Annals of Operations Research 137, 211–227, 2005

The Constrained Equal Awards Rule for Bankruptcy Problems with a Priori Unions

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Abstract. In this paper, we provide two extensions of the constrained equal awards rule for bankruptcy situations to the class of bankruptcy situations with a priori unions. We present some characterisations and relations with corresponding games. The two new extensions are illustrated by a specific application.

Keywords: bankruptcy problems, a priori unions, constrained equal awards rule

1. Introduction

In many situations in which agents interact, they do so in groups. Cooperative game theory studies such situations by taking into account what each particular coalition of players can achieve on its own. These values of the coalitions are subsequently taken into account in determining a fair division of the value of the grand coalition between all players. Often, however, some coalitions play a special role, in that they arise in a natural way from the underlying situation. If these naturally arising groups form a partition of the grand coalition, they are usually referred to as a priori unions.

One interesting class of problems in which the role of a priori unions has been studied is the class of bankruptcy problems. In a bankruptcy problem, there is an estate to be divided among a number of claimants, whose total claim exceeds the estate available. In many situations, these claimants can be divided in a priori unions, based on the nature or cause of their claims. E.g., when a firm goes bankrupt, the creditors can usually be

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grouped in a natural way by distinguishing between claims on the basis of outstanding bonds, equity or commercial transactions. The main focus of the bankruptcy literature is on finding rules assigning to each bankruptcy situation an allocation of the estate, which satisfies some appealing properties. This branch of cooperative game theory was initiated by O'Neill (1982) and has gained in popularity over the years. A recent survey about this topic can be found in Thomson (2003). One natural way to analyse the class of bankruptcy situations with a priori unions is to extend well-known standard bankruptcy rules to this class. E.g., Casas-Méndez et al. (2003) extend the adjusted proportional rule by considering a two-stage procedure in which the estate is first divided among the unions, and subsequently the amount that each union receives is divided among its members.

In this paper, we present two extensions of the constrained equal awards (*CEA*) rule. The first extension involves a similar two-stage procedure as in Casas-Méndez et al. (2003). We relate this extension to the *CEA* solution of a corresponding TU game with a priori unions, which is inspired by Owen (1977). We provide two characterisations of this two-stage extension, inspired by previous results by Dagan (1996) and Herrero and Villar (2002). The second extension of the *CEA* rule is based on the random arrival rule introduced in O' Neill (1982) and it is characterised by a consistency property. We illustrate and compare our two extensions of the *CEA* rule by applying them to the bankruptcy case of the Pacific Gas and Electric Company.

The outline of the paper is as follows. In Section 2, we formally define the class of bankruptcy situations with a priori unions and some related concepts that are used throughout the paper. In Section 3, the problem of extending standard bankruptcy rules is addressed and the first extension is presented. In Section 4, we provide the two characterisations of the two-stage extension of the *CEA* rule. Section 5 contains the second extension and deals with the concept of consistency. Finally, in Section 6 we present the application.

2. Bankruptcy with a priori unions

A bankruptcy problem arises when there is an estate to be divided and this estate is not enough to satisfy all the claims on it. In this kind of problems the question is how to divide the available estate among all the claimants.

We model a *bankruptcy situation* by a triple (N, E, c), where $N = \{1, \dots, n\}$ is the set of players (creditors), $E \in \mathbb{R}_+$ represents the estate (the available resources of the debtor) and $c = (c_1, \dots, c_n) \in \mathbb{R}^N_+$ is the vector of claims of the creditors. We assume $\sum_{i \in N} c_i \geq E$, so the estate is insufficient to meet all the claims.

By B^N we denote the set of all bankruptcy problems with creditor set N. A bankruptcy rule is a function $f : B^N \to \mathbb{R}^N$ that allocates to every bankruptcy problem (N, E, c) a vector $f(N, E, c) \in \mathbb{R}^N$ such that for all $i \in N$, $0 \leq f_i(N, E, c) \leq c_i$ (f is reasonable) and $\sum_{i \in N} f_i(N, E, c) = E$ (f is efficient). In this paper, our main focus is the *constrained equal awards* (CEA) *rule*, which, for $(N, E, c) \in B^N$, is defined for all

$i \in N$ by $CEA_i(N, E, c) = \min\{\lambda, c_i\}$, where λ is such that $\sum_{i \in N} \min\{\lambda, c_i\} = E$. This rule awards the same amount to all claimants with the restriction that no player can get more than his claim. The constrained equal awards is used by different authors, among others Dagan (1996) and Herrero and Villar (2002), who provide different axiomatic characterisations.

A *cooperative game with transferable utility* (or *TU game*) is a pair (N, v), where $N = \{1, \cdots, n\}$ is the set of players, and $v : 2^N \to \mathbb{R}$ is the characteristic function that assigns to each coalition $S \subset N$ its worth $v(S)$. By convention, $v(\emptyset) = 0$. We denote the class of TU games with player set N by TU^N. A solution concept is a function $f : TU^N \to \mathbb{R}^N$ that assigns to every TU game $(N, v) \in TU^N$ an allocation $f(N, v) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(N, v) = v(N)$. The *core* of a game (N, v) is given by $C(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \forall S \subset N : \sum_{i \in S} x_i \geq v(S)\}$ and a game (N, v) is called *exact* (cf. Driessen and Tijs (1985)) if for each $S \subset N$, $S \neq \emptyset$, there exists an $x^S \in C(v)$ such that $\sum_{i \in S} x_i^S = v(S)$.

A cooperative game with transferable utility with a priori unions is a triple $(N, v, \mathcal{P})$ where (N, v) is a standard TU game and $\mathcal{P} = \{P_k\}_{k \in R}$ is a partition of the set of players, R being the set of unions. For $(N, v, \mathcal{P})$, we define the corresponding TU game among the unions $(R, v^{\mathcal{P}})$, the quotient game, where $v^{\mathcal{P}}(L) = v(\cup_{k \in L} P_k)$ for all $L \subset R$.

For every bankruptcy problem (N, E, c), O' Neill (1982) defines an associated *bankruptcy game* $(N, v_{E,c})$. In this game, the value of a coalition S is the part of the estate that remains after paying the creditors in $N \backslash S$ all their claims, that is, $v_{E,c}(S) = \max\{E - \sum_{i \in N \backslash S} c_i, 0\}$ for all $S \subset N$.

Curiel, Maschler, and Tijs (1987) study this class of games. They call a bankruptcy rule *game-theoretic* if the solution of a situation only depends on the game. So, for a game-theoretic $f : B^N \to \mathbb{R}^N$, we can find a function $F : TU^N \to \mathbb{R}^N$ such that $f(N, E, c) = F(N, v_{E,c})$ for all bankruptcy problems $(N, E, c) \in B^N$. In this paper, we only consider game-theoretic bankruptcy rules. It is known that the *CEA* rule is a game-theoretic rule.

We represent a *bankruptcy problem with a priori unions* by $(N, E, c, \mathcal{P})$ where (N, E, c) is a standard bankruptcy problem and $\mathcal{P} = \{P_k\}_{k \in R}$ is a partition of the set of players. We denote by BU^N the set of all bankruptcy problems with a priori unions and player set N.

Our aim is to define bankruptcy with a priori unions rules, that is, functions $\varphi : BU^N \to \mathbb{R}^N$ that assign to each bankruptcy problem with a priori unions $(N, E, c, \mathcal{P})$ a vector $\varphi(N, E, c, \mathcal{P}) \in \mathbb{R}^N$ such that for all $i \in N$, $0 \leq \varphi_i(N, E, c, \mathcal{P}) \leq c_i$ and $\sum_{i \in N} \varphi_i(N, E, c, \mathcal{P}) = E$.

If $(N, E, c, \mathcal{P}) \in BU^N$ is a bankruptcy problem with unions, we can define the corresponding bankruptcy problem among the unions $(R, E, c^{\mathcal{P}})$, the so-called *quotient problem*, where $c^{\mathcal{P}} = (c_k^{\mathcal{P}})_{k \in R}$ is the vector of total claims of the unions, so $c_k^{\mathcal{P}} = \sum_{i \in P_k} c_i$ for each union P_k of creditors. Note that $(R, E, c^{\mathcal{P}})$ is a well defined bankruptcy problem.

Multi-issue allocation situations are introduced in Calleja, Borm, and Hendrickx (2005). The basic idea behind this class of problems is that the agents do not simply have just a single claim on the estate, as in the standard bankruptcy model, but a number of claims, each of which results from a particular *issue*. The basic assumption is that these issues are dealt with in turn: as soon as money is distributed according to one particular issue, this issue must first be completed before the next one is considered.

A *multi-issue allocation situation* is a triple (N, E, C), where $N = \{1, \dots, n\}$ is the set of players, $E \in \mathbb{R}_+$ is the estate and $C \in \mathbb{R}_+^{R \times N}$ is the matrix of claims. Every row in C represents an issue and the set of issues is denoted by $R = \{1, \dots, r\}$. An element $c_{ki} \geq 0$ represents the amount that player $i \in N$ claims according to issue $k \in R$. If a player is not involved in a particular issue, his claim corresponding to that issue equals zero.

The claim matrix C is assumed to satisfy the following properties:

- Every issue gives rise to a claim: $\sum_{i \in N} c_{ki} > 0$ for all $k \in R$.
- Every player is involved in at least one issue: $\sum_{k \in R} c_{ki} > 0$ for all $i \in N$.
- The allocation problem is nontrivial: $\sum_{k \in R} \sum_{i \in N} c_{ki} \geq E$.

We can easily reinterpret a bankruptcy situation with a priori unions $(N, E, c, \mathcal{P})$ as a multi-issue allocation situation (N, E, C), where the issues correspond to the unions and every player is involved in exactly one issue.

An *ordering* of the players in N is a bijection $\sigma : \{1, \dots, n\} \to N$, where $\sigma(i)$ denotes which player in N is at position i. The set of all $n!$ permutations of N is denoted by $\Pi(N)$. Similarly, the set of permutations of the set of issues R is denoted by $\Pi(R)$.

In order to analyse multi-issue allocation situations, Calleja, Borm, and Hendrickx (2005) define two corresponding games, the proportional game and the queue game. In this paper, we consider a variation on the former: instead of dividing the estate proportional to the claims within the final issue to be handled, we apply an arbitrary bankruptcy rule f to this problem. Note that for all f, the resulting game is exact, but not necessarily convex. This procedure is illustrated in the following example.

Example 2.1. Consider the 4-creditor bankruptcy problem (N, E, c) with $E = 10$ and $c = (6, 2, 8, 5)$. Suppose that creditors 1 and 2 form a union and creditors 3 and 4 another one, that is, $\mathcal{P} = \{\{1, 2\}, \{3, 4\}\}$.

This situation gives rise to the 4-player multi-issue allocation problem (N, E, C) with $E = 10$ and the following claim matrix:

$$C = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 0 & 0 & 8 & 5 \end{bmatrix}.$$

Take $S = \{1, 3\}$. In order to determine $v^{CEA}(S)$, we first compute, for both $\tau \in \Pi(R)$, $g_S^{CEA}(\tau)$, the quantity that S receives if the issues are handled in order τ and the final issue is resolved using *CEA*:

τ	$g_S^{CEA}(\tau)$
1, 2	$6 + CEA_3(\{3,4\}, 2, (8,5)) = 7$
2, 1	$CEA_3(\{3,4\}, 10, (8,5)) = 5$

So, $v^{CEA}(S) = \min_{\tau \in \Pi(R)} g_S^{CEA}(\tau) = 5$. Similarly, taking $T = \{1, 4\}$, we obtain $v^{CEA}(T) = 5$, $v^{CEA}(S \cup T) = 8$ and $v^{CEA}(S \cap T) = 0$. Hence, $v^{CEA}(S) + v^{CEA}(T) > v^{CEA}(S \cup T) + v^{CEA}(S \cap T)$. So, although v^{CEA} is exact, it is not convex.

3. Extending bankruptcy rules: A two-step procedure

In this section, we consider a way to extend a bankruptcy rule to a rule for bankruptcy situations with a priori unions. We use the *CEA* rule to illustrate this extension. We also connect our *CEA* solution for a bankruptcy situation with a priori unions to the corresponding TU game with a priori unions.

If we want to divide the total estate among the creditors, one approach is to divide the estate among the unions first and second to divide the allocation of each union among the creditors of this union. Let $f : B^N \to \mathbb{R}^N$. We define the two-stage extension $\bar{f} : BU^N \to \mathbb{R}^N$ as follows. Let $(N, E, c, \mathcal{P}) \in BU^N$ be a bankruptcy problem with a priori unions. First, define $E_k^f = f_k(R, E, c^{\mathcal{P}})$ for all $k \in R$ and secondly, for $i \in P_k$, $\bar{f}_i(N, E, c, \mathcal{P}) = f_i(P_k, E_k^f, (c_j)_{j \in P_k})$.

The $\overline{CEA}$ rule for bankruptcy situations with a priori unions generalises the standard *CEA* rule for bankruptcy situations, in the sense that both $\overline{CEA}(N, E, c, \mathcal{P}^N)$ and $\overline{CEA}(N, E, c, \mathcal{P}^n)$ coincide with $CEA(N, E, c)$, where $\mathcal{P}^n$ is the discrete partition $\mathcal{P}^n = \{\{1\}, \ldots, \{n\}\}$ and $\mathcal{P}^N$ is the trivial partition $\mathcal{P}^N = \{N\}$. Also note that by construction, $\overline{CEA}_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k$ for all $k \in R$.

The $\overline{CEA}$ solution of a bankruptcy situation with a priori unions coincides with the *CEA* solution for a corresponding TU game with a priori unions, which we are going to define next.

First, recall that the *utopia vector* of a TU game (N, v), $M(v)$, is defined by $M_i(v) = v(N) - v(N \backslash \{i\})$ for all $i \in N$. This vector is used to define the *CEA solution* of the game, which, for $(N, v) \in TU^N$, is defined for all $i \in N$ by $CEA_i(N, v) = \min\{\lambda, M_i(v)\}$, where λ is such that $\sum_{i \in N} \min\{\lambda, M_i(v)\} = v(N)$.[1] This solution divides the worth of the total coalition, $v(N)$, among the players in such a way that all of them obtain the same amount with the restriction that no player can get more than his utopia payoff.

Now, let $(N, v, \mathcal{P})$ be a TU-game with a priori unions. The constrained equal awards solution of this game, $CEA(N, v, \mathcal{P})$ is defined in two steps. First, the payoff to each union $P_k \in \mathcal{P}$ equals $CEA(R, v^{\mathcal{P}})$, ie, the constrained equal awards solution of the quotient game.

In the second step, the payoff to each union is divided among its players. To do this, we consider for every player $i \in N$ his cooperation possibilities with the players outside i's union. We should note that a similar idea is used in Owen (1977), where a modification of the Shapley value for TU games with a priori unions is defined.

Let $P_k \in \mathcal{P}$ and let $i \in P_k$. The "claim" of player i is defined as his contribution to the coalition $\cup_{\ell \in R\setminus\{k\}} P_\ell \cup \{i\}$, that is, $M_i(v, \mathcal{P}) = v(\cup_{\ell \in R\setminus\{k\}} P_\ell \cup \{i\}) - v(\cup_{\ell \in R\setminus\{k\}} P_\ell)$. The constrained equal awards solution of the game $(N, v, \mathcal{P})$ for player $i \in P_k$ is then defined by

$$CEA_i(N, v, \mathcal{P}) = CEA_i(P_k, CEA_k(R, v^{\mathcal{P}}), (M_j(v, \mathcal{P}))_{j \in P_k}).$$

The $\overline{CEA}$ rule coincides with the CEA of the game $(N, v^{CEA}, \mathcal{P})$, as is shown in the following proposition, where v^{CEA} is the multi-issue allocation game obtained by applying the CEA rule in the last issue.

Proposition 3.1. For every bankruptcy problem with a priori unions $(N, E, c, \mathcal{P})$ we have that $\overline{CEA}(N, E, c, \mathcal{P}) = CEA(N, v^{CEA}, \mathcal{P})$.

Proof. Let $(N, E, c, \mathcal{P})$ be a bankruptcy problem with a priori unions. First, it follows of the definition of the game v^{CEA} that

$$v^{CEA}(\cup_{k \in L} P_k) = \max\left\{E - \sum_{i \in N \setminus \cup_{k \in L} P_k} c_i, 0\right\}$$

for all $L \subset R$ and hence, the games $(R, (v^{CEA})^{\mathcal{P}})$ and $(R, v_{E,c^{\mathcal{P}}})$ coincide. So,

$$CEA_k(R, (v^{CEA})^{\mathcal{P}}) = CEA_k(R, v_{E,c^{\mathcal{P}}}) = E_k^{CEA} = CEA_k(R, E, c^{\mathcal{P}})$$

for all $k \in R$.

Next, for $i \in P_k$, taking into account the definition of the game v^{CEA},

$$M_i(v^{CEA}, \mathcal{P}) = \begin{cases} CEA_i(P_k, E, (c_j)_{j \in P_k}) & \text{if } E \leq c_k^{\mathcal{P}}, \\ c_i & \text{if } E > c_k^{\mathcal{P}}. \end{cases}$$

From this, we obtain that

$$\begin{aligned} \overline{CEA}_i(N, E, c, \mathcal{P}) &= CEA_i(P_k, E_k^{CEA}, (c_j)_{j \in P_k}) = CEA_i(P_k, E_k^{CEA}, (M_j(v^{CEA}, \mathcal{P}))_{j \in P_k}) \\ &= CEA_i(P_k, CEA_k(R, (v^{CEA})^{\mathcal{P}}), (M_j(v^{CEA}, \mathcal{P}))_{j \in P_k}) = CEA_i(N, v^{CEA}, \mathcal{P}) \end{aligned}$$

for all $i \in P_k$ and the proof is concluded. □

Nevertheless, in general, $\overline{CEA}(N, E, c, \mathcal{P}) \neq CEA(N, v_{E,c}, \mathcal{P})$. This fact is illustrated in the following example.

Example 3.2. Consider the 3-creditor bankruptcy problem (N, E, c) with $E = 400$ and $c = (100, 100, 400)$. Suppose that creditor 1 forms a union and creditors 2 and

3 another one, that is, $\mathcal{P} = \{P_1, P_2\}$ with $P_1 = \{1\}$ and $P_2 = \{2, 3\}$. To find the *CEA* solution of the bankruptcy problem with unions $(N, E, c, \mathcal{P})$, we first consider the bankruptcy problem $(R, E, c^{\mathcal{P}})$ among the unions. We obtain that $CEA(R, E, c^{\mathcal{P}}) = (100, 300)$ and then $CEA(N, E, c, \mathcal{P}) = (100, 100, 200)$. By proposition 3.1. we have that $CEA(N, v^{CEA}, \mathcal{P}) = (100, 100, 200)$. To find the *CEA* solution of the game $(N, v_{E,c}, \mathcal{P})$ we first should consider the corresponding game among the unions $(R, v^{\mathcal{P}}_{E,c})$. It is then easily seen that $CEA(R, v^{\mathcal{P}}_{E,c}) = (100, 300)$ and then $CEA_1(N, v_{E,c}, \mathcal{P}) = 100$. To determine the allocation of $CEA_{23}(R, v^{\mathcal{P}}_{E,c})$ among players 2 and 3, we compute the utopia payoffs $M_2(v_{E,c}, \mathcal{P}) = 0$ and $M_3(v_{E,c}, \mathcal{P}) = 300$. Hence $CEA(N, v_{E,c}, \mathcal{P}) = (100, 0, 300)$.

4. Characterisations of the two-step constrained equal awards rule

In this section we use the axiomatic method to support the two-stage procedure considered in previous section. We provide two different combinations of axioms to characterize the $\overline{CEA}$ rule as defined in Section 3, extending two previous characterizations of the CEA rule for simple bankruptcy problems. Consider the following properties for a rule $\varphi : BU^N \to \mathbb{R}^N$.

Composition (COMP): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$, $\varphi(N, E, c, \mathcal{P}) = \varphi(N, E', c, \mathcal{P}) + \varphi(N, E - E', c - \varphi(N, E', c, \mathcal{P}), \mathcal{P})$ for all $0 \leq E' \leq E$.

This property considers the situation in which after the estate (E') has been divided among the agents, this estate is reevaluated and turns out to be a bigger amount (E). In these cases, we have two options. We can cancel the initial division and apply the rule to the new problem, or we can preserve the initial division and apply the rule to the increment of the estate by considering a new vector of claims, taking into account the quantities already received. The composition property says that both options should lead to the same result.

Path independence (PI): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$, $\varphi(N, E, c, \mathcal{P}) = \varphi(N, E, \varphi(N, E', c, \mathcal{P}), \mathcal{P})$ for all $E' \geq E$.

Here, the opposite situation is considered, one where the estate (E) is actually smaller than the one initially considered (E'). Then, we can apply the rule to the new problem or divide the new value by taking the initial divisions as claim vector. Path independence states that both ways of proceeding should result in the same vector of allocations.

Equal treatment within the unions (ET): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$ and for each two agents i, j of a union $P_k \in \mathcal{P}$ such that $c_i = c_j$, $\varphi_i(N, E, c, \mathcal{P}) = \varphi_j(N, E, c, \mathcal{P})$.

This property requires that agents of the same union with equal claims obtain equal payoffs.

Quotient problem property (QPP): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$ and for each union $P_k \in \mathcal{P}$, $\sum_{i \in P_k} \varphi_i(N, E, c, \mathcal{P}) = \varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$.

In a bankruptcy problem with unions we can consider the associated quotient problem where the unions negotiate about the division of the estate. After this, a negotiation within every union takes place. The quotient problem property states that the total gains of the agents of a union in the initial problem equal the gains of this union in the quotient problem. Note that if φ is the two-step extension $\overline{f}$ of a bankruptcy rule f, then $\varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k^f$. (Recall that $E_k^f = f_k(R, E, c^{\mathcal{P}})$ is the amount that union $k \in R$ gets in the quotient problem according to f.)

Invariance under claims truncation within the unions (ICT): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$ and for every player i of a union $P_k \in \mathcal{P}$ such that $c_i > \sum_{j \in P_k} \varphi_j(N, E, c, \mathcal{P})$, we have $\varphi(N, E, c, \mathcal{P}) = \varphi(N, E, c', \mathcal{P})$, where $c'_j = c_j$ for all $j \in N \setminus \{i\}$ and $c'_i = \sum_{j \in P_k} \varphi_j(N, E, c, \mathcal{P})$.

Suppose that the claim of an agent is greater than the total quantity that his union gets. Then ICT states that the awards of the agents are not affected if we replace the claim of this agent by the total payoff of his union.

Sustainability of creditors within the unions (SUS): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$ and for every player i who is *sustainable* within his union $P_k \in \mathcal{P}$, ie, $\sum_{j \in P_k} \min\{c_i, c_j\} \leq \varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$, we have $\varphi_i(N, E, c, \mathcal{P}) = c_i$.

This property establishes a protective criterion within the unions in the sense that small claims should be completely satisfied. The claim of agent i is considered sustainable within his union if the worth of this union in the quotient problem is enough to pay each agent in this union his claim, truncated by the claim of agent i.

Composition and path independence are in essence identical to the corresponding properties for bankruptcy rules. Equal treatment within the unions is a weak version of equal treatment of bankruptcy rules. Invariance under claims truncation within the unions and sustainability of creditors within the unions are natural extensions of other properties for bankruptcy rules to this context of a priori unions. Note that the quotient problem property implies that the rule involves some two-step procedure to obtain the solution.

In the following theorem we present the first characterisation of the $\overline{CEA}$ rule. This theorem is inspired by a similar result for the *CEA* rule for bankruptcy situations in Dagan (1996).

Theorem 4.1. The $\overline{CEA}$ rule is the unique rule for bankruptcy problems with a priori unions that satisfies equal treatment within the unions, composition, the quotient problem property and invariance under claims truncation within the unions.

Proof. *Existence:* Equal treatment within the unions and the quotient problem property are straightforward to show. We only prove composition. The proof of invariance under claims truncation within the unions follows similar lines.

Let $P_k \in \mathcal{P}$ and let $i \in P_k$. By definition of $\overline{CEA}$ we have that

$$\overline{CEA}_i(N, E, c, \mathcal{P}) = CEA_i\big(P_k, E_k^{CEA}, (c_j)_{j\in P_k}\big).$$

Consider now $0 \leq E' \leq E$. Then

$$\overline{CEA}_i(N, E', c, \mathcal{P}) = CEA_i\big(P_k, E_k^{CEA'}, (c_j)_{j\in P_k}\big),$$

with $E_k^{CEA'} = CEA_k(R, E', c^{\mathcal{P}})$. Define $c' = c - \overline{CEA}(N, E', c, \mathcal{P})$. Then we have

$$\overline{CEA}_i(N, E - E', c', \mathcal{P}) = CEA_i\big(P_k, CEA_k(R, E - E', (c')^{\mathcal{P}}), (c'_j)_{j\in P_k}\big).$$

Because the constrained equal awards rule for bankruptcy problems satisfies composition (Dagan, 1996), we have that

$$\begin{aligned} E_k^{CEA} - E_k^{CEA'} &= CEA_k(R, E, c^{\mathcal{P}}) - CEA_k(R, E', c^{\mathcal{P}}) \\ &= CEA_k(R, E - E', c^{\mathcal{P}} - CEA(R, E', c^{\mathcal{P}})) \\ &= CEA_k(R, E - E', (c')^{\mathcal{P}}). \end{aligned}$$

From the previous, it follows that

$$\begin{aligned} \overline{CEA}_i(N, E, c, \mathcal{P}) &= CEA_i\big(P_k, E_k^{CEA}, (c_j)_{j\in P_k}\big) \\ &= CEA_i\big(P_k, E_k^{CEA'}, (c_j)_{j\in P_k}\big) + CEA_i\big(P_k, E_k^{CEA} - E_k^{CEA'}, (c'_j)_{j\in P_k}\big) \\ &= \overline{CEA}_i(N, E', c, \mathcal{P}) + CEA_i\big(P_k, CEA_k(R, E - E', (c')^{\mathcal{P}}), (c'_j)_{j\in P_k}\big) \\ &= \overline{CEA}_i(N, E', c, \mathcal{P}) + \overline{CEA}_i(N, E - E', c', \mathcal{P}). \end{aligned}$$

Hence, we have that $\overline{CEA}$ satisfies composition.

Uniqueness: Let φ be a rule for BU^N satisfying ET, QPP, COMP and ICT. Let $(N, E, c, \mathcal{P})$ be a bankruptcy problem with unions and consider the quotient problem $(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$. Without loss of generality, suppose that $0 \leq c_1^{\mathcal{P}} \leq \cdots \leq c_r^{\mathcal{P}}$. In Proposition 1 of Dagan (1996) it is established that the constrained equal awards rule is the only rule for bankruptcy problems that satisfies the bankruptcy equivalents of ET, COMP and ICT. Since the quotient problem with $\mathcal{P}^R$ is basically a bankruptcy problem, it follows that $\varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k^{CEA}$ for all $k \in R$.

Now, we consider the first union $P_1 \in \mathcal{P}$. Suppose without loss of generality that $P_1 = \{1, \ldots, n_1\}$ and that $c_{11} \leq \cdots \leq c_{1n_1}$.

Step 1. If $0 \leq E \leq rc_{11}$, then $E_1^{CEA} \leq c_{11}$ and because of ICT, QPP and ET, $\varphi_i(N, E, c, \mathcal{P})$
$= \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$.

If $rc_{11} < E \leq rc_{11} + rc_{11}(1 - \frac{1}{n_1})$, then equality is established using COMP. Repeating the same construction, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$ if $0 \leq E \leq rn_1c_{11}$.

Step 2. If $rn_1c_{11} < E \leq rn_1c_{11} + r(c_{12} - c_{11})$, by COMP and Step 1 we have $\varphi(N, E, c, \mathcal{P}) = x + \varphi(N, E - rn_1c_{11}, c - x, \mathcal{P})$, where $x_i = \varphi_i(N, rn_1c_{11}, c, \mathcal{P}) = \overline{CEA}_i(N, rn_1c_{11}, c, \mathcal{P}) = c_{11}$ for all $i \in P_1$. Furthermore, $E - rn_1c_{11} \leq r(c_{12} - c_{11})$. So because of ICT and ET we have $\varphi_i(N, E - rn_1c_{11}, c - x, \mathcal{P}) = \overline{CEA}_i(N, E - rn_1c_{11}, c - x, \mathcal{P})$ for all $i \in P_1$ and hence, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$.

Repeating the same argument one can prove that $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$ if $0 \leq E \leq rn_1c_{11} + r(n_1 - 1)(c_{12} - c_{11})$.

Using the same arguments, we obtain that $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$ if $0 \leq E \leq rn_1c_{11} + r(n_1 - 1)(c_{12} - c_{11}) + \cdots + r(c_{1n_1} - c_{1,n_1-1}) = r(c_{11} + c_{12} + \cdots + c_{1n_1}) = rc_1^{\mathcal{P}}$.

Now, we consider the second union. We distinguish between two cases. If $E \leq rc_1^{\mathcal{P}}$, we can use the same arguments as in the first union to obtain $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_2$.

So, suppose that $E > rc_1^{\mathcal{P}}$. Because φ satisfies COMP, we have that

$$\varphi(N, E, c, \mathcal{P}) = \varphi\big(N, rc_1^{\mathcal{P}}, c, \mathcal{P}\big) + \varphi\big(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P}\big),$$

where $x = \varphi(N, rc_1^{\mathcal{P}}, c, \mathcal{P})$. By the previous case, $\varphi_i(N, rc_1^{\mathcal{P}}, c, \mathcal{P}) = \overline{CEA}_i(N, rc_1^{\mathcal{P}}, c, \mathcal{P})$ for all $i \in P_2$. With the second term, $\varphi(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P})$, we proceed as with the first union with estate $E - rc_1^{\mathcal{P}}$ and claims $c - x$ and we obtain $\varphi_i(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P}) = \overline{CEA}_i(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P})$ for all $i \in P_2$. Note that in the problem $(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P})$ all the members of P_1 obtain zero. Because $\overline{CEA}$ satisfies COMP, we have $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_2$.

Repeating the same arguments with all the unions, we conclude the statement. □

Our second characterisation is based on Herrero and Villar (2002). In order to give this result, we first present some lemmas.

Lemma 4.2. If φ is a rule for bankruptcy problems with unions that satisfies path independence and sustainability of creditors within the unions then for every bankruptcy problem with unions $(N, E, c, \mathcal{P})$ we have that $\varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k^{CEA}$ for all $k \in R$.

Proof. Let $\varphi : BU^N \to \mathbb{R}^N$ be a rule satisfying PI and SUS and let $(N, E, c, \mathcal{P}) \in BU^N$. Consider the associated quotient problem $(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$. Theorem 1 of Herrero and Villar (2002) states that the constrained equal awards rule is the only rule for bankruptcy problems that satisfies the bankruptcy equivalents of path independence and sustainability. From this, the statement readily follows. □

Lemma 1 of Herrero and Villar (2002) states that if a bankruptcy rule satisfies path independence and sustainability, then it satisfies equal treatment of equals. In a similar way we can establish the next result for a rule for bankruptcy problems with a priori unions.

Lemma 4.3. If a rule for bankruptcy problems with a priori unions satisfies the quotient problem property, path independence and sustainability within the unions then it satisfies equal treatment within the unions.

Now we can give our second axiomatic characterisation of the $\overline{CEA}$ rule.

Theorem 4.4. The $\overline{CEA}$ rule is the unique rule for bankruptcy problems with a priori unions that satisfies path independence, sustainability of creditors within the unions and the quotient problem property.

Proof. *Existence:* Sustainability of creditors within the unions and the quotient problem property are straightforward to show. The proof of path independence follows similar lines to the proof of composition and we omit it.

Uniqueness: Let φ be a rule for BU^N satisfying QPP, PI and SUS and let $(N, E, c, \mathcal{P}) \in BU^N$. Let $P_k \in \mathcal{P}$. We have to show that $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$. We use the following notation: $n_1^k = \max_{i \in P_k} c_i$, $N_1^k = \{i \in P_k \mid c_i = n_1^k\}$, $n_2^k = \max_{i \in P_k \setminus N_1^k} c_i$, $N_2^k = \{i \in P_k \mid c_i = n_2^k\}$.

Step 1. Suppose that, in the union P_k, the claims of the agents in $P_k \setminus N_1^k$ are sustainable. Then $\varphi_i(N, E, c, \mathcal{P}) = c_i$ for all $i \in P_k \setminus N_1^k$ because φ satisfies SUS. Now, we have that $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$ because φ satisfies ET (by Lemma 4.3 and by QPP and Lemma 4.3, $\sum_{i \in P_k} \varphi_i(N, E, c, \mathcal{P}) = E_k^{CEA}$.

Step 2. Suppose now that, in the union P_k, the claims of the agents in $P_k \setminus (N_1^k \cup N_2^k)$ are sustainable. Let $E' > E$ be such that $\varphi_k(R, E', c^{\mathcal{P}}, \mathcal{P}^R)$ is the minimum quantity that sustains the claims of $P_k \setminus N_1^k$ within union P_k, which is possible because of Lemma 4.3 and the basic properties of *CEA*. Let $c' = \varphi(N, E', c, \mathcal{P})$. By step 1, $c'_i = c_i$ for all $i \in P_k \setminus N_1^k$ and $c'_i = c'_j$ for all $i, j \in N_1^k \cup N_2^k$. Because φ and $\overline{CEA}$ satisfy PI, we have that $\varphi_i(N, E, c, \mathcal{P}) = \varphi_i(N, E, c', \mathcal{P})$ and $\overline{CEA}_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c', \mathcal{P})$ for all $i \in N$. By step 1, $\varphi_i(N, E, c', \mathcal{P}) = \overline{CEA}_i(N, E, c', \mathcal{P})$ for all $i \in P_k$ and hence, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$.

Repeating this procedure, we obtain $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$. □

5. Consistent two-step rules

In this section we define the second two-step extension of bankruptcy rules to bankruptcy situations with a priori unions. As in Section 3, we use the *CEA* rule to illustrate this new extension and hence we obtain a second extension of the *CEA* rule for bankruptcy problems to bankruptcy problems with a priori unions, that we will call RA^{CEA}. We also introduce a property of consistency that we subsequently use to characterise this extension. We should mention that the rule and property in this section are clearly inspired by concepts that appear in O'Neill (1982). These concepts are the random arrival rule (which we denote by RA) and the property of consistency.

Let f be a bankruptcy rule and let $(N, E, c, \mathcal{P})$ be a bankruptcy problem with a priori unions. Then we define the f-random arrival rule in the following way:

$$RA_i^f(N, E, c, \mathcal{P}) = \frac{1}{r!}\left[\sum_{\sigma\in\Pi(R)} f_i(P_k, E_\sigma, (c_j)_{j\in P_k})\right]$$

for all $i \in P_k$, where $E_\sigma = \max\{0, E - \sum_{\ell\in R, \sigma^{-1}(\ell)<\sigma^{-1}(k)} c_\ell^{\mathcal{P}}\}$.

The interpretation of this rule is similar to that of other solutions inspired by ideas of random arrival. Here, we suppose that the claims of the different unions are satisfied following a fixed order. If at the moment to allocate money to a particular union, the remaining estate is not enough to satisfy its total claim, we use the rule f to distribute within this union. So, the f-random arrival rule allocates to an agent the average of the amounts he obtains according to the previous procedure over all the possible orders on the unions.

Note that if $\mathcal{P} = \mathcal{P}^n$ we have $RA^f(N, E, c, \mathcal{P}^n) = RA(N, E, c)$, that is, in this boundary case, RA^f coincides with the random arrival rule for bankruptcy problems for every bankruptcy rule f. If $\mathcal{P} = \mathcal{P}^N$, the f-random arrival rule coincides with the rule f.

In the next example, we illustrate the *CEA*-random arrival rule.

Example 5.1. We compute RA^{CEA} in the bankruptcy situation with a priori unions of Example 3.2. If the claims of the union P_1 are satisfied first, then the creditors obtain (100, 100, 200), whereas if the claims of the union P_2 are satisfied first the creditors obtain (0, 100, 300). If we compute the average of the previous amounts we obtain that $RA^{CEA}(N, E, c, \mathcal{P}) = (50, 100, 250)$. Note that $RA^{CEA}(N, E, c, \mathcal{P})$ differs from both $\overline{CEA}(N, E, c, \mathcal{P})$ and $CEA(N, v_{E,c}, \mathcal{P})$.

Now, we define the property of consistency for bankruptcy with a priori unions rules which is based on the consistency property in O'Neill (1982). A bankruptcy with a priori unions rule φ is *consistent* if for every $(N, E, c, \mathcal{P})$, for each union $P_k \in \mathcal{P}$ and for each agent $i \in P_k$ we have

$$\varphi_i(N, E, c, \mathcal{P}) = \frac{1}{r}\left[\varphi_i(P_k, E', (c_j)_{j\in P_k}, \mathcal{P}^{P_k}) + \sum_{\ell\in R, \ell\neq k} \varphi_i(N\backslash P_\ell, E_{-\ell}, c_{-\ell}, \mathcal{P}_{-\ell})\right],$$

where $E' = \min\{E, c_k^{\mathcal{P}}\}$, $c_{-\ell} = (c_j)_{j\in N\backslash P_\ell}$, $E_{-\ell} = \max\{E - c_\ell^{\mathcal{P}}, 0\}$ and $\mathcal{P}_{-\ell}$ is the partition of the set $N\backslash P_\ell$ induced by $\mathcal{P}$.

So, a rule is consistent if in a bankruptcy problem with a priori unions it allocates to an agent the average of what he gets when the rule is applied to the problem restricted to his own union and the solutions of the $r-1$ bankruptcy situations in which the estate is the amount that remains when each of the other unions gets its maximum. Note that if $\mathcal{P} = \mathcal{P}^n$, this definition of consistency corresponds to O'Neill consistency. We should note that the property of consistency we introduce here is a different consistency assumption to the traditional one in the literature, that involves different sets of agents. Our consistency property is involving different sets of a priori unions.

Let f be a bankruptcy rule. We say that a consistent rule φ for bankruptcy problems with a priori unions is *f-consistent* if for every bankruptcy problem (N, E, c) we have that $\varphi(N, E, c, \mathcal{P}^N) = f(N, E, c)$. That is, φ is f-consistent if φ is consistent and it coincides with f when the a priori unions structure $\mathcal{P}$ is the boundary system $\mathcal{P}^N$.

The next theorem establishes, for a fixed bankruptcy rule f, the existence and uniqueness of an f-consistent rule. This result extends the O'Neill result of existence and uniqueness of a bankruptcy consistent rule; this unique rule is the random arrival rule.

Theorem 5.2. The f-random arrival rule RA^f is the unique f-consistent rule for bankruptcy problems with a priori unions.

Proof. Let f be a bankruptcy rule.
Existence: First we show that the f-random arrival rule, RA^f, is f-consistent. We know that for every bankruptcy problem (N, E, c), $RA^f(N, E, c, \mathcal{P}^N) = f(N, E, c)$. So, it remains to be shown that RA^f is consistent. Let $(N, E, c, \mathcal{P})$ a bankruptcy situation with a priori unions. Let $i \in P_k$. Define E_σ, E' and $E_{-\ell}$ as before define and $E_{-\ell,\sigma} = \max\{E_{-\ell} - \sum_{t\in R\backslash\{\ell\}:\sigma^{-1}(t)<\sigma^{-1}(k)} c_t^{\mathcal{P}}, 0\}$ for all $\sigma \in \Pi(R)$, $\ell \in R$. Then,

$$\begin{aligned} RA_i^f(N, E, c, \mathcal{P}) &= \frac{1}{r!}\sum_{\sigma\in\Pi(R)} f_i\big(P_k, E_\sigma, (c_j)_{j\in P_k}\big) \\ &= \frac{1}{r!}\Bigg[(r-1)!\, f_i\big(P_k, E', (c_j)_{j\in P_k}\big) \\ &\qquad + \sum_{\ell\in R, \ell\neq k}\ \sum_{\sigma\in\Pi(R\backslash\{\ell\})} f_i\big(P_k, E_{-\ell,\sigma}, (c_j)_{j\in P_k}\big)\Bigg] \end{aligned}$$

$$= \frac{1}{r}\left[f_i\big(P_k, E', (c_j)_{j\in P_k}\big) + \sum_{\ell\in R,\ell\neq k} \frac{1}{(r-1)!} \times \sum_{\sigma\in\Pi(R\setminus\{\ell\})} f_i\big(P_k, E_{-\ell,\sigma}, (c_j)_{j\in P_k}\big) \right]$$

$$= \frac{1}{r}\left[RA_i^f\big(P_k, E', (c_j)_{j\in P_k}, \mathcal{P}^{P_k}\big) + \sum_{\ell\in R,\,\ell\neq k} RA_i^f\big(N\setminus P_\ell, \max\big\{E - c_\ell^{\mathcal{P}}, 0\big\}, c_{-\ell}, \mathcal{P}_{-\ell}\big) \right].$$

Hence, RA^f is consistent and therefore f-consistent.
Uniqueness: Now we show that if φ is an f- consistent rule for bankruptcy problems with a priori unions then φ coincides with the f-random arrival rule RA^f. We show this by induction on the number of unions. If $r = 1$ then $\varphi(N, E, c, \mathcal{P}^N) = f(N, E, c) = RA^f(N, E, c)$ by the definition of f-consistency. Suppose that this holds for $r = m - 1$. For $r = m$, f-consistency implies that $\varphi(N\setminus P_\ell, \max\{E - c_\ell^{\mathcal{P}}, 0\}, c_{-\ell}, \mathcal{P}_{-\ell})$ is completely determined and hence we conclude that there is a unique f-consistent rule, which is the f-random arrival rule. □

O'Neill (1982) shows that the random arrival rule of a bankruptcy problem coincides with the Shapley value of the bankruptcy game associated. The next theorem extends this result by O'Neill to our context, in the sense that the RA-random arrival rule coincides with the Owen value (cf. Owen, 1977) of the corresponding bankruptcy game with a priori unions. We omit the proof that follows a similar line to the proof of the preceding theorem.

Theorem 5.3. If $(N, E, c, \mathcal{P})$ is a bankruptcy problem with a priori unions, then its RA-random arrival rule coincides with the Owen value of the associated bankruptcy game with a priori unions, that is,

$$RA^{RA}(N, E, c, \mathcal{P}) = Ow(N, v_{E,c}, \mathcal{P}).$$

Now, from the previous two theorems, we immediately obtain the next result.

Theorem 5.4. The only rule for bankruptcy problems with a priori unions satisfying random arrival-consistency is the Owen value of the associated bankruptcy games with a priori unions.

In Winter (1992) and Hamiache (1999), the Owen value is axiomatically characterised on the class of cooperative games with a priori unions by using two different properties of consistency. Note that in the current paper, we characterise the Owen value on the class of bankruptcy situations with a priori unions, using another consistency property that extends the O'Neill consistency property for bankruptcy problems.

6. An application

In this section we apply the two extensions of the *CEA* rule to one particular bankruptcy situation, the Pacific Gas and Electric Company, a fully owned subsidiary of PG&E Corporation and one of the largest combined natural gas and electricity utilities in the United States. Due to negative stocktaking they filed for reorganisation under Chapter 11 of the US Bankruptcy Code in a San Francisco bankruptcy court in 2001.

The debtor's 20 largest unsecured creditors are listed in the Table 1, which is taken from www.bankruptcydata.com

According to Table 1, the creditors claim money on the basis of three issues (nature of claims). So we can analyse this as a bankruptcy situation with three unions of creditors: $P_1 = \{1, 3, 5, 6\}$ related to bank bonds, $P_2 = \{2, 4, 7, 8, 9, 10, 11, 12, 13, 18, 19, 20\}$ related to power purchases and $P_3 = \{14, 15, 16, 17\}$ related to gas purchases. The total estate (E) to be allocated to unsecured creditors equals \$1,060,000,000.

Table 1
The debtor's 20 largest unsecured creditors.

#	Nature of claim	Claim (\$)
1	Bank bonds	2,207,250,000
2	Power purchases	1,966,000,000
3	Bank bonds	1,302,100,000
4	Power purchases	1,228,800,000
5	Bank bonds	938,461,000
6	Bank bonds	310,000,000
7	Power purchases	57,928,385
8	Power purchases	49,452,611
9	Power purchases	48,400,572
10	Power purchases	45,706,378
11	Power purchases	40,147,245
12	Power purchases	40,122,073
13	Power purchases	32,867,878
14	Gas purchases	29,523,530
15	Gas purchases	28,210,551
16	Gas purchases	24,718,334
17	Gas purchases	23,849,455
18	Power purchases	22,576,506
19	Power purchases	21,506,087
20	Power purchases	19,800,248

Table 2
The results of the three constrained equal awards rules.

#	CEA	$\overline{CEA}$	RA^{CEA}	Union
1	95,865,025	119,212,266	128,070,755	P_1
2	95,865,025	52,089,822	161,514,515	P_2
3	95,865,025	119,212,266	128,070,755	P_1
4	95,865,025	52,089,822	161,514,515	P_2
5	95,865,025	119,212,266	128,070,755	P_1
6	95,865,025	119,212,266	128,070,755	P_1
7	57,928,385	52,089,822	28,964,193	P_2
8	49,452,611	49,452,611	24,726,306	P_2
9	48,400,572	48,400,572	24,200,286	P_2
10	45,706,378	45,706,378	22,853,189	P_2
11	40,147,245	40,147,245	20,073,623	P_2
12	40,122,073	40,122,073	20,061,037	P_2
13	32,867,878	32,867,878	16,433,939	P_2
14	29,523,530	29,523,530	9,841,177	P_3
15	28,210,551	28,210,551	9,403,517	P_3
16	24,718,334	24,718,334	8,239,445	P_3
17	23,849,455	23,849,455	7,949,818	P_3
18	22,576,506	22,576,506	11,288,253	P_2
19	21,506,087	21,506,087	10,753,044	P_2
20	19,800,248	19,800,248	9,900,124	P_2

We compute the $\overline{CEA}$ and RA^{CEA} solutions -by using the definitions and a computer application- for the bankruptcy situation with the three unions and compare them with the solution obtained by applying the *CEA* rule to the same situation without the unions. The Table 2 shows the results, where all amounts have been rounded to the nearest integer.

The first conclusion of these results is that all two rules that take the unions into account are more favourable for P_1 and less favourable for P_2 than the *CEA* rule without unions. Since the idea behind constrained equal awards is that the smaller creditors are protected, it is better for the (smaller) claimants in P_2 to be considered as separate creditors than as one big group. Nevertheless, the larger claimants in P_2 are better off when the RA^{CEA} rule is applied, at the expense of the smaller claimants in the same union. Moreover, it is better for the (bigger) claimants in P_1 to be considered as a one group than as separate creditors.

The RA^{CEA} is worst for P_3, which contains only small claimants. The protective aspect of constrained equal awards is partly neutralised by taking averages over a number of extreme outcomes.

The RA^{CEA} solution is an average over more extreme outcomes than *CEA* and $\overline{CEA}$ solutions, and hence, the smaller creditors are again less protected and also bigger creditors are less damaged.

Acknowledgments

Balbina Casas-Méndez and Luisa Carpente acknowledge the financial support of *El Ministerio español de Ciencia y Tecnología*, FEDER and *Xunta de Galicia* through projects BEC2002-04102-C02-02 and PGIDIT03PXIC20701PN. The authors also thank helpful comments of Prof. Ignacio García-Jurado and two anonymous referees.

Note

1. The *CEA* rule for TU games is only well-defined for a subclass of such games. If the game is exact, then the *CEA* rule is well-defined. The same holds for the *CEA* rule for games with a priori unions, which we define later on, where exactness of the underlying game is sufficient.

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