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A $(\phi_{\frac{n}{s}}, \phi)$-POINCARÉ INEQUALITY ON JOHN DOMAINS

S. FENG[1,†] and T. LIANG[2,*,†,‡]

[1]Department of Mathematics, Beijing Normal University, Beijing 100191, P. R. China
e-mail: 202021130044@mail.bnu.edu.cn

[2]School of Mathematics and Statistics, Huizhou University, Guangdong 516007, P. R. China
e-mail: liangtian@hzu.edu.cn

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and $s \in (0,1)$. Assume that $\phi\colon [0,\infty) \to [0,\infty)$ is a Young function obeying the doubling condition with the constant $K_\phi < 2^{\frac{n}{s}}$. We demonstrate that Ω supports a $(\phi_{\frac{n}{s}}, \phi)$-Poincaré inequality if it is a John domain. Alternatively, assume further that Ω is a bounded domain that is quasiconformally equivalent to a uniform domain (for $n \geq 3$) or a simply connected domain (for $n = 2$), then we show that Ω is a John domain if a $(\phi_{\frac{n}{s}}, \phi)$-Poincaré inequality holds.

1. Introduction

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that ϕ is a Young function in $[0,\infty)$, that is, $\phi \in C[0,\infty)$ is convex and satisfies $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. For any s $\in (0,1)$, define the intrinsic fractional Orlicz–Sobolev space $\dot{V}_*^{s,\phi}(\Omega)$ as the collection of all measurable functions u in Ω for which the semi-norm

$$\|u\|_{\dot{V}_*^{s,\phi}(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \int_{|x-y|<\frac{1}{2}d(x,\partial\Omega)} \phi\left(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \leq 1\right\}$$

is finite. Modulo constant functions, $\dot{V}_*^{s,\phi}(\Omega)$ is a Banach space. When $s = 1$, we usually consider the classical Orlicz–Sobolev space $W^{1,\phi}(\Omega)$, whose sharp

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* Corresponding author.

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embedding has been solved in [11] (see also [3] for an alternate formulation of the solution).

Alberico et al. [4] established an imbedding of $\dot{V}_*^{s,\phi}(\mathbb{R}^n)$ into certain an Orlicz target space. Recall that for any Young function ψ, the Orlicz space $L^\psi(\Omega)$ is the collection of all $u \in L^1_{\mathrm{loc}}(\Omega)$ endowed with the norm

$$\|u\|_{L^\psi(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \psi\Big(\frac{|u|}{\lambda}\Big)\, dx \le 1\right\} < \infty.$$

The following is a more thorough description.

THEOREM 1.1. *Let ϕ be a Young function satisfying*

$$\int_0^t \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}}\, d\tau < \infty \quad \text{for } t \in [0,\infty), \tag{1.1}$$

and

$$\int_0^\infty \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}}\, d\tau = \infty. \tag{1.2}$$

Define $\phi_{\frac{n}{s}} := \phi \circ H^{-1}$, where

$$H(t) = \bigg(\int_0^t \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}}\, d\tau\bigg)^{\frac{n-s}{n}} \quad \text{for all } t \ge 0. \tag{1.3}$$

Then we have $V_*^{s,\phi}(\mathbb{R}^n) \subset L^{\phi_{n/s}}(\mathbb{R}^n)$, that is, for any $u \in V_*^{s,\phi}(\mathbb{R}^n)$ with $|\{x \in \mathbb{R}^n : |u(x)| > t\}| < \infty$ for every $t > 0$, one has $u \in L^{\phi_{n/s}}(\mathbb{R}^n)$ with $\|u\|_{L^{\phi_{n/s}}(\mathbb{R}^n)} \le C\|u\|_{V_*^{s,\phi}(\mathbb{R}^n)}$, where C is a constant independent of u.

They also showed that $L^{\phi_{n/s}}(\mathbb{R}^n)$ is an optimal target space for the imbeding of $\dot{V}_*^{s,\phi}(\mathbb{R}^n)$ in the sense that if $\dot{V}_*^{s,\phi}(\mathbb{R}^n) \subset L^A(\mathbb{R}^n)$ holds for another Orlicz space $L^A(\mathbb{R}^n)$, then $L^{\phi_{n/s}}(\mathbb{R}^n) \subset L^A(\mathbb{R}^n)$.

We are interested in bounded domains which support the imbedding $V_*^{s,\phi}(\Omega) \subset L^{\phi_{n/s}}(\Omega)$ or $(\phi_{\frac{n}{s}}, \phi)$-Poincaré inequality, that is, there exists a constant $C \ge 1$ such that

$$\|u - u_\Omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \le C\|u\|_{\dot{V}_*^{s,\phi}(\Omega)}, \tag{1.4}$$

for every $u \in L^1(\Omega)$, where $u_E = \fint_E u = \frac{1}{|E|}\int_E u\, dx$ denotes the average of u in the set of E with $|E| > 0$.

The major aim of this article is to characterize the Orlicz–Sobolev imbedding $V_*^{s,\phi}(\Omega) \subset L^{\phi_{n/s}}(\Omega)$ via John domains under specific doubling assumption in ϕ; see Theorem 1.2 below. Remember that a bounded domain $\Omega \subset \mathbb{R}^n$

is called as a c-John domain with respect to some $x_0 \in \Omega$ for some $c > 0$ if for each $x \in \Omega$, there is a rectifiable curve $\gamma\colon [0, T] \to \Omega$ parameterized by arc-length such that $\gamma(0) = x$, $\gamma(T) = x_0$ and $d(\gamma(t), \Omega^{\complement}) > ct$ for all $t > 0$. For further research on c-John domains, see [6–9,35–37] and references therein. We say that a Young function ϕ has the doubling property ($\phi \in \Delta_2$) if

$$K_\phi := \sup_{t>0} \frac{\phi(2t)}{\phi(t)} < \infty. \tag{1.5}$$

Note that if a Young function $\phi \in \Delta_2$ with $K_\phi < 2^{\frac{n}{s}}$, then ϕ satisfies (1.1) and (1.2); see Lemma 2.3.

MAIN THEOREM 1.2. *Let $0 < s < 1$. Suppose ϕ is a Young function and $\phi \in \Delta_2$ with $K_\phi < 2^{\frac{n}{s}}$ in (1.5).*

(i) *If $\Omega \subset \mathbb{R}^n$ is a c-John domain, then Ω supports the $(\phi_{\frac{n}{s}}, \phi)$-Poincaré inequality (1.4) with the constant C depending on n, s, c and K_ϕ.*

(ii) *Assume further that $\Omega \subset \mathbb{R}^n$ is a bounded simply connected planar domain, or a bounded domain which is a quasiconformally equivalent to some uniform domain when $n \geq 3$. If Ω supports the $(\phi_{\frac{n}{s}}, \phi)$-Poincaré inequality, then Ω is a c-John domain, where the constant c depend on n, s, C, K_ϕ and Ω.*

Theorem 1.2 extends several known results in the literature; for details see the following remark.

REMARK 1.3. (i) For $1 \leq p < n$, c-John domain Ω supports Sobolev $\dot{W}^{1,p}$-imbedding or $(\frac{np}{n-p}, p)$-Poincaré inequality:

$$\|u - u_\Omega\|_{L^{np/(n-p)}(\Omega)} \leq C\|u\|_{\dot{W}^{1,p}(\Omega)} \quad \text{for all } u \in \dot{W}^{1,p}(\Omega), \tag{1.6}$$

where the constant C depends on n, p and c; see Reshetnyak [37] and Martio [36] for $1 < p < n$ and Borjarski [5] (and also Hajlasz [23]) for $p = 1$. Conversely, further assume that Ω is a bounded simply connected planar domain or a domain that is quasiconformally equivalently to some uniform domain when $n \geq 3$. Buckley and Koskela [7] proved that if (1.6) holds, then Ω is a c-John domain.

(ii) For $0 < s < 1$ and $1 \leq p < \infty$, the intrinsic fractional Sobolev space $\dot{W}^{s,p}_*(\Omega)$ consists of all functions $u \in L^1_{\rm loc}(\Omega)$ with the norm

$$\|u\|_{\dot{W}^{s,p}_*(\Omega)} := \left(\int_\Omega \int_{|x-y|<\frac{1}{2}d(x,\partial\Omega)} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy \right)^{1/p} < \infty.$$

In the special case $\phi(t) = t^p$ with $p \geq 1$, $\dot{V}^{s,\phi}_*(\Omega)$ is exactly $\dot{W}^{s,p}_*(\Omega)$.

For $s\in(0,1)$ and $1\le p<n/s$, [17] for $p=1$ and [25] for $1<p<n/s$ proved that a c-John domain Ω supports the following fractional $\big(\frac{np}{n-sp},p\big)_s$-Poincaré inequality (or fractional Sobolev embedding $\dot W^{s,p}_*(\Omega)\hookrightarrow L^{\frac{np}{n-sp}}(\Omega)$), which means that for any $u\in\dot W^{s,p}_*(\Omega)$,

$$\|u-u_\Omega\|_{L^{np/(n-sp)}(\Omega)}\le C\|u\|_{\dot W^{s,p}_*(\Omega)} \tag{1.7}$$

holds, where C depends on n, s, p and c. On the other hand, additionally assume that Ω is a bounded simply connected planar domain or a domain that is quasiconformally equivalently to some uniform domain when $n\ge 3$. They [17,25] also proved that if (1.7) holds, then Ω is a c-John domain.

If $1\le p<\frac{n}{s}$, it is easy to see that $\phi_{\frac{n}{s}}(t)=Ct^{\frac{np}{n-sp}}$ for any $t\ge 0$ and some positive constant C. If $\phi(t)=t^p$ with $p\ge 1$ and $0<s<1$, then the $(\phi_{\frac{n}{s}},\phi)$-Poincaré inequality is the classical fractional $(\frac{np}{n-sp},p)$-Poincaré inequality.

(iii) Analogous results to (ii) were established for the intrinsic fractional Hajlasz–Sobolev space $\dot M^{s,p}_*(\Omega)$; see [41] for details.

(iv) In the above theorem, a domain $\Omega\subset\mathbb{R}^n$ is quasiconformally equivalent to a uniform domain $G\subset\mathbb{R}^n$ means that there exists a homeomorphism $f\colon G\to\Omega$ such that $f\in W^{1,n}_{\rm loc}(G)$ and $|Df(x)|^n\le KJf(x)$ for almost every $x\in G$, where $|Df|$ is the operator norm of the formal derivative Df of f, Jf is the Jacobian determinant of Df, and $K\ge 1$ is a fixed constant.

We also note that imbeddings of the fractional Sobolev space $\dot W^{s,p}(\Omega)$ and fractional Orlicz–Sobolev space $\dot V^{s,\phi}(\Omega)$ were taken into account in the literature (see [4,28,29,40] for examples). Define the fractional Orlicz–Sobolev space $\dot V^{s,\phi}(\Omega)$ consisting of all functions $u\in L^1_{\rm loc}(\Omega)$ with

$$\|u\|_{\dot V^{s,\phi}(\Omega)}:=\inf\left\{\lambda>0:\int_\Omega\int_\Omega\phi\left(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}\le 1\right\}<\infty.$$

The $\dot V^{s,\phi}(\Omega)$-(semi)norm is evidently derived by substituting the whole domain Ω for the range $B(x,\frac12\operatorname{dist}(x,\partial\Omega))$ for the variable y in the $\dot V^{s,\phi}_*(\Omega)$-(semi)norm. It goes without saying that $\dot V^{s,\phi}(\mathbb{R}^n)=\dot V^{s,\phi}_*(\mathbb{R}^n)$. For general domain Ω, one has $\dot V^{s,\phi}(\Omega)\subset\dot V^{s,\phi}_*(\Omega)$ with a normal bound because the following inequality always holds:

$$\begin{aligned}\int_\Omega\int_{|x-y|<\frac12 d(x,\partial\Omega)}&\phi\left(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}\\ &\le\int_\Omega\int_\Omega\phi\left(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}.\end{aligned}$$

But the reverse side $\dot{V}_*^{s,\phi}(\Omega) \subset \dot{V}^{s,\phi}(\Omega)$ is not true necessarily. Just like the example of [17] and [41], the embedding $\dot{W}_*^{s,p}(\Omega) \subset \dot{W}^{s,p}(\Omega)$ fails when a domain $\Omega = B(0,1)\backslash\{(x,0)|x \geq 0\} \subset R^2$ and define u as $u(x) = 1/2$ for $\Omega\cap\{(x,y)|y \geq 0\}$, and $u = 0$ otherwise. In fact, when $\phi(t) = t^p$ with $p \geq 1$, $\dot{V}_*^{s,\phi}(\Omega) = \dot{W}_*^{s,p}(\Omega)$ and $\dot{V}^{s,\phi}(\Omega)$ is the fractional Sobolev space $\dot{W}^{s,p}(\Omega)$, which consists of all functions $u \in L^1_{\rm loc}(\Omega)$ with

$$\|u\|_{\dot{W}^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy\right)^{1/p} < \infty.$$

Remark 1.4. (i) If $\Phi(t) = t^p$ with $s \in (0,1)$ and $1 \leq p < n/s$, we consider the $\dot{W}^{s,p}$-imbedding, that is, for any $u \in \dot{W}^{s,p}(\Omega)$, there exists a constant $C > 0$ such that

$$\|u - u_\Omega\|_{L^{\frac{np}{n-sp}}(\Omega)} \leq C\|u\|_{\dot{W}^{s,p}(\Omega)}.$$

It was shown in [28,29,40] that a domain Ω supports the $\dot{W}^{s,p}$-imbedding if and only if Ω is Ahlfors n-regular, that is, there exists a constant $c > 0$ such that

$$|B(x,r) \cap \Omega| \geq Cr^n \quad \text{for all } x \in \Omega,\ 0 < r < 2\operatorname{diam}\Omega.$$

Note that in the case $|\Omega| = \infty$, we set $u_\Omega = 0$.

(ii) For a general Young function ϕ satisfying (1.1) and (1.2), it was shown in [4] that Lipschitz domain Ω supports the $\dot{V}^{s,\phi}(\Omega)$-imbedding

$$\|u - u_\Omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \leq C\|u\|_{\dot{V}^{s,\phi}(\Omega)} \quad \text{for all } u \in \dot{V}^{s,\phi}(\Omega)$$

whenever $s \in (0,1)$. Note that the $\dot{V}^{s,\phi}(\Omega)$-imbedding is exactly the $\dot{W}^{s,p}$-imbedding if $\Phi(t) = t^p$ with $s \in (0,1)$ and $1 \leq p < n/s$. In this case, $\phi_{\frac{n}{s}}(t) = t^{\frac{np}{n-sp}}$. But it is not clear whether Ahlfors n-regular domains characterize $\dot{V}^{s,\phi}(\Omega)$-imbedding domains.

The paper is organized as follows. The proof of Theorem 1.2(i) is given in Section 2, which uses Boman's chain property, the embedding $\dot{V}_*^{s,\phi}(Q) \hookrightarrow L^{\phi_{n/s}}(Q)$ for cubes $Q \subset \mathbb{R}^n$ and the vector-valued inequality in Orlicz norms for the Hardy–Littlewood maximum operators. We also give some property of $\phi \in \Delta_2$ with $K_\phi < 2^{\frac{n}{s}}$ in Section 2. Conversely, under the condition (2.1), together with the aid of some ideas from [7,24,33,39,40], we obtain the $LLC(2)$ property of Ω, and then prove Theorem 1.2(ii) by a capacity argument; see Section 3 for details.

2. Proof of Theorem 1.2(i)

It is well known that a Young function ϕ satisfies

$$C_\phi := \sup_{t>0} \int_0^t \frac{\phi(\rho)}{\phi(t)} \frac{d\rho}{\rho} < \infty. \tag{2.1}$$

In fact, since for practically all $t \geq 0$, $\phi'(t) \geq 0$ and ϕ' is increasing, we know

$$\frac{\phi(\rho)}{\rho} = \frac{\phi(\rho) - \phi(0)}{\rho} \leq \phi'(\rho).$$

Hence

$$\int_0^t \frac{\phi(\rho)}{\phi(t)} \frac{d\rho}{\rho} \leq \frac{1}{\phi(t)} \int_0^t \phi'(\rho)\, d\rho \leq 1,$$

that is, $C_\phi \leq 1$.

Based on the above fact, it is feasible to obtain the embedding $C_c^\infty(\Omega) \subset \dot{V}_*^{s,\phi}(\Omega)$.

Lemma 2.1. *Let $0 < s < 1$ and ϕ be a Young function. For any bounded domain $\Omega \subset \mathbb{R}^n$, we have $C_c^\infty(\Omega) \subset \dot{V}^{s,\phi}(\Omega) \subset \dot{V}_*^{s,\phi}(\Omega)$.*

Proof. For any $u \in C_c^1(\Omega)$, denote $L := \|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)}$. Letting $W \subset \Omega$ such that $V = \operatorname{supp} u \Subset W \Subset \Omega$, we write

$$H := \int_\Omega \int_\Omega \phi\left(\frac{|u(x) - u(y)|}{\lambda |x-y|^s}\right) \frac{dx\, dy}{|x-y|^n}$$

$$\leq \int_W \int_W \phi\left(\frac{L|x-y|}{\lambda|x-y|^s}\right) \frac{dx\, dy}{|x-y|^n} + 2 \int_V \int_{\Omega \setminus W} \phi\left(\frac{L}{\lambda |x-y|^s}\right) \frac{dx\, dy}{|x-y|^n}.$$

By (2.1), we have

$$\int_W \int_W \phi\Big(\frac{L|x-y|}{\lambda|x-y|^s}\Big) \frac{dx\, dy}{|x-y|^n} \leq \int_W \int_{B(x, 2 \operatorname{diam} W)} \phi\Big(\frac{L|x-y|^{1-s}}{\lambda}\Big) \frac{dy}{|x-y|^n}\, dx$$

$$= n\omega_n \int_W \int_0^{2 \operatorname{diam} W} \phi\Big(\frac{L\rho^{1-s}}{\lambda}\Big) \frac{d\rho}{\rho}\, dx$$

$$= n\omega_n \frac{1}{1-s} \int_W \int_0^{\frac{L(2 \operatorname{diam} W)^{1-s}}{\lambda}} \phi(\mu) \frac{d\mu}{\mu}\, dx$$

$$\leq C_\phi n \omega_n \frac{1}{1-s} \phi\Big(\frac{L(2 \operatorname{diam} W)^{1-s}}{\lambda}\Big) |W|.$$

Using (2.1) once more, we know

$$\begin{aligned}
&\int_V \int_{\Omega\setminus W} \phi\Big(\frac{L}{\lambda|x-y|^s}\Big)\,\frac{dx\,dy}{|x-y|^n}\\
&\le \int_V \int_{\Omega\setminus B(y,\mathrm{dist}(V,W^{\complement}))} \phi\Big(\frac{L}{\lambda|x-y|^s}\Big)\,\frac{dx}{|x-y|^n}\,dy\\
&\le n\omega_n \int_V \int_{\mathrm{dist}(V,W^{\complement})}^{\infty} \phi\Big(\frac{L}{\lambda\rho^s}\Big)\,\frac{d\rho}{\rho}\,dy\\
&= n\omega_n\frac{1}{s}\int_V \int_0^{\frac{L}{\lambda\,\mathrm{dist}(V,W^{\complement})^s}} \phi(\mu)\frac{d\mu}{\mu}\,dy \le C_\phi n\omega_n\frac{1}{s}\phi\Big(\frac{L}{\lambda\,\mathrm{dist}(V,W^{\complement})^s}\Big)|V|.
\end{aligned}$$

Let λ large enough such that $H \le 1$. Then we get $u \in \dot V^{s,\phi}(\Omega)$, and hence $C_c^1(\Omega) \subset \dot V^{s,\phi}(\Omega)$. Combining $C_c^\infty(\Omega) \subset C_c^1(\Omega)$ and $\dot V^{s,\phi}(\Omega) \subset \dot V_*^{s,\phi}(\Omega)$, we get the desired result. $\square$

To prove Theorem 1.2(i), we need the embedding $\dot V^{s,\phi}(Q) \hookrightarrow L^{\phi_{n/s}}(Q)$ in all cubes $Q \subset \mathbb{R}^n$. Hence we give some necessary lemmas.

Lemma 2.2. *Let $\phi \in \Delta_2$ be a Young function. Then for any $c > 1$ and $x > 0$, we have $\phi(cx) \le c^{K_\phi-1}\phi(x)$.*

Proof. By the increasing property of ϕ', for any $x > 0$ we know

$$\phi(2x) - \phi(x) = \int_x^{2x} \phi'(t)\,dt \ge \phi'(x).$$

Moreover, together with $\phi \in \Delta_2$ and $\phi(2x) - \phi(x) \le (K_\phi - 1)\phi(x)$, we get

$$(\ln\phi)'(x) = \frac{\phi'(x)}{\phi(x)} \le \frac{K_\phi - 1}{x}.$$

Hence for any $c > 1$ we have

$$\ln\Big(\frac{\phi(cx)}{\phi(x)}\Big) = \int_x^{cx} (\ln\phi)'(t)\,dt \le \int_x^{cx} \frac{K_\phi - 1}{t}\,dt = \ln(c^{K_\phi-1}).$$

Using the increasing property of ln, we get $\phi(cx) \le c^{K_\phi-1}\phi(x)$ as desired. $\square$

Lemma 2.3. *Let ϕ be a Young function and $\phi \in \Delta_2$ with $K_\phi < 2^{\frac{n}{s}}$. Then ϕ satisfies (1.1) and (1.2).*

PROOF. Applying the definition of the K_ϕ in (1.5), that is, $\phi(2t) \le K_\phi \phi(t)$, we get

$$\int_{\frac{t}{2}}^{t} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau = \int_{\frac{t}{4}}^{\frac{t}{2}} \Big(\frac{2\tau}{\phi(2\tau)}\Big)^{\frac{s}{n-s}} 2\, d\tau \ge \int_{\frac{t}{4}}^{\frac{t}{2}} \Big(\frac{2\tau}{K_\phi \phi(\tau)}\Big)^{\frac{s}{n-s}} 2\, d\tau.$$

Then

$$\int_{\frac{t}{4}}^{\frac{t}{2}} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau \le \frac{K_\phi^{\frac{s}{n-s}}}{2^{\frac{n}{n-s}}} \int_{\frac{t}{2}}^{t} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau.$$

By induction, we have

$$\begin{aligned}\int_{\frac{t}{2^m}}^{\frac{t}{2^{m-1}}} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau &\le \frac{K_\phi^{\frac{s}{n-s}}}{2^{\frac{n}{n-s}}} \int_{\frac{t}{2^{m-1}}}^{\frac{t}{2^{m-2}}} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau \\ &\le \Bigg(\frac{K_\phi^{\frac{s}{n-s}}}{2^{\frac{n}{n-s}}}\Bigg)^{m-1} \int_{\frac{t}{2}}^{t} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau.\end{aligned}$$

If we convert m from 1 to ∞ and add them together, then

$$\int_0^t \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau \le \sum_{m=1}^{\infty} \Bigg(\frac{K_\phi^{\frac{s}{n-s}}}{2^{\frac{n}{n-s}}}\Bigg)^{m-1} \int_{\frac{t}{2}}^{t} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau.$$

Because of the range of the K_ϕ, the series $\sum_{m=1}^{\infty} \Big(\frac{K_\phi^{\frac{s}{n-s}}}{2^{\frac{n}{n-s}}}\Big)^{m-1}$ is convergent. It means that there exists a constant $C > 0$ such that

$$\int_0^t \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau \le C \int_{\frac{t}{2}}^{t} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau. \tag{2.2}$$

Moreover,

$$\Big(\frac{t}{\phi(t)}\Big)' = \frac{\phi(t) - t\phi'(t)}{\phi^2(t)} = \frac{\frac{\phi(t)-\phi(0)}{t} - \phi'(t)}{\phi^2(t)} = \frac{\phi'(\xi) - \phi'(t)}{\phi^2(t)}$$

where $0 < \xi < t$. Using the increasing property of ϕ', we know $\big(\frac{t}{\phi(t)}\big)' \le 0$. Therefore,

$$\int_{\frac{t}{2}}^{t} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau \le \Bigg(\frac{\frac{t}{2}}{\phi(\frac{t}{2})}\Bigg)^{\frac{s}{n-s}} \frac{t}{2} < \infty.$$

Then (1.1) holds.

On the other hand, for any $m \in \mathbb{N}$ we have

$$\int_0^{2^m} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau \geq \int_0^{2^{m-1}} \Big(\frac{2\tau}{K_\phi \phi(\tau)}\Big)^{\frac{s}{n-s}} 2\, d\tau$$

$$= \frac{2^{\frac{n}{n-s}}}{K_\phi^{\frac{s}{n-s}}} \int_0^{2^{m-1}} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau \geq \cdots \geq \Bigg(\frac{2^{\frac{n}{n-s}}}{K_\phi^{\frac{s}{n-s}}}\Bigg)^m \int_0^1 \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau.$$

Letting $m \to \infty$, we get (1.2). $\square$

LEMMA 2.4. *Let ϕ be a Young function and $\phi \in \Delta_2$ with the parameter $K_\phi < 2^{\frac{n}{s}}$. Then for any $A > 0$, there exists a constant $C = C(n, s, K_\phi) > 0$ such that*

$$\frac{H(A)}{A} \leq \frac{C}{\phi(A)^{\frac{s}{n}}}. \tag{2.3}$$

PROOF. Applying (2.2) in Lemma 2.3, we get

$$\int_0^t \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau \leq C \int_{\frac{t}{2}}^t \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau \leq C \Bigg(\frac{\frac{t}{2}}{\phi(\frac{t}{2})}\Bigg)^{\frac{s}{n-s}} \frac{t}{2}.$$

Together with $\phi \in \Delta_2$, we have

$$\frac{H(A)}{A} = \frac{\Big(\int_0^A \big(\frac{\tau}{\phi(\tau)}\big)^{\frac{s}{n-s}} d\tau\Big)^{\frac{n-s}{n}}}{A} \leq \frac{\Bigg(C\Big(\frac{A/2}{\phi(\frac{A}{2})}\Big)^{\frac{s}{n-s}} \frac{A}{2}\Bigg)^{\frac{n-s}{n}}}{A}$$

$$\leq \frac{\Bigg(C\Big(\frac{A/2}{\frac{1}{K_\phi}\phi(A)}\Big)^{\frac{s}{n-s}} \frac{A}{2}\Bigg)^{\frac{n-s}{n}}}{A} \leq \frac{C^{\frac{n-s}{n}} K_\Phi^{\frac{s}{n}}/2}{\phi(A)^{\frac{s}{n}}}. \quad \square$$

With above lemmas, we proved $\dot{V}^{s,\phi}(Q) \hookrightarrow L^{\phi_{n/s}}(Q)$.

LEMMA 2.5. *Let $0 < s < 1$ and a Young function $\phi \in \Delta_2$ with $K_\phi < 2^{\frac{n}{s}}$. Then there exists a constant $C_1 = C_1(n, s)$ such that for any cube $Q \subset \mathbb{R}^n$ and $u \in \dot{V}^{s,\phi}(Q)$, letting $\lambda \geq C_1 \|u\|_{\dot{V}^{s,\phi}(Q)}$, the following inequality holds:*

$$\int_Q \phi_{\frac{n}{s}}\Big(\frac{u(x) - u_Q}{\lambda}\Big) dx \leq \int_Q \int_Q \phi\Big(\frac{C_1|u(x) - u(y)|}{\lambda |x - y|^s}\Big) \frac{dx\, dy}{|x - y|^n} \leq 1. \tag{2.4}$$

PROOF. Denote a cube centered at the origin with sides of length 2 paralleled to the axes by $Q(0, 1)$. At first we prove that for $u \in \dot{V}^{s,\phi}(Q(0, 1))$,

$$\int_{Q(0,1)} \phi_{\frac{n}{s}}\Big(\frac{|u(x) - u_{Q(0,1)}|}{\lambda}\Big) dx \tag{2.5}$$

$$\leq \int_{Q(0,1)} \int_{Q(0,1)} \phi\Big(\frac{C_1|u(x)-u(y)|}{\lambda|x-y|^s}\Big)\frac{dx\,dy}{|x-y|^n}.$$

By Lemma 2.3 and [4], we know that for $u \in \dot{V}_{\perp}^{s,\phi}(Q(0,1))$ there exists a constant $C_1 = C_1(n,s)$ such that

$$\|u\|_{L^{\phi_{\frac{n}{s}}}(Q(0,1))} \leq C_1 \|u\|_{\dot{V}^{s,\phi}(Q(0,1))}$$

where

$$\dot{V}_{\perp}^{s,\phi}(Q(0,1)) := \big\{u \in \dot{V}^{s,\phi}(Q(0,1)) : u_{Q(0,1)} = 0\big\}.$$

Replacing u by $u - u_{Q(0,1)}$, for any $u \in \dot{V}^{s,\phi}(Q(0,1))$, we have

$$\|u - u_{Q(0,1)}\|_{L^{\phi_{\frac{n}{s}}}(Q(0,1))} \leq C_1 \|u - u_{Q(0,1)}\|_{\dot{V}^{s,\phi}(Q(0,1))}.$$

Without loss of generality, suppose that $\|u\|_{\dot{V}^{s,\phi}(Q(0,1))} \neq 0$. Letting $\lambda > C_1\|u\|_{\dot{V}^{s,\phi}(Q(0,1))}$, we have

$$\int_{Q(0,1)} \phi_{\frac{n}{s}}\Big(\frac{|u - u_{Q(0,1)}|}{\lambda}\Big)\,dx \leq 1.$$

Otherwise, if $\|u\|_{\dot{V}^{s,\phi}(Q(0,1))} = 0$, then $u - u_{Q(0,1)}$ is constant in $Q(0,1)$. The inequality holds obviously.

Fixed $u_0 \in \dot{V}^{s,\phi}(Q(0,1))$, write

$$M := \int_{Q(0,1)} \int_{Q(0,1)} \phi\Big(\frac{C_1|u_0(x)-u_0(y)|}{|x-y|^s}\Big)\frac{dx\,dy}{|x-y|^n} \neq 0.$$

Letting $\overline{\phi} = \frac{\phi}{M}$, then $\overline{\phi}_{\frac{n}{s}}(t) = \frac{1}{M}\phi_{\frac{n}{s}}\big(\frac{t}{M^{\frac{s}{n}}}\big)$. Therefore,

$$\|u - u_{Q(0,1)}\|_{L^{\overline{\phi}_{\frac{n}{s}}}(Q(0,1))} \leq C_1 \|u\|_{\dot{V}^{s,\overline{\phi}}(Q(0,1))}.$$

Then we know

$$\int_{Q(0,1)} \phi_{\frac{n}{s}}\Bigg(\frac{|u - u_{Q(0,1)}|}{C_1 M^{\frac{s}{n}}\|u\|_{\dot{V}^{s,\overline{\phi}}(Q(0,1))}}\Bigg)\,dx \leq M$$

and $C_1\|u_0\|_{\dot{V}^{s,\overline{\phi}}(Q(0,1))} \leq 1$. Otherwise, we have

$$1 < \int_{Q(0,1)} \int_{Q(0,1)} \overline{\phi}\Big(\frac{C_1|u_0(x)-u_0(y)|}{|x-y|^s}\Big)\frac{dx\,dy}{|x-y|^n}$$

$$= \frac{1}{M} \int_{Q(0,1)} \int_{Q(0,1)} \phi\Big(\frac{C_1|u_0(x) - u_0(y)|}{|x-y|^s} \Big) \frac{dx\,dy}{|x-y|^n} = 1,$$

that is, a contradiction.

Specially, if $u = u_0$, we have

$$\int_{Q(0,1)} \phi_{\frac{n}{s}} \left(\frac{|u_0 - u_{0Q(0,1)}|}{\big(\int_{Q(0,1)} \int_{Q(0,1)} \phi\big(\frac{C_1|u_0(x)-u_0(y)|}{|x-y|^s} \big) \frac{dx\,dy}{|x-y|^n} \big)^{\frac{s}{n}}} \right) dx$$
$$\le \int_{Q(0,1)} \phi_{\frac{n}{s}} \left(\frac{|u_0 - u_{0Q(0,1)}|}{C_1 M^{\frac{s}{n}} \|u_0\|_{\dot{V}^{s,\overline{\phi}}(Q(0,1))}} \right) dx$$
$$\le \int_{Q(0,1)} \int_{Q(0,1)} \phi\Big(\frac{C_1|u_0(x) - u_0(y)|}{|x-y|^s} \Big) \frac{dx\,dy}{|x-y|^n}.$$

By the arbitrariness of u_0, we have

$$\int_{Q(0,1)} \phi_{\frac{n}{s}} \left(\frac{|u - u_{Q(0,1)}|}{\big(\int_{Q(0,1)} \int_{Q(0,1)} \phi\big(\frac{C_1|u(x)-u(y)|}{|x-y|^s} \big) \frac{dx\,dy}{|x-y|^n} \big)^{\frac{s}{n}}} \right) dx$$
$$\le \int_{Q(0,1)} \int_{Q(0,1)} \phi\Big(\frac{C_1|u(x) - u(y)|}{|x-y|^s} \Big) \frac{dx\,dy}{|x-y|^n}.$$

Replacing u by $\frac{u}{\lambda}$,

$$\int_{Q(0,1)} \phi_{\frac{n}{s}} \left(\frac{|u - u_{Q(0,1)}|}{\lambda\big(\int_{Q(0,1)} \int_{Q(0,1)} \phi\big(\frac{C_1|u(x)-u(y)|}{\lambda|x-y|^s} \big) \frac{dx\,dy}{|x-y|^n} \big)^{\frac{s}{n}}} \right) dx \tag{2.6}$$
$$\le \int_{Q(0,1)} \int_{Q(0,1)} \phi\Big(\frac{C_1|u(x) - u(y)|}{\lambda|x-y|^s} \Big) \frac{dx\,dy}{|x-y|^n}.$$

Putting $\lambda \ge C_1\|u\|_{\dot{V}^{s,\phi}(Q(0,1))}$, we know

$$\int_{Q(0,1)} \int_{Q(0,1)} \phi\Big(\frac{C_1|u(x) - u(y)|}{\lambda|x-y|^s} \Big) \frac{dx\,dy}{|x-y|^n} \le 1.$$

Hence

$$\int_{Q(0,1)} \phi_{\frac{n}{s}} \Big(\frac{|u - u_{Q(0,1)}|}{\lambda} \Big) dx$$
$$\le \int_{Q(0,1)} \phi_{\frac{n}{s}} \left(\frac{|u - u_{Q(0,1)}|}{\lambda\big(\int_{Q(0,1)} \int_{Q(0,1)} \phi\big(\frac{C_1|u(x)-u(y)|}{\lambda|x-y|^s} \big) \frac{dx\,dy}{|x-y|^n} \big)^{\frac{s}{n}}} \right) dx$$

$$\leq \int_{Q(0,1)} \int_{Q(0,1)} \phi\bigg(\frac{C_1|u(x)-u(y)|}{\lambda|x-y|^s}\bigg) \frac{dx\,dy}{|x-y|^n}.$$

Now we prove the case of general cube Q. Let Q be a cube with a as the center and $2l$ as the side length, then there is an orthogonal transformation T, and $T(Q-a)=Q(0,l)$. For any $u \in \dot{V}^{s,\phi}(Q)$ and u is not a constant. Let $v(x)=\frac{u(T^{-1}(lx)+a)}{l^s}$ where $x \in Q(0,1)$. Then $v \in \dot{V}^{s,\phi}(Q(0,1))$, $v_{Q(0,1)}=\frac{u_Q}{l^s}$, and

$$\begin{aligned}
&\int_{Q(0,1)} \int_{Q(0,1)} \phi\bigg(\frac{C_1|v(x)-v(y)|}{\lambda|x-y|^s}\bigg) \frac{dx\,dy}{|x-y|^n} \\
&= \int_{Q(0,1)} \int_{Q(0,1)} \phi\Bigg(\frac{C_1\big|\frac{u(T^{-1}(lx)+a)}{l^s}-\frac{u(T^{-1}(ly)+a)}{l^s}\big|}{\lambda|x-y|^s}\Bigg) \frac{dx\,dy}{|x-y|^n}.
\end{aligned}$$

By transformation $z_1=T^{-1}(lx)+a, z_2=T^{-1}(ly)+a$, we have $|x-y|=\big|\frac{T(z_1-a)}{l}-\frac{T(z_2-a)}{l}\big|=\frac{|z_1-z_2|}{l}$, so we have

$$\begin{aligned}
&\int_{Q(0,1)} \int_{Q(0,1)} \phi\bigg(\frac{C_1|v(x)-v(y)|}{\lambda|x-y|^s}\bigg) \frac{dx\,dy}{|x-y|^n} \\
&= \int_Q \int_Q \phi\bigg(\frac{C_1|u(z_1)-u(z_2)|}{\lambda|z_1-z_2|^s}\bigg) \frac{dz_1\,dz_2}{l^n|z_1-z_2|^n},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{Q(0,1)} \phi_{\frac{n}{s}}\Bigg(\frac{|v-v_{Q(0,1)}|}{\lambda\big(\int_{Q(0,1)}\int_{Q(0,1)}\phi\big(\frac{C_1|u(x)-u(y)|}{\lambda|x-y|^s}\big)\frac{dx\,dy}{|x-y|^n}\big)^{\frac{s}{n}}}\Bigg)dx \\
&= \int_{Q(0,1)} \phi_{\frac{n}{s}}\Bigg(\frac{|v-v_{Q(0,1)}|}{\lambda\big(\int_Q\int_Q\phi\big(\frac{C_1|u(z_1)-u(z_2)|}{\lambda|z_1-z_2|^s}\big)\frac{dz_1\,dz_2}{l^n|z_1-z_2|^n}\big)^{\frac{s}{n}}}\Bigg)dx.
\end{aligned}$$

By transformation $y=T^{-1}(lx)+a$, we get

$$\begin{aligned}
&\int_{Q(0,1)} \phi_{\frac{n}{s}}\Bigg(\frac{|v-v_{Q(0,1)}|}{\lambda\big(\int_{Q(0,1)}\int_{Q(0,1)}\phi\big(\frac{C_1|u(x)-u(y)|}{\lambda|x-y|^s}\big)\frac{dx\,dy}{|x-y|^n}\big)^{\frac{s}{n}}}\Bigg)dx \\
&= \int_Q \phi_{\frac{n}{s}}\Bigg(\frac{|u(y)-u_Q|}{\lambda\big(\int_Q\int_Q\phi\big(\frac{C_1|u(z_1)-u(z_2)|}{\lambda|z_1-z_2|^s}\big)\frac{dz_1\,dz_2}{|z_1-z_2|^n}\big)^{\frac{s}{n}}}\Bigg)\frac{dy}{l^n}.
\end{aligned}$$

Applying (2.6), we have

$$\int_Q \phi_{\frac{n}{s}}\Bigg(\frac{|u(y)-u_Q|}{\lambda\big(\int_Q\int_Q \phi\big(\frac{C_1|u(z_1)-u(z_2)|}{\lambda|z_1-z_2|^s}\big)\frac{dz_1\,dz_2}{|z_1-z_2|^n}\big)^{\frac{s}{n}}}\Bigg)\,dy \leq \int_Q\int_Q \phi\Bigg(\frac{C_1|u(z_1)-u(z_2)|}{\lambda|z_1-z_2|^s}\Bigg)\frac{dz_1\,dz_2}{|z_1-z_2|^n}.$$

Letting $\lambda \geq C_1\|u\|_{\dot V^{s,\phi}(Q)}$, we get

$$\int_Q\int_Q \phi\Bigg(\frac{C_1|u(x)-u(y)|}{\lambda|x-y|^s}\Bigg)\frac{dx\,dy}{|x-y|^n} \leq 1.$$

Hence,

$$\int_Q \phi_{\frac{n}{s}}\Big(\frac{|u-u_Q|}{\lambda}\Big)\,dx \leq \int_Q \phi_{\frac{n}{s}}\Bigg(\frac{|u-u_Q|}{\lambda\big(\int_Q\int_Q \phi\big(\frac{C_1|u(x)-u(y)|}{\lambda|x-y|^s}\big)\frac{dx\,dy}{|x-y|^n}\big)^{\frac{s}{n}}}\Bigg)\,dx \leq \int_Q\int_Q \phi\Bigg(\frac{C_1|u(x)-u(y)|}{\lambda|x-y|^s}\Bigg)\frac{dx\,dy}{|x-y|^n} \leq 1. \quad \square$$

We also need the Fefferman–Stein type vector-valued inequality for Hardy–Littlewood maximum operator in Orlicz space. Denote by $\mathcal{M}$ the Hardy–Littlewood maximum operator,

$$\mathcal{M}(g)(x) = \sup_{x\in Q} \fint_Q |g|\,dx$$

with the supremum taken over all cubes $Q \subset \mathbb{R}^n$ containing x. The Young function ϕ is in ∇_2 if there exist an $a > 1$, such that

$$\phi(x) \leq \frac{1}{2a}\phi(ax) \quad \text{for all } x \geq 0.$$

Lemma 2.6. *If a Young function $\phi \in \Delta_2$ with $K_\phi < 2^{\frac{n}{s}}$, we have $\phi_{\frac{n}{s}} \in \Delta_2 \cap \nabla_2$ with the parameter $K_{\phi^{n/s}}, a$ depending on n, s, K_Φ.*

Proof. Since $\phi \in \Delta_2$, we write

$$H(2t) = \Bigg(\int_0^{2t}\Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}}\,d\tau\Bigg)^{\frac{n-s}{n}} \geq \Bigg(\int_0^{t}\Big(\frac{2\tau}{K_\phi\phi(\tau)}\Big)^{\frac{s}{n-s}}2\,d\tau\Bigg)^{\frac{n-s}{n}} = \frac{2}{K_\phi^{\frac{s}{n}}}H(t).$$

Putting $2y = H(2t)$, we have $K_\phi^{\frac{s}{n}}y \geq H\big(\frac{H^{-1}(2y)}{2}\big)$. Therefore,

$$H^{-1}(2y) \leq 2H^{-1}(K_\phi^{\frac{s}{n}}y) \leq 2^2H^{-1}\Bigg(K_\phi^{\frac{s}{n}}\frac{K_\phi^{\frac{s}{n}}}{2}y\Bigg)$$

$$\leq \cdots \leq 2^{m+1} H^{-1}\Bigg(K_\phi^{\frac{s}{n}}\bigg(\frac{K_\phi^{\frac{s}{n}}}{2}\bigg)^m y\Bigg).$$

Because of the range of K, we get $\frac{K_\phi^{\frac{s}{n}}}{2} < 1$. Letting m large enough such that $K_\phi^{\frac{s}{n}}\big(\frac{K_\phi^{\frac{s}{n}}}{2}\big)^m < 1$, then we have $H^{-1}(2y) < CH^{-1}(y)$. Hence $H^{-1} \in \Delta_2$ and $\phi_{\frac{n}{s}} = \phi \circ H^{-1} \in \Delta_2$.

On the other hand, using the decreasing property of $\frac{\tau}{\phi(\tau)}$, we get

$$H(2^{\frac{n}{s}}x) = \bigg(\int_0^{2^{\frac{n}{s}}x} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} d\tau\bigg)^{\frac{n-s}{n}} = \bigg(\int_0^{x} \Big(\frac{2^{\frac{n}{s}}\tau}{\phi(2^{\frac{n}{s}}\tau)}\Big)^{\frac{s}{n-s}} 2^{\frac{n}{s}}\, d\tau\bigg)^{\frac{n-s}{n}}$$

$$\leq \bigg(\int_0^{x} \Big(\frac{\tau}{\phi(\tau)}\Big)^{\frac{s}{n-s}} 2^{\frac{n}{s}}\, d\tau\bigg)^{\frac{n-s}{n}} = 2^{\frac{n-s}{s}} H(x).$$

Hence $2^{\frac{n}{s}}x \leq H^{-1}(2^{\frac{n-s}{s}}H(x))$, that is, $2^{\frac{n}{s}}H^{-1}(x) \leq H^{-1}(2^{\frac{n-s}{s}}x)$. It means that

$$2^{\frac{n}{s}}\phi \circ H^{-1}(x) \leq \phi(2^{\frac{n}{s}}H^{-1}(x)) \leq \phi \circ H^{-1}(2^{\frac{n-s}{s}}x).$$

Letting $a = 2^{\frac{n-s}{s}} > 1$, we have $\phi_{\frac{n}{s}}(x) \leq \frac{1}{2a}\phi_{\frac{n}{s}}(ax)$ and $\phi_{\frac{n}{s}} \in \nabla_2$. $\square$

Remark 2.7. If $K_\phi \geq 2^{\frac{n}{s}}$, then there exists $\phi \in \Delta_2$ such that $\phi_{\frac{n}{s}} \notin \Delta_2$ (see [4, Example 6.4]). Let ϕ be a Young function with

$$\phi(t) = \begin{cases} t^{\frac{n}{s}}(\log\frac{1}{t})^{\alpha_0} & \text{near } 0, \\ t^{\frac{n}{s}}(\log t)^{\alpha} & \text{near } \infty, \end{cases}$$

where $\alpha_0 > \frac{n}{s} - 1$, $\alpha \leq \frac{n}{s} - 1$. And connected by a convex function,

$$\phi_{\frac{n}{s}}(t) \text{ is equivalent to } \begin{cases} e^{-t^{-\frac{n}{s(\alpha_0+1)-n}}} & \text{near } 0, \\ e^{t^{\frac{n}{n-s(\alpha+1)}}} & \text{near } \infty,\ \alpha < \frac{n}{s} - 1, \\ e^{e^{t^{\frac{n}{n-s}}}} & \text{near } \infty,\ \alpha = \frac{n}{s} - 1, \end{cases}$$

so $\phi_{\frac{n}{s}} \notin \Delta_2$.

Then we present a few lemmas that might be utilized to support the assertion of Theorem 1.2(i).

Lemma 2.8 [16]. *Let $\psi \in \Delta_2 \cap \nabla_2$ be a Young function. For any $0 < q < \infty$, there exists a constant $C > 1$ depending on n, q, K_ψ and a such that for* *all sequences* $\{f_j\}_{j\in\mathbb{N}}$*, we have*

$$\int_{\mathbb{R}^n} \psi\bigg(\bigg[\sum_{j\in\mathbb{N}} (\mathcal{M}(f_j))^2\bigg]^{\frac{1}{q}}\bigg)\,dx \le C(n, K_\psi, a) \int_{\mathbb{R}^n} \psi\bigg(\bigg[\sum_{j\in\mathbb{N}} (f_j)^2\bigg]^{\frac{1}{q}}\bigg)\,dx.$$

LEMMA 2.9. *For any constant* $k \ge 1$*, sequence* $\{a_j\}_{j\in\mathbb{N}}$*, and cubes* $\{Q_j\}_{j\in\mathbb{N}}$ *with* $\sum_j \chi_{Q_j} \le k$*, we have*

$$\sum_j |a_j|\chi_{kQ_j} \le C(k,n) \sum_j \big[\mathcal{M}(|a_j|^{\frac{1}{2}}\chi_{Q_j})\big]^2.$$

PROOF. By the definition of $\mathcal{M}$ we know that $\chi_{kQ_j} \le k^n \mathcal{M}(\chi_{Q_j})$, so

$$\sum_j |a_j|\chi_{kQ_j} = \sum_j \big(|a_j|^{\frac{1}{2}}\chi_{kQ_j}\big)^2 \le k^{2n} \sum_j \big[\mathcal{M}(|a_j|^{\frac{1}{2}}\chi_{Q_j})\big]^2. \quad \square$$

Now we are in the position to prove Theorem 1.2(i).

PROOF OF THEOREM 1.2(i). Let Ω be a c-John domain. Applying Boman [6] and Buckley [9], Ω enjoys the following chain property: for every integer $\kappa > 1$, there exist a positive constant $C_{\kappa,\Omega}$ and a collection $\mathcal{F}$ of the cubes satisfying the following conditions:

(i) $Q \subset \kappa Q \subset \Omega$ for all $Q \in \mathcal{F}$, $\Omega = \bigcup_{Q\in\mathcal{F}} Q$ and

$$\sum_{Q\in\mathcal{F}} \chi_{\kappa Q} \le C_{\kappa,c}\chi_\Omega.$$

(ii) Let $Q_0 \in \mathcal{F}$ be the fixed cube with center x_0 by the definition of the John domain. For any other $Q \in \mathcal{F}$, there exists a subsequence $\{Q_j\}_{j=1}^N \subset \mathcal{F}$, satisfying $Q = Q_N \subset C_{\kappa,c}Q_j$, $C_{\kappa,c}^{-1}|Q_{j+1}| \le |Q_j| \le C_{\kappa,c}|Q_{j+1}|$ and $|Q_j \cap Q_{j+1}| \ge C_{\kappa,c}^{-1}\min\{|Q_j|, |Q_{j+1}|\}$ for all $j = 0, \dots, N-1$.

Let $\kappa = 5n$. By $Q \subset 5nQ \subset \Omega$ for each $Q \in \mathcal{F}$ in (i), we know

$$d(Q, \partial\Omega) \ge d(Q, \partial(5nQ) \ge \frac{5n-1}{2} l(Q) \ge 2nl(Q).$$

Hence for any $x, y \in Q \in \mathcal{F}$,

$$|x-y| \le \sqrt{n}l(Q) \le nl(Q) \le \frac{1}{2}d(Q,\partial\Omega) \le \frac{1}{2}d(x,\partial\Omega).$$

Let $u \in \dot{V}_*^{s,\phi}(\Omega)$. Up to approximating by $\min\{\max\{u, -N\}, N\}$, we can assume that $u \in L^\infty(\Omega)$. Remember the boundedness of Ω, we know $u \in L^1(\Omega)$. Since $|x-y| \le \frac{1}{2}d(x,\partial\Omega)$ for any $x, y \in Q$, we know that

$$\int_Q \int_Q \phi\bigg(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\bigg)\frac{dx\,dy}{|x-y|^n} \tag{2.7}$$

$$\leq \int_Q \int_{B(x,\frac{1}{2}d(x,\partial\Omega))} \phi\Big(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\Big)\frac{dy\,dx}{|x-y|^n}.$$

Hence $\|u\|_{\dot{V}^{s,\phi}(Q)} \leq \|u\|_{\dot{V}_*^{s,\phi}(\Omega)}$. It means that if $\lambda \geq \|u\|_{\dot{V}_*^{s,\phi}(\Omega)}$, then $\lambda \geq \|u\|_{\dot{V}^{s,\phi}(Q)}$.

Because of the convexity of $\phi_{\frac{n}{s}}$, we have

$$I := \int_\Omega \phi_{\frac{n}{s}}\Big(\frac{|u(z)-u_\Omega|}{\lambda}\Big)\,dz \leq \int_\Omega \phi_{\frac{n}{s}}\Big(\frac{1}{2}\Big(\frac{2|u(z)-u_{Q_0}|+2|u_\Omega-u_{Q_0}|}{\lambda}\Big)\Big)\,dz$$
$$\leq \frac{1}{2}\Big[\int_\Omega \phi_{\frac{n}{s}}\Big(\frac{2|u(z)-u_{Q_0}|}{\lambda}\Big)\,dz + |\Omega|\phi_{\frac{n}{s}}\Big(\frac{2|u_\Omega-u_{Q_0}|}{\lambda}\Big)\Big].$$

By Jensen inequality,

$$|\Omega|\phi_{\frac{n}{s}}\Big(\frac{2|u_\Omega-u_{Q_0}|}{\lambda}\Big) \leq \int_\Omega \phi_{\frac{n}{s}}\Big(\frac{2|u(z)-u_{Q_0}|}{\lambda}\Big)\,dz.$$

Applying $\chi_\Omega \leq \sum_{Q\in\mathcal{F}} \chi_Q$ in (i),

$$I \leq \int_\Omega \phi_{\frac{n}{s}}\Big(\frac{2|u(z)-u_{Q_0}|}{\lambda}\Big)\,dz \leq \sum_{Q\in\mathcal{F}} \int_Q \phi_{\frac{n}{s}}\Big(\frac{2|u(z)-u_{Q_0}|}{\lambda}\Big)\,dz$$
$$\leq \frac{1}{2}\sum_{Q\in\mathcal{F}} \int_Q \phi_{\frac{n}{s}}\Big(\frac{4|u(z)-u_Q|}{\lambda}\Big)\,dz + \frac{1}{2}\sum_{Q\in\mathcal{F}\setminus\{Q_0\}} |Q|\phi_{\frac{n}{s}}\Big(\frac{4|u_Q-u_{Q_0}|}{\lambda}\Big)$$
$$=: \frac{1}{2}I_1 + \frac{1}{2}I_2.$$

By the inequalities (2.4) and (2.7), we get

$$I_1 \leq \sum_{Q\in\mathcal{F}} \int_Q \int_Q \phi\Big(\frac{|u(x)-u(y)|}{\frac{\lambda}{4C_1}|x-y|^s}\Big)\frac{dx\,dy}{|x-y|^n}$$
$$\leq \sum_{Q\in\mathcal{F}} \int_Q \int_{B(x,\frac{1}{2}d(x,\partial\Omega))} \phi\Big(\frac{|u(x)-u(y)|}{\frac{\lambda}{4C_1}|x-y|^s}\Big)\frac{dy\,dx}{|x-y|^n}.$$

Using the $\sum_{Q\in\mathcal{F}} \chi_{\kappa Q} \leq C_{\kappa,c}\chi_\Omega$ in (i) above,

$$I_1 \leq C_{\kappa,c} \int_\Omega \int_{B(x,\frac{1}{2}d(x,\partial\Omega))} \phi\Big(\frac{|u(x)-u(y)|}{\frac{\lambda}{4C_1}|x-y|^s}\Big)\frac{dy\,dx}{|x-y|^n}.$$

If $\lambda > 4C_1 \max\{C_{\kappa,c}, 1\}\|u\|_{\dot{V}_*^{s,\phi}(\Omega)}$ where $C_1 = C(n,s)$, we have $I_1 \leq 1$.

For I_2, for each $Q \in \mathcal{F}$ and $Q \neq Q_0$, by the chain property in (ii), we know that there exist a chain connecting $Q = Q_N$ and Q_0. Write

$$|u_Q - u_{Q_0}| \leq \sum_{j=0}^{N-1} |u_{Q_j} - u_{Q_{j+1}}|$$

$$\leq \sum_{j=0}^{N-1} \big(|u_{Q_j} - u_{Q_{j+1}\cap Q_j}| + |u_{Q_{j+1}} - u_{Q_{j+1}\cap Q_j}|\big).$$

For adjacent cubes Q_j, Q_{j+1}, one has

$$|Q_j - Q_{j+1}| \geq C_{\kappa,c}^{-1} \min\{|Q_j|, |Q_{j+1}|\}, \quad C_{\kappa,c}^{-1}|Q_{j+1}| \leq |Q_j| \leq C_{\kappa,c}|Q_{j+1}|,$$

so

$$|u_{Q_j} - u_{Q_{j+1}\cap Q_j}| \leq \frac{1}{|Q_{j+1} \cap Q_j|} \int_{Q_{j+1}\cap Q_j} |u(v) - u_{Q_j}|\, dv$$

$$\leq \frac{C_{\kappa,c}^2}{|Q_j|} \int_{Q_j} |u(v) - u_{Q_j}|\, dv.$$

Similarly,

$$|u_{Q_{j+1}} - u_{Q_{j+1}\cap Q_j}| \leq \frac{C_{\kappa,c}^2}{|Q_{j+1}|} \int_{Q_{j+1}} |u(v) - u_{Q_{j+1}}|\, dv.$$

As a result, we get

$$|u_Q - u_{Q_0}| \leq 2C_{\kappa,c}^2 \sum_{j=0}^{N} \fint_{Q_j} |u(v) - u_{Q_j}|\, dv.$$

For each Q_j, by the convexity of $\phi_{\frac{n}{s}}$ and the Jensen inequality,

$$\fint_{Q_j} \frac{|u(v) - u_{Q_j}|}{\lambda}\, dv = \phi_{\frac{n}{s}}{}^{-1} \circ \phi_{\frac{n}{s}}\Big(\fint_{Q_j} \frac{|u(v) - u_{Q_j}|}{\lambda}\, dv\Big)$$

$$\leq \phi_{\frac{n}{s}}{}^{-1}\Big(\fint_{Q_j} \phi_{\frac{n}{s}}\Big(\frac{|u(v) - u_{Q_j}|}{\lambda}\Big)\, dv\Big).$$

Using inequalities (2.4) and (2.7),

$$\int_{Q_j} \phi_{\frac{n}{s}}\Big(\frac{|u(v) - u_{Q_j}|}{\lambda}\Big)\, dv \leq \int_{Q_j}\int_{Q_j} \phi\Big(\frac{|u(v) - u(w)|}{\frac{\lambda}{C_1}|v - w|^s}\Big) \frac{dv\, dw}{|v - w|^n}$$

$$\leq \int_{Q_j}\int_{B(v,\frac{1}{2}d(v,\partial\Omega))}\phi\bigg(\frac{|u(v)-u(w)|}{\frac{\lambda}{C_1}|v-w|^s}\bigg)\frac{dw\,dv}{|v-w|^n}=:\int_{Q_j}f(v)\,dv.$$

Then we have

$$⨍_{Q_j}\frac{|u(v)-u_{Q_j}|}{\lambda}\,dv\leq {\phi_{\frac{n}{s}}}^{-1}\bigg(⨍_{Q_j}f(v)\,dv\bigg).$$

Hence

$$\frac{4|u_Q-u_{Q_0}|}{\lambda}\leq 8\,C_{\kappa,c}^2\sum_{j=0}^{N}{\phi_{\frac{n}{s}}}^{-1}\bigg(⨍_{Q_j}f(v)\,dv\bigg).$$

Together with Lemma 2.2, we get

$$\phi_{\frac{n}{s}}\bigg(8\,C_{\kappa,c}^2\sum_{j=0}^{N}{\phi_{\frac{n}{s}}}^{-1}\bigg(⨍_{Q_j}f(v)\,dv\bigg)\bigg)$$
$$\leq C(n,s,C_{\kappa,c},K_\Phi)\phi_{\frac{n}{s}}\bigg(\sum_{j=0}^{N}{\phi_{\frac{n}{s}}}^{-1}\bigg(⨍_{Q_j}f(v)\,dv\bigg)\bigg).$$

Applying $Q=Q_N\subset C_{\kappa,c}Q_j$ given in (ii),

$$|Q|\phi_{\frac{n}{s}}\bigg(\sum_{j=0}^{N}{\phi_{\frac{n}{s}}}^{-1}\bigg(⨍_{Q_j}f(v)\,dv\bigg)\bigg)$$
$$\leq\int_Q\phi_{\frac{n}{s}}\bigg(\sum_{P\in\mathcal{F}}{\phi_{\frac{n}{s}}}^{-1}\bigg(⨍_P f(v)\,dv\bigg)\chi_{C_{\kappa,c}P}\bigg)(x)\,dx.$$

Using the $\sum_{Q\in\mathcal{F}}\chi_Q\leq\sum_{Q\in\mathcal{F}}\chi_{\kappa Q}\leq C_{\kappa,c}\chi_\Omega$ in (i) above,

$$I_2\leq C(n,s,C_{\kappa,c},K_\Phi)\sum_{Q\in\mathcal{F}}\int_Q\phi_{\frac{n}{s}}\bigg(\sum_{P\in\mathcal{F}}{\phi_{\frac{n}{s}}}^{-1}\bigg(⨍_P f(v)\,dv\bigg)\chi_{C_{\kappa,c}P}\bigg)(x)\,dx$$
$$\leq C(n,s,C_{\kappa,c},K_\Phi)\int_\Omega\phi_{\frac{n}{s}}\bigg(\sum_{P\in\mathcal{F}}{\phi_{\frac{n}{s}}}^{-1}\bigg(⨍_P f(v)\,dv\bigg)\chi_{C_{\kappa,c}P}\bigg)(x)\,dx.$$

By Lemma 2.9,

$$I_2\leq C(n,s,C_{\kappa,c},K_\Phi)$$
$$\times\int_\Omega\phi_{\frac{n}{s}}\bigg(\sum_{P\in\mathcal{F}}\bigg\{\mathcal{M}\bigg[\bigg({\phi_{\frac{n}{s}}}^{-1}(⨍_P f(v)\,dv)\bigg)^{\frac{1}{2}}\chi_P\bigg]\bigg\}^2\bigg)(x)\,dx.$$

Set $\psi(t) := \phi_{\frac{n}{s}}(t^2)$. Applying $\phi_{\frac{n}{s}} \in \Delta_2 \cap \nabla_2$ in Lemma 2.6 gives $\psi \in \Delta_2 \cap \nabla_2$. Together with Lemma 2.8 to $q = 2$ and ψ, we have

$$I_2 \le C(n, s, C_{\kappa,c}, K_\Phi) \int_\Omega \phi_{\frac{n}{s}} \Bigg(\sum_{P \in \mathcal{F}} \Bigg({\phi_{\frac{n}{s}}}^{-1} \Bigg(\fint_P f(v)\, dv \Bigg) \Bigg) \chi_P \Bigg)(x)\, dx.$$

Denote $a_P = \fint_P f(v)\, dv$. For each $x \in \Omega$, using the convexity and the increasing property of $\phi_{\frac{n}{s}}$,

$$\begin{aligned} \phi_{\frac{n}{s}} \Bigg(\sum_{P \in \mathcal{F}} ({\phi_{\frac{n}{s}}}^{-1}(a_P))\chi_P(x) \Bigg) &= \phi_{\frac{n}{s}} \Bigg(\frac{\sum_{P \in \mathcal{F}} \chi_P(x)}{\sum_{P \in \mathcal{F}} \chi_P(x)} \sum_{P \in \mathcal{F}} ({\phi_{\frac{n}{s}}}^{-1}(a_P))\chi_P(x) \Bigg) \\ &\le \phi_{\frac{n}{s}} \Bigg(\frac{C_{\kappa,c}}{\sum_{P \in \mathcal{F}} \chi_P(x)} \sum_{P \in \mathcal{F}} ({\phi_{\frac{n}{s}}}^{-1}(a_P))\chi_P(x) \Bigg) \\ \le \sum_{P \in \mathcal{F}} \frac{\chi_P(x)}{\sum_{P \in \mathcal{F}} \chi_P(x)} \phi_{\frac{n}{s}}(C_{\kappa,c}{\phi_{\frac{n}{s}}}^{-1}(a_P)) &\le C(n, s, C_{\kappa,c}, K_\Phi) \sum_{P \in \mathcal{F}} \chi_P(x) a_P. \end{aligned}$$

Therefore,

$$\begin{aligned} I_2 &\le C \int_\Omega \sum_{P \in \mathcal{F}} a_P \chi_P(x)\, dx \le C \sum_{P \in \mathcal{F}} a_P |P| = C \sum_{P \in \mathcal{F}} \int_P f(v)\, dv \\ &\le C(n, s, C_{\kappa,c}, K_\phi) \int_\Omega \int_{B(v, \frac{1}{2}d(v, \partial\Omega))} \phi \Bigg(\frac{C|u(v) - u(y)|}{\lambda |u - w|^s} \Bigg) \frac{dw\, dv}{|u - w|^n}. \end{aligned}$$

Combining I_1 and I_2 gives

$$I \le C(n, s, C_{\kappa,c}, K_\phi) \int_\Omega \int_{B(v, \frac{1}{2}d(v, \partial\Omega))} \phi \Bigg(\frac{C|u(v) - u(y)|}{\lambda |u - w|^s} \Bigg) \frac{dw\, dv}{|u - w|^n}.$$

Letting $\lambda > 4C(n, s, C_{\kappa,c}, K_\phi) C_1 \|u\|_{{\dot{V}_*}^{s,\phi}(\Omega)}$, we have $I \le 1$. ☐

3. Proof of Theorem 1.2(ii)

To prove Theorem 1.2(ii), the most important method is getting the fact which Lemma 3.5 expressed. We first need to choose a special test function to estimate the relationship between its norms and its radius.

Let $z \in \Omega$, $d(z, \partial\Omega) \le m < \operatorname{diam} \Omega$. Denote by $\Omega_{z,m}$ a component of $\Omega \setminus \overline{B_\Omega(z, m)}$. For $t > r \ge m$ with $\Omega_{z,m} \ne \varnothing$, define $u_{z,r,t}$ in Ω as

$$u_{z,r,t}(y) = \begin{cases} 0 & y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z, r)] \\ \frac{|y - z| - r}{t - r} & y \in \Omega_{z,m} \cap [B(z, t) \setminus B(z, r)], \\ 1 & y \in \Omega_{z,m} \setminus B_\Omega(z, t), \end{cases} \tag{3.1}$$

where $B_\Omega(z,t) = B(z,t) \cap \Omega$.

It's not difficult to prove the following property.

Lemma 3.1. *$u_{z,r,t}$ is Lipschitz with the Lipschitz constant $\frac{1}{t-r}$.*

Proof. We split the proof into three cases.

Case 1. For $x \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z,r)]$, we have $u_{z,r,t}(x) = 0$. Since $u_{z,r,t}(y) = u_{z,r,t}(x) = 0$ when $y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z,r)]$, we only need to consider $y \in \Omega_{z,m} \cap [B(z,t) \setminus B(z,r)]$ or $y \in \Omega_{z,m} \setminus B_\Omega(z,t)$. If $y \in \Omega_{z,m} \cap [B(z,t) \setminus B(z,r)]$, we know $|x-z| \le r$. Hence

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = \frac{|y-z|-r}{t-r} \le \frac{|y-z|-|x-z|}{t-r} \le \frac{|x-y|}{t-r}.$$

If $y \in \Omega_{z,m} \setminus B_\Omega(z,t)$, we get $|x-y| \ge t-r$. Therefore,

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = 1 \le \frac{|x-y|}{t-r}.$$

Case 2. For $x \in \Omega_{z,m} \cap [B(z,t) \setminus B(z,r)]$, we have $u_{z,r,t}(x) = \frac{|x-z|-r}{t-r}$. If $y \in \Omega_{z,m} \cap [B(z,t) \setminus B(z,r)]$ with $u_{z,r,t}(y) = \frac{|y-z|-r}{t-r}$,

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = \left| \frac{|x-z|-r}{t-r} - \frac{|y-z|-r}{t-r} \right| \le \frac{|x-z|-|y-z|}{t-r} \le \frac{|x-y|}{t-r}.$$

If $y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z,r)]$ with $u_{z,r,t}(y) = 0$, we have $|y-z| \le r$. Then

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = \frac{|x-z|-r}{t-r} \le \frac{|y-z|-|x-z|}{t-r} \le \frac{|x-y|}{t-r}.$$

If $y \in \Omega_{z,m} \setminus B_\Omega(z,t)$ with $u_{z,r,t}(y) = 1$, then $|y-z| \ge t$. Together with $|x-z| \le t$, we have

$$\big|u_{z,r,t}(x) - u_{z,r,t}(y)\big| = \left| \frac{|x-z|-r}{t-r} - 1 \right| = \left| \frac{|x-z|-t}{t-r} \right| = \frac{t-|x-z|}{t-r} \le \frac{|y-z|-|x-z|}{t-r} \le \frac{|x-y|}{t-r}.$$

Case 3. For $x \in \Omega_{z,m} \setminus B_\Omega(z,t)$, we have $u_{z,r,t}(x) = 1$. Since $u_{z,r,t}(y) = u_{z,r,t}(x) = 1$ when $y \in \Omega_{z,m} \setminus B_\Omega(z,t)$, we only need to consider $y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z,r)]$ or $y \in \Omega_{z,m} \cap [B(z,t) \setminus B(z,r)]$. If $y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z,r)]$ with $u_{z,r,t}(y) = 0$, together with $|x-y| \ge t-r$, we know

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = 1 \le \frac{|x-y|}{t-r}.$$

If $y \in \Omega_{z,m} \cap [B(z,t) \setminus B(z,r)]$ with $u_{z,r,t}(y) = \frac{|y-z|-r}{t-r}$, then $|y-z| \le t$. Moreover, $|x-z| \ge t$. Hence

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = \left|1 - \frac{|y-z|-r}{t-r}\right| \le \frac{|x-z|-|y-z|}{t-r} \le \frac{|x-y|}{t-r}.$$

Combining the above cases, we get that $u_{z,r,t}$ is Lipschitz with the Lipschitz constant $\frac{1}{t-r}$. □

Next we provide an estimation of the test function.

Lemma 3.2. *Let $s \in (0,1)$ and ϕ be a Young function. For any bounded domain $\Omega \subset \mathbb{R}^n$ and $z \in \Omega$ with $d(z, \partial\Omega) \le m < \operatorname{diam} \Omega$. For $t > r \ge m$, we have $u_{z,r,t} \in \dot{V}_*^{s,\phi}(\Omega)$ with*

$$\|u_{z,r,t}\|_{\dot{V}_*^{s,\phi}(\Omega)} \le C\left[\phi^{-1}\Big(\frac{1}{|\Omega_{z,m} \setminus B(z,r)|}\Big)\right]^{-1} \frac{1}{(t-r)^s},$$

where $C = C(n, s, C_\phi) \ge 1$.

Proof. For any $x \in \Omega$ and $y \in B(x, \frac{1}{2}d(x,\partial\Omega)) \subset \Omega$, $|u_{z,r,t}(x) - u_{z,r,t}(y)| \neq 0$ means that either x or y in $\Omega_{z,m} \setminus B(z,r)$.

$$\begin{aligned}
H &:= \int_\Omega \int_{|x-y|<\frac{1}{2}d(x,\partial\Omega)} \phi\Big(\frac{|u_{z,r,t}(x) - u_{z,r,t}(y)|}{\lambda|x-y|^s}\Big) \frac{dy\,dx}{|x-y|^n} \\
&\le 2\int_{\Omega_{z,m}\setminus B(z,r)} \int_\Omega \phi\Big(\frac{|u_{z,r,t}(x) - u_{z,r,t}(y)|}{\lambda|x-y|^s}\Big) \frac{dy\,dx}{|x-y|^n} \\
&\le 2\int_{\Omega_{z,m}\setminus B(z,r)} \int_{B(x,t-r)} \phi\Big(\frac{|x-y|^{1-s}}{\lambda(t-r)}\Big) \frac{dy\,dx}{|x-y|^n} \\
&\quad +2\int_{\Omega_{z,m}\setminus B(z,r)} \int_{\mathbb{R}^n\setminus B(x,t-r)} \phi\Big(\frac{1}{\lambda|x-y|^s}\Big) \frac{dy\,dx}{|x-y|^n} := 2H_1 + 2H_2.
\end{aligned}$$

Using change of variable and (2.1), we have

$$\begin{aligned}
H_1 &= \int_{\Omega_{z,m}\setminus B(z,r)} \int_0^{t-r} n\omega_n \phi\Big(\frac{\rho^{1-s}}{\lambda(t-r)}\Big) \frac{d\rho}{\rho}\,dx \\
&= \int_{\Omega_{z,m}\setminus B(z,r)} \int_0^{\frac{1}{\lambda(t-r)^s}} n\omega_n \frac{1}{1-s}\phi(\mu)\frac{d\mu}{\mu}\,dx \\
&\le \int_{\Omega_{z,m}\setminus B(z,r)} \frac{C_\phi n\omega_n}{1-s}\phi\Big(\frac{1}{\lambda(t-r)^s}\Big)\,dx = \frac{C_\phi n\omega_n}{1-s}\phi\Big(\frac{1}{\lambda(t-r)^s}\Big)|\Omega_{z,m}\setminus B(z,r)|.
\end{aligned}$$

Similarly, for I_2, we get

$$H_2 = \int_{\Omega_{z,m}\setminus B(z,r)} \int_{t-r}^{\infty} n\omega_n \phi\Big(\frac{1}{\lambda\rho^s}\Big)\frac{d\rho}{\rho}\,dx$$
$$= \int_{\Omega_{z,m}\setminus B(z,r)} \int_0^{\frac{1}{\lambda(t-r)^s}} n\omega_n \frac{1}{s}\phi(\mu)\frac{d\mu}{\mu}\,dx$$
$$\le \int_{\Omega_{z,m}\setminus B(z,r)} \frac{C_\phi n\omega_n}{s}\phi\Big(\frac{1}{\lambda(t-r)^s}\Big)\,dx$$
$$= \frac{C_\phi n\omega_n}{s}\phi\Big(\frac{1}{\lambda(t-r)^s}\Big)\big|\Omega_{z,m}\setminus B(z,r)\big|.$$

Let

$$\lambda = M\Big[\phi^{-1}\Big(\frac{1}{|\Omega_{z,m}\setminus B(z,r)|}\Big)\Big]^{-1}\frac{1}{(t-r)^s}$$

with $M \ge \max\big\{\frac{4C_\phi n\omega_n}{1-s}, \frac{4C_\phi n\omega_n}{s}, 1\big\}$. Then $H_1 \le \frac{1}{4}$ and $H_2 \le \frac{1}{4}$. In other words, $H \le 1$. As a result, we have

$$\|u_{z,r,t}\|_{\dot{V}_*^{s,\phi}(\Omega)} \le C\Big(\phi^{-1}\Big(\frac{1}{|\Omega_{z,m}\setminus B(z,r)|}\Big)\Big)^{-1}\frac{1}{(t-r)^s}$$

as desired. □

For $x_0, z \in \Omega$, let $r > 0$ such that $d(z,\partial\Omega) < r < |x_0 - z|$. Define

$$\omega_{x_0,z,r}(y) := \frac{1}{r}\inf_{\gamma(x_0,y)} l(\gamma \cap B(z,r)) \quad \text{for all } y \in \Omega,$$

where the infimum is taken over all rectifiable curves γ joining x_0 and y.

LEMMA 3.3. *Let $s \in (0,1)$ and ϕ be a Young function. For any bounded domain $\Omega \subset \mathbb{R}^n$ and $x_0, z \in \Omega$ and $r > 0$ with $d(z,\partial\Omega) \le r < |x_0 - z|$, we know $\omega_{x_0,z,r} \in \dot{V}_*^{s,\phi}(\Omega)$ and there exist a constant $C = C(n,s,C_\phi) \ge 1$ such that*

$$\|\omega_{x_0,z,r}\|_{\dot{V}_*^{s,\phi}(\Omega)} \le C\Big[\phi^{-1}\Big(\frac{1}{r^n}\Big)\Big]^{-1}\frac{1}{r^s}.$$

PROOF. For $x \in \Omega \setminus B(z,6r), y \in B(x, \frac{1}{2}d(x,\partial\Omega))$, we have

$$d(x,\partial\Omega) \le |x-z| + d(z,\partial\Omega) \le |x-z| + r,$$

and

$$|y-z| \ge |x-z| - |y-x| \ge |x-z| - \frac{1}{2}(|x-z|+r)$$

$$= \frac{1}{2}|x-z| - \frac{r}{2} \geq 3r - \frac{r}{2} \geq 2r.$$

So $B(x, \partial\Omega)) \cap B(z, 2r) = \varnothing$.

Let $\gamma_{x,y}$ be the segment joining x, y contained in $B(x, \frac{1}{2}d(x, \partial\Omega))$, then $\gamma_{x,y} \subset \Omega \setminus B(z, r)$. For any $\gamma(x_0, x)$, we know $\gamma(x_0, x) \cup \gamma_{x,y}$ is a curve joining x_0 and y, with

$$l((\gamma(x_0, x) \cup \gamma_{x,y}) \cap B(z, r)) = l(\gamma(x_0, x) \cap B(z, r)).$$

Hence $\omega_{x_0,z,r}(y) \leq \omega_{x_0,z,r}(x)$. Similarly, we could also get $\omega_{x_0,z,r}(x) \leq \omega_{x_0,z,r}(y)$. Hence for any $x \in \Omega \setminus B(z, 6r), y \in B(x, \frac{1}{2}d(x, \partial\Omega))$, we have

$$\omega_{x_0,z,r}(x) = \omega_{x_0,z,r}(y).$$

For any $x \in \Omega$ and $|x-y| < \frac{1}{2}d(x, \partial\Omega)$, it is easy to know $l(\gamma_{x,y} \cap B(z, r)) \leq |x-y|$. Since $\gamma(x_0, x) \cup \gamma_{x,y}$ is a curve joining x_0 and y, we get

$$\omega_{x_0,z,r}(y) \leq \omega_{x_0,z,r}(x) + \frac{1}{r}|x-y|.$$

Likewise, $\omega_{x_0,z,r}(x) \leq \omega_{x_0,z,r}(y) + \frac{1}{r}|x-y|$. So

$$|\omega_{x_0,z,r}(y) - \omega_{x_0,z,r}(x)| \leq \frac{1}{r}|x-y|.$$

For $x \in \Omega \cap B(z, 6r)$, we have $d(x, \partial\Omega) \leq 6r + d(z, \partial\Omega) < 8r$. Hence

$$H := \int_\Omega \int_{|x-y|<\frac{1}{2}d(x,\partial\Omega)} \phi\bigg(\frac{|\omega_{x_0,z,r}(x) - \omega_{x_0,z,r}(y)|}{\lambda|x-y|^s}\bigg)\frac{dy\,dx}{|x-y|^n}$$
$$= \int_{\Omega\cap B(z,6r)} \int_{|x-y|<\frac{1}{2}d(x,\partial\Omega)} \phi\bigg(\frac{|\omega_{x_0,z,r}(x) - \omega_{x_0,z,r}(y)|}{\lambda|x-y|^s}\bigg)\frac{dy\,dx}{|x-y|^n}$$
$$\leq \int_{\Omega\cap B(z,6r)} \int_0^{4r} n\,\omega_n \phi\bigg(\frac{\rho^{1-s}}{\lambda}\bigg)\frac{d\rho}{\rho}\,dx$$
$$\leq \int_{\Omega\cap B(z,6r)} \frac{C_\phi n\,\omega_n}{1-s}\phi\bigg(\frac{4^{1-s}}{\lambda r^s}\bigg)\,dx \leq \frac{C_\phi n\,\omega_n^2}{1-s}\phi\bigg(\frac{4^{1-s}}{\lambda r^s}\bigg)(6r)^n,$$

Letting $\lambda = M\big[\phi^{-1}(\frac{1}{r^n})\big]^{-1}\frac{1}{r^s}$ with $M > \max\big\{\frac{C_\phi n\,\omega_n^2 4^{1-s}}{1-s}6^n, 4^{1-s}\big\}$, we get $H \leq 1$. So

$$\|\omega_{x_0,z,r}\|_{\dot{V}_*^{s,\phi}(\Omega)} \leq C\bigg[\phi^{-1}\bigg(\frac{1}{r^n}\bigg)\bigg]^{-1}\frac{1}{r^s}. \quad \square$$

LEMMA 3.4. *Let $s \in (0,1)$ and a Young function $\phi \in \Delta_2$ with $K_\phi < 2^{\frac{n}{s}}$. Assume a bounded domain $\Omega \subset \mathbb{R}^n$ supports the $(\phi_{\frac{n}{s}}, \phi)$-Poincaré inequality* (1.4). *Fix a point x_0 so that $r_0 := \max\{d(x,\partial\Omega) : x \in \Omega\} = d(x_0, \partial\Omega)$. If $x, x_0 \in \Omega \setminus \overline{B(z,r)}$ for some $z \in \Omega$ and $r \in (0, 2\operatorname{diam}\Omega)$, then there exists a positive constant b_0 independent of x, z, r such that x, x_0 are contained in the same component of $\Omega \setminus \overline{B(z, b_0 r)}$.*

PROOF. Denote

$$b_{x,z,r} := \sup\Big\{c \in (0,1] : x, x_0 \text{ are in the same component of } \Omega \setminus \overline{B(z,cr)}\Big\}.$$

It suffices to prove that $b_{x,z,r}$ has the positive low bound independent of x, z, r. Since let $b_0 = \frac{b_{x,z,r}}{2}$, we could get the conclusion. Because it is a infimum problem, we may assume $b_{x,z,r} \le \frac{1}{10}$. Then we only need to prove that there exists a constant $C \ge 1$ such that

$$\frac{r}{C}\Big(\frac{1}{2} - 2b_{x,z,r}\Big) \le |\Omega_x|^{\frac{1}{n}} \le 2Cb_{x,z,r}r,$$

where Ω_x is the component of $\Omega \setminus \overline{B(z, 2b_{x,z,r}r)}$ containing x. Hence $b_{x,z,r} \ge \frac{1}{4(C^2+1)}$, that is, $b > 0$.

First for fixed x, z, r, we get $b_{x,z,r} > 0$. In fact, applying $z \in \Omega$, then there exists $0 < \delta < 1$ such that $B(z,\delta r) \subset \Omega$, and $x_0 \notin B(z,\delta r)$. Let $h = \frac{\delta}{2}$ and a curve $\gamma(x,x_0)$ connecting x and x_0. If $\gamma(x,x_0) \cap \overline{B(z,hr)} = \varnothing$, then x, x_0 are contained in the same component of $\Omega \setminus \overline{B(z,hr)}$. If $\gamma(x,x_0) \cap \overline{B(z,hr)} \ne \varnothing$, we make some notations. Put

$$t_0 := \inf\Big\{t \in [0,1] : \gamma(x,x_0)(t) \in \partial B(z,\delta r)\Big\},$$

$$t_1 := \sup\Big\{t \in [0,1] : \gamma(x,x_0)(t) \in \partial B(z,\delta r)\Big\},$$

$A := \gamma(x,x_0)(t_0)$ and $B := \gamma(x,x_0)(t_1)$. Then we have

$$\tilde{\gamma} = \gamma(x,x_0)|_{t\in(0,t_0)} \cup \widehat{AB} \cup \gamma(x,x_0)|_{t\in(t_0,1)} \subset \Omega \setminus \overline{B(z,hr)},$$

where x, x_0 are contained in the same component of $\Omega \setminus \overline{B(z,hr)}$. Therefore, we have $b_{x,z,r} \ge h > 0$.

Set $c_0 = 2b_{x,z,r} \le \frac{1}{5}$, then $x_0 \notin \overline{B(z,c_0 r)}$. Denote by Ω_{x_0} the component of $\Omega \setminus \overline{B(z,c_0 r)}$ containing x_0. Using $b_{x,z,r} < \frac{2}{3}c_0 < 1$, we have x, x_0 are not contained in the same component of $\Omega \setminus \overline{B(z, \frac{2}{3}c_0 r)}$. Now we prove that $B(z,c_0 r) \cap \partial\Omega \ne \varnothing$. If not, by $z \in \Omega$, we have $B(z, \frac{2}{3}c_0 r) \subset B(z,c_0 r) \subset \Omega$. Then x, x_0 are contained in the same component of $\Omega \setminus \overline{B(z, \frac{2}{3}c_0 r)}$, which we get contradiction with location between x and x_0. So $B(z, c_0 r) \cap \partial\Omega \neq \varnothing$. Noting that

$$\begin{aligned} r_0 = d(x_0, \partial\Omega) &\le \max_{y \in B(z, c_0 r)} |x_0 - y| \le r + c_0 r + d(x_0, B(z,r)) \\ &\le \frac{6}{5} r + d(x_0, B(z,r)), \end{aligned}$$

we know

$$\begin{aligned} d(x_0, B(z, c_0 r)) &\ge |x_0 - z| - \frac{r}{5} = d(x_0, B(z,r)) + \frac{4}{5} r \\ &\ge \frac{1}{2}\Big(d(x_0, B(z,r)) + \frac{6}{5} r\Big) = \frac{r_0}{2}, \end{aligned}$$

that is, $d(x_0, B(z, c_0 r)) \ge \frac{r_0}{2}$. Therefore,

$$B\Big(x_0, \frac{r_0}{2}\Big) \subset \Omega_{x_0} \subset \Omega \setminus \Omega_x. \tag{3.2}$$

For any $y \in \Omega$, define

$$\omega(y) := \frac{1}{c_0 r} \inf_{\gamma(x_0, y)} l(\gamma \cap B(z, c_0 r)).$$

Since $B(z, c_0 r) \cap \partial\Omega \neq \varnothing$ and $x_0 \notin \overline{B(z, c_0 r)}$, we have $d(z, \partial\Omega) < c_0 r < |x_0 - z|$. Using Lemma 3.3, we know

$$\|\omega\|_{\dot{V}_*^{s,\phi}(\Omega)} \le C \Big[\phi^{-1}\Big(\frac{1}{(c_0 r)^n}\Big)\Big]^{-1} \frac{1}{(c_0 r)^s}.$$

Applying the $(\phi_{\frac{n}{s}}, \phi)$-Poincaré inequality (1.4) ,

$$\|\omega - \omega_\Omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \le C \|\omega\|_{\dot{V}^{s,\phi}(\Omega)} \le C \Big[\phi^{-1}\Big(\frac{1}{(c_0 r)^n}\Big)\Big]^{-1} \frac{1}{(c_0 r)^s}.$$

On the other hand, we have $\omega(y) = 0$ for $y \in B(x_0, \frac{1}{2} r_0)$ by (3.2). Since Ω is bounded and $r_0 > 0$, we have $\frac{|\operatorname{diam}\Omega|}{r_0^n} \le C$. Using the convexity of $\phi_{\frac{n}{s}}$,

$$\begin{aligned} &\int_\Omega \phi_{\frac{n}{s}}\Big(\frac{|\omega(x)|}{\lambda}\Big)\, dx \\ &\le \frac{1}{2} \int_\Omega \phi_{\frac{n}{s}}\Big(\frac{|\omega(x) - \omega_\Omega|}{\lambda}\Big)\, dx + \frac{|\Omega|}{2} \phi_{\frac{n}{s}}\Big(\frac{|\omega_{B(x_0, \frac{1}{2} r_0)} - \omega_\Omega|}{\lambda}\Big). \end{aligned}$$

By the Jensen inequality once more,

$$|\Omega|\phi_{\frac{n}{s}}\left(\frac{|\omega_{B(x_0,\frac{1}{2}r_0)}-\omega_\Omega|}{\lambda}\right)\leq|\Omega|\fint_{B(x_0,\frac{1}{2}r_0)}\phi_{\frac{n}{s}}\left(\frac{|\omega(x)-\omega_\Omega|}{\lambda}\right)dx$$

$$\leq\frac{|\Omega|}{|B(x_0,\frac{1}{2}r_0)|}\int_\Omega\phi_{\frac{n}{s}}\left(\frac{|\omega(x)-\omega_\Omega|}{\lambda}\right)dx\leq 2^nC^n\int_\Omega\phi_{\frac{n}{s}}\left(\frac{|\omega(x)-\omega_\Omega|}{\lambda}\right)dx.$$

As a result,

$$\int_\Omega\phi_{\frac{n}{s}}\Big(\frac{|\omega(x)|}{\lambda}\Big)dx\leq C\int_\Omega\phi_{\frac{n}{s}}\Big(\frac{|\omega(x)-\omega_\Omega|}{\lambda}\Big)dx,$$

that is,

$$\|\omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)}\leq C\|\omega-\omega_\Omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)}. \tag{3.3}$$

Since $\omega(y)\geq 1$ for any $y\in\Omega_x$, we have

$$\int_\Omega\phi_{\frac{n}{s}}\Big(\frac{|\omega(x)|}{\lambda}\Big)dx\geq\phi_{\frac{n}{s}}\Big(\frac{1}{\lambda}\Big)|\Omega_x|\quad\text{and}\quad\|\omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)}\geq\left[\phi_{\frac{n}{s}}{}^{-1}\Big(\frac{1}{|\Omega_x|}\Big)\right]^{-1}.$$

Therefore,

$$C\phi^{-1}\Big[\frac{1}{(c_0r)^n}\Big](c_0r)^s\leq\phi_{\frac{n}{s}}{}^{-1}\Big[\frac{1}{|\Omega_x|}\Big].$$

By $\frac{H(A)}{A}\leq\frac{C}{\phi(A)^{\frac{s}{n}}}$ in (2.3), letting $A=\phi^{-1}\big[\frac{1}{(c_0r)^n}\big]$, we get

$$\frac{\phi_{\frac{n}{s}}{}^{-1}\big[\frac{1}{(c_0r)^n}\big]}{\phi^{-1}\big[\frac{1}{(c_0r)^n}\big]}\leq C(c_0r)^s.$$

Hence

$$\phi_{\frac{n}{s}}{}^{-1}\Big[\frac{1}{(c_0r)^n}\Big]\leq C\phi_{\frac{n}{s}}{}^{-1}\Big[\frac{1}{|\Omega_x|}\Big].$$

Together Lemma 2.6, $\phi_{\frac{n}{s}}\in\Delta_2$ and Lemma 2.2, we have

$$\frac{1}{(c_0r)^n}\leq C\frac{1}{|\Omega_x|},$$

and

$$|\Omega_x|^{\frac{1}{n}}\leq C(c_0r). \tag{3.4}$$

Set $c_j > c_{j-i}$ for $j \geq 1$ such that

$$\left|\Omega_x \setminus B(z, c_j r)\right| = \frac{1}{2}\left|\Omega_x \setminus B(z, c_{j-1} r)\right| = 2^{-j}|\Omega_x|.$$

For $j \geq 0$ with $\Omega_x \setminus \overline{B(z, c_j r)} \neq \varnothing$, define v_j in Ω by

$$v_j(y) = \begin{cases} 0, & y \in \Omega \setminus [\Omega_x \setminus B_\Omega(z, c_{j+1} r)] \\ \frac{|y-z|-c_j r}{c_{j+1} r - c_j r}, & y \in \Omega_x \cap [B(z, c_j r) \setminus B(z, c_{j+1} r)], \\ 1, & y \in \Omega_x \setminus B_\Omega(z, c_j r), \end{cases}$$

If $\Omega_{z,x} = \Omega_x$, $r = c_j r$ and $t = c_{j+1} r$, then

$$v_j(y) = u_{z, c_j r, c_{j+1} r}(y)$$

where $u_{z, c_j r, c_{j+1} r}(y)$ is defined in (3.1). Using Lemma 3.2, we know

$$\|v_j\|_{\dot{V}_*^{s,\phi}(\Omega)} \leq C\left[\phi^{-1}\Big(\frac{1}{|\Omega_x \setminus B(z, c_j r)|}\Big)\right]^{-1} \frac{1}{(c_{j+1} r - c_j r)^s}.$$

By (3.2), we have $v_j(y) = 0$ for $y \in B(x_0, \frac{1}{2} r_0)$. In a manner similar to (3.3), we have

$$\|v_j\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \leq C\|v_j - v_{j\Omega}\|_{L^{\phi_{\frac{n}{s}}}(\Omega)}. \tag{3.5}$$

Together with $v_j(y) = 1$ for $y \in \Omega_x \setminus B_\Omega(z, c_j r)$, we have

$$\|v_j\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \geq \left[\phi_{\frac{n}{s}}^{-1}\Big(\frac{1}{|\Omega_x \setminus B_\Omega(z, c_j r)|}\Big)\right]^{-1}.$$

Using $(\phi_{\frac{n}{s}}, \phi)$-Poincaré inequality (1.4), we have

$$\phi_{\frac{n}{s}}^{-1}\Big(\frac{1}{|\Omega_x \setminus B_\Omega(z, c_j r)|}\Big) \geq C\phi^{-1}\Big(\frac{1}{|\Omega_x \setminus B(z, c_j r)|}\Big)(c_{j+1} r - c_j r)^s.$$

By (2.3) once again and letting $A = \phi^{-1}\big(\frac{1}{|\Omega_x \setminus B(z, c_j r)|}\big)$, we get

$$(c_{j+1} r - c_j r)^s \leq C\left|\Omega_x \setminus B(z, c_j r)\right|^{\frac{s}{n}}.$$

Hence $c_{j+1} r - c_j r \leq C|\Omega_x \setminus B(z, c_j r)|^{\frac{1}{n}} \leq C 2^{-\frac{j}{n}}|\Omega_x|^{\frac{1}{n}}$.

Now we prove that $\sup\{c_j\} > 1$. Otherwise, we have $c_j \leq 1$ for all j. Since $x \in \Omega \setminus \overline{B(z, r)}$, then there exists $\delta > 0$ such that

$$B(x, \delta) \subset \Omega \setminus \overline{B(x, r)} \subset \Omega \setminus \overline{B(x, c_0 r)}.$$

By the connectivity of the $B(x,\delta)$, we have $B(x,\delta)\subset\Omega_x$.

$$B(x,\delta)\subset\Omega_x\setminus\overline{B(x,r)}\subset\Omega_x\setminus B(x,c_jr),$$

and

$$0<|B(x,\delta)|\le\big|\Omega_x\setminus\overline{B(x,r)}\big|\le|\Omega_x\setminus B(x,c_jr)|=2^{-j}|\Omega_x|.$$

Letting $j\to\infty$, we get a contradiction. Hence we get $\sup\{c_j\}>1$ as desired. It means that there exists c_j such that $c_j\ge\frac{1}{2}$. Put $j_0=\inf\left\{j\ge 1: c_j\le\frac{1}{2}\right\}$, then

$$\Big(\frac{1}{2}-c_0\Big)r\le(c_{j_0}-c_0)r=\sum_{j=0}^{j_0-1}(c_{j+1}-c_j)r\le C\sum_{j=0}^{j_0-1}2^{-\frac{j}{n}}|\Omega_x|^{\frac{1}{n}}\le 2C|\Omega_x|^{\frac{1}{n}}.$$

So $\frac{r}{C}(\frac{1}{2}-2b_{x,z,r})\le|\Omega_x|^{\frac{1}{n}}$. Applying the (3.4), there exists a constant $C\ge 1$ such that

$$\frac{r}{C}\Big(\frac{1}{2}-2b_{x,z,r}\Big)\le|\Omega_x|^{\frac{1}{n}}\le C2b_{x,z,r}r.$$

Then $b_{x,z,r}\ge\frac{1}{4(C^2+1)}$ which implies $b>0$. □

LEMMA 3.5. *Let $s\in(0,1)$ and a Young function $\phi\in\Delta_2$ with $K_\phi<2^{\frac{n}{s}}$ in (1.5). If a bounded domain $\Omega\subset\mathbb{R}^n$ supports the $(\phi_{\frac{n}{s}},\phi)$-Poincaré inequality (1.4), then the Ω has the LLC(2) property, that is, there exists a constant $b\in(0,1)$ such that for all $z\in\mathbb{R}^n$ and $r>0$, any pair of points in $\Omega\setminus\overline{B(z,r)}$ can be joined in $\Omega\setminus\overline{B(z,br)}$.*

PROOF. Fix x_0 so that $r_0:=\max(d(x,\partial\Omega):x\in\Omega)=d(x_0,\partial\Omega))$ and b_0 is the constant in Lemma 3.4. Then we spilt into three cases to prove it.

Case 1. For $z\notin B\big(x_0,\frac{r_0}{8\operatorname{diam}\Omega}r\big)$, we consider the radius r.

If $r>\frac{16(\operatorname{diam}\Omega)^2}{r_0}$, then for any $y\in\overline{B\big(z,\frac{r_0}{16\operatorname{diam}\Omega}r\big)}$, we have

$$|y-x_0|\ge|z-x_0|-|z-y|\ge\frac{r_0}{16\operatorname{diam}\Omega}r>\operatorname{diam}\Omega.$$

By $\Omega\subset B(x_0,\operatorname{diam}\Omega)$, we get $\Omega\cap\overline{B\big(z,\frac{r_0}{16\operatorname{diam}\Omega}r\big)}=\varnothing$. Here, any pair of points in $\Omega\setminus\overline{B(z,r)}$ can be joined in $\Omega\setminus\overline{B(z,\frac{r_0}{16\operatorname{diam}\Omega}r)}=\Omega$.

If $r\le\frac{16(\operatorname{diam}\Omega)^2}{r_0}$ and $d(z,\partial\Omega)>\frac{b_0r_0}{32\operatorname{diam}\Omega}r$. When $z\notin\Omega$, then any pair of points in $\Omega\setminus\overline{B(z,r)}$ can be joined in $\Omega\setminus\overline{B\big(z,\frac{b_0r_0}{32\operatorname{diam}\Omega}r\big)}=\Omega$. When $z\in\Omega$, then $B\big(z,\frac{b_0r_0}{64\operatorname{diam}\Omega}r\big)\subset B\big(z,\frac{b_0r_0}{32\operatorname{diam}\Omega}r\big)\subset\Omega$. Similar to the process of proving $b_{x,z,r}>0$ in Lemma 3.4, we know $\Omega\setminus\overline{B\big(z,\frac{b_0r_0}{64\operatorname{diam}\Omega}r\big)}$ is a connected set. Here, any pair of points in $\Omega\setminus\overline{B(z,r)}$ can be joined in $\Omega\setminus\overline{B\big(z,\frac{b_0r_0}{64\operatorname{diam}\Omega}r\big)}$.

If $r \leq \frac{16(\operatorname{diam}\Omega)^2}{r_0}$ and $d(z,\partial\Omega) \leq \frac{b_0 r_0}{32 \operatorname{diam}\Omega} r$. Let $y \in B\big(z, \frac{b_0 r_0}{16\operatorname{diam}\Omega} r\big) \cap \Omega$. By $B\big(y, (1-\frac{b_0}{2})\frac{r_0}{8\operatorname{diam}\Omega} r\big) \subset B\big(z, \frac{r_0}{8\operatorname{diam}\Omega} r\big) \subset B(z,r)$, for any $x \in \Omega \setminus \overline{B(z,r)}$, we know

$$x, x_0 \in \Omega \setminus \overline{B\Big(y, \Big(1-\frac{b_0}{2}\Big)\frac{r_0}{8\operatorname{diam}\Omega} r\Big)}.$$

Applying Lemma 3.4, x, x_0 are in the same component of

$$\Omega \setminus \overline{B\Big(y, b_0\Big(1-\frac{b_0}{2}\Big)\frac{r_0}{8\operatorname{diam}\Omega} r\Big)}.$$

Since for any $w \in B\big(z, \frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega} r\big)$, we have

$$\begin{aligned} |w-y| &\leq |w-z| + |z-y| \\ &< \frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega} r + \frac{b_0 r_0}{16\operatorname{diam}\Omega} r = b_0\Big(1-\frac{b_0}{2}\Big)\frac{r_0}{8\operatorname{diam}\Omega} r. \end{aligned}$$

Then we get

$$B\Big(z, \frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega} r\Big) \subset B\Big(y, b_0\Big(1-\frac{b_0}{2}\Big)\frac{r_0}{8\operatorname{diam}\Omega} r\Big),$$

and $\Omega \setminus \overline{B\big(y, b_0(1-\frac{b_0}{2})\frac{r_0}{8\operatorname{diam}\Omega} r\big)} \subset \Omega \setminus \overline{B\big(z, \frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega} r\big)}$. Here, any pair of points in $\Omega \setminus \overline{B(z,r)}$ can be joined in $\Omega \setminus B\big(z, \frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega} r\big)$.

Case 2. If $z \in B\big(x_0, \frac{r_0}{8\operatorname{diam}\Omega} r\big)$, for any $x \in \Omega \setminus \overline{B(z,r)}$,

$$r - \frac{r_0}{8\operatorname{diam}\Omega} r \leq |x-z| - |x_0 - z| \leq |x - x_0| \leq \operatorname{diam}\Omega,$$

so

$$r \leq \frac{\operatorname{diam}\Omega}{1 - \frac{r_0}{8\operatorname{diam}\Omega}} \leq 2\operatorname{diam}\Omega.$$

Then

$$B\Big(z, \frac{r_0}{8\operatorname{diam}\Omega} r\Big) \subset B\Big(x_0, \frac{r_0}{4\operatorname{diam}\Omega} r\Big) \subset B\Big(x_0, \frac{r_0}{2}\Big) \subset B(x_0, r_0) \subset \Omega$$

Similar to the process of proving $b_{x,z,r} > 0$ in Lemma 3.4, we have $\Omega \setminus \overline{B\big(z, \frac{r_0}{8\operatorname{diam}\Omega} r\big)}$ is a connected set. And by

$$\Omega \setminus \overline{B(z,r)} \subset \Omega \setminus \overline{B\Big(z, \frac{r_0}{8\operatorname{diam}\Omega} r\Big)},$$

we know any pair of points in $\Omega \setminus \overline{B(z,r)}$ can be joined in $\Omega \setminus \overline{B\big(z, \frac{r_0}{8\operatorname{diam}\Omega} r\big)}$.

Combining the above cases, we get the desired result with

$$b = \min\left\{\frac{r_0}{16\operatorname{diam}\Omega}, \frac{b_0 r_0}{64\operatorname{diam}\Omega}, \frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega}\right\}. \quad \square$$

Proof of Theorem 1.2(ii). Suppose $\Omega\subset\mathbb{R}^n$ be a simply connected planar domain, or a bounded domain that is quasiconformally equivalent to some uniform domain when $n\geq 3$. Assume that Ω is in agreement with the $(\phi_{\frac{n}{s}},\phi)$-Poincaré inequality.

According to [7,8], Ω has a separation property with $x_0\in\Omega$ and some constant $C_0\geq 1$. It means that for any $x\in\Omega$, there exists a curve $\gamma(t):[0,1]\to\Omega$ with $\gamma(0)=x,\gamma(1)=x_0$ such that for any $t\in[0,1]$, either $\gamma([0,1])\subset\overline{B}:=\overline{B(\gamma(t),C_0 d(\gamma(t),\Omega^\complement))}$ or for any $y\in\gamma([0,1])\setminus\overline{B}$ belongs to the different component of $\Omega\setminus\overline{B}$. For any $x\in\Omega$, let γ be the above curve. By the arguments in [35], it suffices to prove there exists a constant $C>0$ so that

$$d\big(\gamma(t),\Omega^\complement\big)\geq C\operatorname{diam}\gamma([0,t]) \quad \text{for all } t\in[0,1]. \tag{3.6}$$

Indeed, (3.6) could modify γ to get a John curve for x.

Applying Lemma 3.5, Ω has the LLC(2) property. Let $a=2+\frac{C_0}{b}$, where b is the constant in Lemma 3.5. Next we spilt into two cases.

(i) If $d(\gamma(t),\Omega^\complement)\geq\frac{d(x_0,\Omega^\complement)}{a}$, then we know

$$\gamma([0,t])\subset\Omega\subset B\Big(\gamma(t),\frac{ad(\gamma(t),\Omega^\complement)}{d(x_0,\Omega^\complement)}\operatorname{diam}\Omega\Big).$$

Therefore,

$$\operatorname{diam}\ \gamma([0,t])\leq\frac{2ad(\gamma(t),\Omega^\complement)}{d(x_0,\Omega^\complement)}\operatorname{diam}\Omega.$$

In other words, it means

$$d(\gamma(t),\Omega^\complement)\geq\frac{d(x_0,\Omega^\complement)}{2a\operatorname{diam}\Omega}\operatorname{diam}\ \gamma([0,t]).$$

(ii) If $d(\gamma(t),\Omega^\complement)<\frac{d(x_0,\Omega^\complement)}{a}$, we declare that

$$\gamma([0,t])\subset\overline{B\big(\gamma(t),(a-1)d(\gamma(t),\Omega^\complement)\big)}.$$

Otherwise, there exists a $y\in\gamma([0,t])\setminus\overline{B\big(\gamma(t),(a-1)d(\gamma(t),\Omega^\complement)\big)}$. Because of

$$|x_0-\gamma(t)|\geq d(x_0,\Omega^\complement)-d(\gamma(t),\Omega^\complement)>(a-1)d(\gamma(t),\Omega^\complement),$$

we know $x_0,y\in\Omega\setminus\overline{B\left(\gamma(t),(a-1)d(\gamma(t),\Omega^\complement)\right)}$.

Together with Lemma 3.5, we know x_0 and y are contained in the same complement of $\Omega \setminus \overline{B\big(\gamma(t), b(a-1)d(\gamma(t), \Omega^{\complement})\big)}$. Since $b(a-1) \geq C_0$, then x_0 and y are contained in the same complement of $\Omega \setminus \overline{B\big(\gamma(t), C_0 d(\gamma(t), \Omega^{\complement})\big)}$, which is in contradiction with the separation property. Hence we have $\gamma([0,t]) \subset \overline{B\big(\gamma(t), (a-1)d(\gamma(t), \Omega^{\complement})\big)}$ as desired.

With above claim, we get

$$\operatorname{diam}\ \gamma([0,t]) \leq 2(a-1)d(\gamma(t), \Omega^{\complement}).$$

It means that

$$d(\gamma(t), \Omega^{\complement}) \geq \frac{1}{2(a-1)} \operatorname{diam}\ \gamma([0,t]).$$

Combining two cases, (3.6) holds if $C = \min\big\{\frac{d(x_0, \Omega^{\complement})}{2a \operatorname{diam} \Omega}, \frac{1}{2(a-1)}\big\}$. The proof is completed. $\square$

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