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AN ASYMPTOTIC EQUALITY OF CARTAN'S SECOND MAIN THEOREM AND SOME GENERALIZATIONS

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Abstract. Motivated by [19] and [10], we define the modified proximity function $\overline{m}_q(f, r)$ for entire curves in complex projective space $\mathbf{P}^n\mathbf{C}$, and establish an asymptotic equality of Cartan's Second Main Theorem. This is a generalization of [19, Theorem 1.6] for transcendental meromorphic functions. Moreover, we strengthen the result to entire curves of finite order and holomorphic mappings over multiple variables.

1. Introduction

Nevanlinna theory is a generalization of the fundamental theorem of algebra to meromorphic maps between complex spaces. Classical Nevanlinna theory consists of two fundamental theorems, which study the relation between the proximity function $m_f(r, a)$, counting function $N_f(r, a)$ and characteristic function $T_f(r)$. (We will give the definitions later.) The First Main Theorem (FMT) is just a reformulation of Poisson–Jensen formula and can be derived directly from the definitions, while the Second Main Theorem (SMT) is much deeper and more complicated. In some sense, Nevanlinna's SMT can be considered as a generalization of Riemann–Hurwitz formula. However, the later is an equality while SMT is just an inequality. This inspires a question that whether one can modify the SMT to an equality. There are some early results about this question in [17]. But in previous research, the SMT equality can only hold for certain restricted meromorphic functions. Owing to the compatibility conditions in [3], it was believed that the form of equality for SMT can not be literally true for all meromorphic functions.

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In 2013, Yamanoi [19] proved the Gol'dberg conjecture and Mues' conjecture. His innovative technique consists of Ahlfors' covering theory, holomorphic motions for quasimeromorphic functions and the tree theory for points configurations. In his article, he modified the proximity function (with moving targets) and obtained an asymptotic equality of SMT for transcendental meromorphic functions. Soon later, Eremenko [3] applied the potential theory and discussed the possibility of an asymptotic equality for higher-dimensional cases. However, Eremenko's modification is not optimal so that the asymptotic equality only holds for a class of holomorphic curves defined by solutions for linear differential equations.

Our work is motivated by the oscillation methods in [19] and the general form of SMT in [10]. We define the modified proximity function $\overline{m}_q(f, r)$, which is a special case of multidivisor proximity function in [14], and generalize the asymptotic equality of SMT to the holomorphic curve in $\mathbf{P}^n\mathbf{C}$. Our main result reads as follows.

THEOREM 1.1 (main theorem). *Let $f = [f_0 : \dots : f_n]\colon \mathbf{C} \to \mathbf{P}^n\mathbf{C}$ be a holomorphic curve that is non-degenerate. Let $v\colon \mathbf{R}_{>e} \to \mathbf{N}_{>0}$ be a positive function satisfying that $v(r) \sim \left(\log^+ \frac{T(r)}{\log r}\right)^{20}$. Then for any $\varepsilon > 0$, we have*

$$\overline{m}_{v(r)+n+1}(f, r) + N_{W_f}(r, 0) = (n + 1)T_f(r) + \varepsilon(T_f(r)), \tag{1.1}$$

for all $r \to \infty$ outside an exceptional set of logarithmic density 0.

Here is the outline of our paper. After recalling the definitions and main results in Nevanlinna theory, Section 2 introduces the asymptotic equality of SMT for transcendental meromorphic functions in [19]. Section 3 defines the modified proximity function $\overline{m}_q(f, r)$ in higher-dimensional case and gives the reversion of Cartan's Second Main Theorem, which proves our Theorem 1.1. Assuming $v(r)$ to be arbitrarily slow growth, we also exhibit the asymptotic SMT for entire curves of finite order. In Section 4, we describe the main Theorem in the setting of several complex variables.

2. Holomorphic curves in projective space

2.1. Notations of Nevanlinna theory. We start to recall some notations and definitions in Nevanlinna theory. Under homogenous coordinates, let $f = [f_0 : \dots : f_n]\colon \mathbf{C} \to \mathbf{P}^n\mathbf{C}$ be a holomorphic map where $f_0, \dots, f_n$ are holomorphic functions having no common zeros. Denote by $\mathbf{f} = (f_0, \dots, f_n)$ the reduced representation of the entire curve f. Cartan's characteristic function $T_f(r)$ is defined by

$$T_f(r) = \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \,\frac{d\theta}{2\pi},$$

where $\|\mathbf{f}(z)\| = \max_{k=0,\dots,n} |f_k(z)|$.

A hyperplane H in $\mathbf{P}^n\mathbf{C}$ is given by

$$H = \left\{ [x_0 : \ldots : x_n] \in \mathbf{P}^n\mathbf{C} \;\middle|\; \sum_{k=0}^{n} a_k x_k = 0 \right\},$$

where $\mathbf{a} = (a_0, \ldots, a_n)$ is the nonzero vector associated with H. The Weil function $\lambda_H(f(z))$ of f with respect to H is defined by

$$\lambda_H(f(z)) = \log \frac{\|\mathbf{f}(z)\| \cdot \|\mathbf{a}\|}{|\langle \mathbf{f}(z), \mathbf{a} \rangle|},$$

where $|\langle \mathbf{f}(z), \mathbf{a} \rangle|$ is the inner product in $\mathbf{C}^{n+1}$. We define the proximity function $m_f(r, H)$ of f with respect to H as

$$m_f(r, H) = \int_0^{2\pi} \lambda_H(f(re^{i\theta})) \, \frac{d\theta}{2\pi}.$$

The counting function $N_f(r, H)$ of f with respect to H is defined by

$$N_f(r, H) = \int_0^{r} (n_f(t, H) - n_f(0, H)) \, \frac{dt}{t} + n_f(0, H) \log r,$$

where $n_f(t, H)$ is the number (counting multiplicity) of zeros of $|\langle \mathbf{f}(z), \mathbf{a} \rangle|$ in the disk $\{|z| < t\}$. And the truncated counting function $N_f^{[k]}(r, H)$ is given by

$$N_f(r, H) = \int_0^{2\pi} \left(n_f^{[k]}(t, H) - n_f^{[k]}(0, H) \right) \frac{dt}{t} + n_f^{[k]}(0, H) \log r,$$

where $n_f^{[k]}(t, H)$ is the number of zeros of $|\langle \mathbf{f}(z), \mathbf{a} \rangle|$ in the disk $\{|z| < t\}$ with multiplicity counted at most k times.

Remark 1. Note that the above functions are all independent of the choice of homogenous coordinates.

Theorem 2.1 (First Main Theorem). *Following the definitions above, we derive from the Poincaré–Lelong formula that*

$$T_f(r) = m_f(r, H) + N_f(r, H) + O(1).$$

2.2. A general form of Second Main Theorem. There have been many generalizations of Nevanlinna's second main theorem, see [1], [15], [10] and [12] etc. The following presentation of SMT appeared in [10]. It was firstly introduced by Vojta [16] as an analogy of Schmidt's subspace theorem in number theory. Compared with the standard SMT, the author does not make the assumption of "in the general position" for hyperplanes.

THEOREM 2.2 (Second Main Theorem of general form, [10, Theorem 2.1]). *Let* $f = [f_0 : \dots : f_n] \colon \mathbf{C} \to \mathbf{P}^n\mathbf{C}$ *be a holomorphic curve whose image is not contained in any proper subspaces. Let* $H_1, \dots, H_q$ *be arbitrary hyperplanes in* $\mathbf{P}^n\mathbf{C}$. *Then for any* $\varepsilon > 0$, *we have*

$$\int_0^{2\pi} \max_K \sum_{k\in K} \lambda_{H_k}(f(re^{i\theta}))\frac{d\theta}{2\pi} + N_{W_f}(r,0) \le (n+1)T_f(r) + \varepsilon T_f(r) \quad \|, \tag{2.1}$$

where the maximum is taken over all subsets K *of* $\{1, \dots, q\}$ *such that* H_k, $k \in K$ *are linearly independent, and the ramification term* $N_{W_f}(r,0)$ *is the counting function of the Wronskian fucntion* $W(f_0, \dots, f_n)$ *with respect to* f.

Here and for the rest of the paper, the notation $\|$ at the end of the inequality or equality means that it holds for all $r > e$ outside a set of finite Lebesgue measure.

The protagonist of the proof of Theorem 2.2 is the lemma of logarithmic derivatives. Since we will use the same strategies to examine the error term in next section, we list the lemmas here; for references, see [2] and [12].

LEMMA 2.3 [12, Lemma A5.1.4]. *Let* f *be a non-const meromorphic function.* l *is a non-negative integer. For arbitrary* α *with* $0 < \alpha l < 1/2$, *there exists constants* C, C_1, C_2 *such that for any* $r < \rho < R$,

$$\int_0^{2\pi}\left|\frac{f^{(l)}(re^{i\theta})}{f(re^{i\theta})}\right|^{\alpha}\frac{d\theta}{2\pi} \le C\Big(\frac{\rho}{r(\rho - r)}\Big)^{\alpha l}\Big[C_1T_f(\rho) + C_2\log\frac{R}{\rho(R-\rho)}T_f(\rho)\Big]^{\alpha l}.$$

LEMMA 2.4 [12, Lemma A3.2.4]. *Let* F *be a non-decreasing, positive, continuous function defined on* $[e, \infty)$ *such that* $F(r) \ge e$. *Then for arbitrary* $\varepsilon > 0$, *there exists a closed exceptional set* $E \subset [e, \infty)$ *of finite Lebesgue measure, satisfying that if we take* $\rho = r + \frac{1}{\log^{1+\varepsilon} F(r)}$ *for all* $r > e$ *and not in* E, *we have*

$$\log F(\rho) \le \log F(r) + 1,$$

and

$$\log^+\frac{\rho}{r(\rho - r)} \le (1+\varepsilon)\log^+\log F(r) + O(1).$$

From the concavity of $\log^+$, Lemma 2.3 and Lemma 2.4, it follows that

LEMMA 2.5 (logarithmic derivative lemma). *Let* f *be meromorphic function on* $\mathbb{C}$. *Then for arbitrary* $k \ge 1$ *and arbitrary* $\varepsilon > 0$, *we have*

$$\int_0^{2\pi}\log^+\left|\frac{f^{(k)}(re^{i\theta})}{f(re^{i\theta})}\right|\frac{d\theta}{2\pi} \le \varepsilon T_f(r) \quad \|.$$

2.3. Asymptotic equality of SMT for meromorphic functions. In this subsection, we pass to the asymptotic equality of SMT for meromorphic functions. This is also the origin of our story. The reader is referred to [19] for more details. Let f be a transcendental meromorphic function on $\mathbf{C}$. That is, f is a holomorphic map from $\mathbf{C}$ to $\mathbf{P}^1\mathbf{C}$.

Definition 1 (modified proximity function, [19, p. 708]). Given a positive integer q and a real number $r > e$, we define

$$\overline{m}_{0,q}(f,r) = \sup_{(a_1,\dots,a_q)\in\mathbf{C}^q} \int_0^{2\pi} \max_{1\le j\le q} \log \frac{\sqrt{1+|f(re^{i\theta})|^2}\sqrt{1+|a_j|^2}}{|f(re^{i\theta})-a_j|}\,\frac{d\theta}{2\pi}.$$

It is a natural question that whether Definition 1 will make sense or not. We will show it in the next section (See also [19, Lemma 2.2]). Indeed, Yamanoi gave the definition more general when $a_j(z), j = 1\dots,q$, are rational functions of degree less than or equal to d. But here, we only deal with the case of $d = 0$, which means that a_j are complex constants.

Yamanoi showed the lower estimate of $\overline{m}_{0,q}(f,r)$ with the characteristic function $T_f(r)$, which makes Definition 1 very interesting. This is a reversion of the standard SMT. The following theorem is a simple case of the original result in [19].

Theorem 2.6 [19, Theorem 1.3]. *Let f be a transcendental meromorphic function on $\mathbf{C}$. Let $v\colon \mathbf{R}_{>e} \to \mathbf{N}_{>0}$ be a positive function satisfying that $v(r) \sim \big(\log^+ \frac{T_f(r)}{\log r}\big)^{20}$. Then for any $\varepsilon > 0$, we have*

$$2T_f(r) \le \overline{m}_{0,v(r)} + N_{f'}(r,0) + N_1(f,r,\infty) + \varepsilon T_f(r), \tag{2.2}$$

where $r \to \infty$ outside a set of logarithmic density 0.

Here, $N_1(f,r,\infty) = N_f(r,\infty) - N_f^{[1]}(r,\infty)$ is the counting function with multiplicity greater than 1, and $N_{f'}(r,0)$ is the counting function of the derivative f'.

Remark 2. We say a set E is of logarithmic density 0, if it satisfies that

$$\limsup_{r\to\infty} \frac{\int_E \frac{dt}{t}}{\log r} = 0.$$

It is not difficult to verify that the condition of finite Lebesgue measure implies the condition of logarithmic density 0.

Remark 3. We would like to emphasize the history that the question of reversal of SMT is not newly proposed. Many mathematicians contributed to these results; see [7], [17] and [4]. The innovative point here is the definition of modified approximation function.

The proof of Theorem 2.6 is based on the oscillation estimate of the meromorphic function on small arcs of the circle $\{|z|=r\}$. For a meromorphic function f, we put the oscillation function as

$$u(r,f,\theta)=\sup_{\tau\in[0,2\pi]}\Big(\sup_{t\in[\tau,\tau+\theta]}\log|f(re^{it})|-\inf_{t\in[\tau,\tau+\theta]}\log|f(re^{it})|\Big),$$

with respect to the parameters r, f, θ. The next proposition announces the relationship between the oscillation function and the characteristic function.

Proposition 2.7 [19, Proposition 3.1]. *Let f be a transcendental meromorphic function in the complex plane. Let $\varepsilon>0$, then we have*

$$u(r,f,\lambda(r)^{20})\le\varepsilon T_f(r),$$

for all $r>e$ outside a set of logarithmic density zero. Here

$$\lambda(r)=\min\Big\{1,\Big(\log^+\frac{T_f(r)}{\log r}\Big)^{-1}\Big\}.$$

This is a very important and useful tool in the sequel. However, we will not collect this wonderful but long proof in our article. The reader who are interested can refer to [19, Section 3], which applies Poisson–Jensen formula. Provided with Proposition 2.7, Yamanoi equi-divide the the circle $\{|z|=r\}$ into $v(r)$ parts, where the oscillations hardly contribute. Together with Taylor expansion of $f(z)$, the characteristic function is bounded from above by $\overline{m}$.

Combining Theorem 2.6 and Theorem 2.2 (see also inequality (1.10) in [19]), one can obtain the asymptotic equality of SMT for meromorphic functions as follows.

Theorem 2.8 [19, Theorem 1.6]. *Let f and $v(r)$ be as above. Then for any $\varepsilon>0$, we have*

$$\overline{m}_{0,v(r)}+\sum_{a\in\hat{\mathbf{C}}}N_1(f,r,a)=2T_f(r)+\varepsilon T_f(r),\tag{2.3}$$

where $r\to\infty$ outside a set of logarithmic density 0.

To end this subsection, we would like to insert more explanations on Theorem 2.6. By some direct computations, we can check that

$$N_{f'}(r,0)+N_1(f,r,\infty)=\sum_{a\in\mathbf{P}^1\mathbf{C}}N_1(f,r,a).$$

For the transcendental meromorphic function $f(z)$, we can regard $f(z)$ as a holomorphic map from $\mathbf{C}$ to $\mathbf{P}^1\mathbf{C}$. Let $f(z) = [f_0(z) : f_1(z)]$. The Wronskian function of f is defined by

$$W_f(z) = \begin{vmatrix} f_0(z) & f_1(z) \\ f_0'(z) & f_1'(z) \end{vmatrix}.$$

Then we have

$$\sum_{a\in\mathbf{P}^1\mathbf{C}} N_1(r,a,f) = N_{W_f}(r,0).$$

For any $a \in \mathbf{P}^1\mathbf{C}$, we can define the hyperplane H_a like that

$$H_a = \begin{cases} [1 : -a], & a \in \mathbf{C}, \\ [0 : 1], & a = \infty. \end{cases}$$

Then the inequality 2.2 becomes that

$$2T_f(r) \le \sup_{H_{a_1},\dots,H_{a_{v(r)}}} \int_0^{2\pi} \max_{k\in K} \log \frac{\|f\|\cdot\|H_{a_k}\|}{|\langle f, H_{a_k}\rangle|}(re^{i\theta})\,\frac{d\theta}{2\pi} + N_{W_f}(r,0) + \varepsilon T_f(r),$$

where the maximum is taken over all subsets K of $\{1,\dots,v(r)\}$ such that a_k, $k \in K$, are distinct complex numbers. This inspires us to generalize Theorem 2.6 to higher dimension.

3. Asymptotic equality of SMT in $\mathbf{P}^n\mathbf{C}$

In this section, we will imitate Definition 1 to modify the proximity function of the holomorphic curve in $\mathbf{P}^n\mathbf{C}$. As explained in the end paragraph of Section 2, we will generalize Theorem 2.6 to higher dimension and obtain the asymptotic equality as a direct consequence. Readers who are familiar with [10] and [19] will understand our techniques without any obstacles.

3.1. Preliminaries.

Definition 2 (modified proximity function). Let $f = [f_0 : \dots : f_n]\colon \mathbf{C} \to \mathbf{P}^n\mathbf{C}$ be a holomorphic map. For any positive integer q and $r > e$, we define

$$\overline{m}_q(f,r) = \sup_{H_1,\dots,H_q} \int_0^{2\pi} \max_K \sum_{k\in K} \lambda_{H_k}(f(re^{i\theta}))\,\frac{d\theta}{2\pi},$$

where the maximum is taken over all subsets K of $\{1,\dots,q\}$ such that the hyperplanes H_k, $k \in K$ are linearly independent and the superior is taken over all sets of q arbitrary hyperplanes in $\mathbf{P}^n\mathbf{C}$.

As foreshowed in Section 2, we firstly show the finiteness of Definition 2. It clearly follows from [19, Remark 2.3]. For readers' convenience, we repeat and refine the proof here.

LEMMA 3.1. *Let $f = [f_0 : \ldots : f_n] \colon \mathbf{C} \to \mathbf{P}^n\mathbf{C}$ be a holomorphic curve which is non-degenerate. Let H be an arbitrary hyperplane in $\mathbf{P}^n\mathbf{C}$. Then we have*

$$m_f(1, H) \leq C$$

for some positive constant C which only depends on f.

PROOF. Assume on the contrary that there is a sequence of hyperplanes $\{H_1, H_2, \ldots\}$ in $\mathbf{P}^n\mathbf{C}$, such that $m_f(1, H_k) \to \infty$ as $k \to \infty$. For each k, denote by $\mathbf{a_k} = (a_{0,k}, \ldots, a_{n,k})$ the associated vectors of hyperplanes H_k. We may select a suitable subsequence from $\{\mathbf{a_k}\}_{k=1}^{\infty}$ that converges to an $(n+1)$-vector $\mathbf{a} = (a_0, \ldots, a_n) \in \overline{\mathbf{C}}^{n+1}$. For some l, $0 \leq l \leq n$, it is possible that the l-th component a_l of $\mathbf{a}$ reveals to be ∞. Under homogenous coordinate, we may assign a_l to be 1 and other finite components to be 0. In this setting, we can reduce that the limit vector $\mathbf{a} \in \mathbf{C}^{n+1}$. Denote by H the hyperplane associated to the vector $\mathbf{a}$. Since f is non-degenerate, we can take a constant $0 < \delta < 1$ satisfying that

$$\min_{0 \leq \theta \leq 2\pi} |\langle \mathbf{f}(\delta e^{i\theta}), \mathbf{a} \rangle| > 0.$$

It implies that

$$\sup_k m_f(\delta, H_k) < \infty.$$

On the other hand, by Jensen formula, we have

$$\int_\delta^1 \left(\int_{|z| \leq t} f^*\omega_{FS} \right) \frac{dt}{t} + O(1) = \int_\delta^1 n_f(r, H_k) \frac{dt}{t} + m_f(1, H_k) - m_f(\delta, H_k),$$

where ω_{FS} is the Fubini–Study metric defined in $\mathbf{P}^n\mathbf{C}$. Since $\int_\delta^1 n_f(r, H) \frac{dt}{t} \geq 0$, we have

$$m_f(\delta, H_k) \geq m_f(1, H_k) - \int_\delta^1 \left(\int_{|z| \leq t} f^*\omega_{FS} \right) \frac{dt}{t} + O(1).$$

By our assumption at the very beginning, we have

$$\limsup_{k \to \infty} m_f(\delta, H_k) \to \infty.$$

This gives a contradiction. □

REMARK 4. Let $C_f = \sup_H m_f(1,H)$. For any hyperplane H, we have

$$m_f(r,H) \le C_{f(rz)}.$$

Let $H_1, \dots, H_q$ be q arbitrary hyperplanes in $\mathbf{P}^n\mathbf{C}$. Thus for $r > e$, we have

$$\int_0^{2\pi} \max_K \sum_{k\in K} \lambda_{H_k}(f(re^{i\theta}))\frac{d\theta}{2\pi} \le \sum_{j=1}^q m_f(r,H_j)$$
$$= \sum_{j=1}^q m_{f(rz)}(1,H_j) \le qC_{f(rz)}.$$

We can assert that $\overline{m}_q(f,r)$ is finite.

3.2. Proof of the main result. Now we will introduce the main result of this article. It gives a lower estimate of $\overline{m}_q(f,r)$ by $T_f(r)$, and thus infers an extension of Theorem 2.6.

THEOREM 3.2. *Let $f = [f_0 : \dots : f_n]\colon \mathbf{C} \to \mathbf{P}^n\mathbf{C}$ be a holomorphic curve that is non-degenerate. Let $v\colon \mathbf{R}_{>e} \to \mathbf{N}_{>0}$ satisfies that $v(r) \sim \big(\log^+ \frac{T(r)}{\log r}\big)^{20}$. Then for arbitrary $0 < \varepsilon < 1$, we have*

$$\overline{m}_{v(r)+n+1}(f,r) + N_{W_f}(r,0) + \varepsilon(T_f(r)) \ge (n+1)T_f(r) + O(1), \tag{3.1}$$

for all $r \to \infty$ outside an exceptional set of logarithmic density 0.

We firstly prove Theorem 1.1 as a corollary of Theorem 3.2 and Theorem 2.2.

PROOF OF THEOREM 1.1. Taking the superior over $v(r)+n+1$ hyperplanes in $\mathbf{P}^n\mathbf{C}$ in Theorem 2.2, we have

$$\overline{m}_{v(r)+n+1}(f,r) + N_{W_f}(r,0) \le (n+1)T_f(r) + \varepsilon(T_f(r)) \quad \|.$$

Accompanied with Theorem 3.2, we complete the proof of Theorem 1.1. □

Before giving the direct proof of Theorem 3.2, we would like to describe the main idea briefly. It is a combination of [19] and [10]. We equi-divide the circle $\{|z| = r\}$ into some small arcs, and then the estimate of oscillation in Proposition 2.7 makes a great difference. On each arc, we select $n+1$ appropriate hyperplanes in general position according to the Wronskian function W_f. Using integral formula, we bound $\overline{m}_q(f,r)$ from below by $N_{W_f}(r,0)$, $T_f(r)$ as well as the error terms. Lemmas 2.3–2.5 and Proposition 2.7 can help us to control the errors. These techniques will produce a reversion of Cartan's SMT.

PROOF OF THEOREM 3.2. We will divide our proof into three steps.

Step 1: Select $n+1$ hyperplanes. We fix an $r>e$ and work on the circle $\{|z|=r\}$. Given a positive integer q, for $j=1,\dots,q$, we put $\theta_j=\frac{2\pi j}{q}$ and $z_j=re^{i\theta_j}$. In this way, we equi-divide the circle $\{|z|=r\}$ into q parts and $\{z_j\}_{j=1}^q$ are q break points. Recall the definition of the Wronskian function $W_f(z)$ associated to f that

$$W_f(z)=\begin{vmatrix} f_0(z) & \dots & f_n(z) \\ f_0'(z) & \dots & f_n'(z) \\ \vdots & \dots & \vdots \\ f_0^{(n)}(z) & \dots & f_n^{(n)}(z) \end{vmatrix}.$$

For each $j=1,\dots,q$, we will restrict our proof to the j-th arc $\Theta_j=\{z=re^{i\theta}\mid\theta\in[\theta_{j-1},\theta_j]\}$. We define the hyperplanes D^j in $\mathbf{P}^n\mathbf{C}$ as

$$D^j=\left\{[x_0:\dots:x_n]\in\mathbf{P}^n\mathbf{C}:\sum_{k=0}^n a_k^j x_k=0\right\},$$

where the associated vector $\mathbf{a}^j=(a_0^j,\dots,a_n^j)$ satisfies that each component a_k^j, $k=0,\dots,n$, is the cofactor of the Wronskian $W_f(z_j)$ with respect to the entries $f_k'(z_j)$. More precisely, for $k=0,\dots,n$, we define

$$a_k^j=(-1)^{k+1}\begin{vmatrix} f_0(z_j) & \dots & f_{k-1}(z_j) & f_{k+1}(z_j) & \dots & f_n(z_j) \\ f_0^{(2)}(z_j) & \dots & f_{k-1}^{(2)}(z_j) & f_{k+1}^{(2)}(z_j) & \dots & f_n^{(2)}(z_j) \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ f_0^{(n)}(z_j) & \dots & f_{k-1}^{(n)}(z_j) & f_{k+1}^{(n)}(z_j) & \dots & f_n^{(n)}(z_j) \end{vmatrix}.$$

Then we have

$$\langle\mathbf{f}(z),\mathbf{a}^j\rangle=\begin{vmatrix} f_0(z_j) & \dots & f_n(z_j) \\ f_0(z) & \dots & f_n(z) \\ f_0^{(2)}(z_j) & \dots & f_n^{(2)}(z_j) \\ \vdots & \dots & \vdots \\ f_0^{(n)}(z_j) & \dots & f_n^{(n)}(z_j) \end{vmatrix}.$$

Moreover, we have

$$\langle\mathbf{f}(z),\mathbf{a}^j\rangle\big|_{z=z_j}=0,$$

and

$$(\langle \mathbf{f}(z), \mathbf{a}^j \rangle)'\big|_{z=z_j} = \begin{vmatrix} f_0(z_j) & \dots & f_n(z_j) \\ f_0'(z_j) & \dots & f_n'(z_j) \\ \vdots & \dots & \vdots \\ f_0^{(n)}(z_j) & \dots & f_n^{(n)}(z_j) \end{vmatrix} = W_f(z_j).$$

Thus, using the elementary integral formula, for any $\theta \in [\theta_{j-1}, \theta_j]$, we have

$$\langle \mathbf{f}(re^{i\theta}), \mathbf{a}^j \rangle = \int_{\theta_{j-1}}^{\theta} \langle \mathbf{f}'(re^{is}), \mathbf{a}^j \rangle \, d(re^{is}). \tag{3.2}$$

Then for any $\theta \in [\theta_{j-1}, \theta_j]$, we obtain that

$$|\langle \mathbf{f}(re^{i\theta}), \mathbf{a}^j \rangle| \le e^{\tau_j}(2\pi r), \tag{3.3}$$

where

$$\tau_j = \max_{s \in (\theta_{j-1}, \theta]} \log|W_f(re^{is})| + \max_{s \in (\theta_{j-1}, \theta]} \log \frac{|\langle \mathbf{f}'(re^{is}), \mathbf{a}^j \rangle|}{|W_f(re^{is})|}.$$

By the definition of $\mathbf{a}^j$, we have

$$\begin{aligned} \log \frac{|\langle \mathbf{f}'(re^{is}), \mathbf{a}^j \rangle|}{|W_f(re^{is})|} &= \log|\langle \mathbf{f}'(re^{is}), \mathbf{a}^j \rangle| - \log|W_f(re^{is})| \\ &\le O(\log|W_f(re^{is})| - \log|W_f(z_j)|) \le O\Big(u\Big(r, W_f, \frac{2\pi}{q}\Big)\Big), \end{aligned} \tag{3.4}$$

where $u\big(r, W_f, \frac{2\pi}{q}\big)$ is the oscillation function defined before Proposition 2.7. Combining (3.2), (3.3) and (3.4), we have

$$\begin{aligned} \log \frac{1}{|\langle \mathbf{f}(re^{i\theta}), \mathbf{a}^j \rangle|} &\ge -\tau_j - \log(2\pi r) \\ &\ge - \max_{s \in (\theta_{j-1}, \theta]} \log|W_f(re^{is})| - O\Big(u\Big(r, W_f, \frac{2\pi}{q}\Big)\Big) - \log(2\pi r) \\ &\ge \log \frac{1}{|W_f(re^{i\theta})|} + \Big(\log|W_f(re^{i\theta})| - \max_{s \in (\theta_{j-1}, \theta]} \log|W_f(re^{is})|\Big) \\ &\quad - O\Big(u\Big(r, W_f, \frac{2\pi}{q}\Big)\Big) - \log(2\pi r) \\ &\ge \log \frac{1}{|W_f(re^{i\theta})|} - O\Big(u\Big(r, W_f, \frac{2\pi}{q}\Big)\Big) - \log(2\pi r), \end{aligned}$$

for any $\theta \in [\theta_{j-1}, \theta_j]$.

Summing up q parts together, we have

$$\int_0^{2\pi} \log \frac{1}{|W_f(re^{i\theta})|} \frac{d\theta}{2\pi} \tag{3.5}$$

$$\leq \sum_{j=1}^{q} \int_{\theta_{j-1}}^{\theta_j} \log \frac{1}{|\langle \mathbf{f}(re^{i\theta}), \mathbf{a}^j \rangle|} \frac{d\theta}{2\pi} + O\Big(u\Big(r, W_f, \frac{2\pi}{q}\Big)\Big) + \log(2\pi r).$$

Denote by $D_k, k = 0, \dots, n$, the $n+1$ coordinate hyperplanes in $\mathbf{P}^n\mathbf{C}$ with

$$D_k = \{[x_0 : \dots : x_n] | x_k = 0\},$$

and the associated vectors

$$\mathbf{a}_k = [a_{0,k} : \dots : a_{k,k} : \dots : a_{n,k}] = [0 : \dots : 1 : \dots : 0].$$

Assume that

$$\|\mathbf{a}^j\| = \max_{0 \leq k \leq n} |a_k^j| = |a_{k_j}^j|$$

$$= \left| \det \begin{pmatrix} f_0(z_j) & \dots & f_{k_j-1}(z_j) & f_{k_j+1}(z_j) & \dots & f_n(z_j) \\ f_0^{(2)}(z_j) & \dots & f_{k_j-1}^{(2)}(z_j) & f_{k_j+1}^{(2)}(z_j) & \dots & f_n^{(2)}(z_j) \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ f_0^{(n)}(z_j) & \dots & f_{k_j-1}^{(n)}(z_j) & f_{k_j+1}^{(n)}(z_j) & \dots & f_n^{(n)}(z_j) \end{pmatrix} \right|.$$

Then we can see that $n+1$ hyperplanes $\{D^j, D_0, \dots, D_{k_j-1}, D_{k_j+1} \dots, D_n\}$ are in general position.

Now using the definitions of Weil function and modified proximity function, it follows from the inequality (3.5) that

$$\int_0^{2\pi} \Big(\log \frac{1}{|W_f(re^{i\theta})|} + (n+1) \log \|\mathbf{f}(re^{i\theta})\|\Big) \frac{d\theta}{2\pi} \tag{3.6}$$

$$\leq \sum_{j=1}^{q} \int_{\theta_{j-1}}^{\theta_j} \Big(\log \frac{1}{|\langle \mathbf{f}(re^{i\theta}), \mathbf{a}^j \rangle|} + (n+1) \log \|\mathbf{f}(re^{i\theta})\|\Big) \frac{d\theta}{2\pi}$$

$$+ O\Big(u\Big(r, W_f, \frac{2\pi}{q}\Big)\Big) + \log(2\pi r)$$

$$\leq \sum_{j=1}^{q} \int_{\theta_{j-1}}^{\theta_j} \Big(\log \frac{\|\mathbf{f}(re^{i\theta})\| \cdot \|\mathbf{a}^j\|}{|\langle \mathbf{f}(re^{i\theta}), \mathbf{a}^j \rangle|} + \log \frac{\|\mathbf{f}(re^{i\theta})\|}{|\langle \mathbf{f}(re^{i\theta}), \mathbf{a}_0 \rangle|} + \cdots$$

$$+ \log \frac{\|\mathbf{f}(re^{i\theta})\|}{|\langle \mathbf{f}(re^{i\theta}), \mathbf{a}_{k_j-1} \rangle|} + \log \frac{\|\mathbf{f}(re^{i\theta})\|}{|\langle \mathbf{f}(re^{i\theta}), \mathbf{a}_{k_j+1} \rangle|} + \cdots + \log \frac{\|\mathbf{f}(re^{i\theta})\|}{|\langle \mathbf{f}(re^{i\theta}), \mathbf{a}_n \rangle|}\Big) \frac{d\theta}{2\pi}$$

$$+\sum_{j=1}^{q}\int_{\theta_{j-1}}^{\theta_j}\log\frac{|f_0\cdots\widehat{f_{k_j}}\cdots f_n|(re^{i\theta})}{\|\mathbf{a}^j\|}\,\frac{d\theta}{2\pi}+O\Big(u\Big(r,W_f,\frac{2\pi}{q}\Big)\Big)+\log(2\pi r)$$

$$\le\sum_{j=1}^{q}\int_{\theta_{j-1}}^{\theta_j}\Big(\lambda_{D^j}(f)+\lambda_{D_0}(f)+\cdots+\widehat{\lambda_{D_{k_j}}(f)}+\cdots+\lambda_{D_n}(f)\Big)(re^{i\theta})\,\frac{d\theta}{2\pi}$$

$$+\sum_{j=1}^{q}\int_{\theta_{j-1}}^{\theta_j}\log\frac{|f_0\cdots\widehat{f_{k_j}}\cdots f_n|(re^{i\theta})}{\|\mathbf{a}^j\|}\,\frac{d\theta}{2\pi}+O\Big(u\Big(r,W_f,\frac{2\pi}{q}\Big)\Big)+\log(2\pi r)$$

$$\le\overline{m}_{q+n+1}(f,r)+\sum_{j=1}^{q}\int_{\theta_{j-1}}^{\theta_j}\log\frac{|f_0\cdots\widehat{f_{k_j}}\cdots f_n|(re^{i\theta})}{\|\mathbf{a}^j\|}\,\frac{d\theta}{2\pi}$$
$$+O\Big(u\Big(r,W_f,\frac{2\pi}{q}\Big)\Big)+\log(2\pi r).$$

Step 2: Estimate the error terms. Firstly, we consider the left-hand side of (3.6). By Poincaré–Lelong formula, it is easy to see that

$$\int_0^{2\pi}\Big(\log\frac{1}{|W_f(re^{i\theta})|}+(n+1)\log\|\mathbf{f}(re^{i\theta})\|\Big)\,\frac{d\theta}{2\pi}$$
$$=(n+1)T_f(r)-N_{W_f}(r,0)+O(1). \tag{3.7}$$

Next, we will estimate the right-hand side of (3.6). For a fixed j, k_j is also a fixed integer in $\{0,\dots,n\}$. We define the coefficient functions

$$|a^j_{k_j}|(z):=\left|\det\begin{pmatrix} f_0(z) & \dots & f_{k_j-1}(z) & f_{k_j+1}(z) & \dots & f_n(z)\\ f_0^{(2)}(z) & \dots & f_{k_j-1}^{(2)}(z) & f_{k_j+1}^{(2)}(z) & \dots & f_n^{(2)}(z)\\ \vdots & \dots & \vdots & \vdots & \dots & \vdots\\ f_0^{(n)}(z) & \dots & f_{k_j-1}^{(n)}(z) & f_{k_j+1}^{(n)}(z) & \dots & f_n^{(n)}(z)\end{pmatrix}\right|.$$

Then by previous assumption, we have

$$|a^j_{k_j}|(z_j)=\|\mathbf{a}^j\|,$$

the norm of the vector $\mathbf{a}^j$.

Claim 3.3. *We have*

$$\frac{\log\|\mathbf{a}^j\|}{q}\le\int_{\theta_{j-1}}^{\theta_j}\log\big(|a^j_{k_j}|(re^{i\theta})\big)\,\frac{d\theta}{2\pi}+\frac{1}{q}\,u\Big(r,|a^j_{k_j}|(z),\frac{2\pi}{q}\Big).$$

PROOF. We consider the following term:

$$U_j = \frac{\log\|\mathbf{a}^j\|}{q} - \int_{\theta_{j-1}}^{\theta_j} \log\big(|a^j_{k_j}|(re^{i\theta})\big)\,\frac{d\theta}{2\pi}.$$

Since

$$u\Big(r, |a^j_{k_j}|(z), \frac{2\pi}{q}\Big)$$
$$= \sup_{\tau\in[0,2\pi]}\Big(\sup_{t\in[\tau,\tau+\frac{2\pi}{q}]} \log\big(|a^j_{k_j}|(re^{it})\big) - \inf_{t\in[\tau,\tau+\frac{2\pi}{q}]} \log\big(|a^j_{k_j}|(re^{it})\big)\Big),$$

we have

$$|U_j| = \bigg|\int_{\theta_{j-1}}^{\theta_j} \big(\log\|\mathbf{a}^j\| - \log(|a^j_{k_j}|(re^{i\theta}))\big)\,\frac{d\theta}{2\pi}\bigg|$$
$$\le \int_{\theta_{j-1}}^{\theta_j} \Big|\log\big(|a^j_{k_j}|(z_j)\big) - \log\big(|a^j_{k_j}|(re^{i\theta})\big)\Big|\,\frac{d\theta}{2\pi} \le \frac{1}{q}\cdot u\Big(r, |a^j_{k_j}|(z), \frac{2\pi}{q}\Big). \quad \square$$

Now we go back to our proof. We use Claim 3.3 to assert that

$$\sum_{j=1}^{q} \int_{\theta_{j-1}}^{\theta_j} \log \frac{|f_0\cdots\widehat{f_{k_j}}\cdots f_n|(re^{i\theta})}{\|\mathbf{a}^j\|}\,\frac{d\theta}{2\pi}$$
$$\le \bigg|\sum_{j=1}^{q} \int_{\theta_{j-1}}^{\theta_j} \log \frac{\|\mathbf{a}^j\|}{|f_0\cdots\widehat{f_{k_j}}\cdots f_n|(re^{i\theta})}\,\frac{d\theta}{2\pi}\bigg|$$
$$\le \bigg|\sum_{j=1}^{q} \int_{\theta_{j-1}}^{\theta_j} \log \frac{|a^j_{k_j}|(re^{i\theta})}{|f_0\cdots\widehat{f_{k_j}}\cdots f_n|(re^{i\theta})}\,\frac{d\theta}{2\pi}\bigg| + \sum_{j=1}^{q} \frac{1}{q} u\Big(r, |a^j_{k_j}|(z), \frac{2\pi}{q}\Big).$$

Observing that

$$|a^j_{k_j}|(z) = \bigg|\sum_{l=0}^{k_j-1} (-1)^l f_l(z)$$
$$\times \Bigg(\sum_{i_0+\cdots+i_n=\frac{(n-1)(n+2)}{2}} f_0^{(i_0)}(z)\cdots\widehat{f_l(z)}\cdots\widehat{f_{k_j}(z)}\cdots f_n^{(i_n)}(z)\Bigg)$$
$$+ \sum_{l=k_j+1}^{n} (-1)^l f_l(z)\Bigg(\sum_{i_0+\cdots+i_n=\frac{(n-1)(n+2)}{2}} f_0^{(i_0)}(z)\cdots\widehat{f_{k_j}(z)}\cdots\widehat{f_l(z)}\cdots f_n^{(i_n)}(z)\Bigg)\bigg|,$$

we have

$$\begin{aligned}
&\sum_{j=1}^{q}\int_{\theta_{j-1}}^{\theta_j}\log\frac{|a^j_{k_j}|(re^{i\theta})}{|f_0\cdots\widehat{f_{k_j}}\cdots f_n|(re^{i\theta})}\,\frac{d\theta}{2\pi}\\
&=\sum_{j=1}^{q}\int_{\theta_{j-1}}^{\theta_j}\log\frac{|a^j_{k_j}|(re^{i\theta})\cdot|f_{k_j}(re^{i\theta})|}{|f_0\cdots f_{k_j}\cdots f_n|(re^{i\theta})}\,\frac{d\theta}{2\pi}\\
&\le\sum_{j=1}^{q}\int_{\theta_{j-1}}^{\theta_j}\log^+\Big(\frac{|a^j_{k_j}|(re^{i\theta})\cdot|f_{k_j}(re^{i\theta})|}{|f_0\cdots f_n|(re^{i\theta})}\Big)\,\frac{d\theta}{2\pi}\\
&\le\int_0^{2\pi}\log^+\Big(\sum_{i_0+\cdots+i_n=\frac{(n+2)(n-1)}{2}}\Big|\frac{f_0^{(i_0)}}{f_0}\Big|\cdots\Big|\frac{f_n^{(i_n)}}{f_n}\Big|(re^{i\theta})\Big)\,\frac{d\theta}{2\pi}.
\end{aligned}$$

By the definition of the characteristic function, we have

$$T_{f_l}(r)\le T_f(r)+O(1)$$

for each $l=0,\dots,n$. Let $\alpha>0$ and $\alpha n(n+1)<\frac{1}{2}$. From the concavity of the logarithm, Hölder inequality, Lemma 2.3 and the inequality $\big(\sum_i a_i\big)^\alpha \le C\sum_i a_i^\alpha$, we deduce that

$$\begin{aligned}
&\int_0^{2\pi}\log^+\Big(\sum_{i_0+\cdots+i_n=\frac{(n+2)(n-1)}{2}}\Big|\frac{f_0^{(i_0)}}{f_0}\Big|\cdots\Big|\frac{f_n^{(i_n)}}{f_n}\Big|(re^{i\theta})\Big)\,\frac{d\theta}{2\pi}\\
&\le\frac{1}{\alpha}\int_0^{2\pi}\log^+\Big(\sum_{i_0+\cdots+i_n=\frac{(n+2)(n-1)}{2}}\Big|\frac{f_0^{(i_0)}}{f_0}\Big|\cdots\Big|\frac{f_n^{(i_n)}}{f_n}\Big|(re^{i\theta})\Big)^\alpha\,\frac{d\theta}{2\pi}\\
&\le\frac{1}{\alpha}\log^+\Big\{\int_0^{2\pi}\sum_{i_0+\cdots+i_n=\frac{(n+2)(n-1)}{2}}\Big(\Big|\frac{f_0^{(i_0)}}{f_0}\Big|\cdots\Big|\frac{f_n^{(i_n)}}{f_n}\Big|(re^{i\theta})\Big)^\alpha\,\frac{d\theta}{2\pi}\Big\}\\
&\le\frac{1}{\alpha}\log^+\Big\{\sum_{i_0+\cdots+i_n=\frac{(n+2)(n-1)}{2}}\prod_{l=0}^{n}\Big(\int_0^{2\pi}\Big|\frac{f_l^{(i_l)}}{f_l}(re^{i\theta})\Big|^{\alpha(n+1)}\frac{d\theta}{2\pi}\Big)^{\frac{1}{n+1}}\Big\}+O(1)\\
&\le\frac{1}{\alpha}\log^+\Big\{\sum_{i_0+\cdots+i_n=\frac{(n+2)(n-1)}{2}}\prod_{l=0}^{n}\Big(\Big(\frac{\rho}{r(\rho-r)}\Big)^{i_l\alpha}\\
&\quad\times\Big[C_1T_{f_l}(\rho)+C_2\log^+\Big(\frac{R}{\rho(R-\rho)}T_{f_l}(\rho)\Big)\Big]^{i_l\alpha}\Big)\Big\}+O(1)
\end{aligned}$$

$$\le \frac{(n+2)(n-1)}{2}\log^+\Big\{\frac{\rho}{r(\rho-r)}\Big[C_1T_f(\rho)+C_2\log\Big(\frac{R}{\rho(R-\rho)}T_f(\rho)\Big)\Big]\Big\}+O(1).$$

For arbitrary $\varepsilon > 0$, if we put

$$R = r+\frac{1}{\log^{1+\varepsilon}T_f(r)} \quad \text{and} \quad \rho=\frac{R+r}{2}=r+\frac{1}{2\log^{1+\varepsilon}T_f(r)},$$

then for r large enough, $\frac{\rho}{r}\le 2$, $\frac{R}{\rho}\le 2$, $\frac{1}{\rho-r}\le 2\log^{1+\varepsilon}T_f(r)$ and $\frac{1}{R-\rho}\le 4\log^{1+\varepsilon}T_f(r)$. In addition, Lemma 2.4 implies that

$$T_f(\rho)\le T_f(r)+O(1) \quad \|.$$

Combining above inequalities, we have

$$\tag{3.8} \sum_{j=1}^{q}\int_{\theta_{j-1}}^{\theta_j}\log\frac{|f_0\cdots\widehat{f_{k_j}}\cdots f_n|(re^{i\theta})}{\|\mathbf{a}^j\|}\frac{d\theta}{2\pi}$$

$$\le \int_0^{2\pi}\log^+\Big(\sum_{i_0+\cdots+i_n=\frac{(n+2)(n-1)}{2}}\Big|\frac{f_0^{(i_0)}}{f_0}\Big|\cdots\Big|\frac{f_n^{(i_n)}}{f_n}\Big|(re^{i\theta})\Big)\frac{d\theta}{2\pi}$$

$$+\sum_{j=1}^{q}\frac{1}{q}u\Big(r,|a^j_{k_j}|(z),\frac{2\pi}{q}\Big)$$

$$\le \frac{(n+2)(n-1)}{2}\log^+\Big\{\frac{\rho}{r(\rho-r)}\Big[C_1T_f(\rho)+C_2\log\Big(\frac{R}{\rho(R-\rho)}T_f(\rho)\Big)\Big]\Big\}$$

$$+\sum_{j=1}^{q}\frac{1}{q}u\Big(r,|a^j_{k_j}|(z),\frac{2\pi}{q}\Big)+O(1)$$

$$\le \frac{(n+2)(n-1)}{2}\big(\log T_f(r)+(1+\varepsilon)\log^+\log T_f(r)\big)$$

$$+\sum_{j=1}^{q}\frac{1}{q}u\Big(r,|a^j_{k_j}|(z),\frac{2\pi}{q}\Big)+O(1) \quad \|.$$

It follows from (3.6), (4.9) and (3.8) that

$$\tag{3.9} (n+1)T_f(r)-N_{W_f}(r,0)$$

$$\le \overline{m}_{q+n+1}(f,r)+\sum_{j=1}^{q}\frac{1}{q}u\Big(r,|a^j_{k_j}|(z),\frac{2\pi}{q}\Big)+O\Big(u\Big(r,W_f,\frac{2\pi}{q}\Big)\Big)$$

$$+\frac{(n+2)(n-1)}{2}\big(\log T_f(r)+(1+\varepsilon)\log^+\log T_f(r)\big)+\log(2\pi r)+O(1)\quad\|.$$

Step 3: End of the proof. In order to bound the error terms in (3.9), we can always find a constant r_0 such that

$$\frac{(n+2)(n-1)}{2}\big(\log T_f(r)+(1+\varepsilon)\log^+\log T_f(r)\big)+\log(2\pi r)+O(1)\le \varepsilon T_f(r)$$

for all $r>r_0$.

For above $\varepsilon>0$, by Proposition 2.7, we have

$$u(r,W_f,\lambda_{W_f}(r)^{20})\le \varepsilon T_{W_f}(r)$$

for all $r>e$ outside a set E_1 of logarithmic density zero. Here $\lambda_{W_f}(r)=\min\big\{1,\big(\log^+\frac{T_{W_f}(r)}{\log r}\big)^{-1}\big\}$. On the other hand, by Lemma 2.5 (logarithmic derivative lemma), we have

$$T_{W_f}(r)\le C_n\cdot T_f(r)$$

outside a set E_2 of finite Lebesgue measure, where C_n is a constant dependent of n. Hence we obtain

$$u(r,W_f,\lambda_{W_f}(r)^{20})\le \varepsilon T_f(r)$$

for all $r>e$ outside a set $E_1\cup E_2$ of logarithmic density zero.

Again by Proposition 2.7 and Lemma 2.5, we have

$$u(r,|a^j_{k_j}|,\lambda_{k_j}(r)^{20})\le \varepsilon T_f(r),\quad j=1,\dots,q,$$

for all $r>e$ outside a set E_3 of logarithmic density zero. Here

$$\lambda_{k_j}(r)=\min\left\{1,\left(\log^+\frac{T_{|a^j_{k_j}|}(r)}{\log r}\right)^{-1}\right\}.$$

Recall the condition that $v(r)\sim\big(\log^+\frac{T(r)}{\log r}\big)^{20}$. Hence for r sufficiently large, we have $\frac{2\pi}{v(r)}<\lambda(r)^{20}$, where $\lambda(r)=\min\big\{1,\big(\log^+\frac{T_f(r)}{\log r}\big)^{-1}\big\}$. We can find a constant r_1 such that $\lambda(r)^{20}<\lambda_{W_f}(r)^{20}$ and $\lambda(r)^{20}<\lambda_{k_j}(r)^{20}$ for $r>r_1$ outside a set $E_1\cup E_2\cup E_3$ of logarithmic density zero. Hence taking $q=v(r)$ in (3.9), we have

$$u\Big(r,W_f,\frac{2\pi}{v(r)}\Big)\le \varepsilon T_f(r)$$

and

$$u\Big(r, |a_{k_j}^j|, \frac{2\pi}{v(r)}\Big) \le \varepsilon T_f(r), \quad j = 1, \ldots, q,$$

for all $r > e$ outside a larger set $E_4 = (e, r_1] \cup E_1 \cup E_2 \cup E_3$ of logarithmic density zero.

Now we put $E = [e, r_0] \cup E_4$, which is a set of logarithmic density zero. Combining above estimates, we have

$$(n+1)T_f(r) - N_{W_f}(r, 0) \le \overline{m}_{v(r)+n+1}(f, r) + \varepsilon T_f(r)$$

for all $r > e$ outside a larger set E of logarithmic density zero. Then we finish. □

3.3. Application for holomorphic map of finite order. In the following contexts, we will prove a homologous theorem, in which the term $v(r)$ in Theorem 1.1 is assumed to be arbitrary slow growth provided that $f(z)$ is of finite order. This is a parallel generalization of [20].

Let f be a holomorphic curve from $\mathbf{C}$ into the n-dimensional complex projective space $\mathbf{P}^n\mathbf{C}$. We define the order of f as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

Like Theorem 3.2, the major step is to prove the reversal of Cartan's SMT. This is based on the oscillation estimate as Proposition 2.7.

PROPOSITION 3.4 [20, Proposition 1]. *Let f be a transcendental meromorphic function of finite order λ. Let $\varepsilon > 0$, then there exists a positive constant $\theta_{\lambda,\varepsilon}$ such that*

$$u(r, f, \theta_{\lambda,\varepsilon}) \le \varepsilon T_f(r),$$

for all $r > e$ outside a set $E_{\lambda,\varepsilon}$ with $\overline{\log\operatorname{dens}} E_{\lambda,\varepsilon} < \varepsilon$. Here $u(r, f, \theta)$ is the oscillation function defined as Proposition 2.7.

THEOREM 3.5 (finite-order version of Theorem 3.2). *Let $f = [f_0 : \ldots : f_n]\colon \mathbf{C} \to \mathbf{P}^n\mathbf{C}$ be a holomorphic curve that is non-degenerate and that is of finite order λ. For $0 < \varepsilon < 1$, there exists an integer $q_{\lambda,\varepsilon}$ and a set $E_{\lambda,\varepsilon} \subset [e, \infty)$ with $\overline{\log\operatorname{dens}} E_{\lambda,\varepsilon} < \varepsilon$ such that the inequality*

$$\overline{m}_{q_{\lambda,\varepsilon}+n+1}(f, r) + N_{W_f}(r, 0) + o(T_f(r)) \ge (n+1)T_f(r) \tag{3.10}$$

holds for all $r > e$ outside $E_{\lambda,\varepsilon}$, where $q_{\lambda,\varepsilon}$ depends only on λ and ε.

REMARK 5. Yamanoi [20] showed that $\theta_{\lambda,\varepsilon} = \varepsilon^{20}/(2^{140}2^{120\frac{\lambda}{\varepsilon^2}})$ and $q_{\lambda,\varepsilon} = \lceil 2^{203}2^{7680\frac{\lambda}{\varepsilon^2}}/\varepsilon^{20}\rceil$, where $\lceil x\rceil$ is the smallest integer which is not less than x.

Proof. Granted with Proposition 3.4, the proof here is quite similar to Theorem 3.2. We omit the details of first two steps to avoid the repetition. Now let us begin with an analogue of inequality (3.9). Let $q > 0$ be a positive integer. We assert that

$$(3.11)\qquad (n+1)T_f(r) - N_{W_f}(r,0)$$
$$\leq \overline{m}_{q+n+1}(f,r) + \sum_{j=1}^{q} \frac{1}{q} u\Big(r, |a^j_{k_j}|(z), \frac{2\pi}{q}\Big) + O\Big(u\Big(r, W_f, \frac{2\pi}{q}\Big)\Big)$$
$$+ \frac{(n+2)(n-1)}{2}\big(\log T_f(r) + (1+\varepsilon)\log^+\log T_f(r)\big) + \log(2\pi r) + O(1) \quad \|.$$

For $0 < \varepsilon < 1$, we set $q = q_{\lambda,\varepsilon} = \lceil \frac{2\pi}{\theta_{\lambda,\varepsilon}} \rceil$. Note that Wronskian function $W_f(z)$ and the coefficient function $|a^j_{k_j}|(z)$ have the same order λ. Combined with lemma of logarithmic derivative, it yields that there exists a set E_1 with $\overline{\log \operatorname{dens}} E_1 < \varepsilon/4$ such that

$$u\big(r, W_f, \theta_{\lambda, \frac{2\pi}{q_{\lambda,\varepsilon}}}\big) \leq \frac{\varepsilon}{4} T_f(r)$$

and

$$u\big(r, |a^j_{k_j}|(z), \theta_{\lambda, \frac{2\pi}{q_{\lambda,\varepsilon}}}\big) \leq \frac{\varepsilon}{4} T_f(r)$$

hold for all $r > e$ outside E_1. We can always find a constant r_0 such that

$$\frac{(n+2)(n-1)}{2}\big(\log T_f(r) + (1+\varepsilon)\log^+\log T_f(r)\big) + \log(2\pi r) + O(1) \leq \frac{\varepsilon}{2} T_f(r)$$

for all $r > r_0$. Then we put $E_{\lambda,\varepsilon} = [e, r_0] \cup E_1$ with $\overline{\log \operatorname{dens}} E_{\lambda,\varepsilon} < \varepsilon$. We conclude (3.10) as desired. $\square$

Corollary 3.6. *Let $f = [f_0 : \ldots : f_n] \colon \mathbf{C} \to \mathbf{P}^n\mathbf{C}$ be a holomorphic curve that is non-degenerate and that is of finite order. Let $v \colon \mathbf{R}_{>e} \to \mathbf{N}_{>0}$ be a positive function satisfying that $v(r) \to \infty$ and $\log v(r) = o(T_f(r))$ as $r \to \infty$. Then we have*

$$(3.12)\qquad \overline{m}_{v(r)+n+1}(f,r) + N_{W_f}(r,0) = (n+1)T_f(r) + o(T_f(r)),$$

for all $r \to \infty$ outside an exceptional set of logarithmic density 0.

Proof. There is still a narrow gap between our corollary and Theorem 3.5. In Theorem 3.5, we take $\varepsilon = \frac{1}{2^n}$ for any positive integer n. There exists a $r_n > e$ such that $v(r) > q_{\lambda,\frac{1}{2^n}}$ if $r > r_n$. We define a set $F_{\frac{1}{2^n}} \subset [e,\infty)$ such that for any $r \in F_{\frac{1}{2^n}}$, we have

$$\overline{m}_{v(r)+n+1}(f,r) + N_{W_f}(r,0) + \frac{1}{2^n} T_f(r) < (n+1)T_f(r).$$

Then Theorem 3.5 yields that $\overline{\log\operatorname{dens}} F_{\frac{1}{2^n}} < \frac{1}{2^n}$. That is, we can select $r_n > e$ such that for all $r > r_n$, we have

$$\frac{\int_{F_{\frac{1}{2^n}}\cap[e,r]} \frac{dt}{t}}{\log r} < \frac{1}{2^n}.$$

Thereby we obtain a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ as n tends to ∞. For $r \in [r_n, r_{n+1})$, we define $\varepsilon(r) = \frac{1}{2^n}$. Define a set $F \in [r_1, \infty)$ such that for any $r \in F$, we have

$$\overline{m}_{v(r)+n+1}(f, r) + N_{W_f}(r, 0) + \varepsilon(r)T_f(r) < (n+1)T_f(r).$$

Similarly we have

$$\frac{\int_{F\cap[r_1,r]} \frac{dt}{t}}{\log r} < \frac{1}{2^n}$$

for any $r \in [r_n, r_{n+1})$. It implies that

$$\lim_{r\to\infty} \frac{\int_{F\cap[e,r]} \frac{dt}{t}}{\log r} = 0$$

as $\lim_{r\to\infty} \varepsilon(r) = 0$. Thus the inequality (3.12) holds when $r \to \infty$ outside the set F of logarithmic density 0. □

4. Asymptotic equality of SMT for holomorphic mappings over $\mathbf{C}^p$

We fix p a positive integer for what follows. Let $f\colon \mathbf{C}^p \to \mathbf{P}^n\mathbf{C}$ be a linearly non-degenerate holomorphic mapping. The main purpose of this section is to strengthen the asymptotic equality of SMT for holomorphic mappings by implementing the definition of geometric generalized Wronskians in [5].

THEOREM 4.1. *Given $1 \le p \le n$, let $f = [f_0 : \dots : f_n]\colon \mathbf{C}^p \to \mathbf{P}^n\mathbf{C}$ be a holomorphic mapping that is non-degenerate. Let $v\colon \mathbf{R}_{>e} \to \mathbf{N}_{>0}$ satisfies that $v(r) \sim \left(\log^+ \frac{T_f(r)}{\log r}\right)^{20}$. Then there exists a full set $\mathcal{S}$ and the associated geometric generalized Wronskian $W_{\mathcal{S}}$ such that for arbitrary $0 < \varepsilon < 1$, we have*

$$\overline{m}_{pv(r)+n+1}(f, r) + N_{W_{\mathcal{S}},f}(r, 0) + \varepsilon(T_f(r)) = (n+1)T_f(r) + O(1), \tag{4.1}$$

for all $r \to \infty$ outside an exceptional set of logarithmic density 0.

Theorem 4.1 is obviously derived from Theorem 4.4 (general form of SMT) and Theorem 4.7 (recersion of SMT).

4.1. Preliminaries of Nevanlinna theory in several complex variables. First we recall some notations. For $z = (z_1, \ldots, z_p)$ in $\mathbf{C}^p$, we define the norm as $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. Denote by

$$\omega(z) = dd^c \log|z|^2$$

the homogeneous metric form on $\mathbf{C}^p$ and denote by

$$v(z) = dd^c|z|^2, \quad \sigma_p(z) = d^c \log|z|^2 \wedge \omega(z)^{p-1}$$

where $d = \partial + \overline{\partial}$ and $d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$. Put

$$\mathbf{B}_p(r) = \{z \in \mathbf{C}^p \mid |z| < r\}$$

the ball of radius r in $\mathbf{C}^p$, and the sphere

$$\mathbf{S}_p(r) = \partial \mathbf{B}_p(r) = \{z \in \mathbf{C}^p \mid |z| = r\}.$$

It is clear that the total measure of the form $\sigma_p(z)$ along $\mathbf{S}_p(r)$ will be 1.

Let $f = [f_0 : \ldots : f_n] \colon \mathbf{C}^p \to \mathbf{P}^n\mathbf{C}$ be a holomorphic mapping under the homogeneous coordinate $[w_0 : \ldots : w_n]$ of $\mathbf{P}^n\mathbf{C}$. For each hyperplane $H = \{[w_0 : \ldots : w_n] \in \mathbf{P}^n\mathbf{C} \mid a_0 w_0 + \cdots + a_n w_n = 0\}$ associated with the non-zero vector $\mathbf{a} = (a_0, \ldots, a_n)$ in $\mathbf{C}^{n+1}$, we suppose that the image of f does not degenerate into $\operatorname{supp}(H)$. Then we can define the counting function of f with respect to H as

$$N_f(r, H) = \int_0^r \frac{dt}{t^{2p-1}} \int_{f^{-1}(H) \cap \mathbf{B}_p(t)} v(z)^{p-1}.$$

And the approximation function of f with respect to H is given by

$$m_f(r, H) = \int_{\mathbf{S}_p(r)} \log \frac{\|f(z)\| \cdot \|H\|}{|\langle f(z), \mathbf{a} \rangle|} \sigma_p(z),$$

where $\|\mathbf{f}(z)\|$ and $\|H\|$ are defined as in Section 2.1. And the characteristic function is given by

$$T_{f,\omega_{FS}}(r) = \int_0^r \frac{dt}{t^{2p-1}} \int_{\mathbf{B}_p(t)} f^* \omega_{FS} \wedge v(z)^{p-1} = \int_{\mathbf{S}_p(r)} \log \|f(z)\| \, \sigma_p(z),$$

where ω_{FS} is the Fubini–Study metric on $\mathbf{P}^n\mathbf{C}$. Similarly, we also follow Poincaré–Lelong formular and Jensen fomular to deduce the First Main Theorem:

$$T_{f,\omega_{FS}}(r) = m_f(r, H) + N_f(r, H) + O(1). \tag{4.2}$$

Remark 6. The equality (4.2) is also valid if one replaces the hyperplane H by any hypersurface D in $\mathbf{P}^n\mathbf{C}$.

4.2. Geometric Generalized Wronskians. Wronskian is a fundamental tool to check the linear dependence. Given $n+1$ holomorphic functions $f_0, \dots, f_n \in \mathcal{O}_{\mathbf{C}}$, they are linearly independent if and only if their Wronskian $W(f_0, \dots, f_n)$ does not vanish identically. It is generalized to multi-variables case by many mathematicians. In this subsection, we will introduce the geometric generalized Wronskians defined by Etesse [5].

Denote $\mathcal{W}_p$ the set of words written in the lexicographic order with the alphabet $\{1, \dots, p\}$. That is, any word $\overline{u} \in \mathcal{W}_p$ is defined as

$$\overline{u} = 1^{\alpha_1(\overline{u})} \cdots p^{\alpha_p(\overline{u})},$$

where $\alpha_i(\overline{u})$ is the number of occurences of the letter i in the word $\overline{u}$. Then we define

$$\partial_{\overline{u}} = \frac{\partial^{\alpha_1(\overline{u})+\cdots+\alpha_p(\overline{u})}}{\partial z_1^{\alpha_1(\overline{u})} \cdots \partial z_p^{\alpha_p(\overline{u})}}.$$

Let $\mathcal{U} = \{\overline{u}_1, \dots, \overline{u}_n\}$ be a finite set in $\mathcal{W}_p$ of size $n = card(\mathcal{U})$. We say $\mathcal{U}$ is admissible if there exists an ordering of words $\{\overline{u}_1, \dots, \overline{u}_n\}$ in $\mathcal{U}$ such that $l(\overline{u}_i) \leq i$ for $i = 1, \dots, n$. Let $f_0, \dots, f_n \in \mathcal{O}_{\mathbf{C}^p}$ be $n+1$ holomorphic functions over $\mathbf{C}^p$. The associated Wronskian $W_{\mathcal{U}}$ is defined as follows:

$$W_{\mathcal{U}}(f_0, \dots, f_n) = \begin{vmatrix} f_0 & \dots & f_n \\ \partial_{\overline{u}_1}(f_0) & \dots & \partial_{\overline{u}_1}(f_n) \\ \vdots & \dots & \vdots \\ \partial_{\overline{u}_n}(f_0) & \dots & \partial_{\overline{u}_n}(f_n) \end{vmatrix}.$$

W is called a geometric generalized Wronskian if for any $g, f_0, \dots, f_n \in \mathbf{C}^p$, it satisfies that $W(gf_0, \dots, gf_n) = g^{n+1} W(f_0, \dots, f_n)$. Etesse [5] proved that $W_{\mathcal{U}}$ is a geometric generalized Wronskian with respect to an admissible set $\mathcal{U}$ if and only if $\mathcal{U}$ is a full set, which means that for any word $\overline{u} \in \mathcal{U}$, each subword of $\overline{u}$ also belongs to $\mathcal{U}$.

There are two important applications of geometric generalized Wronskian. Firstly, it can imply the linear dependence.

THEOREM 4.2 [5, Theorem1.4.1]. *The holomorphic functions $f_0, \dots, f_n$ on $\mathbf{C}^p$ are linearly independent if and only if there exists a full set $\mathcal{U}$ such that $W_{\mathcal{U}}(f_0, \dots, f_n)$ does not vanish identically.*

Secondly, it induces the global invariant jet differentials on projective varieties.

THEOREM 4.3 [5, Theorem 1.2.7]. *Let X be a projective variety equipped with a line bundle $L \to X$. Let $s_0, \dots, s_n$ be global sections of L. Consider the local chart U of X and $x \in U$. Let $\gamma\colon (\mathbf{C}^p, 0) \to (X, x)$ be a holomorphic* *germ through x. Then the geometric generalized Wronskian W induces a global section $W(s_0,\dots,s_n)$ of $E_{p,k,w}\otimes L^{n+1}$ locally defined as*

$$W(s_0,\dots,s_n)(\gamma)=W(s_{0,U}\circ\gamma,\dots,s_{n,U}\circ\gamma).$$

Here n,k,w are respectively the size, order and weight of W defined in [5].

In the next subsection, we will utilize Theorem 4.3 when X is the n-dimensional projective complex space, $L=\mathcal{O}(1)$ is the hyperplane line bundle and $w_0,\dots,w_n$ are the coordinate sections.

4.3. General form of Cartan's SMT for holomorphic mappings over $\mathbf{C}^p$. This subsection is devoted to the several variables version of Theorem 2.2. We follow closely the proof in [10] with some suitable adaptions for several variables settings. Thus we will just indicate the differences and we invite readers to refer to [10] and [12].

THEOREM 4.4. *Let $f=[f_0:\ldots:f_n]\colon \mathbf{C}^p\to\mathbf{P}^n\mathbf{C}$ be a holomorphic mapping which is non-degenerated. Let $H_1,\ \ldots,\ H_q$ be arbitrary hyperplanes in $\mathbf{P}^n\mathbf{C}$. Then there exists a geometric generalized Wronskian $W_{\mathcal{S}}$ such that for arbitrary $\varepsilon>0$, we have*

$$\int_{\mathbf{S}_p(r)}\max_K\sum_{k\in K}\lambda_{H_k}(f(re^{i\theta}))\,\sigma_p(z)+N_{W_{\mathcal{S}},f}(r,0)\le(n+1)T_f(r)+\varepsilon T_f(r)\quad\|,\tag{4.3}$$

where the notations serve as the ones in Theorem 2.2.

PROOF. Denote by $\mathbf{a}_1,\dots,\mathbf{a}_q$ q vectors in $\mathbf{C}^{n+1}$ associated with the hyperplanes $H_1,\dots,H_q$ respectively. Without loss of generality, we assume that there are always $n+1$ linearly independent hyperplanes. Let $\mu:\{0,1,\dots,n\}\to\{1,\dots,q\}$ be an injective map such that $\mathbf{a}_{\mu(0)},\ \ldots,\ \mathbf{a}_{\mu(n)}$ are linearly independent. Since f is non-degenerated, there exists a geometric generalized Wronskian $W_{\mathcal{S}}$ associated with a full set $\mathcal{S}=\{\overline{u}_1,\dots,\overline{u}_n\}$ such that $W_{\mathcal{S}}(f)\not\equiv 0$. Thus we have

$$\begin{aligned}\int_{\mathbf{S}_p(r)}\max_K\sum_{k\in K}\lambda_{H_k}(f(re^{i\theta}))\,\sigma_p(z)&=\int_{\mathbf{S}_p(r)}\max_\mu\sum_{j=0}^n\log\left(\frac{\|f(z)\|\cdot\|H_{\mu(j)}\|}{|\langle f(z),\mathbf{a}_{\mu(j)}\rangle|}\right)\sigma_p(z)\\ &\le\int_{\mathbf{S}_p(r)}\max_\mu\log\frac{|W_{\mathcal{S}}(\langle f(z),\mathbf{a}_{\mu(0)}\rangle,\dots,\langle f(z),\mathbf{a}_{\mu(n)}\rangle)|}{\prod_{j=0}^n|\langle f(z),\mathbf{a}_{\mu(j)}\rangle|}\,\sigma_p(z)\\ &\quad+\int_{\mathbf{S}_p(r)}\log\frac{\|f(z)\|^{n+1}}{|W_{\mathcal{S}}(f)|}\,\sigma_p(z)+O(1).\end{aligned}$$

Here $W_{\mathcal{S}}\big(\langle f(z),\mathbf{a}_{\mu(0)}\rangle,\dots,\langle f(z),\mathbf{a}_{\mu(n)}\rangle\big)$ is the geometric generalized Wronskian of functions $\langle f(z),\mathbf{a}_{\mu(0)}\rangle$, ..., $\langle f(z),\mathbf{a}_{\mu(n)}\rangle$, which is also not identically zero since the hyperplanes are in general position.

To estimate the first term, we recall the following lemmas on logarithmic derivative for several complex variables.

LEMMA 4.5 [12, Theorem A8.1.4]. *Let $g=[g_0:g_1]\colon \mathbf{C}^p\to\mathbf{P}^1\mathbf{C}$ be a meromorphic function, and let $\overline{u}$ be a word in $\mathcal{W}_p$. Then for any α with $0<\alpha l(\overline{u})<\frac{1}{2}$, there exist positive constants C_0, C_1, C_2 such that for any $r<\rho<R$, we have*

$$\int_{\mathbf{S}_p(r)}\log^+\left|\frac{\partial_{\overline{u}}(\frac{g_1}{g_0})}{\frac{g_1}{g_0}}\right|^{\alpha}\sigma_p(z)\le C_0\Big(\frac{\rho}{r}\Big)^{\alpha l(\overline{u})(2n-2)}\Big(\frac{\rho}{r(\rho-r)}\Big)^{\alpha l(\overline{u})}$$
$$\times\Big[C_1T_g(\rho)+C_2\log\Big(\Big(\frac{R}{\rho}\Big)^{\alpha(2n-2)}\frac{R}{\rho(R-\rho)}T_g(R)\Big)\Big]^{\alpha l(\overline{u})}.$$

LEMMA 4.6 ([5, Theorem B.0.3], [12, Theorem A8.1.5]). *Let $g=[g_0:g_1]:\mathbf{C}^p\to\mathbf{P}^1\mathbf{C}$ be a meromorphic function, and let $\overline{u}$ be a word in $\mathcal{W}_p$. Then for any $\varepsilon>0$ we have*

$$\int_{\mathbf{S}_p(r)}\log^+\left|\frac{\partial_{\overline{u}}(g)}{g}\right|\sigma_p(z)\le l(\overline{u})\log T_g(r)+(1+\varepsilon)\log^+\log T_g(r)\quad\|,$$

where C_0,C_1,C_2 are positive constants.

For $j=0,\dots,n$, denote by $g_{\mu(j)}(z)=\frac{|\langle f(z),\mathbf{a}_{\mu(j)}\rangle|}{|\langle f(z),\mathbf{a}_{\mu(0)}\rangle|}$. It is clear that $T_{g_{\mu(j)}}(z)\le T_{f,\omega_{FS}}(r)+O(1)$ for each j. Selecting an α with $0<\alpha l(\overline{u})<1/2$ for any word $\overline{u}\in\mathcal{S}$, we yields that

$$\int_{\mathbf{S}_p(r)}\max_{\mu}\log\frac{|W_{\mathcal{S}}(\langle f(z),\mathbf{a}_{\mu(0)}\rangle,\dots,\langle f(z),\mathbf{a}_{\mu(n)}\rangle)|}{\prod_{j=0}^n|\langle f(z),\mathbf{a}_{\mu(j)}\rangle|}\sigma_p(z)$$
$$\le\frac{1}{\alpha}\int_{\mathbf{S}_p(r)}\max_{\mu}\Big\{\log\Big(\frac{|W_{\mathcal{S}}(1,g_{\mu(1)},\dots,g_{\mu(n)})|}{|g_{\mu(1)},\dots,g_{\mu(n)}|}(z)\Big)^{\alpha}\Big\}\sigma_p(z)+O(1)$$
$$\le\frac{1}{\alpha}\log^+\int_{\mathbf{S}_p(r)}$$
$$\max_{\mu}\Big(\sum_{k_1+\dots+k_n\le n(n+1)/2}\left|\frac{\partial_{\overline{u}_{k_1}}(g_{\mu(1)})}{g_{\mu(1)}}\cdots\frac{\partial_{\overline{u}_{k_n}}(g_{\mu(n)})}{g_{\mu(n)}}\right|^{\alpha}(z)\Big)\sigma_p(z)+O(1)$$
$$\le\frac{1}{\alpha}\log^+\Big\{\max_{\mu}\Big[\sum_{k_1+\dots+k_n\le n(n+1)/2}\prod_{l=1}^n\Big(\int_{\mathbf{S}_p(r)}\left|\frac{\partial_{\overline{u}_{k_l}}(g_{\mu(l)})}{g_{\mu(l)}}(z)\right|^{\alpha\cdot l(\overline{u}_{k_l})}\sigma_p(z)\Big)\Big]\Big\}$$

$$+ O(1) \leq l(\mathcal{S})\big(\log T_{f,\omega_{FS}}(r) + (1+\varepsilon)\log^+ \log T_{f,\omega_{FS}}(r)\big) + O(1) \quad \|.$$

By definitions, the second term becomes

$$\int_{\mathbf{S}_p(r)} \log \frac{\|f(z)\|^{n+1}}{|W_{\mathcal{S}}(f)|}\, \sigma_p(z) = (n+1)T_{f,\omega_{FS}}(r) - N_{W_{\mathcal{S}}}(r,0).$$

Now we combine with the previous inequalities and conclude (4.3). □

4.4. End of proof. Tracking the same ideas in Section 3.3, we establish the reversal of Theorem 4.4.

Definition 3 (modified proximity function). Let $f = [f_0 : \dots : f_n] \colon \mathbf{C}^p \to \mathbf{P}^n\mathbf{C}$ be a holomorphic map. For any positive integer q and rational number $r > e$, we define

$$\overline{m}_q(f,r) = \sup_{H_1,\dots,H_q} \int_{\mathbf{S}_p(r)} \max_K \sum_{k\in K} \lambda_{H_k}(f(re^{i\theta}))\, \sigma_p(z),$$

where the maximum is taken over all subsets K of $\{1,\dots,q\}$ such that the hyperplanes H_k, $k \in K$ are linearly independent, and the superior is taken over all sets of q arbitrary hyperplanes in $\mathbf{P}^n\mathbf{C}$.

Theorem 4.7. *Given $1 \leq p \leq n$, let $f = [f_0 : \dots : f_n] \colon \mathbf{C}^p \to \mathbf{P}^n\mathbf{C}$ be a holomorphic mapping that is non-degenerate. Let $v \colon \mathbf{R}_{>e} \to \mathbf{N}_{>0}$ satisfies that $v(r) \sim (\log^+ \frac{T(r)}{\log r})^{20}$. Then there exists a geometric generalized Wronskian $W_{\mathcal{S}}$ such that for arbitrary $0 < \varepsilon < 1$, we have*

$$\overline{m}_{pv(r)+n+1}(f,r) + N_{W_{\mathcal{S}},f}(r,0) + \varepsilon(T_f(r)) \geq (n+1)T_f(r) + O(1), \tag{4.4}$$

for all $r \to \infty$ outside an exceptional set of logarithmic density 0.

Proof. We also follow the three steps exposed in Theorem 3.2.

Step 1: Select $n+1$ hyperplanes.

We fix an $r > e$ and work on the sphere $\{\|z\| = r\}$. Given a positive integer q, for all $k = 1,\dots,p$, we put $j_k \in \{1,\dots,q\}$, $\theta_{j_k} = \frac{2\pi j}{q}$ and $z_k^{j_k} = r_k e^{i\theta_{j_k}}$. In this way, we equi-divide the sphere $\{\|z\| = r\}$ into pq parts and

$$\big\{z_{j_1,\dots,j_p} = (r_1 e^{i\theta_{j_1}}, \dots, r_p e^{i\theta_{j_p}}) : r_1^2 + \dots + r_p^2 = r^2\big\}_{(j_1,\dots,j_p)\in\{1,\dots,q\}^p}$$

are pq break points. According to the construction in [5], there exists some geometric generalized Wronskian $W_{\mathcal{S}}$ associated with the full set $\mathcal{S} = \{\overline{u}_1,\dots,\overline{u}_n\}$, such that $W_{\mathcal{S}}(f_0,\dots,f_n) \not\equiv 0$. Without loss of generality, we assume that $\partial_{\overline{u}_1} = \frac{\partial}{\partial z_p}$, the partial derivative of order 1 with respect to the last variable. Denote by

$$W_{\mathcal{S}}(f) = \begin{vmatrix} f_0 & \dots & f_n \\ \partial_{\overline{u}_1}(f_0) & \dots & \partial_{\overline{u}_1}(f_n) \\ \vdots & \dots & \vdots \\ \partial_{\overline{u}_n}(f_0) & \dots & \partial_{\overline{u}_n}(f_n) \end{vmatrix}.$$

For $(j_1, \dots, j_p) \in \{1, \dots, q\}^p$, we define the hyperplane $D^{j_1,\dots,j_p}$ in $\mathbf{P}^n\mathbf{C}$ as

$$D^{j_1,\dots,j_p} = \Big\{ [w_0 : \dots : w_n] \in \mathbf{P}^n\mathbf{C} : \sum_{k=0}^{n} a_k^{j_1,\dots,j_p} w_k = 0 \Big\},$$

where the associated vector $\mathbf{a}^{j_1,\dots,j_p} = (a_0^{j_1,\dots,j_p}, \dots, a_n^{j_1,\dots,j_p})$ satisfies that each component $a_k^{j_1,\dots,j_p}$ is the cofactor of the Wronskian $W_{\mathcal{S}}(f(z_{j_1,\dots,j_p}))$ with respect to the entries $\partial_{\overline{u}_1}(f_k(z_{j_1,\dots,j_p}))$. More precisely, we define the hyperplane $D^{j_1,\dots,j_p}$ such that

$$\begin{vmatrix} f_0(z_{j_1,\dots,j_p}) & \dots & f_n(z_{j_1,\dots,j_p}) \\ w_0 & \dots & w_n \\ \partial_{\overline{u}_2}(f_0(z_{j_1,\dots,j_p})) & \dots & \partial_{\overline{u}_2}(f_n(z_{j_1,\dots,j_p})) \\ \vdots & \dots & \vdots \\ \partial_{\overline{u}_n}(f_0(z_{j_1,\dots,j_p})) & \dots & \partial_{\overline{u}_n}(f_n(z_{j_1,\dots,j_p})) \end{vmatrix} = 0.$$

Then we have

$$\langle \mathbf{f}(z), \mathbf{a}^{j_1,\dots,j_p} \rangle = \begin{vmatrix} f_0(z_{j_1,\dots,j_p}) & \dots & f_n(z_{j_1,\dots,j_p}) \\ f_0(z_1, \dots, z_p) & \dots & f_n(z_1, \dots, z_p) \\ \partial_{\overline{u}_2}(f_0(z_{j_1,\dots,j_p})) & \dots & \partial_{\overline{u}_2}(f_n(z_{j_1,\dots,j_p})) \\ \vdots & \dots & \vdots \\ \partial_{\overline{u}_n}(f_0(z_{j_1,\dots,j_p})) & \dots & \partial_{\overline{u}_n}(f_n(z_{j_1,\dots,j_p})) \end{vmatrix}.$$

Moreover, we have

$$\langle \mathbf{f}(z), \mathbf{a}^{j_1,\dots,j_p} \rangle \big|_{z=z_{j_1,\dots,j_p}} = 0,$$

and

$$\partial_{\overline{u}_1}\big(\langle \mathbf{f}(z), \mathbf{a}^{j_1,\dots,j_p} \rangle\big)\big|_{z=z_{j_1,\dots,j_p}} = W_{\mathcal{S}}(f(z_{j_1,\dots,j_p})).$$

Recalling the oscillation function $u(r, f, \theta)$ and the estimation in Proposition 2.7, we bound the difference between the multivariate function

$\mathbf{f}(z_1,\dots,z_p)$ and univariate function $\mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},z_p)$ for fixed components $(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}})$ in $\mathbf{C}^{p-1}$. This is the first distinction between single variable and multiple variables. Define

$$
\begin{aligned}
(4.5)\quad I^1_{j_1,\dots,j_p} &\triangleq \big|\big\langle \mathbf{f}(z_1,\dots,z_p),\mathbf{a}^{j_1,\dots,j_p}\big\rangle-\big\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},z_p),\mathbf{a}^{j_1,\dots,j_p}\big\rangle\big|\\
&\le \big|\big\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},z_p),\mathbf{a}^{j_1,\dots,j_p}\big\rangle-\big\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-2}^{j_{p-2}},z_{p-1},z_p),\mathbf{a}^{j_1,\dots,j_p}\big\rangle\big|\\
&\quad+\big|\big\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-2}^{j_{p-2}},z_{p-1},z_p),\mathbf{a}^{j_1,\dots,j_p}\big\rangle\\
&\quad-\big\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-3}^{j_{p-3}},z_{p-2},z_{p-1},z_p),\mathbf{a}^{j_1,\dots,j_p}\big\rangle\big|\\
&\quad+\dots+\big|\big\langle \mathbf{f}(z_1^{j_1},z_2,\dots,z_p),\mathbf{a}^{j_1,\dots,j_p}\big\rangle-\big\langle \mathbf{f}(z_1,\dots,z_p),\mathbf{a}^{j_1,\dots,j_p}\big\rangle\big|\\
&\le u\Big(\sqrt{r^2-(r_1^2+\dots+r_{p-2}^2+|\xi_p|^2)},\big\langle \mathbf{f}\big|_{z_1^{j_1},\dots,z_{p-2}^{j_{p-2}},\xi_p}(z_{p-1}),\mathbf{a}^{j_1,\dots,j_p}\big\rangle,\frac{2\pi}{q}\Big)\\
&\quad+u\Big(\sqrt{r^2-(r_1^2+\dots+r_{p-3}^2+|\xi_{p-1}|^2+|\xi_p|^2)},\\
&\quad\big\langle \mathbf{f}\big|_{z_1^{j_1},\dots,z_{p-3}^{j_{p-3}},\xi_{p-1},\xi_p}(z_{p-2}),\mathbf{a}^{j_1,\dots,j_p}\big\rangle,\frac{2\pi}{q}\Big)\\
&\quad+\dots+u\Big(\sqrt{r^2-(|\xi_2|^2+\dots+|\xi_p|^2)},\big\langle \mathbf{f}\big|_{\xi_2,\dots,\xi_p}(z_1),\mathbf{a}^{j_1,\dots,j_p}\big\rangle,\frac{2\pi}{q}\Big)\\
&\le O\Big(u\Big(\sqrt{r^2-(r_1^2+\dots+r_{p-2}^2+|\xi_p|^2)},f\big|_{z_1^{j_1},\dots,z_{p-2}^{j_{p-2}},\xi_p}(z_{p-1}),\frac{2\pi}{q}\Big)\\
&\quad+u\Big(\sqrt{r^2-(r_1^2+\dots+r_{p-3}^2+|\xi_{p-1}|^2+|\xi_p|^2)},f\big|_{z_1^{j_1},\dots,z_{p-3}^{j_{p-3}},\xi_{p-1},\xi_p}(z_{p-2}),\frac{2\pi}{q}\Big)\\
&\quad+\dots+u\Big(\sqrt{r^2-(|\xi_2|^2+\dots+|\xi_p|^2)},f\big|_{\xi_2,\dots,\xi_p}(z_1),\frac{2\pi}{q}\Big)\Big).
\end{aligned}
$$

Here $f\big|_{z_1^{j_1},\dots,z_{k-1}^{j_{k-1}},\xi_{k+1},\dots,\xi_p}(z_k)$, $k=2,\dots,p$, denotes the complex function with fixed parameters

$$(\xi_{k+1},\dots,\xi_p)\in\big((\theta_{j_{k+1}-1},\theta_{j_{k+1}}],\dots,(\theta_{j_p-1},\theta_{j_p}]\big)$$

and fixed components $(z_1^{j_1},\dots,z_{k-1}^{j_{k-1}})$ which are evaluated at the very beginning. Thus, using the elementary integral formula for

$$\big\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},z_p),\mathbf{a}^{j_1,\dots,j_p}\big\rangle,$$

for any $\theta_p\in[\theta_{j_p-1},\theta_{j_p}]$, we have

$$\big\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},r_pe^{i\theta_p}),\mathbf{a}^{j_1,\dots,j_p}\big\rangle$$

$$= \int_{\theta_{j_p-1}}^{\theta_p} \partial_{\overline{u}_1}\big(\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},r_pe^{is}),\mathbf{a}^{j_1,\dots,j_p}\rangle\big)\,d(r_pe^{is}).$$

Then for any $\theta\in[\theta_{j-1},\theta_j]$, we obtain that

$$\big|\big\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},z_p),\mathbf{a}^{j_1,\dots,j_p}\big\rangle\big| \le e^{\tau^{j_1,\dots,j_p}}(2\pi r),$$

where

$$\tau^{j_1,\dots,j_p} = \max_{\substack{\theta'_k\in(\theta_{j_k-1},\theta_k]\\ k=1,\dots,p}} \log\big|W_{\mathcal{S}}(f(r_1e^{i\theta'_1},\dots,r_pe^{i\theta'_p}))\big|$$
$$+ \max_{\substack{\theta'_k\in(\theta_{j_k-1},\theta_k]\\ k=1,\dots,p}} \log\frac{\big|\partial_{\overline{u}_1}\big(\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},r_pe^{i\theta'_p}),\mathbf{a}^{j_1,\dots,j_p}\rangle\big)\big|}{\big|W_{\mathcal{S}}(f(r_1e^{i\theta'_1},\dots,r_pe^{i\theta'_p}))\big|}.$$

By the definition of $\mathbf{a}^{j_1,\dots,j_p}$, we have

$$I^2_{j_1,\dots,j_p} \triangleq \log\frac{\big|\partial_{\overline{u}_1}\big(\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},r_pe^{i\theta'_p}),\mathbf{a}^{j_1,\dots,j_p}\rangle\big)\big|}{\big|W_{\mathcal{S}}(f(r_1e^{i\theta'_1},\dots,r_pe^{i\theta'_p}))\big|} \tag{4.6}$$
$$\le \Big[\log\big|\partial_{\overline{u}_1}\big(\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},r_pe^{i\theta'_p}),\mathbf{a}^{j_1,\dots,j_p}\rangle\big)\big| - \log\big|W_{\mathcal{S}}(f(z_1^{j_1},\dots,z_p^{j_p}))\big|\Big]$$
$$+ \Big[\log\big|W_{\mathcal{S}}(f(z_1,\dots,z_{p-1},z_p^{j_p}))\big| - \log\big|W_{\mathcal{S}}(f(z_1,\dots,z_{p-1},z_p))\big|\Big]$$
$$+ \Big[\log\big|W_{\mathcal{S}}(f(z_1,\dots,z_{p-2},z_{p-1}^{j_{p-1}},z_p^{j_p}))\big| - \log\big|W_{\mathcal{S}}(f(z_1,\dots,z_{p-2},z_{p-1},z_p^{j_p}))\big|\Big]$$
$$+\cdots+ \Big[\log\big|W_{\mathcal{S}}(f(z_1^{j_1},\dots,z_p^{j_p}))\big| - \log\big|W_{\mathcal{S}}(f(z_1,z_2^{j_2},\dots,z_p^{j_p}))\big|\Big]$$
$$\le O\Big(u\Big(\sqrt{r^2-(r_1^2+\cdots+r_{p-1}^2)},\partial_{\overline{u}_1}\Big(f\big|_{z_1^{j_1},\dots,z_{p-1}^{j_{p-1}}}(z_p)\Big),\frac{2\pi}{q}\Big)\Big)$$
$$+u\big(\sqrt{r^2-(|\xi_1|^2+\cdots+|\xi_{p-1}|^2)},W_{\mathcal{S}}(f\big|_{\xi_1,\dots,\xi_{p-1}}(z_p)),\frac{2\pi}{q}\big)$$
$$+u\Big(\sqrt{r^2-(|\xi_1|^2+\cdots+|\xi_{p-2}|^2+r_p^2)},W_{\mathcal{S}}\big(f\big|_{\xi_1,\dots,\xi_{p-2},z_p^{j_p}}(z_{p-1})\big),\frac{2\pi}{q}\Big)$$
$$+\cdots+u\Big(\sqrt{r^2-(r_2^2+\cdots+r_p^2)},W_{\mathcal{S}}\big(f\big|_{z_2^{j_2},\dots,z_p^{j_p}}(z_1)\big),\frac{2\pi}{q}\Big).$$

Combining (4.5) and (4.6), for any $\theta_k\in[\theta_{j_k-1},\theta_{j_k}], k=1,\dots,p$, we have

$$\log\frac{1}{|\langle \mathbf{f}(z_1,\dots,z_p),\mathbf{a}^{j_1,\dots,j_p}\rangle|}$$

$$\geq \log \frac{1}{|\langle \mathbf{f}(z_1^{j_1},\dots,z_{p-1}^{j_{p-1}},z_p),\mathbf{a}^{j_1,\dots,j_p}\rangle|} - I^1_{j_1,\dots,j_p}$$

$$\geq - \max_{\substack{\theta'_k\in(\theta_{j_k-1},\theta_k]\\ k=1,\dots,p}} \log\big|W_{\mathcal{S}}(f(r_1e^{i\theta'_1},\dots,r_pe^{i\theta'_p}))\big| - I^2_{j_1,\dots,j_p} - \log(2\pi r) - I^1_{j_1,\dots,j_p}$$

$$\geq \log \frac{1}{|W_{\mathcal{S}}(f(r_1e^{i\theta_1},\dots,r_pe^{i\theta_p}))|} - I^3_{j_1,\dots,j_p} - I^2_{j_1,\dots,j_p} - I^1_{j_1,\dots,j_p} - \log(2\pi r),$$

where

$$I^3_{j_1,\dots,j_p} = \max_{\substack{\theta'_k\in(\theta_{j_k-1},\theta_k]\\ k=1,\dots,p}} \log\big|W_{\mathcal{S}}(f(r_1e^{i\theta'_1},\dots,r_pe^{i\theta'_p}))\big|$$

$$- \log\big|W_{\mathcal{S}}(f(r_1e^{i\theta_1},\dots,r_pe^{i\theta_p}))\big|$$

$$\leq u\Big(\sqrt{r^2-(|\xi_1|^2+\dots+|\xi_{p-1}|^2)}, W_{\mathcal{S}}\big(f\big|_{\xi_1,\dots,\xi_{p-1}}(z_p)\big), \frac{2\pi}{q}\Big)$$

$$+ u\Big(\sqrt{r^2-(|\xi_1|^2+\dots+|\xi_{p-2}|^2+r_p^2)}, W_{\mathcal{S}}\big(f\big|_{\xi_1,\dots,\xi_{p-2},r_pe^{i\theta'_p}}(z_{p-1})\big), \frac{2\pi}{q}\Big)$$

$$+ \dots + u\Big(\sqrt{r^2-(r_2^2+\dots+r_p^2)}, W_{\mathcal{S}}\big(f\big|_{r_2e^{i\theta'_2},\dots,r_pe^{i\theta'_p}}(z_1)\big), \frac{2\pi}{q}\Big).$$

Integrating along the sphere $\mathbf{S}_p(r)$, we have

$$\int_{\mathbf{S}_p(r)} \log \frac{1}{|W_{\mathcal{S}}(f(r_1e^{i\theta_1},\dots,r_pe^{i\theta_p}))|}\,\sigma_p(z) \tag{4.7}$$

$$\leq \sum_{\substack{j=1,\dots,q\\ k=1,\dots,p}} \int_{\theta_k\in(\theta_{j_k-1},\theta_{j_k}]} \log \frac{1}{|\langle \mathbf{f}(z),\mathbf{a}^{j_1,\dots,j_p}\rangle|}\,\sigma_p(z)$$

$$+ \int_{\mathbf{S}_p(r)} \big(I^1_{j_1,\dots,j_p}+I^2_{j_1,\dots,j_p}+I^3_{j_1,\dots,j_p}\big)\,\sigma_p(z) + \log(2\pi r).$$

Denote by D_l, $l=0,\dots,n$, the $n+1$ coordinate hyperplanes in $\mathbf{P}^n\mathbf{C}$ with

$$D_l = \{[w_0:\dots:w_n]\big|w_l=0\},$$

and the associated vectors

$$\mathbf{a}_l = [a_{0,l}:\dots:a_{l,l}:\dots:a_{n,l}] = [0:\dots:1:\dots:0].$$

Assume that

$$\|\mathbf{a}^{j_1,\dots,j_p}\| = \max_{0\leq l\leq n}|a_l^{j_1,\dots,j_p}| = |a^{j_1,\dots,j_p}_{l_{j_1,\dots,j_p}}|$$

$$= \left| \det \begin{pmatrix} f_0(z_{j_1,\dots,j_p}) & \dots & \widehat{f_{l_{j_1,\dots,j_p}}(z_{j_1,\dots,j_p})} & \dots & f_n(z_{j_1,\dots,j_p}) \\ \partial_{\overline{u}_2}(f_0(z_{j_1,\dots,j_p})) & \dots & \partial_{\overline{u}_2}(\widehat{f_{l_{j_1,\dots,j_p}}(z_{j_1,\dots,j_p})}) & \dots & \partial_{\overline{u}_2}(f_n(z_{j_1,\dots,j_p})) \\ \vdots & \dots & \vdots & & \\ \partial_{\overline{u}_n}(f_0(z_{j_1,\dots,j_p})) & \dots & \partial_{\overline{u}_n}(\widehat{f_{l_{j_1,\dots,j_p}}(z_{j_1,\dots,j_p})}) & \dots & \partial_{\overline{u}_n}(f_n(z_{j_1,\dots,j_p})) \end{pmatrix} \right|,$$

where $\widehat{\ }$ means omitting this term in the matrix. Then we can see that $n+1$ hyperplanes $\{D^{j_1,\dots,j_p}, D_0, \dots, D_{l_{j_1,\dots,j_p}-1}, D_{l_{j_1,\dots,j_p}+1} \dots, D_n\}$ are in general position.

Now using the definitions of Weil function and modified proximity function, it follows from the inequality (4.7) that

$$\begin{aligned}
(4.8)\qquad & \int_{\mathbf{S}_p(r)} \Big(\log \frac{1}{|W_{\mathcal{S}}(f(z))|} + (n+1)\log\|\mathbf{f}(z)\|\Big)\,\sigma_p(z) \\
& \le \sum_{\substack{j=1,\dots,q\\k=1,\dots,p}} \int_{\theta_k\in(\theta_{j_k-1},\theta_{j_k}]} \left(\log \frac{1}{|\langle \mathbf{f}(z), \mathbf{a}^{j_1,\dots,j_p}\rangle|} + (n+1)\log\|\mathbf{f}(z)\|\right)\sigma_p(z) \\
& \qquad + \int_{\mathbf{S}_p(r)} \big(I^1_{j_1,\dots,j_p} + I^2_{j_1,\dots,j_p} + I^3_{j_1,\dots,j_p}\big)\,\sigma_p(z) + \log(2\pi r) \\
& \le \sum_{\substack{j=1,\dots,q\\k=1,\dots,p}} \int_{\theta_k\in(\theta_{j_k-1},\theta_{j_k}]} \left(\log \frac{\|\mathbf{f}(z)\|\cdot\|\mathbf{a}^{j_1,\dots,j_p}\|}{|\langle \mathbf{f}(z), \mathbf{a}^{j_1,\dots,j_p}\rangle|} + \log\frac{\|\mathbf{f}(z)\|}{|\langle \mathbf{f}(z), \mathbf{a}_0\rangle|} + \cdots\right. \\
& \qquad + \log\frac{\|\mathbf{f}(z)\|}{|\langle \mathbf{f}(z), \mathbf{a}_{l_{j_1,\dots,j_p}-1}\rangle|} + \log\frac{\|\mathbf{f}(z)\|}{|\langle \mathbf{f}(z), \mathbf{a}_{l_{j_1,\dots,j_p}+1}\rangle|} \\
& \qquad \left. + \cdots + \log\frac{\|\mathbf{f}(z)\|}{|\langle \mathbf{f}(z), \mathbf{a}_n >|}\right)\sigma_p(z) \\
& \qquad + \sum_{\substack{j=1,\dots,q\\k=1,\dots,p}} \int_{\theta_k\in(\theta_{j_k-1},\theta_{j_k}]} \log\frac{|f_0\cdots \widehat{f_{l_{j_1,\dots,j_p}}}\cdots f_n|(z)}{\|\mathbf{a}^{j_1,\dots,j_p}\|}\,\sigma_p(z) \\
& \qquad + \int_{\mathbf{S}_p(r)} \big(I^1_{j_1,\dots,j_p} + I^2_{j_1,\dots,j_p} + I^3_{j_1,\dots,j_p}\big)\,\sigma_p(z) + \log(2\pi r) \\
& \le \sum_{\substack{j=1,\dots,q\\k=1,\dots,p}} \int_{\theta_k\in(\theta_{j_k-1},\theta_{j_k}]} \Big(\lambda_{D^{j_1,\dots,j_p}}(f) + \lambda_{D_0}(f) \\
& \qquad + \cdots + \lambda_{\widehat{D_{l_{j_1,\dots,j_p}}}}(f) + \cdots + \lambda_{D_n}(f)\Big)(z)\,\sigma_p(z)
\end{aligned}$$

$$+\sum_{\substack{j=1,\dots,q\\k=1,\dots,p}}\int_{\theta_k\in(\theta_{j_k-1},\theta_{j_k}]}\log\frac{|f_0\cdots\widehat{f_{l_{j_1,\dots,j_p}}}\cdots f_n|(z)}{\|\mathbf{a}^{j_1,\dots,j_p}\|}\,\sigma_p(z)$$

$$+\int_{\mathbf{S}_p(r)}(I^1_{j_1,\dots,j_p}+I^2_{j_1,\dots,j_p}+I^3_{j_1,\dots,j_p})\,\sigma_p(z)+\log(2\pi r)$$

$$\le\overline{m}_{pq+n+1}(f,r)+\sum_{\substack{j=1,\dots,q\\k=1,\dots,p}}\int_{\theta_k\in(\theta_{j_k-1},\theta_{j_k}]}\log\frac{|f_0\cdots\widehat{f_{l_{j_1,\dots,j_p}}}\cdots f_n|(z)}{\|\mathbf{a}^{j_1,\dots,j_p}\|}\,\sigma_p(z)$$

$$+\int_{\mathbf{S}_p(r)}(I^1_{j_1,\dots,j_p}+I^2_{j_1,\dots,j_p}+I^3_{j_1,\dots,j_p})\,\sigma_p(z)+\log(2\pi r).$$

Step 2: Estimate the error terms. By Poincaré–Lelong formula, we firstly handle the left-hand side of (4.8).

$$\int_{\mathbf{S}_p(r)}\Big(\log\frac{1}{|W_{\mathcal{S}}(f(z))|}+(n+1)\log\|\mathbf{f}(z)\|\Big)\,\sigma_p(z)$$
$$=(n+1)T_f(r)-N_{W_{\mathcal{S}},f}(r,0)+O(1). \tag{4.9}$$

For the right-hand side of (3.6), we estimate in the same methods as in Section 3.3. For fixed $j_1,\dots,j_p$, $l_{j_1,\dots,j_p}$ is also a fixed integer in $\{0,\dots,n\}$. We define the coefficient functions

$$\left|a^{j_1,\dots,j_p}_{l_{j_1,\dots,j_p}}\right|(z):=\left|\det\begin{pmatrix} f_0(z) & \dots & \widehat{f_{l_{j_1,\dots,j_p}}}(z) & \dots & f_n(z)\\ \partial_{\overline{u}_2}(f_0(z)) & \dots & \partial_{\overline{u}_2}(\widehat{f_{l_{j_1,\dots,j_p}}}(z)) & \dots & \partial_{\overline{u}_2}(f_n(z))\\ \vdots & & \vdots & & \vdots\\ \partial_{\overline{u}_n}(f_0(z)) & \dots & \partial_{\overline{u}_n}(\widehat{f_{l_{j_1,\dots,j_p}}}(z)) & \dots & \partial_{\overline{u}_n}(f_n(z))\end{pmatrix}\right|.$$

We also have

$$\big|a^{j_1,\dots,j_p}_{l_{j_1,\dots,j_p}}\big|(z_{j_1,\dots,j_p})=\|\mathbf{a}^{j_1,\dots,j_p}\|.$$

Depending on the same strategies in (4.5) and (4.6), it yields that, for any $j_1,\dots,j_p$,

$$I^4_{j_1,\dots,j_p}=\log\big|a^{j_1,\dots,j_p}_{l_{j_1,\dots,j_p}}\big|(z)-\log\|\mathbf{a}^{j_1,\dots,j_p}\|$$

$$\le O\Big(u\Big(\sqrt{r^2-(|\xi_1|^2+\dots+|\xi_{p-1}|^2)},W_{\mathcal{S}}(f\big|_{\xi_1,\dots,\xi_{p-1}}(z_p)),\frac{2\pi}{q}\Big)$$

$$+u\Big(\sqrt{r^2-(|\xi_1|^2+\dots+|\xi_{p-2}|^2+r_p^2)},W_{\mathcal{S}}\big(f\big|_{\xi_1,\dots,\xi_{p-2},z_p^{j_p}}(z_{p-1})\big),\frac{2\pi}{q}\Big)$$

$$+\cdots+u\Big(\sqrt{r^2-(r_2^2+\cdots+r_p^2)},W_{\mathcal{S}}\big(f\big|_{z_2^{j_2},\ldots,z_p^{j_p}}(z_1)\big),\frac{2\pi}{q}\Big)\Big).$$

Thus we assert that

$$\sum_{\substack{j=1,\ldots,q\\k=1,\ldots,p}}\int_{\theta_k\in(\theta_{j_k-1},\theta_{j_k}]}\log\frac{\big|f_0\cdots\widehat{f_{l_{j_1,\ldots,j_p}}}\cdots f_n\big|(z)}{\|\mathbf{a}^{j_1,\ldots,j_p}\|}\,\sigma_p(z)$$
$$\le\Bigg|\sum_{\substack{j=1,\ldots,q\\k=1,\ldots,p}}\int_{\theta_k\in(\theta_{j_k-1},\theta_{j_k}]}\log\frac{\big|a^{j_1,\ldots,j_p}_{l_{j_1,\ldots,j_p}}\big|(z)}{|f_0\cdots\widehat{f_{l_{j_1,\ldots,j_p}}}\cdots f_n|(z)}\,\sigma_p(z)\Bigg|$$
$$+\int_{\mathbf{S}_p(r)}I^4_{j_1,\ldots,j_p}\,\sigma_p(z).$$

Deriving from the definition of $\left|a^{j_1,\ldots,j_p}_{l_{j_1,\ldots,j_p}}\right|(z)$, we can see that

$$\sum_{\substack{j=1,\ldots,q\\k=1,\ldots,p}}\int_{\theta_k\in(\theta_{j_k-1},\theta_{j_k}]}\log\frac{\big|a^{j_1,\ldots,j_p}_{l_{j_1,\ldots,j_p}}\big|(z)}{|f_0\cdots\widehat{f_{l_{j_1,\ldots,j_p}}}\cdots f_n|(z)}\,\sigma_p(z)$$
$$\le\int_{\mathbf{S}_p(r)}\log^+\Bigg(\sum_{i_1+\cdots+i_n=n(n+1)/2-1}\left|\frac{\partial_{\overline{u}_{i_1}}(f_0)}{f_0}\right|\cdots\left|\frac{\partial_{\overline{u}_{i_n}}(f_n)}{f_n}\right|(z)\Bigg)\,\sigma_p(z)$$
$$\le\big(l(\mathcal{S})-l(\overline{u}_1)\big)\big(\log T_f(r)+(1+\varepsilon)\log^+\log T_f(r)\big)+O(1).$$

Similar to (3.8), the last inequality refers to Lemma 4.5, Lemma 4.6 and Hölder inequality.

Combining the statements above, it follows that

$$(n+1)T_f(r)-N_{W_{\mathcal{S}},f}(r,0)\le\overline{m}_{pq+n+1}(f,r)\tag{4.10}$$
$$+\int_{\mathbf{S}_p(r)}\big(I^1_{j_1,\ldots,j_p}+I^2_{j_1,\ldots,j_p}+I^3_{j_1,\ldots,j_p}+I^4_{j_1,\ldots,j_p}\big)\,\sigma_p(z)$$
$$+\,(l(\mathcal{S})-l(\partial_{\overline{u}_1}))(\log T_f(r)+(1+\varepsilon)\log^+\log T_f(r))+\log(2\pi r)+O(1)\quad\|.$$

Step 3: End of the proof. Firstly, we can always find a constant r_0 such that

$$(l(\mathcal{S})-l(\partial_{\overline{u}_1}))\big(\log T_f(r)+(1+\varepsilon)\log^+\log T_f(r)\big)+\log(2\pi r)+O(1)\le\varepsilon T_f(r),$$

for all $r>r_0$.

In the right-hand side of (4.5), let

$$G_k(z) = f\big|_{z_1^{j_1},\ldots,z_{k-1}^{j_{k-1}},\xi_{k+1},\ldots,\xi_p}(z_k),$$

and

$$r'_k = \sqrt{r^2 - (r_1^2 + \cdots + r_{k-1}^2 + |\xi_{k+1}|^2 + \cdots + |\xi_p|^2)},$$

for each $k = 2, \ldots, p$. By Proposition 2.7, for above $\varepsilon > 0$, we have

$$u\big(r'_k, G_k, \lambda_{G_k}(r'_k)^{20}\big) \le \varepsilon T_{G_k}(r'_k)$$

for all $r > e$ outside a set E_k of logarithmic density zero. Recall that $\lambda_{G_k}(r'_k)^{20} = \min\{1, \big(\log^+ \frac{T_{G_k}(r'_k)}{\log r'_k}\big)^{-1}\}$. Indeed, $\log^+\big(\frac{T_{G_k}(r'_k)}{\log r'_k}\big) \le O\big(\log^+ \frac{T_f(r)}{\log r}\big)$ for r sufficiently large. Hence we obtain that there exists some integer $q \ge \frac{2\pi}{\lambda_f(r)^{20}}$ satisfying that

$$\begin{aligned}
\int_{\mathbf{S}_p(r)} I^1_{j_1,\ldots,j_p}\,\sigma_p(z) &= \frac{1}{r^{2p-2}}\int_{\mathbf{C}_{p-1}(r)}\Big(\int_0^{2\pi} I^1_{j_1,\ldots,j_p}\,\frac{d\theta}{2\pi}\Big)[v(z)]^{p-1}\\
&\le \sum_{k=2}^{p}\frac{1}{r^{2p-2}}\int_{\mathbf{C}_{p-1}(r)}\Big(\int_0^{2\pi}\varepsilon T_{G_k}(r'_k)\,\frac{d\theta}{2\pi}\Big)[v(z)]^{p-1}\\
&= \sum_{k=2}^{p}\frac{1}{r^{2p-2}}\int_{\mathbf{C}_{p-1}(r)}\varepsilon\Big(\int_0^{2\pi}\log\|G_k(r'_k e^{i\theta})\|\,\frac{d\theta}{2\pi}\Big)[v(z)]^{p-1}\\
&\le \sum_{k=2}^{p}\frac{1}{r^{2p-2}}\int_{\mathbf{S}_p(r)}\varepsilon T_f(r)\,\sigma_p(z) = (p-1)\varepsilon T_f(r),
\end{aligned}$$

for all $r > e$ outside a set $\mathcal{E}_1 = \bigcup_{k=2}^{p} E_k$ of logarithmic density zero. Again by Proposition 2.7 and Lemma 2.5, we have

$$\int_{\mathbf{S}_p(r)}\big(I^2_{j_1,\ldots,j_p} + I^3_{j_1,\ldots,j_p} + I^4_{j_1,\ldots,j_p}\big)\,\sigma_p(z) \le C_p\cdot\varepsilon T_f(r)$$

for all $r > e$ outside a set $\mathcal{E}_2$ of logarithmic density zero and some integer $q \ge \frac{2\pi}{\lambda_f(r)^{20}}$. Here C_p is a constant dependent of p. Note that we omit the details of comparing $\lambda_f(r)$ and $\lambda_{W_{\mathcal{S},f}}(r)$, since it is extremely close to step 3 in the proof of Theorem 3.2.

Recall the condition that $v(r) \sim \big(\log^+ \frac{T(r)}{\log r}\big)^{20}$. For r sufficiently large, we have $\frac{2\pi}{v(r)} < \lambda(r)^{20}$, where $\lambda(r) = \min\{1, \big(\log^+ \frac{T_f(r)}{\log r}\big)^{-1}\}$. Hence taking $q = v(r)$ and $E = [e, r_0] \cup \mathcal{E}_1 \cup \mathcal{E}_2$ a set of logarithmic density zero, we have

$$(n+1)T_f(r) - N_{W_{\mathcal{S},f}}(r,0) \le \overline{m}_{pv(r)+n+1}(f,r) + \varepsilon T_f(r)$$

for all $r > e$ outside E. We complete the proof. □

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