^3| < |M^2|$ and a set $\mathcal{P}_{M^3}$ of MB-feasible paths that is optimum for $(C_{i,n}, B_{i,n} \backslash \{u\})$; this implies that $M^1 \cup M^3$ and the set $\mathcal{P}_{M^1} \cup \{P_{u,v_h+1}\} \cup P_{M^3}$ is feasible for (C, B), leading to a contradiction since $|M^1 \cup M^3| < |M^1 \cup M^2| = |M|$. Thus, applying Lemma 2 to $(M^2, \mathcal{P}_{M^2})$, we can obtain, as for *Case A*, a new couple $(\hat{M}, \mathcal{P}_{\hat{M}})$ which is optimal for (C, B) and that contains exactly k spinal vertices.

If $i > j$ (Fig. 20) then consider the bound vertex $z \in \{v_l, v'_l\}$, $l > i$, covered by the adjacent vertex v_j, i.e. consider the MB-feasible path P_{z,v_j}. Let us define $M^{2*} = (\{x_k, \ldots, x_m\} \backslash \{x_k\}) \cup \{v'_l\}$ and the set $\mathcal{P}_{M^{2*}} = (\mathcal{P}_{M^2} \backslash \{P_{z,v_j}\}) \cup \{P_{z,z}\}$, where $P_{z,z}$ is added only if $z = v_l$. Thus, the couple $(M^{2*}, \mathcal{P}_{M^{2*}})$ is optimal for $(C_{i,n}, B^2)$. Applying Lemma 2 to $(M^{2*}, \mathcal{P}_{M^{2*}})$, we obtain,

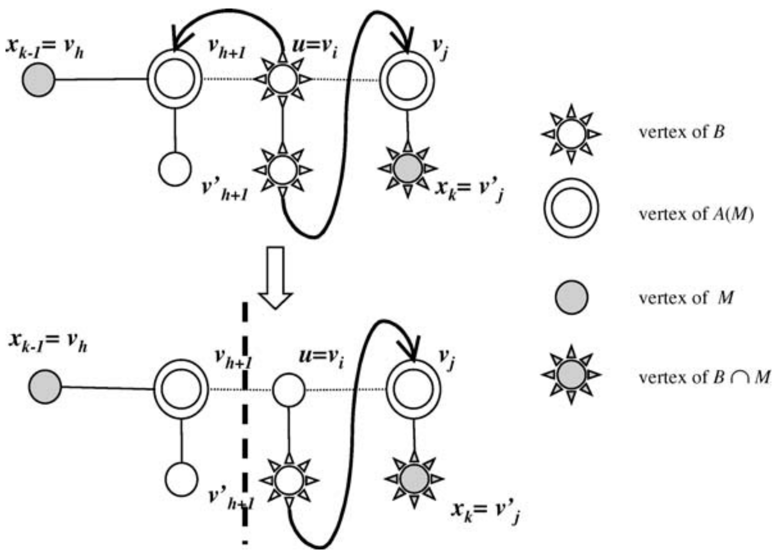


Fig. 19 Example of Case B of Theorem 8 when $h + 1 \leq i \leq j$.

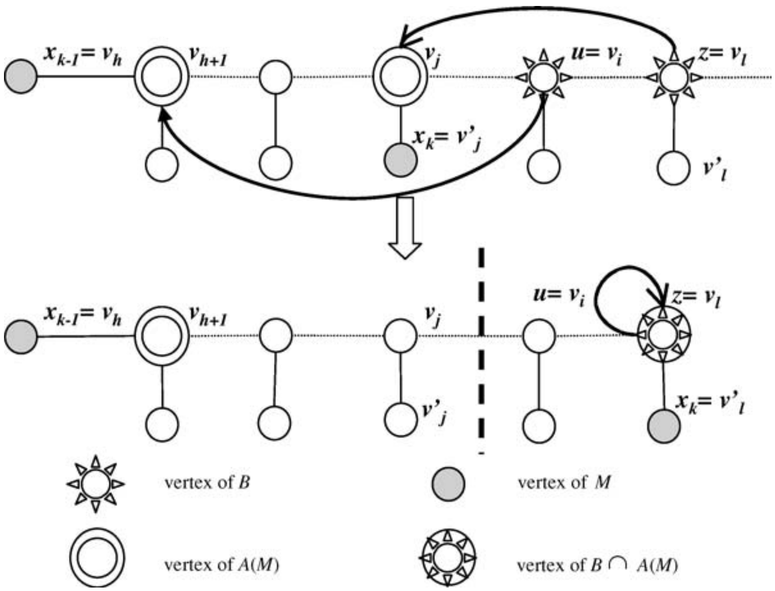


Fig. 20 Example of Case B of Theorem 8 when $i > j$.

as for *Case A* a new couple $(\hat{M}, \mathcal{P}_{\hat{M}})$ which is optimal for *DPP* on (C, B) and that contains k spinal vertices. $\square$

Acknowledgments The authors thank the two anonymous referees whose comments led to a significant improvement in the contents and the presentation of the paper.

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