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ON STRONG EXTENSION GROUPS OF CUNTZ–KRIEGER ALGEBRAS

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Abstract. In this paper, we study the strong extension groups of Cuntz–Krieger algebras, and present a formula to compute the groups. We also detect the position of the Toeplitz extension of a Cuntz–Krieger algebra in the strong extension group and in the weak extension group to see that the weak extension group with the position of the Toeplitz extension is a complete invariant of the isomorphism class of the Cuntz–Krieger algebra associated with its transposed matrix.

1. Preliminary

There are several kinds of extension groups $\mathrm{Ext}_*(\mathcal{A})$ for a C^*-algebra $\mathcal{A}$. Among them two extension groups $\mathrm{Ext}_{\mathrm{w}}(\mathcal{A})$ and $\mathrm{Ext}_{\mathrm{s}}(\mathcal{A})$ for a unital nuclear separable C^*-algebra $\mathcal{A}$ have been studying in many papers (see [2,4,7–9,13,14,16–18], etc.). In this paper, we study the strong extension groups $\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A)$ of Cuntz–Krieger algebras $\mathcal{O}_A$, and present a formula to compute the groups. We also detect the position of the Toeplitz extension $\mathcal{T}_A$ of a Cuntz–Krieger algebra $\mathcal{O}_A$ in the weak extension group $\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A)$ to show that it is a complete invariant of the isomorphism class of the Cuntz–Krieger algebra $\mathcal{O}_{A^t}$ for the transposed matrix A^t of A by using Rørdam's classification result.

In what follows, H stands for a separable infinite dimensional Hilbert space. Let us denote by $K(H)$ the C^*-algebra of compact operators on H. It is a closed two-sided ideal of the C^*-algebra $B(H)$ of bounded linear operators on H. The quotient C^*-algebra $B(H)/K(H)$ is called the Calkin algebra, denoted by $Q(H)$. The quotient map $B(H) \longrightarrow Q(H)$ is denoted by π.

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Let $\mathcal{A}$ be a unital separable C^*-algebra. Throughout the paper, a unital $*$-monomorphism $\tau\colon \mathcal{A} \longrightarrow Q(H)$ is called an extension. Two extensions $\tau_1, \tau_2\colon \mathcal{A} \longrightarrow Q(H)$ are said to be *strongly equivalent*, written $\tau_1 \sim_s \tau_2$, if there exists a unitary $U \in B(H)$ such that $\tau_1(a) = \pi(U)\tau_2(a)\pi(U^*)$ in $Q(H)$ for all $a \in \mathcal{A}$. They are said to be *weakly equivalent*, written $\tau_1 \sim_w \tau_2$, if there exists a unitary $u \in Q(H)$ such that $\tau_1(a) = u\tau_2(a)u^*$ in $Q(H)$ for all $a \in \mathcal{A}$. The strong equivalence class of an extension $\tau\colon \mathcal{A} \longrightarrow Q(H)$ is denoted by $[\tau]_s$, and similarly the weak equivalence class is denoted by $[\tau]_w$. We note that weakly equivalent extensions are strongly equivalent if one may take a unitary $u \in Q(H)$ of Fredholm index zero such that $\tau_1(a) = u\tau_2(a)u^*$ in $Q(H)$ for all $a \in \mathcal{A}$. An extension $\tau\colon \mathcal{A} \longrightarrow Q(H)$ is said to be *trivial* if there exists a unital $*$-monomorphism $\rho\colon \mathcal{A} \longrightarrow B(H)$ such that $\tau = \pi \circ \rho$. We regard $Q(H) \oplus Q(H) \subset Q(H \oplus H)$ in a natural way and identify $H \oplus H$ with H, so that $Q(H) \oplus Q(H) \subset Q(H)$. The sum of extensions $\tau_1, \tau_2\colon \mathcal{A} \longrightarrow Q(H)$ are defined by

$$(\tau_1 + \tau_2)(a) = \tau_1(a) \oplus \tau_2(a) \in Q(H) \oplus Q(H) \subset Q(H), \quad a \in \mathcal{A}$$

that gives rise to an extension $\tau_1 \oplus \tau_2\colon \mathcal{A} \longrightarrow Q(H)$. Let us denote by $\mathrm{Ext_s}(\mathcal{A})$ the set of strong equivalence classes of extensions. Similarly the set of weak equivalence classes is denoted by $\mathrm{Ext_w}(\mathcal{A})$. Both $\mathrm{Ext_s}(\mathcal{A})$ and $\mathrm{Ext_w}(\mathcal{A})$ have commutative semigroup structure by the above sums. There is a canonical surjective homomorphism $q_{\mathcal{A}}\colon \mathrm{Ext_s}(\mathcal{A}) \longrightarrow \mathrm{Ext_w}(\mathcal{A})$ of commutative semigroups defined by $q_{\mathcal{A}}([\tau]_s) = [\tau]_w$.

By virtue of Voiculescu's theorem in [20], the following basic lemma holds:

LEMMA 1.1 [20]. *Let $\mathcal{A}$ be a unital separable C^*-algebra. For any two trivial extensions $\tau_1, \tau_2\colon \mathcal{A} \longrightarrow Q(H)$, there exists a unitary $U \in B(H)$ such that $\tau_2 = \mathrm{Ad}(\pi(U)) \circ \tau_1$, that is, $\tau_1 \sim_s \tau_2$. The strong (resp. weak) equivalence class of a trivial extension is the neutral element of $\mathrm{Ext_s}(\mathcal{A})$ (resp. $\mathrm{Ext_w}(\mathcal{A})$).*

Choi–Effros in [5] (cf. [1]) proved that if $\mathcal{A}$ is nuclear, the semigroups $\mathrm{Ext_s}(\mathcal{A})$, $\mathrm{Ext_w}(\mathcal{A})$ become groups, that is, any element has its inverse. The following lemma is seen in [17].

LEMMA 1.2. *Let $\mathcal{A}$ be a unital separable nuclear C^*-algebra. For $m \in \mathbb{Z}$, take a unitary $u_m \in Q(H)$ of Fredholm index m. Take a trivial extension $\tau\colon \mathcal{A} \longrightarrow Q(H)$. Consider the extension $\sigma_m = \mathrm{Ad}(u_m) \circ \tau\colon \mathcal{A} \longrightarrow Q(H)$. Then the map $\iota_{\mathcal{A}}\colon m \in \mathbb{Z} \longrightarrow [\sigma_m] \in \mathrm{Ext_s}(\mathcal{A})$ gives rise to a homomorphism of groups such that the sequence*

$$\mathbb{Z} \xrightarrow{\iota_{\mathcal{A}}} \mathrm{Ext_s}(\mathcal{A}) \xrightarrow{q_{\mathcal{A}}} \mathrm{Ext_w}(\mathcal{A}). \tag{1.1}$$

is exact at the middle, that is, $\iota_A(\mathbb{Z}) = \mathrm{Ker}(q_{\mathcal{A}})$, so that

$$\mathrm{Ext_s}(\mathcal{A})/\iota_{\mathcal{A}}(\mathbb{Z}) \cong \mathrm{Ext_w}(\mathcal{A}).$$

The groups $\mathrm{Ext}_{\mathrm{s}}(\mathcal{A})$ and $\mathrm{Ext}_{\mathrm{w}}(\mathcal{A})$ for a unital separable nuclear C^*-algebra $\mathcal{A}$ are called the *strong extension group* for $\mathcal{A}$ and the *weak extension group* for $\mathcal{A}$, respectively.

Let $e \in Q(H)$, $E \in B(H)$ be projections such that $e = \pi(E)$. For an element $x \in Q(H)$ such that $exe \in eQ(H)e$ is invertible in $eQ(H)e$, one may denote by $\mathrm{ind}_e\, x$ the Fredholm index $\mathrm{ind}_E\, X$ for $X \in B(EH)$ satisfying $x = \pi(X)$. As the Fredholm index is invariant under compact perturbations, the integer $\mathrm{ind}_e\, x$ does not depend on the choice of E and X as long as $e = \pi(E)$, $x = \pi(X)$. The following lemma is well-known (cf. [8, Lemma 5.1]).

LEMMA 1.3. *Let $e, f \in Q(H)$ be projections. Suppose that $x \in Q(H)$ commutes with e and f, and exe, fxf are invertible in $eQ(H)e$ and $fQ(H)f$, respectively.*

(i) *If $ef = 0$, then $\mathrm{ind}_{e+f}\, x = \mathrm{ind}_e\, x + \mathrm{ind}_f\, x$.*

(ii) *If $x, y \in eQ(H)e$ are both invertible in $eQ(H)e$, then $\mathrm{ind}_e\, xy = \mathrm{ind}_e\, x + \mathrm{ind}_e\, y$.*

2. Ext-groups for Cuntz–Krieger algebras

Let $A = [A(i,j)]_{i,j=1}^N$ be an irreducible non permutation matrix with entries in $\{0,1\}$ with $N > 1$. The Cuntz–Krieger algebra $\mathcal{O}_A$ is defined to be the universal C^*-algebra generated by N partial isometries $S_1, \ldots, S_N$ subject to the operator relations (see [8]):

$$\sum_{j=1}^N S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \quad i = 1, \ldots N \tag{2.1}$$

It is a nuclear C^*-algebra uniquely determined by the operator relations (2.1) (see [8]). If the entries of A are all one's, the C^*-algebra $\mathcal{O}_A$ is called the Cuntz algebra written $\mathcal{O}_N$ ([6]).

In [8], Cuntz–Krieger pointed out the C^*-algebras $\mathcal{O}_A$ are closely related to dynamical properties of underlying topological Markov shifts. Among other things, they proved that the weak extension group $\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A)$ is isomorphic to the abelian group $\mathbb{Z}^N/(I - A)\mathbb{Z}^N$. The group $\mathbb{Z}^N/(I - A)\mathbb{Z}^N$ is known as the Bowen–Franks group that is a crucial invariant under flow equivalence class of the underlying two-sided topological Markov shift (see [3]). We note that the group $\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A)$ was written as $\mathrm{Ext}(\mathcal{O}_A)$ in the Cuntz–Krieger's paper [8]. For the Cuntz algebra $\mathcal{O}_N$, both of the groups $\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_N)$ and $\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_N)$ had been computed as $\mathbb{Z}$ and $\mathbb{Z}/(1 - N)\mathbb{Z}$, respectively by Pimsner–Popa [17] and Paschke–Salinas [16].

In this paper, we will compute the strong extension group $\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A)$ for $\mathcal{O}_A$ and present a formula (2.3) stated in the theorem below. For $n = 1, \dots, N$, let $R_n = [R_n(i,j)]_{i,j=1}^N$ be the $N \times N$ matrix defined by

$$R_n(i,j) = \begin{cases} 1 & \text{if } i = n, \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

meaning that the only nth row is the vector $[1,\dots,1]$ but the other rows are zero vectors. The homomorphisms $\iota_{\mathcal{A}} \colon \mathbb{Z} \longrightarrow \mathrm{Ext}_{\mathrm{s}}(\mathcal{A})$ and $q_{\mathcal{A}} \colon \mathrm{Ext}_{\mathrm{s}}(\mathcal{A}) \longrightarrow \mathrm{Ext}_{\mathrm{w}}(\mathcal{A})$ in (1.1) for $\mathcal{A} = \mathcal{O}_A$ are denoted by $\iota_A \colon \mathbb{Z} \longrightarrow \mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A)$ and $q_A \colon \mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A) \longrightarrow \mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A)$, respectively.

THEOREM 2.1 (Theorem 2.6 and Theorem 3.3). (i) *The strong extension group* $\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A)$ *for the Cuntz–Krieger algebra* $\mathcal{O}_A$ *is*

$$\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A) = \mathbb{Z}^N/(1 - \widehat{A})\mathbb{Z}^N \tag{2.3}$$

where the matrix $\widehat{A}$ *is* $\widehat{A} = A + R_1 - AR_1$.

(ii) *The homomorphism* $\iota_A \colon \mathbb{Z} \longrightarrow \mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A)$ *in* (1.1) *is injective if*

$$\det(I - A) \neq 0.$$

Hence the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota_A} \mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A) \xrightarrow{q_A} \mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A) \longrightarrow 0 \tag{2.4}$$

holds if $\det(I - A) \neq 0$.

The given proof in this paper for the formula (2.3), presented as Theorem 2.6, basically follows the proof of [8, Theorem 5.3] that showed the formula $\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A) = \mathbb{Z}^N/(I - A)\mathbb{Z}^N$.

Among various extensions of the Cuntz–Krieger algebra $\mathcal{O}_A$, there is one specific extension called the Toeplitz extension $\sigma_{\mathcal{T}_A}$ of $\mathcal{O}_A$. It arises from the short exact sequence

$$0 \longrightarrow K(H_A) \xrightarrow{\iota} \mathcal{T}_A \xrightarrow{q} \mathcal{O}_A \longrightarrow 0 \tag{2.5}$$

of the Toeplitz algebra $\mathcal{T}_A$ on the sub Fock space H_A (cf. [10,12]). We will detect the positions of the Toeplitz extension $\sigma_{\mathcal{T}_A}$ of $\mathcal{O}_A$ in the strong extension group $\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A)$ and in the weak extension group $\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A)$ (Theorem 4.4). As a result, we will know that the group $\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A)$ with the position $[\mathcal{T}_A]_w$ of the Toeplitz extension $\sigma_{\mathcal{T}_A}$ in $\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A)$ is a complete invariant of the isomorphism class of the Cuntz–Krieger algebra $\mathcal{O}_{A^t}$ for its transposed matrix A^t of A by using Rørdam's classification result (Corollary 4.5).

Let us denote by P_i the projection $S_iS_i^*$. Let $\sigma\colon \mathcal{O}_A \longrightarrow Q(H)$ be an extension. Put $e_i = \sigma(P_i)$. There exists a trivial extension $\tau\colon \mathcal{O}_A \longrightarrow Q(H)$ such that $\tau(P_i) = \sigma(P_i)$, $i = 1, \dots, N$. As the partial isometry $\sigma(S_i)\tau(S_i^*)$ commutes with e_i, $e_i\sigma(S_i)\tau(S_i^*)e_i$ becomes a unitary in $e_iQ(H)e_i$. One may define $\mathrm{ind}_{e_i}\, \sigma(S_i)\tau(S_i^*)$ denoted by $d_i(\sigma,\tau)$, that is

$$d_i(\sigma,\tau) = \mathrm{ind}_{e_i}\, \sigma(S_i)\tau(S_i^*), \quad i = 1, \dots, N.$$

The proof of [8, Proposition 5.2] describes the following lemma. We give its proof for the sake of completeness.

LEMMA 2.2 [8, Proposition 5.2]. *Let $\sigma\colon \mathcal{O}_A \longrightarrow Q(H)$ be an extension. Put $e_i = \sigma(P_i)$. Let $\tau_1, \tau_2\colon \mathcal{O}_A \longrightarrow Q(H)$ be trivial extensions such that $\tau_j(P_i) = \sigma(P_i)$, $j = 1, 2$, $i = 1, \dots, N$. Then there exists a vector $[k_i]_{i=1}^N \in \mathbb{Z}^N$ such that*

(i) $d_i(\sigma,\tau_2) = d_i(\sigma,\tau_1) - k_i + \sum_{j=1}^N A(i,j)k_j$,

(ii) $\sum_{i=1}^N k_i = 0$.

PROOF. By Lemma 1.1, one may find a unitary $U \in B(H)$ such that $\tau_2(x) = \pi(U)\tau_1(x)\pi(U^*)$, $x \in \mathcal{O}_A$. Put $u = \pi(U) \in Q(H)$. Since

$$(e_iue_i)(e_iue_i)^* = \tau_2(P_i)\pi(U)\tau_1(P_i)\pi(U^*)\tau_2(P_i) = \tau_2(P_i)\tau_2(P_i)\tau_2(P_i) = e_i$$

and similarly $(e_iue_i)^*(e_iue_i) = e_i$, we see that e_iue_i is a unitary in $e_iQ(H)e_i$. By putting $k_i = \mathrm{ind}_{e_i}\, u$, the equality

$$d_i(\sigma,\tau_2) = d_i(\sigma,\tau_1) - k_i + \sum_{j=1}^N A(i,j)k_j \tag{2.6}$$

holds, following the proof of [8, Proposition 5.2]. In fact, we see that

$$\begin{aligned} d_i(\sigma,\tau_2) &= \mathrm{ind}_{e_i}\, \sigma(S_i)\tau_2(S_i^*) = \mathrm{ind}_{e_i}\, \sigma(S_i)\sigma(S_i^*S_i)u\tau_1(S_i^*S_i)\tau_1(S_i^*)u^* \\ &= \mathrm{ind}_{e_i}\, \sigma(S_i)\tau_1(S_i^*S_i)u\tau_1(S_i^*S_i)\tau_1(S_i^*)\tau_1(S_iS_i^*)u^* \\ &= \mathrm{ind}_{e_i}\, \sigma(S_i)\tau_1(S_i^*)\Big(\tau_1(S_i)\sum_{j=1}^N A(i,j)u\tau_1(S_jS_j^*)\tau_1(S_i^*)\Big)e_iu^* \\ &= \mathrm{ind}_{e_i}\, \sigma(S_i)\tau_1(S_i^*)\Big(\tau_1(S_i)\sum_{j=1}^N A(i,j)e_jue_j\tau_1(S_i^*)\Big)e_iu^*e_i \\ &= \mathrm{ind}_{e_i}\, \sigma(S_i)\tau_1(S_i^*) + \mathrm{ind}_{e_i}\Big(\tau_1(S_i)\sum_{j=1}^N A(i,j)e_jue_j\tau_1(S_i^*)\Big) + \mathrm{ind}_{e_i}\, u^* \end{aligned}$$

$$= d_i(\sigma, \tau_1) + \sum_{j=1}^{N} A(i,j)\,\mathrm{ind}_{e_i}\,\tau_1(S_i)e_j u e_j \tau_1(S_i^*) - k_i.$$

As $\mathrm{ind}_{e_i}\tau_1(S_i)e_j u e_j \tau_1(S_i^*) = \mathrm{ind}_{e_j}\, u = k_j$ whenever $A(i,j) = 1$, we obtain equality (2.6).

Since $u = \pi(U)$ for some unitary U on H, Lemma 1.3 tells us

$$\sum_{i=1}^{N} k_i = \sum_{i=1}^{N} \mathrm{ind}_{e_i}\, u = \mathrm{ind}_{\sum_{i=1}^{N} e_i}\, u = \mathrm{ind}\, U = 0. \quad \square$$

Define a subgroup $\mathrm{Im}(I - A)_0 \subset \mathbb{Z}^N$ by setting

$$\mathrm{Im}(I - A)_0 = \Big\{ (I - A)[k_i]_{i=1}^N \in \mathbb{Z}^N \mid [k_i]_{i=1}^N \in \mathbb{Z}^N \text{ with } \sum_{i=1}^{N} k_i = 0 \Big\}.$$

We thus see that an extension $\sigma\colon \mathcal{O}_A \longrightarrow Q(H)$ defines an element of $\mathbb{Z}^N/\mathrm{Im}(I - A)_0$ in a unique way by

$$d_\sigma := [d_i(\sigma, \tau)]_{i=1}^N \in \mathbb{Z}^N/\mathrm{Im}(I - A)_0$$

for a trivial extension $\tau\colon \mathcal{O}_A \longrightarrow Q(H)$ satisfying $\tau(P_i) = \sigma(P_i)$, $i = 1, \dots, N$.

LEMMA 2.3. *Let $\sigma_1, \sigma_2\colon \mathcal{O}_A \longrightarrow Q(H)$ be extensions. If $\sigma_1 \sim_s \sigma_2$, then $d_{\sigma_1} = d_{\sigma_2}$ in $\mathbb{Z}^N/\mathrm{Im}(I - A)_0$.*

PROOF. Assume that $\sigma_1 \sim_s \sigma_2$ so that, by Lemma 1.1, one may find a unitary V in $B(H)$ such that $\sigma_2 = \mathrm{Ad}(\pi(V)) \circ \sigma_1$. Put $e_i^1 = \sigma_1(P_i)$, $e_i^2 = \sigma_2(P_i)$, $i = 1, \dots, N$ and $v = \pi(V) \in Q(H)$, and hence $v e_i^1 v^* = e_i^2$. Take a trivial extension $\tau_1\colon \mathcal{O}_A \longrightarrow Q(H)$ such that $\tau_1(P_i) = e_i^1$, $i = 1, \dots, N$. We set $\tau_2 = \mathrm{Ad}(v) \circ \tau_1$ so that $\tau_2(P_i) = v\tau_1(P_i)v^* = e_i^2$. We then have

$$d_i(\sigma_2, \tau_2) = \mathrm{ind}_{e_i^2}\, \sigma_2(S_i)\tau_2(S_i^*) = \mathrm{ind}_{v e_i^1 v^*}\, v\sigma_1(S_i)\tau_1(S_i^*)v^* = d_i(\sigma_1, \tau_1). \quad \square$$

Let us define $d_s\colon \mathrm{Ext_s}(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N/\mathrm{Im}(I - A)_0$ by setting

$$d_s([\sigma]_s) = \big[[d_i(\sigma, \tau)]_{i=1}^N \big] \in \mathbb{Z}^N/\mathrm{Im}(I - A)_0$$

for a trivial extension $\tau\colon \mathcal{O}_A \longrightarrow Q(H)$ satisfying $\tau(P_i) = \sigma(P_i)$, $i = 1, \dots, N$.

PROPOSITION 2.4. *$d_s\colon \mathrm{Ext_s}(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N/\,\mathrm{Im}(I - A)_0$ is an isomorphism of groups.*

PROOF. It is obvious that $d_s\colon \mathrm{Ext_s}(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N/\,\mathrm{Im}(I - A)_0$ is a homomorphism of groups. It remains to show that d_s is bijective. We will first show that d_s is injective. Let $\sigma\colon \mathcal{O}_A \longrightarrow Q(H)$ be an extension such that $d_s([\sigma]_s) = 0$ in $\mathbb{Z}^N/\operatorname{Im}(I-A)_0$. Take a trivial extension τ such that $\tau(P_i) = \sigma(P_i)$, $i = 1, \dots, N$. Put $d_i = d_i(\sigma, \tau) \in \mathbb{Z}$. Let $\rho_\tau \colon \mathcal{O}_A \longrightarrow B(H)$ be a unital $*$-monomorphism such that $\tau = \pi \circ \rho_\tau$. By the assumption, there exists $[k_i]_{i=1}^N \in \mathbb{Z}^N$ such that

$$[d_i]_{i=1}^N = (I-A)[k_i]_{i=1}^N, \quad \sum_{i=1}^N k_i = 0.$$

Put $e_i = \tau(P_i)$ and $E_i = \rho_\tau(P_i)$ so that $\pi(E_i) = e_i$. Take an isometry or coisometry $V_i \in B(E_i H)$ such that $\operatorname{ind}(V_i) = -k_i$. Put $V = \sum_{i=1}^N V_i \in B(H)$ and $v = \pi(V)$. Since v is a unitary in $Q(H)$ such that $\operatorname{ind}(v) = \sum_{i=1}^N \operatorname{ind}_{E_i}(V_i) = -\sum_{i=1}^N k_i = 0$, one may take a unitary U in $B(H)$ such that $v = \pi(U)$. By following the proof of [8, Theorem 5.3], we have

$$\begin{aligned}
&\operatorname{ind}_{e_i} \pi(U)\sigma(S_i)\pi(U^*)\tau(S_i^*) \\
&= \operatorname{ind}_{e_i} \pi(V_i)\sigma(S_i)\sigma(S_i^* S_i)\pi\Big(\sum_{n=1}^N V_n^*\Big)\tau(S_i^*) \\
&= \operatorname{ind}_{e_i} \pi(V_i)\sigma(S_i)\Big(\sum_{j=1}^N A(i,j)\pi(E_j)\Big)\pi\Big(\sum_{n=1}^N E_n V_n^*\Big)\tau(S_i^*) \\
&= \operatorname{ind}_{e_i} \pi(V_i)\sigma(S_i)\sigma(S_i^* S_i)\pi\Big(\sum_{j=1}^N A(i,j)V_j^*\Big)\tau(S_i^*) \\
&= \operatorname{ind}_{e_i} \pi(V_i)\sigma(S_i)\tau(S_i^*)\Big(\tau(S_i)\pi\Big(\sum_{j=1}^N A(i,j)V_j^*\Big)\tau(S_i^*)\Big) \\
&= \operatorname{ind}_{e_i} \pi(V_i) + \operatorname{ind}_{e_i} \sigma(S_i)\tau(S_i^*) + \operatorname{ind}_{e_i} \tau(S_i)\pi\Big(\sum_{j=1}^N A(i,j)V_j^*\Big)\tau(S_i^*) \\
&= -k_i + d_i + \sum_{j=1}^N A(i,j) \operatorname{ind}_{e_i} \tau(S_i)\pi(V_j^*)\tau(S_i^*).
\end{aligned}$$

Since $\operatorname{ind}_{e_i} \tau(S_i)\pi(V_j^*)\tau(S_i^*) = \operatorname{ind}_{e_j} \pi(V_j^*) = k_j$ whenever $A(i,j) = 1$, we have

$$\operatorname{ind}_{e_i} \pi(U)\sigma(S_i)\pi(U^*)\tau(S_i^*) = -k_i + d_i + \sum_{j=1}^N A(i,j)k_j = 0$$

so that there exists a unitary $W_i \in B(E_iH)$ on E_iH such that

$$\pi(U)\sigma(S_i)\pi(U^*)\tau(S_i^*) = \pi(W_i), \quad i = 1, \dots, N.$$

By putting $T_i = W_i\rho_\tau(S_i)$, $i = 1, \dots, N$, we have

$$\sum_{j=1}^N T_jT_j^* = \sum_{j=1}^N W_j\rho_\tau(S_j)\rho_\tau(S_j^*)W_j^* = \sum_{j=1}^N W_jW_j^* = \sum_{j=1}^N E_j = 1,$$

and

$$T_i^*T_i = \rho_\tau(S_i^*)W_i^*W_i\rho_\tau(S_i) = \rho_\tau(S_i^*)\rho_\tau(S_iS_i^*)\rho_\tau(S_i) = \sum_{j=1}^N A(i,j)\rho_\tau(S_jS_j^*).$$

As $\rho_\tau(S_jS_j^*) = T_jT_j^*$, we see that $T_i^*T_i = \sum_{j=1}^N A(i,j)T_jT_j^*$. Since $\pi(U) = \pi(V) = \sum_{k=1}^N \pi(V_k)$ and $V_k \in B(E_kH)$, we have

$$\pi(U^*)\tau(S_i^*S_i) = \pi(V^*)\sum_{j=1}^N A(i,j)\tau(S_jS_j^*)$$

$$= \sum_{j=1}^N A(i,j)\sum_{k=1}^N \pi(V_k^*E_j) = \sum_{j=1}^N A(i,j)\pi(V_j^*)$$

and

$$\tau(S_i^*S_i)\pi(U^*) = \sum_{j=1}^N A(i,j)\tau(S_jS_j^*)\pi(V^*)$$

$$= \sum_{j=1}^N A(i,j)\sum_{k=1}^N \pi(E_jV_k^*) = \sum_{j=1}^N A(i,j)\pi(V_j^*)$$

so that $\pi(U^*)\tau(S_i^*S_i) = \tau(S_i^*S_i)\pi(U^*)$.

Define $\rho_\sigma(S_i) = T_i \in B(H)$, $i = 1, \dots, N$ so that $\rho_\sigma\colon \mathcal{O}_A \longrightarrow Q(H)$ is a unital $*$-monomorphism such that

$$\begin{aligned}\pi \circ \rho_\sigma(S_i) &= \pi(W_i\rho_\tau(S_i)) = \pi(U)\sigma(S_i)\pi(U^*)\tau(S_i^*)\pi(\rho_\tau(S_i))\\ &= \pi(U)\sigma(S_i)\pi(U^*)\tau(S_i^*)\tau(S_i)\\ &= \pi(U)\sigma(S_i)\tau(S_i^*S_i)\pi(U^*) = \pi(U)\sigma(S_i)\pi(U^*).\end{aligned}$$

Hence we have $\mathrm{Ad}(\pi(U)) \circ \sigma = \pi \circ \rho_\sigma$. This shows that σ is strongly equivalent to the trivial extension $\pi \circ \rho_\sigma$ proving $[\sigma]_s = 0$ in $\mathrm{Ext_s}(\mathcal{O}_A)$.

We will next show that d_s is surjective. We will show that there exists an extension $\sigma\colon \mathcal{O}_A \longrightarrow Q(H)$ and a trivial extension $\tau\colon \mathcal{O}_A \longrightarrow Q(H)$ such that $\tau(P_i) = \sigma(P_i)$ denoted by e_i and

$$\operatorname{ind}_{e_i} \sigma(S_i)\tau(S_i^*) = \begin{cases} -1 & \text{if } i = 1, \\ 0 & \text{otherwise}. \end{cases} \tag{2.7}$$

Decompose the Hilbert space H as $H = H_1 \oplus \cdots \oplus H_N$ such that $\dim H_i = \dim H$, $i = 1, \ldots, N$. Take a nonzero vector $v_1 \in H_1$ and put its orthogonal complement $H_1^0 = \{\mathbb{C}v_1\}^\perp \cap H_1$ in H_1. Let E_i be the orthogonal projection onto H_i, $i = 1, \ldots, N$. The orthogonal projection onto H_1^0 is denoted by E_1^0, so that $\sum_{i=1}^N E_i = 1$ and $E_1 - E_1^0$ is the projection onto $\mathbb{C}v_1$. Take partial isometries $T_1, \ldots, T_N$ and $V_1, \ldots, V_N$ on H such that

$$T_1T_1^* = E_1^0, \quad T_iT_i^* = E_i, \quad i = 2, \ldots, N, \quad V_iV_i^* = E_i, \quad i = 1, \ldots, N,$$

$$T_i^*T_i = V_i^*V_i = \sum_{j=1}^N A(i,j)E_j, \quad i = 1, \ldots, N.$$

We know that

$$\pi(T_i)\pi(T_i)^* = \pi(V_i)\pi(V_i)^* = \pi(E_i), \quad \sum_{i=1}^N \pi(E_i) = 1,$$

$$\pi(T_i)^*\pi(T_i) = \pi(V_i)^*\pi(V_i) = \sum_{j=1}^N A(i,j)\pi(E_j), \quad i = 1, \ldots, N.$$

By setting $\sigma(S_i) = \pi(T_i)$, $\tau(S_i) = \pi(V_i)$, $i = 1, \ldots, N$, we have extensions $\sigma, \tau\colon \mathcal{O}_A \longrightarrow Q(H)$ such that τ is a trivial extension. Put $e_i = \pi(E_i)$, $i = 1, \ldots, N$. Since $\sigma(S_i)\tau(S_i^*) = \pi(T_iV_i^*)$, $i = 1, \ldots, N$, we have

$$\operatorname{ind}_{e_i} \sigma(S_i)\tau(S_i^*) = \operatorname{ind}_{E_i} T_iV_i^*$$

so that the equality (2.7) holds. Therefore we have $d_s([\sigma]_s) = [(-1, 0, \ldots, 0)]$ in $\mathbb{Z}^N / \operatorname{Im}(I - A)_0$. One may show that $d_s\colon \operatorname{Ext_s}(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N / \operatorname{Im}(I - A)_0$ is surjective by a similar fashion. □

Recall that the $N \times N$ matrix R_n for $n = 1, \ldots, N$ is defined in (2.2).

Lemma 2.5. *For $n = 1, 2, \ldots, N$, put $\widehat{A}_n = A + R_n - AR_n$. Then we have*

$$\operatorname{Im}(I - A)_0 = (I - \widehat{A}_n)\mathbb{Z}^N. \tag{2.8}$$

In particular for $n = 1$, we put $\widehat{A} = \widehat{A}_1$ so that we have $\operatorname{Im}(I - A)_0 = (I - \widehat{A})\mathbb{Z}^N$.

Proof. As $\operatorname{Im}(I-A)_0 = \big\{(I-A)[k_i]_{i=1}^N \mid \sum_{i=1}^N k_i = 0\big\}$, a vector $[k_i]_{i=1}^N \in \mathbb{Z}^N$ satisfies $\sum_{i=1}^N k_i = 0$ if and only if $[k_i]_{i=1}^N = (I-R_n)[k_i]_{i=1}^N$. Hence we have

$$\operatorname{Im}(I-A)_0 = (I-A)(I-R_n)\mathbb{Z}^N.$$

Since $(I-A(I-R_n) = I-\widehat{A}_n$, we have $\operatorname{Im}(I-A)_0 = (I-\widehat{A}_n)\mathbb{Z}^N$. $\square$

Therefore we reach the following theorem.

Theorem 2.6. $\operatorname{Ext_s}(\mathcal{O}_A) \cong \mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N$, *where* $\widehat{A} = A + R_1 - AR_1$.

3. The homomorphism $\iota_A\colon \mathbb{Z} \longrightarrow \operatorname{Ext}_s(\mathcal{O}_A)$

For $m \in \mathbb{Z}$, take $k_1,\dots,k_N \in \mathbb{Z}$ such that $m = \sum_{j=1}^N k_j$. Take trivial extensions $\tau,\tau'\colon \mathcal{O}_A \longrightarrow Q(H)$ such that $\tau(P_i) = \tau'(P_i)$ denoted by e_i, $i = 1,\dots,N$. Let $\rho_\tau, \rho_{\tau'}\colon \mathcal{O}_A \longrightarrow B(H)$ be unital $*$-monomorphisms such that $\tau = \pi\circ\rho_\tau$, $\tau' = \pi\circ\rho_{\tau'}$, respectively. Put $E_i = \rho_\tau(P_i)$ so that $\pi(E_i) = e_i$. Take an isometry or coisometry $V_i \in B(E_iH)$ such that $\operatorname{ind}_{E_i} V_i = k_i$ and put $V = \sum_{i=1}^N V_i \in B(H)$. Hence we see that

$$\operatorname{ind}_{e_i}\pi(V) = k_i, \quad i = 1,\dots,N.$$

Recall that the extension $\sigma_m\colon \mathcal{O}_A \longrightarrow Q(H)$ is defined by setting $\sigma_m = \operatorname{Ad}(\pi(V))\circ\tau\colon \mathcal{O}_A \longrightarrow Q(H)$. Put $d_i = d_i(\sigma_m,\tau') = \operatorname{ind}_{e_i}\sigma_m(S_i)\tau'(S_i^*)$. Then

$$d_s([\sigma_m]_s) = [(d_1,\dots,d_N)] \in \mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N$$

does not depend on the choice of trivial extensions τ,τ', because of Lemma 2.2 and Lemma 2.3.

Proposition 3.1. *Define* $\hat{\iota}_A\colon \mathbb{Z} \longrightarrow \mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N$ *by setting* $\hat{\iota}_A(m) = \big[(I-A)[k_i]_{i=1}^N\big]$ *for* $m = \sum_{i=1}^N k_i$. *Then we have*

(i) $\hat{\iota}_A(m) = [(I-A)[k_i]_{i=1}^N]$ *does not depend on the choice of* $[k_i]_{i=1}^N$ *as long as* $m = \sum_{i=1}^N k_i$.

(ii) *The diagram*

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\ \iota_A\ } & \operatorname{Ext_s}(\mathcal{O}_A) \\ \Big\downarrow = & & \Big\downarrow d_s \\ \mathbb{Z} & \xrightarrow{\ \hat{\iota}_A\ } & \mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N \end{array}$$

is commutative, that is $d_s(\iota_A(m)) = \hat{\iota}_A(m)$, *where* $\iota_A(m) = [\sigma_m]_s$.

(iii) *The position* $\hat{\iota}_A(1)$ *in* $\mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N$ *is invariant under the isomorphism class of* $\mathcal{O}_A$.

(iv) *If* $\det(I-A)\neq 0$, *then we have a short exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota_A} \operatorname{Ext_s}(\mathcal{O}_A) \xrightarrow{q_A} \operatorname{Ext_w}(\mathcal{O}_A) \longrightarrow 0. \tag{3.1}$$

Proof. (i) Suppose that $m=\sum_{i=1}^N k_i=\sum_{i=1}^N k_i'$ for some $k_i,k_i'\in\mathbb{Z}$. Put $l_i=k_i-k_i'$ so that $\sum_{i=1}^N l_i=0$ and

$$(I-A)[k_i]_{i=1}^N-(I-A)[k_i']_{i=1}^N=(I-A)[l_i]_{i=1}^N\in(I-\widehat{A})\mathbb{Z}^N$$

by Lemma 2.5. This shows that

$$[(I-A)[k_i]_{i=1}^N]=[(I-A)[k_i']_{i=1}^N]$$

in $\mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N$.

(ii) Keep the notation stated before Proposition 3.1. Since

$$d_i=\operatorname{ind}_{e_i}\sigma_m(S_i)\tau'(S_i^*)$$

does not depend on the choice of a trivial extension $\tau'\colon\mathcal{O}_A\longrightarrow Q(H)$ as long as $\tau(P_i)=\tau'(P_i)$, we may take τ' as τ. We then have

$$\begin{aligned}d_i&=\operatorname{ind}_{e_i}\sigma_m(S_i)\tau(S_i^*)=\operatorname{ind}_{e_i}\pi(V)\tau(S_i)\pi(V^*)\tau(S_i^*)\\&=\operatorname{ind}_{e_i}\pi(V)+\operatorname{ind}_{e_i}\tau(S_i)\pi(V^*)\tau(S_i^*)=k_i+\operatorname{ind}_{\tau(S_i^*P_iS_i)}\pi(V^*)\\&=k_i+\sum_{j=1}^N A(i,j)\operatorname{ind}_{\tau(P_j)}\pi(V^*)=k_i-\sum_{j=1}^N A(i,j)k_j\end{aligned}$$

so that we obtain

$$d_s(\iota_A(m))=d_s([\sigma_m]_s)=[d_i]_{i=1}^N=[(I-A)[k_i]_{i=1}^N]=\hat{\iota}_A(m).$$

(iii) By construction, the map $\iota_{\mathcal{A}}\colon m\in\mathbb{Z}\longrightarrow[\sigma_m]_s\in\operatorname{Ext_s}(\mathcal{A})$ as well as the position $\iota_{\mathcal{A}}(1)$ in $\operatorname{Ext_s}(\mathcal{A})$ is invariant under the isomorphism class of a C^*-algebra $\mathcal{A}$. For $\mathcal{A}=\mathcal{O}_A$, the assertion (ii) says that

$$\big(\operatorname{Ext_s}(\mathcal{O}_A),\iota_A(1)\big)\cong\big(\mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N,\hat{\iota}_A(1)\big)$$

so that the position of $\hat{\iota}_A(1)$ in the group $\mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N$ is invariant under the isomorphism class of $\mathcal{O}_A$.

(iv) Assume that $\det(I-A)\neq 0$. Let $m\in\mathbb{Z}$ satisfy $\iota_A(m)=0$. Take $k_1,\dots,k_N\in\mathbb{Z}$ such that $m=\sum_{i=1}^N k_i$ and hence $\hat{\iota}_A(m)=[(I-A)[k_i]_{i=1}^N]$.

As $\hat{\iota}_A(m) = d_s(\iota_A(m)) = 0$, there exists $[n_i]_{i=1}^N \in \mathbb{Z}^N$ such that $\sum_{i=1}^N n_i = 0$ and $\hat{\iota}_A(m) = (I - A)[n_i]_{i=1}^N$. We then have

$$(I - A)\big[k_i\big]_{i=1}^N = (I - A)\big[n_i\big]_{i=1}^N.$$

By the assumption $\det(I - A) \neq 0$, we have $[n_i]_{i=1}^N = [k_i]_{i=1}^N$ so that $m = \sum_{i=1}^N n_i = 0$. $\square$

Since $I - \widehat{A} = (I - A)(I - R_1)$, the inclusion relation

$$(I - \widehat{A})\mathbb{Z}^N \subset (I - A)\mathbb{Z}^N$$

holds. There exists a natural quotient map

$$\hat{q}_A \colon \mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N \longrightarrow \mathbb{Z}^N/(I - A)\mathbb{Z}^N.$$

In [8], Cuntz–Krieger proved that the map

$$d_w \colon \operatorname{Ext}_{\mathrm{w}}(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N/(I - A)\mathbb{Z}^N$$

defined by $d_w([\sigma]_w) = [(d_1, \dots, d_N)] \in \mathbb{Z}^N/(I - A)\mathbb{Z}^N$ yields an isomorphism of groups.

Let us denote by $\operatorname{Ker}(I - A)$, $\operatorname{Ker}(I - \widehat{A})$ the subgroups of $\mathbb{Z}^N$ defined by the kernels in $\mathbb{Z}^N$ of the matrices $I - A$ and of $I - \widehat{A}$, respectively. Define homomorphisms of groups

$$i_1 \colon \mathbb{Z} \longrightarrow \operatorname{Ker}(I - \widehat{A}), \quad j_A \colon \operatorname{Ker}(I - \widehat{A}) \longrightarrow \operatorname{Ker}(I - A),$$
$$s_A \colon \operatorname{Ker}(I - A) \longrightarrow \mathbb{Z}$$

by setting

$$i_1 : n \longrightarrow \begin{bmatrix} n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad j_A \colon [l_i]_{i=1}^N \longrightarrow \begin{bmatrix} -\sum_{i=2}^N l_i \\ l_2 \\ \vdots \\ l_N \end{bmatrix}, \quad s_A \colon [l_i]_{i=1}^N \longrightarrow \sum_{i=1}^N l_i.$$

LEMMA 3.2. *We have the following long exact sequence*:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\ i_1\ } & \operatorname{Ker}(I - \widehat{A}) & \xrightarrow{\ j_A\ } & \operatorname{Ker}(I - A) \\ & & & & & & \big\downarrow s_A \\ 0 & \longleftarrow & \mathbb{Z}^N/(I - A)\mathbb{Z}^N & \xleftarrow{\ \hat{q}_A\ } & \mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N & \xleftarrow{\ \hat{\iota}_A\ } & \mathbb{Z} \end{array}$$

PROOF. It suffices to show the exactness at the lower right corner

$$\mathrm{Ker}(I-A) \xrightarrow{s_A} \mathbb{Z} \xrightarrow{\hat{\iota}_A} \mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N. \tag{3.2}$$

Suppose that $m \in \mathbb{Z}$ satisfies $\hat{\iota}_A(m) = 0$. Take $k_1, \dots, k_N \in \mathbb{Z}$ such that $m = \sum_{i=1}^N k_i$ and hence $(I-A)[k_i]_{i=1}^N$ belongs to $\mathrm{Im}(I-A)_0$. There exists $[n_i]_{i=1}^N \in \mathbb{Z}^N$ such that $(I-A)[k_i]_{i=1}^N = (I-A)[n_i]_{i=1}^N$ and $\sum_{i=1}^N n_i = 0$. Put $l_i = k_i - n_i$. Hence $[l_i]_{i=1}^N \in \mathrm{Ker}(I-A)$ and $\sum_{i=1}^N l_i = \sum_{i=1}^N k_i = m$ so that $s_A([l_i]_{i=1}^N) = m$, proving $\mathrm{Ker}(\hat{\iota}_A) \subset s_A(\mathrm{Ker}(I-A))$.

Conversely, for $[l_i]_{i=1}^N \in \mathrm{Ker}(I-A)$, we have

$$\hat{\iota}_A\big(s_A([l_i]_{i=1}^N)\big) = \hat{\iota}_A\left(\sum_{i=1}^N l_i\right) = \big[(I-A)[l_i]_{i=1}^N\big] = 0,$$

so that $s_A(\mathrm{Ker}(I-A)) \subset \mathrm{Ker}(\hat{\iota}_A)$. Hence the sequence (3.2) is exact at the middle. Exactness at the other places is easily seen. □

THEOREM 3.3. (i) *The isomorphisms*

$$d_w\colon \mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N/(I-A)\mathbb{Z}^N, \quad d_s\colon \mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N$$

of groups and a homomorphism $\hat{\iota}_A\colon \mathbb{Z} \longrightarrow \mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N$ *defined by* $\hat{\iota}_A(m) = (I-A)[k_i]_{i=1}^N$ *with* $m = \sum_{i=1}^N k_i$ *yield the commutative diagram:*

$$\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{\iota_A} & \mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A) & \xrightarrow{q_A} & \mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A) \\
\big\downarrow = & & \big\downarrow d_s & & \big\downarrow d_w \\
\mathbb{Z} & \xrightarrow{\hat{\iota}_A} & \mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N & \xrightarrow{\hat{q}_A} & Z^N/(I-A)\mathbb{Z}^N.
\end{array}$$

(ii) *The pair* $\big(\mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N, \hat{\iota}_A(1)\big)$ *showing the position*

$$\hat{\iota}_A(1) = \left[(I-A)\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right]$$

in the group $\mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N$ *is invariant under the isomorphism class of* $\mathcal{O}_A$.

(iii) *The homomorphism* $\iota_A\colon \mathbb{Z} \longrightarrow \mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A)$ *is injective if* $\det(I-A) \neq 0$. *In this case, we have a short exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota_A} \mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_A) \xrightarrow{q_A} \mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_A) \longrightarrow 0. \tag{3.3}$$

4. Toeplitz extension

Among various extensions of $\mathcal{O}_A$, there is a specific extension $\sigma_{\mathcal{T}_A}$ of $\mathcal{O}_A$ called the Toeplitz extension (cf. [10,12]). We fix an irreducible non permutation matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0,1\}$. Let $\mathbb{C}^N$ be an N-dimensional Hilbert space with orthonormal basis $\{\xi_1, \dots, \xi_N\}$. Let H_0 be a one-dimensional Hilbert space with unit vector v_0. Let $H^{\otimes n}$ be the n-fold tensor product $\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N$. Consider the full Fock space $F_N = H_0 \oplus \big(\bigoplus_{n=1}^\infty H^{\otimes n}\big)$. Define a sub Fock space H_A to be the closed linear span of vectors

$$\{v_0\} \cup \big\{\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \mid A(i_j, i_{j+1}) = 1 \text{ for } j = 1, \dots, n-1,\ n = 1, 2, \dots\big\}.$$

Define creation operators T_i for $i = 1, \dots, N$ on H_A by

$$T_i v_0 = \xi_i,$$

$$T_i(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \begin{cases} \xi_i \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} & \text{if } A(i, i_1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by E_0 the rank one projection onto the subspace H_0 on H_A. The operators T_i, $i = 1, \dots, N$ on H_A are partial isometries satisfying the relations

$$\sum_{j=1}^N T_j T_j^* = 1 - E_0, \quad T_i^* T_i = \sum_{j=1}^N A(i,j) T_j T_j^* + E_0, \quad i = 1, \dots N \tag{4.1}$$

(see [10,12]). The Toeplitz algebra for the matrix A is defined to be the C^*-algebra $C^*(T_1, \dots, T_N)$ on H_A generated by the partial isometries T_i, $i = 1, \dots, N$. By (4.1), we know that the correspondence $S_i \in \mathcal{O}_A \longrightarrow \pi(T_i) \in Q(H_A) = B(H_A)/K(H_A)$ gives rise to a unital $*$-monomorphism, that is called the Toeplitz extension denoted by $\sigma_{\mathcal{T}_A}$. In this section, we will detect the positions $d_s([\sigma_{\mathcal{T}_A}]_s)$ in $\mathrm{Ext_s}(\mathcal{O}_A)$ and $d_w([\sigma_{\mathcal{T}_A}]_w)$ in $\mathrm{Ext_w}(\mathcal{O}_A)$, respectively. The classes $[\sigma_{\mathcal{T}_A}]_s$ and $[\sigma_{\mathcal{T}_A}]_w$ are simply denoted by $[\mathcal{T}_A]_s$ and $[\mathcal{T}_A]_w$, respectively.

For $j = 1, \dots, N$, let $H_{A,j}$ be the closed linear subspace of H_A spanned by the vectors $\{\xi_j \otimes \eta \in H_A \mid \eta \in H_A\}$, so that $H_A = H_0 \oplus H_{A,1} \oplus \cdots \oplus H_{A,N}$. Let us denote by $E_{A,i}$ the projection on H_A onto the subspace $H_{A,i}$. We then see that $E_0 + \sum_{j=1}^N E_{A,j} = 1$ and

$$T_i T_i^* = E_{A,i}, \quad T_i^* T_i = E_0 + \sum_{j=1}^N A(i,j) E_{A,j}, \quad i = 1, \dots N. \tag{4.2}$$

We fix $m \in \{1, \dots, N\}$ for a while. By setting

$$H_j := \begin{cases} H_{A,j} \oplus H_0 & \text{if } j = m, \\ H_{A,j} & \text{if } j \neq m, \end{cases}$$

we have a decomposition $H_A = H_1 \oplus \cdots \oplus H_N$ of H_A depending on m. Let us denote by E_i the orthogonal projection on H_A onto the subspace H_i, so that we have $\sum_{j=1}^N E_j = 1$. Take a family of partial isometries $V_1, \dots, V_N$ on H_A satisfying the relations

$$V_i V_i^* = E_i, \quad V_i^* V_i = \sum_{j=1}^N A(i,j) E_j, \quad i = 1, \dots N. \tag{4.3}$$

Lemma 4.1. *For a fixed $m \in \{1, \dots, N\}$, we have for $i = 1, \dots, N$,*

$$E_i = \begin{cases} E_{A,i} + E_0 & \textit{if } i = m, \\ E_{A,i} & \textit{if } i \neq m, \end{cases} \quad V_i^* V_i = \begin{cases} T_i^* T_i & \textit{if } A(i,m) = 1, \\ T_i^* T_i - E_0 & \textit{if } A(i,m) = 0. \end{cases}$$

For $i = 1, \dots, N$, the operator $T_i E_0 T_i^*$ on H_A is a rank one projection on H_A onto the one-dimensional subspace spanned by the vector ξ_i. We note that the operator $T_i V_i^* \colon H_i \longrightarrow H_i$ is a (not necessarily onto) partial isometry. We then have

Lemma 4.2. *For $i = 1, \dots, N$, we have $(T_i V_i^*)^* T_i V_i^* = V_i V_i^* = E_i$ and*

$$T_i V_i^* (T_i V_i^*)^* = \begin{cases} E_i - E_0 & \textit{if } i = m, \ A(i,m) = 1, \\ E_i - E_0 - T_i E_o T_i^* & \textit{if } i = m, \ A(i,m) = 0, \\ E_i & \textit{if } i \neq m, \ A(i,m) = 1, \\ E_i - T_i E_o T_i^* & \textit{if } i \neq m, \ A(i,m) = 0. \end{cases} \tag{4.4}$$

Since the partial isometries V_i, $i = 1, \dots, N$ on H_A satisfy (2.1), there exists a unital $*$-monomorphism $\tau_m \colon \mathcal{O}_A \longrightarrow B(H_A)$ satisfying $\tau_m(S_i) = V_i$, $i = 1, \dots, N$, so that $\pi \circ \tau_m \colon \mathcal{O}_A \longrightarrow Q(H_A)$ is a trivial extension. The above lemma says the following proposition.

Proposition 4.3. *For a fixed $m \in \{1, \dots, N\}$, we have*

$$d_i(\sigma_{\mathcal{T}_A}, \tau_m) = \begin{cases} -1 & \textit{if } i = m, \ A(i,m) = 1, \\ -2 & \textit{if } i = m, \ A(i,m) = 0, \\ 0 & \textit{if } i \neq m, \ A(i,m) = 1, \\ -1 & \textit{if } i \neq m, \ A(i,m) = 0. \end{cases} \tag{4.5}$$

Proof. As $H_i = E_i H_A$ and

$$\begin{aligned} d_i(\sigma_{\mathcal{T}_A}, \tau_m) &= \operatorname{ind}_{E_i} T_i V_i^* \\ &= \dim(\operatorname{Ker}(T_i V_i^*) \text{ in } H_i) - \dim(\operatorname{Coker}(T_i V_i^*) \text{ in } H_i) \\ &= -\dim(H_i / T_i V_i^* (T_i V_i^*)^* H_i), \end{aligned}$$

we get the formula (4.5) by (4.4). □

Therefore we have

Theorem 4.4. *Let us denote by* $[\mathcal{T}_A]_*$ *the class in* $\operatorname{Ext}_*(\mathcal{O}_A)$ *of the Toeplitz extension* $\sigma_{\mathcal{T}_A}$ *of* $\mathcal{O}_A$. *We then have*

(i) $d_s([\mathcal{T}_A]_s) = -\hat{\iota}_A(1) - [1_N]$ *in* $\mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N$,

(ii) $d_w([\mathcal{T}_A]_w) = -[1_N]$ *in* $\mathbb{Z}^N/(I - A)\mathbb{Z}^N$,

where $[1_N] = [(1, \dots, 1)]$ *means the class of the vector* $(1, \dots, 1) \in \mathbb{Z}^N$

Proof. Let us denote by $v(m) \in \mathbb{Z}^N$ the column vector in $\mathbb{Z}^N$ whose mth component is one and the other components are zero's. Denote by $(1, \dots, 1)^t$ the column vector defined by the transpose of the row vector whose components are all one's. By (4.5), we have

$$\begin{aligned} \big[d_i(\sigma_{\mathcal{T}_A}, \tau_m)\big]_{i=1}^N &= -(1, \dots, 1)^t - v(m) + \big[A(i, m)\big]_{i=1}^N \\ &= -(I - A)v(m) - (1, \dots, 1)^t. \end{aligned}$$

Since $[(I - A)v(m)] = \hat{\iota}_A(1)$ in $\mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N$, we have

$$d_s([\mathcal{T}_A]_s) = -\hat{\iota}_A(1) - [1_N]$$

in $\mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N$. As $\hat{\iota}_A(1) = 0$ in $\mathbb{Z}^N/(I - A)\mathbb{Z}^N$, we have $d_w([\mathcal{T}_A]_w) = -[1_N]$ in $\mathbb{Z}^N/(I - A)\mathbb{Z}^N$. □

By virtue of the Rørdam's classification theorem for Cuntz–Krieger algebras [19] (cf. [7,11]) showing that the K_0-group $K_0(\mathcal{O}_A)$ with the position of the class [1] of the unit 1 of $\mathcal{O}_A$ in $K_0(\mathcal{O}_A)$ is a complete invariant of the isomorphism class of the algebra $\mathcal{O}_A$, we obtain the following corollary.

Corollary 4.5. *The pair* $(\operatorname{Ext_w}(\mathcal{O}_A), [\mathcal{T}_A]_w)$ *of the weak extension group* $\operatorname{Ext_w}(\mathcal{O}_A)$ *and the weak equivalence class* $[\mathcal{T}_A]_w$ *of the Toeplitz extension* $\sigma_{\mathcal{T}_A}$ *of the Cuntz–Krieger algebra* $\mathcal{O}_A$ *is a complete invariant of the isomorphism class of the Cuntz–Krieger algebra* $\mathcal{O}_{A^t}$ *for the transposed matrix* A^t *of the matrix* A. *This shows that two Cuntz–Krieger algebras* $\mathcal{O}_A$ *and* $\mathcal{O}_B$ *are isomorphic if and only if there exists an isomorphism* $\varphi\colon \operatorname{Ext_w}(\mathcal{O}_{A^t}) \longrightarrow \operatorname{Ext_w}(\mathcal{O}_{B^t})$ *of groups such that* $\varphi([\mathcal{T}_{A^t}]_w) = [\mathcal{T}_{B^t}]_w$.

Proof. As $K_0(\mathcal{O}_{A^t}) \cong \mathbb{Z}^N/(I-A)\mathbb{Z}^N$ and $(\mathbb{Z}^N/(I-A)\mathbb{Z}^N, -[1_N]) \cong (\mathbb{Z}^N/(I-A)\mathbb{Z}^N, [1_N])$, we have

$$(\mathrm{Ext_w}(\mathcal{O}_A), [\mathcal{T}_A]_w) \cong (\mathbb{Z}^N/(I-A)\mathbb{Z}^N, [1_N]) \cong (K_0(\mathcal{O}_{A^t}), [1]).$$

By virtue of the Rørdam's classification result for Cuntz–Krieger algebras [19] ([7], cf. [11]), we obtain the desired assertion. □

Remark 4.6. (i) The position $[\mathcal{T}_A]_*$ in $\mathrm{Ext}_*(\mathcal{O}_A)$ is not necessarily invariant under the isomorphism class of $\mathcal{O}_A$ (see Example 2 in the next section).

(ii) The abelian groups $\mathrm{Ext_w}(\mathcal{O}_A)$ and $K_0(\mathcal{O}_A)$ are isomorphic, and two C^*-algebras $\mathcal{O}_A \otimes K(H)$ and $\mathcal{O}_{A^t} \otimes K(H)$ are always isomorphic for every matrix A. There is however an example of an irreducible non permutation matrix A such that $\mathcal{O}_A$ is not isomorphic to $\mathcal{O}_{A^t}$ as in the classification table in [11] of the Cuntz–Krieger algebras for 3×3 matrices (see also [11, Example 2.1], or Example 4 in the next section).

5. Examples

Example 1. Let $A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ be the $N \times N$ matrix whose entries are each one with $N > 1$. The Cuntz–Krieger algebra $\mathcal{O}_A$ is nothing but the Cuntz algebra $\mathcal{O}_N$ (see [6]). The element $\iota_A(1)$ in $\mathrm{Ext_s}(\mathcal{O}_N)$ is denoted by $\iota_N(1)$. The Toeplitz algebra $\mathcal{T}_A$ is also denoted by $\mathcal{T}_N$. As $AR_1 = A$, we have $\widehat{A} = A + R_1 - AR_1 = R_1$, so that

$$I - \widehat{A} = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Define

$$L_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{so that} \quad L_N(I-\widehat{A}) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Hence L_N induces an isomorphism

$$\mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N \longrightarrow L_N\mathbb{Z}^N/L_N(I-\widehat{A})\mathbb{Z}^N \cong \mathbb{Z}$$

such that

$$[v] \in \mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N \longrightarrow [L_N v] \in L_N\mathbb{Z}^N/L_N(I-\widehat{A})\mathbb{Z}^N \longrightarrow (L_N v)_1 \in \mathbb{Z}.$$

For

$$[v] = \hat{\iota}_N(1) = \left[(I-A)\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right],$$

we see that

$$L_N v = L_N(I-A)\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1-N \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

so that $(L_N v)_1 = 1-N$. Therefore we have $(\mathrm{Ext_s}(\mathcal{O}_N), \iota_N(1)) \cong (\mathbb{Z}, 1-N)$ and hence the exact sequence (3.3) goes to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times(1-N)} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/(1-N)\mathbb{Z} \longrightarrow 0. \tag{5.1}$$

By using Theorem 4.4, one may easily compute that

$$(\mathrm{Ext_w}(\mathcal{O}_N), [\mathcal{T}_N]_w) \cong (\mathbb{Z}/(1-N)\mathbb{Z}, -1),$$
$$(\mathrm{Ext_s}(\mathcal{O}_N), [\mathcal{T}_N]_s, \iota_N(1)) \cong (\mathbb{Z}, -1, 1-N).$$

Example 2. Let us denote by F the Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. It is well-known that the Cuntz–Krieger algebra $\mathcal{O}_F$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$. Hence we have $\mathrm{Ext_w}(\mathcal{O}_F) \cong \mathrm{Ext_w}(\mathcal{O}_2) \cong \{0\}$, and $\mathrm{Ext_s}(\mathcal{O}_F) \cong \mathrm{Ext_s}(\mathcal{O}_2) \cong \mathbb{Z}$. By the formula in Theorem 4.4 together with the above Example 1, we see

$$(\mathrm{Ext_s}(\mathcal{O}_F), [\mathcal{T}_F]_s, \iota_F(1)) = (\mathbb{Z}, -2, -1),$$
$$(\mathrm{Ext_s}(\mathcal{O}_2), [\mathcal{T}_2]_s, \iota_2(1)) = (\mathbb{Z}, -1, -1).$$

Hence the position $[\mathcal{T}_F]_s$ in $\mathrm{Ext_s}(\mathcal{O}_F)$ is different from the position $[\mathcal{T}_2]_s$ in $\mathrm{Ext_s}(\mathcal{O}_2)$.

EXAMPLE 3. The weak extension groups $\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_{A_i})$, $i=1,2,3,4$ of $\mathcal{O}_{A_i}$, $i=1,2,3,4$ for the following list of matrices A_i, $i=1,2,3,4$ have been presented in [8, Remark 3.4]. Their strong extension groups $\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_{A_i})$ with the positions of the element $\iota_{A_i}(1)$, $i=1,2,3,4$ are easily computed by using Theorem 3.3. We also easily know the positions $[\mathcal{T}_{A_i}]_*$ in $\mathrm{Ext}_*(\mathcal{O}_{A_i})$ by Theorem 4.4. We present the list in the following, computed without difficulty by hand.

- $A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $(\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_{A_1}), [\mathcal{T}_{A_1}]_w) \cong (\mathbb{Z}/3\mathbb{Z}, 2)$,
$$(\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_{A_1}), [\mathcal{T}_{A_1}]_s, \iota_{A_1}(1)) \cong (\mathbb{Z}, 4, 3).$$
- $A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $(\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_{A_2}), [\mathcal{T}_{A_2}]_w) \cong (\mathbb{Z}/4\mathbb{Z}, 2)$,
$$(\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_{A_2}), [\mathcal{T}_{A_2}]_s, \iota_{A_2}(1)) \cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, -2 \oplus 0, 2 \oplus 1).$$
- $A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $(\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_{A_3}), [\mathcal{T}_{A_3}]_w) \cong (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, 0 \oplus 0)$,
$$(\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_{A_3}), [\mathcal{T}_{A_3}]_s, \iota_{A_3}(1)) \cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, -2 \oplus 0 \oplus 0, 1 \oplus 1 \oplus 1).$$
- $A_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $(\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_{A_4}), [\mathcal{T}_{A_4}]_w) \cong (\mathbb{Z}, -1)$,
$$(\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_{A_4}), [\mathcal{T}_{A_4}]_s, \iota_{A_4}(1)) \cong (\mathbb{Z} \oplus \mathbb{Z}, -2 \oplus (-1), 1 \oplus 0).$$

EXAMPLE 4. The matrices
$$A_5 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_6 = A_5^t = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$
are examples presented in [11, Example 2.1] such that $(K_0(\mathcal{O}_{A_5}), [1]) \cong (\mathbb{Z}/2\mathbb{Z}, 1)$ and $(K_0(\mathcal{O}_{A_6}), [1]) \cong (\mathbb{Z}/2\mathbb{Z}, 0)$, so that $\mathcal{O}_{A_5}$ is not isomorphic to $\mathcal{O}_{A_6}$. We then see that
$$(\mathrm{Ext}_w(\mathcal{O}_{A_5}), [\mathcal{T}_{A_5}]_w) \cong (\mathbb{Z}/2\mathbb{Z}, 0), \quad (\mathrm{Ext}_w(\mathcal{O}_{A_6}), [\mathcal{T}_{A_6}]_w) \cong (\mathbb{Z}/2\mathbb{Z}, 1).$$
We also easily see that
$$(\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_{A_5}), [\mathcal{T}_{A_5}]_s, \iota_{A_5}(1)) \cong (\mathbb{Z}, -2, -2),$$

$$(\mathrm{Ext_s}(\mathcal{O}_{A_6}), [\mathcal{T}_{A_6}]_s, \iota_{A_6}(1)) \cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, -1 \oplus 0, -1 \oplus (-1)),$$

and hence $\mathrm{Ext_s}(\mathcal{O}_{A_5})$ is not isomorphic to $\mathrm{Ext_s}(\mathcal{O}_{A_6})$.

Some of the results in this paper will be generalized to more general setting in a class of C $*$-algebras associated with symbolic dynamical systems in [15].

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