Anal Math., **50** (**2**) (2024), 717–730
https://doi.org/10.1007/s10476-024-00018-9
First published online April 30, 2024

ON FUNCTIONS OF BOUNDED MEAN OSCILLATION WITH BOUNDED NEGATIVE PART

H. ZHAO[1] and D. WANG[2,*]

[1]Department of Mathematics, Zhejiang University of Science and Technology, Hangzhou 310023, People's Republic of China
e-mail: zhaohuanmath1994@163.com

[2]School of Mathematics and Statistics, Anhui Normal University, Wuhu 241000, People's Republic of China
e-mail: Wangdh1990@126.com

(Received July 14, 2023; accepted November 6, 2023)

Abstract. Let b be a locally integrable function and $\mathfrak{M}$ be the bilinear maximal function

$$\mathfrak{M}(f,g)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)g(2x-y)|dy.$$

In this paper, characterization of the BMO function in terms of commutator $\mathfrak{M}_b^{(1)}$ is established. Also, we obtain the necessary and sufficient conditions for the boundedness of the commutator $[b,\mathfrak{M}]_1$. Moreover, some new characterizations of Lipschitz and non-negative Lipschitz functions are obtained.

1. Introduction

The commutator operators play an important role in studying the regularity of solutions of elliptic and parabolic partial differential equations of second order, and the boundedness results are used to characterize certain function spaces [2,3,5,7,8,12,20]. Due to its interest, the singular integral operators were replaced by maximal operators as an object of study.

---

* Corresponding author.

This work was supported by National Natural Science Foundation of China (No. 12101010) and Natural Science Foundation of China of Anhui Province (No. 2108085QA19).

Key words and phrases: BMO function, boundedness, commutator, characterization, maximal function.

Mathematics Subject Classification: 42B20, 42B35, 47G99.

In this paper, we will be concerned with the following family of bilinear maximal operators in $\mathbb{R}^n$. Define

$$\mathcal{M}(f,g)(x) = \sup_{r>0} \frac{1}{|B(O,r)|} \int_{B(O,r)} |f(x-y)g(x+y)|\,dy = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)g(2x-y)|\,dy, \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $O = (0,0,\dots,0) \in \mathbb{R}^n$. More general,

$$\mathfrak{M}(f,g)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)g(2x-y)|\,dy.$$

The maximal functions $\mathcal{M}f$ of f and $\mathfrak{M}f$ are obviously pointwise equivalent each other. In 2000, Lacey in the remarkable paper [15] showed that the family of one-dimensional bilinear maximal operators defined by (1.1) maps $L^p \times L^q$ into L^r provided $1 < p,q < \infty$, $1/p + 1/q = 1/r$ and $2/3 < r \le 1$, solving a conjecture posed by A. Calderón in 1964.

An interesting question is raised. Is $b \in \mathrm{BMO}$ necessary and sufficient for the boundedness of commutators of the bilinear maximal operator $\mathfrak{M}$.

We briefly summarize some classical and recent works in the literature. A locally integrable function b belongs to the BMO space if b satisfies

$$\|b\|_{\mathrm{BMO}} := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q|\,dx < \infty,$$

where $b_Q := \frac{1}{|Q|}\int_Q b(x)\,dx$ and the supremum is taken over all cubes (or balls) Q in $\mathbb{R}^n$. There is a number of classical results which demonstrate that BMO functions are the right collections to do harmonic analysis on the boundedness of commutators. A well-known result of Coifman, Rochberg and Weiss [4] states that the commutator $[b,T](f) = bT(f) - T(bf)$ is bounded on some L^p, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}$, where T be the classical Calderón–Zygmund operator. The theory was then extended and generalized to several directions. In 1991, García-Cuerva, Harboure, Segovia and Torrea [9] showed that the maximal commutator

$$M_b(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)|\,|f(y)|\,dy$$

is bounded on L^p, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}$. In 2000, Bastero, Milman and Ruiz [1] studied the necessary and sufficient conditions for the boundedness of commutator of Hardy–Littlewood maximal function on L^p

spaces when $1 < p < \infty$, where the Hardy–Littlewood maximal function is defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy.$$

They proved that the commutator $[b, M]$ is bounded on L^p, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}$ with $b^- \in L^\infty$, where $b^-(x) = -\min\{b(x), 0\}$. We note that the operators M_b and $[b, M]$ essentially differ from each other. For example, M_b is positive and sublinear in each entry, but $[b, M]$ is neither positive nor sublinear. In the past ten years, Zhang and his collaborators studied the boundedness of different commutators of maximal operators and the necessity of the symbol [25–30]. Recently, the further research about the functions of BMO with bounded negative part were given in [11,21].

In this paper, we show that the question above has an affirmative solution. Now, we give the definitions of commutators of the bilinear maximal function. Let b be a locally integrable function and $\mathfrak{M}$ be the bilinear maximal function. The commutators of the bilinear maximal function are defined by

$$[b, \mathfrak{M}]_1(f, g)(x) = b(x)\mathfrak{M}(f, g)(x) - \mathfrak{M}(bf, g)(x)$$

and

$$[b, \mathfrak{M}]_2(f, g)(x) = b(x)\mathfrak{M}(f, g)(x) - \mathfrak{M}(f, bg)(x).$$

The maximal commutators of the bilinear maximal function are defined by

$$\mathfrak{M}_b^{(1)}(f, g)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)g(2x - y)|\,dy$$

and

$$\mathfrak{M}_b^{(2)}(f, g)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(2x - y)g(y)|\,dy.$$

Now, we show the necessary and sufficient conditions for the bounded commutators of $\mathfrak{M}$ as follows. We note that by symmetry, it is enough to prove this for $\mathfrak{M}_b^{(1)}$ and $[b, \mathfrak{M}]_1$.

THEOREM 1.1. *Let* $1 < p, p_1, p_2 < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ *and* $b \in L^1_{loc}(\mathbb{R}^n)$. *Then the following statements are equivalent*:

(1) $b \in \mathrm{BMO}$;

(2) $\mathfrak{M}_b^{(1)}$ *is bounded from* $L^{p_1} \times L^{p_2}$ *to* L^p;

(3) $\mathfrak{M}_b^{(1)}$ *is bounded from* $L^{p_1} \times L^{p_2}$ *to* $L^{p,\infty}$.

THEOREM 1.2. *Let* $1 < p, p_1, p_2 < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ *and* $b \in L^1_{loc}(\mathbb{R}^n)$. *Then the following statements are equivalent*:

(1) $b \in \mathrm{BMO}$ *and* $b^- \in L^\infty$;

(2) $[b, \mathfrak{M}]_1$ *is bounded from* $L^{p_1} \times L^{p_2}$ *to* L^p;

(3) $[b, \mathfrak{M}]_1$ *is bounded from* $L^{p_1} \times L^{p_2}$ *to* $L^{p,\infty}$.

Let $0 < q < \infty$ and $-n/q < \alpha < n$. A locally integrable function f is said to belong to the Campanato space $\mathcal{C}_{\alpha,q}$ if there exists a constant $C > 0$ such that for any cube $Q \subset \mathbb{R}^n$ we have

$$\frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^q \, dx \right)^{1/q} \le C,$$

where $f_Q = \frac{1}{|Q|} \int_Q f(x)\, dx$ and the minimal constant C is defined by $\|f\|_{\mathcal{C}_{\alpha,q}}$.

The Lipschitz (Hölder) and Campanato spaces are related by the equivalences

$$\|f\|_{\mathrm{Lip}_\alpha} := \sup_{x,h \in \mathbb{R}^n, h \ne 0} \frac{|f(x+h) - f(x)|}{|h|^\alpha} \approx \|f\|_{\mathcal{C}_{\alpha,q}}, \quad 0 < \alpha < 1.$$

The equivalence can be found in [6] for $q = 1$, [13] for $1 < q < \infty$ and [23] for $0 < q < 1$.

Specially, $\mathcal{C}_{0,q} = \mathrm{BMO}$, the spaces of bounded mean oscillation. The crucial property of BMO functions is the John–Nirenberg inequality [14],

$$\big|\{x \in Q : |f(x) - f_Q| > \lambda\}\big| \le c_1 |Q| e^{-\frac{c_2 \lambda}{\|f\|_{\mathrm{BMO}}}},$$

where c_1 and c_2 depend only on the dimension. A well-known immediate corollary of the John–Nirenberg inequality reads as follows:

$$\|f\|_{\mathrm{BMO}} \approx \sup_Q \frac{1}{|Q|} \left(\int_Q |f(x) - f_Q|^p \, dx \right)^{1/p}$$

for all $1 < p < \infty$. In fact, the equivalence also holds for $0 < p < 1$. See, for example, the work of Strömberg [19] (or [10] and [24] for the general case).

THEOREM 1.3. *Let* $0 < \alpha < 1$, $1 < q, p_1, p_2 < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $b \in L^1_{loc}(\mathbb{R}^n)$. *Then the following statements are equivalent*:

(1) $b \in \mathrm{Lip}_\alpha$;

(2) $\mathfrak{M}^{(1)}_b$ *is bounded from* $L^{p_1} \times L^{p_2}$ *to* L^q;

(3) $\mathfrak{M}^{(1)}_b$ *is bounded from* $L^{p_1} \times L^{p_2}$ *to* $L^{q,\infty}$.

THEOREM 1.4. *Let* $0 < \alpha < 1$, $1 < q, p_1, p_2 < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $b \in L^1_{loc}(\mathbb{R}^n)$. *Then the following statements are equivalent*:

(1) $b \in \mathrm{Lip}_\alpha$ *with* $b \geq 0$;

(2) $[b, \mathfrak{M}]_1$ *is bounded from* $L^{p_1} \times L^{p_2}$ *to* L^q;

(3) $[b, \mathfrak{M}]_1$ *is bounded from* $L^{p_1} \times L^{p_2}$ *to* $L^{q,\infty}$.

Let $|E|$ denote the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. Throughout this paper, the letter C denotes constants which are independent of main variables and may change from one occurrence to another. $Q(x, r)$ denotes a cube centered at x, with side length r, and sides parallel to the axes.

2. Main lemmas

To prove Theorems 1.1–1.4, we need the following results.

LEMMA 2.1. *Let* b *be a locally integral function. Then, for any cube* $Q_0 = Q(x_0, r_0)$,

$$\mathfrak{M}(\chi_{Q_0}, \chi_{3\sqrt{n}Q_0})(x) \equiv 1, \tag{2.1}$$

$$\mathfrak{M}(b\chi_{Q_0}, \chi_{3\sqrt{n}Q_0})(x) = M_{Q_0}(b)(x), \tag{2.2}$$

where $M_{Q_0}(b)(x) = \sup_{Q_0 \supset Q \ni x} \frac{1}{|Q|} \int_Q |b(y)| \, dy$.

PROOF. We only give the proof of the equality (2.2), since the equality (2.1) follows from (2.2) for $b \equiv 1$. For any $Q \subset Q_0$, it follows from $x \in Q$ and $y \in Q_0$ that

$$|2x - y - x_0| \leq 2|x - x_0| + |y - x_0| \leq 3\sqrt{n}r_0$$

and $2x - y \in 3\sqrt{n}Q_0$, one has $\chi_{3\sqrt{n}Q_0}(2x - y) \equiv 1$ and

$$\frac{1}{|Q|} \int_Q |b(y)|\chi_{Q_0}(y) \cdot \chi_{3\sqrt{n}Q_0}(2x - y) \, dy = \frac{1}{|Q|} \int_Q |b(y)| \, dy.$$

According to the arbitrariness of the cube Q, we arrive at

$$\mathfrak{M}(b\chi_{Q_0}, \chi_{3\sqrt{n}Q_0})(x) \geq M_{Q_0}(b)(x).$$

For any cube $Q \subset \mathbb{R}^n$, we can construct a cube Q_1 such that

$$Q_0 \supset Q_1 \supset Q_0 \cap Q$$

and $|Q_1| \leq |Q|$. Therefore, for $x \in Q$,

$$\frac{1}{|Q|} \int_{Q \cap Q_0} |b(y)| \, dy \leq \frac{1}{|Q_1|} \int_{Q_1} |b(y)| \, dy \leq M_{Q_0}(b)(x).$$

Using the fact that $\chi_{3\sqrt{n}Q_0}(2x-y)\le 1$, we get

$$\mathfrak{M}(b\chi_{Q_0},\chi_{3\sqrt{n}Q_0})(x)\le M_{Q_0}(b)(x).$$

Then (2.2) is proved. □

Bastero, Milman and Ruiz proved that

LEMMA 2.2 [1]. *Let $1\le p<\infty$ and b be a locally integrable function. Then the following statements are equivalent*:

(1) $b\in \mathrm{BMO}$ *with* $b^-\in L^\infty$;

(2) $\|b\|_{\mathrm{BMO}_p^-}:=\sup\limits_Q \frac{1}{|Q|}\int_Q |b(x)-M_Q(b)(x)|^p\,dx<\infty$.

Furthermore, we can obtain the following result.

LEMMA 2.3. *Let $1<p<\infty$ and b be a locally integrable function. Then the following statements are equivalent*:

(1) $b\in \mathrm{BMO}$ *with* $b^-\in L^\infty$;

(2) $\|b\|_{\mathrm{BMO}_{p,\infty}^-}:=\sup\limits_Q\sup\limits_{\lambda>0}\frac{\lambda}{|Q|^{1/p}}\big|\{x\in Q:|b(x)-M_Q(b)(x)|>\lambda\}\big|^{1/p}<\infty$.

PROOF. By a direct computation, we arrive at $\|\cdot\|_{\mathrm{BMO}_{p,\infty}^-}\le\|\cdot\|_{\mathrm{BMO}_p^-}$. By Lemma 2.2, we need only to prove that $\|\cdot\|_{\mathrm{BMO}_q^-}\lesssim\|\cdot\|_{\mathrm{BMO}_{p,\infty}^-}$ with $1\le q<p<\infty$.

Let $b\in \mathrm{BMO}_{p,\infty}^-$. Given a fixed cube $Q\subset\mathbb{R}^n$ and for any $\lambda>0$, one has

$$\frac{\lambda}{|Q|^{1/p}}\big|\{x\in Q:|b(x)-M_Q(b)(x)|>\lambda\}\big|^{1/p}\le\|b\|_{\mathrm{BMO}_{p,\infty}^-};$$

that is,

$$\big|\{x\in Q:|f(x)-M_Q(b)(x)|>\lambda\}\big|\le\|b\|_{\mathrm{BMO}_{p,\infty}^-}^p\lambda^{-p}|Q|.$$

It follows that

$$\begin{aligned}\int_Q|b(x)-M_Q(b)(x)|^q\,dx&=q\int_0^\infty\lambda^{q-1}\big|\{x\in Q:|b(x)-M_Q(b)(x)|>\lambda\}\big|\,d\lambda\\&\le q\int_0^N\lambda^{q-1}|Q|\,d\lambda+q\int_N^\infty\lambda^{q-1}\|b\|_{\mathrm{BMO}_{p,\infty}^-}^p\lambda^{-p}|Q|\,d\lambda\\&=|Q|N^q+\frac{q}{p-q}\|b\|_{\mathrm{BMO}_{p,\infty}^-}^p|Q|N^{q-p}.\end{aligned}$$

Choose

$$N=\|b\|_{\mathrm{BMO}_{p,\infty}^-}\Big(\frac{q}{p-q}\Big)^{1/p},$$

which gives

$$\Big(\frac{1}{|Q|}\int_Q |b(y)-M_Q(b)(x)|^q\,dy\Big)^{1/q} \le 2\Big(\frac{q}{p-q}\Big)^{1/p}\|b\|_{\mathrm{BMO}^-_{p,\infty}}.$$

Then

$$\|b\|_{\mathrm{BMO}^-_q} \le 2\Big(\frac{q}{p-q}\Big)^{1/p}\|b\|_{\mathrm{BMO}^-_{p,\infty}}$$

and the lemma follows. □

Similarly, we can obtain the result for non-negative Lipschitz functions as follows.

LEMMA 2.4. *Let $0<\alpha<1$, $1<p<\infty$ and b be a locally integrable function. Then the following statements are equivalent*:

(1) $b\in \mathrm{Lip}_\alpha$ *with* $b\ge 0$;

(2) $\|b\|_{\mathrm{Lip}^-_{\alpha,p,\infty}} := \sup\limits_Q \sup\limits_{\lambda>0} \frac{\lambda}{|Q|^{1/p-\alpha}}\big|\{x\in Q : |b(x)-M_Q(b)(x)|>\lambda\}\big|^{\frac{1}{p}}<\infty.$

Standard real analysis tools as the maximal function $M(f)$, the sharp maximal function $M^\sharp(f)$ carries over to this context, namely,

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

$$M^\sharp(f)(x)=\sup_{Q\ni x}\inf_c\frac{1}{|Q|}\int_Q |f(y)-c|\,dy \approx \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy.$$

A variant of maximal function and sharp maximal operator $M_s(f)(x)=\big(M(|f|^s)\big)^{1/s}$ and $M^\sharp_\delta(f)(x)=\big(M^\sharp(|f|^s)(x)\big)^{1/s}$ with $0<s<\infty$, which will become the main tool in our scheme. Let $1<s<\infty$ and b be a locally integrable function. The operator $M_{b,s}$ is defined by

$$M_{b,s}(f)(x)=\sup_{Q\ni x}\Big(\frac{1}{|Q|}\int_Q |b(x)-b(y)|^s|f(y)|^s\,dy\Big)^{1/s}.$$

LEMMA 2.5. *Let $0<\delta<\varepsilon<1<s<\bar{s}<\infty$ and $b\in\mathrm{BMO}$. Then*

$$M^\sharp_\delta\big(M_{b,s}(f)\big)(x)\lesssim \|b\|_{\mathrm{BMO}}\big(M_\varepsilon(M_s(f))(x)+M_{\bar{s}}(f)(x)\big),$$

for the bounded compact supported function f.

PROOF. First of all, we give the definition of the following auxiliary maximal function, which has been studied in [17] and [18] for the linear case.

Let $\varphi(x) \geq 0$ be a smooth function such that $\varphi_\varepsilon(t) = \varepsilon^{-n}\varphi(\frac{t}{\varepsilon})$, $|\varphi'(t)| \lesssim t^{-1}$ and $\chi_{[0,1]}(t) \leq \varphi(t) \leq \chi_{[0,2]}(t)$. Then

$$M(f)(x) \approx \Phi(f)(x) := \sup_{\varepsilon>0} \int_{\mathbb{R}^n} \varphi_\varepsilon(|x-y|)|f(y)|\,dy.$$

Define $\Phi_s(f)(x) = \big(\Phi(|f|^s)(x)\big)^{1/s}$ and

$$\Phi_{b,s}(f)(x) = \sup_{\varepsilon>0} \bigg(\int_{\mathbb{R}^n} \varphi_\varepsilon(|x-y|)|b(x)-b(y)|^s|f(y)|^s\,dy \bigg)^{1/s}.$$

Obviously, $\Phi_{b,s}(f)(x) \approx M_{b,s}(f)(x)$. In fact, let $B_\varepsilon = \{y \in \mathbb{R}^n : |x-y| \leq \varepsilon\}$. The bounded compact supported condition of φ gives

$$\begin{aligned} \Phi^s_{b,s}(f)(x) &= \sup_{\varepsilon>0} \int_{\mathbb{R}^n} \varphi_\varepsilon(|x-y|)|b(x)-b(y)|^s|f(y)|^s\,dy \\ &\leq \sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int_{B_\varepsilon} \varphi\Big(\frac{|x-y|}{\varepsilon}\Big)|b(x)-b(y)|^s|f(y)|^s\,dy \lesssim M^s_{b,s}(f)(x) \end{aligned}$$

and

$$\Phi^s_{b,s}(f)(x) \geq \sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int_{B_{\frac{\varepsilon}{2}}} \varphi\Big(\frac{|x-y|}{\varepsilon}\Big)|b(x)-b(y)|^s|f(y)|^s\,dy \gtrsim M^s_{b,s}(f)(x).$$

Now, we shall estimate the sharp maximal function of the auxiliary maximal function. Let Q be a cube and $x \in Q$. For any $z \in Q$, we have

$$\begin{aligned} &\big|\Phi_{b,s}(f)(z) - c_Q\big| \\ &\lesssim \big|b(z) - b_Q\big|\Phi_s(|f|)(z) + \big|\Phi_s(|b-b_Q||f^0|)(z)\big| + \big|\Phi_s(|b-b_Q||f^\infty|)(z) - c_Q\big| \\ &=: A_1^Q(z) + A_2^Q(z) + A_3^Q(z), \end{aligned}$$

where $c_Q = \big(\Phi_s(|b-b_Q||f^\infty|)\big)_Q$ and $f = f^0 + f^\infty$ with $f^0 = f\chi_{2Q}$. Therefore,

$$\begin{aligned} &\bigg(\frac{1}{|Q|}\int_Q \big|\big|\Phi_{b,s}(f)(z)\big|^\delta - |c_Q|^\delta\big|\,dz\bigg)^{1/\delta} \\ &\lesssim \bigg(\frac{1}{|Q|}\int_Q \big|\Phi_{b,s}(f)(z) - c_Q\big|^\delta\bigg)^{1/\delta} \lesssim A_1 + A_2 + A_3, \end{aligned}$$

where $A_j = \big(\frac{1}{|Q|}\int_Q \big(A_j^Q(z)\big)^\delta\,dz\big)^{1/\delta}$, $j = 1,2,3$.

Let us consider first the term A_1. By averaging A_1^Q over Q, we get

$$A_1 = \left(\frac{1}{|Q|}\int_Q \left(\left|b(z) - b_Q\right|\Phi_s(f)(z)\right)^\delta dz\right)^{1/\delta} \lesssim \|b\|_{\mathrm{BMO}} M_\varepsilon\big(M_s(f)\big)(x),$$

where $0 < \delta < \varepsilon < 1$. To estimate A_2, by Kolmogorov's inequality and the weak $(1,1)$ boundedness of M, we have

$$\begin{aligned} A_2 &\lesssim |Q|^{-1/s}\|M_s((b-b_Q)f^0\|_{L^{s,\infty}} \lesssim |Q|^{-1/s}\|M(|b-b_Q|^s|f^0|^s\|_{L^{1,\infty}}^{1/s} \\ &\lesssim |Q|^{-1/s}\||b-b_Q|^s|f^0|^s\|_{L^1}^{1/s} \lesssim \|b\|_{\mathrm{BMO}} M_{\bar{s}}(f)(x) \end{aligned}$$

for $1 < s < \bar{s} < \infty$. Now, we consider the term A_3. For $|z-z'| \le \frac{1}{2}\max\{|z-y_1|, |z-y_2|\}$ we have

$$\left|\varphi_\varepsilon(|z-y|) - \varphi_\varepsilon(|z'-y|)\right| \lesssim \frac{|z-z'|}{|z-y|^{n+1}}.$$

Therefore,

$$\begin{aligned} &\left|\Phi_s((b-b_Q)f^\infty)(z) - \Phi_s((b-b_Q)f^\infty)(z')\right|^s \\ &\lesssim \sup_{\varepsilon>0}\int_{\mathbb{R}^n\setminus 2Q}\left|\varphi_\varepsilon(|z-y|) - \varphi_\varepsilon(|z'-y|)\right||b(y)-b_Q|^s|f(y)|^s\,dy \\ &\lesssim \int_{\mathbb{R}^n\setminus 2Q}\frac{|z-z'|}{|z-y|^{n+1}}|b(y)-b_Q|^s|f(y)|^s\,dy \\ &\lesssim \sum_{k=1}^\infty \frac{-2^{kn}}{|2^kQ|}\int_{2^kQ}|b(y)-b_Q|^s|f(y)|^s\,dy \lesssim \|b\|_{\mathrm{BMO}}^s M_{\bar{s}}^s(f)(x). \end{aligned}$$

Collecting our estimates, we have shown that

$$M_\delta^\sharp\big(M_{b,s}(f)\big)(x) \lesssim \|b\|_{\mathrm{BMO}}\big(M_\varepsilon(M_s(f))(x) + M_{\bar{s}}(f)(x)\big)$$

for the bounded compact supported functions f. □

LEMMA 2.6. *Let $0 < \delta < 1 < s < \bar{s} < p < \infty$ and $b \in \mathrm{BMO}$. Then $M_{b,s}$ is a bounded operator from L^p to L^p.*

PROOF. We observe that to use the Fefferman–Stein inequality, one needs to verify that certain terms in the left-hand side of the inequalities are finite. Applying a similar argument as in [16, pp. 32-33], the boundedness properties of M and Fatou's lemma, one gets the desired result.

Using Lemma 2.5 and the condition $0<\delta<\varepsilon<1<s<\bar{s}<p<\infty$, from a standard argument that we can obtain

$$\|M_{b,s}(f)\|_{L^p} \lesssim \big\|M_\delta(M_{b,s}(f))\big\|_{L^p} \lesssim \big\|M_\delta^\sharp(M_{b,s}(f))\big\|_{L^p}$$
$$\lesssim \|b\|_{\mathrm{BMO}}\Big(\big\|M_\varepsilon(M_s(f))\big\|_{L^p}+\big\|M_{\bar{s}}(f)\big\|_{L^p}\Big) \lesssim \|b\|_{\mathrm{BMO}}\|f\|_{L^p}.$$

Thus, the proof of Lemma 2.6 is completed. □

3. Proofs of Theorems 1.1–1.4

Proof of Theorem 1.1. (1) ⇒ (2). For any pair of conjugate exponents $1/r+1/s=1$, Hölder's inequality yields

$$\mathfrak{M}_b^{(1)}(f,g)(x) \le M_{b,s}(f)(x)M_r(g)(x).$$

If we let $r=p_1/p$ and $s=p_2/p$, then $r,s>1$ and $1/r+1/s=1$. Using the fact that $M_r\colon L^{p_2}\to L^{p_2}$ and Lemma 2.6, we arrive at

$$\Big(\int_{\mathbb{R}^n}\mathfrak{M}_b^{(1)}(f,g)(x)^p\,dx\Big)^{1/p}$$
$$\le\Big(\int_{\mathbb{R}^n}M\big(|b(x)-b|^s|f|^s\big)(x)^{p/s}M(|g|^r)(x)^{p/r}\,dx\Big)^{1/p}$$
$$\le\Big(\int_{\mathbb{R}^n}M\big(|b(x)-b|^s|f|^s\big)(x)^p\,dx\Big)^{1/sp}\Big(\int_{\mathbb{R}^n}M(|g|^r)(x)^p\,dx\Big)^{1/rp}$$
$$=\|M_{b,s}(f)\|_{L^{p_1}}\|M_r(g)\|_{L^{p_2}} \lesssim \|b\|_{\mathrm{BMO}}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

Then, $\mathfrak{M}_b^{(1)}$ is bounded from $L^{p_1}\times L^{p_2}$ to L^p. Moreover, (2) ⇒ (3) is obvious.

(3) ⇒ (1). Let $Q=Q(x_Q,r_Q)$ be any fixed cube. For any $x,y\in Q$, we have

$$|2x-y-x_Q|\le 2|x-x_Q|+|y-x_Q|\le 3\sqrt{n}\,r_Q.$$

it follows that for any $x\in Q$,

$$|b(x)-b_Q|\le\frac{1}{|Q|}\int_Q|b(x)-b(y)|\chi_Q(y)\chi_{3\sqrt{n}Q}(2x-y)\,dy$$
$$\lesssim \mathfrak{M}_b(\chi_Q,3\sqrt{n}\,\chi_Q)(x).$$

Suppose that $\mathfrak{M}_b^{(1)}$ is bounded from $L^{p_1}\times L^{p_2}$ into $L^{p,\infty}$. Then

$$\big\|\mathfrak{M}_b^{(1)}(\chi_Q,3\sqrt{n}\,\chi_Q)\big\|_{L^{p,\infty}}\lesssim\|\chi_Q\|_{L^{p_1}}\|\chi_Q\|_{L^{p_2}}\lesssim|Q|^{1/p},$$

which implies that for any $\lambda > 0$,

$$\begin{aligned}
(3.1)\qquad & \frac{\lambda}{|Q|^{1/p}}\big|\{x \in Q : |b(x) - b_Q| > \lambda\}\big|^{1/p} \\
&\lesssim \frac{\lambda}{|Q|^{1/p}}\big|\{x \in Q : \mathfrak{M}_b^{(1)}(\chi_Q, 3\sqrt{n}\chi_Q) > \lambda\}\big|^{1/p} \\
&\lesssim \|\mathfrak{M}_b^{(1)}(\chi_Q, 3\sqrt{n}\chi_Q)\|_{L^{p,\infty}} \lesssim 1.
\end{aligned}$$

In [22], Wang and Zhou proved that $b \in \mathrm{BMO}$ if and only if

$$\sup_Q \sup_\lambda \frac{\lambda}{|Q|^{1/q}}\big|\{x \in Q : |b(x) - b_Q| > \lambda\}\big|^{1/q}$$

for any $0 < q < \infty$. Therefore, the inequality (3.1) implies that $b \in \mathrm{BMO}$. □

Proof of Theorem 1.2. (1) $\Rightarrow$ (2). By the definitions of $\mathfrak{M}_b^{(1)}$, $[b, \mathfrak{M}]_1$ and $\big||a| - |c|\big| \le |a - c|$ for any real numbers a and c, we have

$$\big|[|b|, \mathfrak{M}]_1(f, g)(x)\big| \le \mathfrak{M}_b^{(1)}(f, g)(x)$$

and

$$\begin{aligned}
&\big|[b, \mathfrak{M}]_1(f, g)(x) - [|b|, \mathfrak{M}]_1(f, g)(x)\big| \\
&\lesssim \big|b(x)\mathfrak{M}(f, g)(x) - \mathfrak{M}(bf)(x) - |b(x)|\mathfrak{M}(f, g)(x) + \mathfrak{M}(bf, g)(x)\big| \\
&\lesssim b^-(x)\mathfrak{M}(f, g)(x).
\end{aligned}$$

This shows that

$$(3.2)\qquad \big|[b, \mathfrak{M}]_1(f, g)(x)\big| \lesssim \mathfrak{M}_b^{(1)}(f, g)(x) + b^-(x)\mathfrak{M}(f, g)(x).$$

Applying (3.2) and Theorem 1.1 we have

$$\begin{aligned}
\big\|[b, \mathfrak{M}]_1(f, g)\big\|_{L^p} &\lesssim \big\|\mathfrak{M}_b^{(1)}(f, g)\big\|_{L^p} + \big\|b^-\mathfrak{M}(f, g)\big\|_{L^p} \\
&\lesssim \big(\|b^-\|_{L^\infty} + \|b\|_{\mathrm{BMO}}\big)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.
\end{aligned}$$

Therefore, $b \in \mathrm{BMO}$ with $b^- \in L^\infty$ implies that $[b, \mathfrak{M}]_1$ is bounded from $L^{p_1} \times L^{p_2}$ to L^p.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Let Q_0 be any fixed cube. By Lemma 2.1, for any $x \in Q_0$,

$$b(x) = b(x)\mathfrak{M}(\chi_{Q_0}, 3\sqrt{n}\chi_{Q_0})(x), \quad M_{Q_0}(b)(x) = \mathfrak{M}(b\chi_{Q_0}, 3\sqrt{n}\chi_{Q_0})(x),$$

Then,

$$|b(x) - M_{Q_0}(b)(x)| = \big|[b, \mathfrak{M}]_1(\chi_{Q_0}, 3\sqrt{n}\,\chi_{Q_0})(x)\big|$$

and

$$\frac{\lambda}{|Q_0|^{1/p}}\big|\{x \in Q_0 : |b(x) - M_{Q_0}(b)(x)| > \lambda\}\big|^{1/p}$$
$$\lesssim \frac{1}{|Q_0|^{1/p}}\big\|[b, \mathfrak{M}]_1(\chi_{Q_0}, 3\sqrt{n}\,\chi_{Q_0})\big\|_{L^{p,\infty}} \lesssim \|[b, \mathfrak{M}]_1\|_{L^{p_1}\times L^{p_2}\to L^{p,\infty}},$$

which implies that $b \in \mathrm{BMO}$ by Lemma 2.3. □

Proof of Theorem 1.3. (1) ⇒ (2). Let $b \in \mathrm{Lip}_\alpha$. For any pair of conjugate exponents $1/r + 1/s = 1$, Hölder's inequality yields$1/r + 1/s = 1$, we get

$$\mathfrak{M}_b^{(1)}(f, g)(x) \le \|b\|_{\mathrm{Lip}_\alpha} \int_{\mathbb{R}^n} \frac{|f(x-y)||g(x+y)|}{|y|^{n-\alpha}}\,dy$$
$$\le \|b\|_{\mathrm{Lip}_\alpha}\big(I_\alpha(|f|^r)(x)\big)^{1/r}\big(I_\alpha(|g|^s)(x)\big)^{1/s},$$

where

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Using the fact that $I_\alpha\colon L^p \to L^q$ and set $p_1 = q/r, p_2 = q/s$, we have

$$\|\mathfrak{M}_b^{(1)}(f, g)\|_{L^q} \le \|b\|_{\mathrm{Lip}_\alpha}\|I_\alpha(|f|^r)\|_{L^q}^{1/r}\|\,I_\alpha(|g|^s)\|_{L^q}^{1/s} \lesssim \|b\|_{\mathrm{Lip}_\alpha}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

The implications (2) ⇒ (3) follows directly.

Next, we give the proof of (3) ⇒ (1). Similar to the proof of Theorem 1.1, one has

$$|b(x) - b_Q| \le \mathfrak{M}_b^{(1)}\big(\chi_Q, 3\sqrt{n}\,\chi_Q\big)(x),$$

for any Q and $x \in Q$. Suppose that $\mathfrak{M}_b^{(1)}$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^{q,\infty}$, then

$$\big\|\mathfrak{M}_b^{(1)}(\chi_Q, 3\sqrt{n}\,\chi_Q)\big\|_{L^{q,\infty}} \lesssim \|\chi_Q\|_{L^{p_1}}\|\chi_Q\|_{L^{p_2}} \lesssim |Q|^{1/p},$$

which implies that for any $\lambda > 0$,

$$\frac{\lambda}{|Q|^{1/q}}\big|\{x \in Q : |b(x) - b_Q| > \lambda\}\big|^{1/q}$$
$$\lesssim \frac{\lambda}{|Q|^{1/q}}\big|\{x \in Q : \mathfrak{M}_b^{(1)}(\chi_Q, 3\sqrt{n}\,\chi_Q) > \lambda\}\big|^{1/q}$$

$$\lesssim \|\mathfrak{M}_b^{(1)}(\chi_Q, 3\sqrt{n}\chi_Q)\|_{L^{q,\infty}} \lesssim |Q|^{\alpha/n}.$$

We conclude that $b \in \mathrm{Lip}_\alpha$. $\square$

Proof of Theorem 1.4. By the same arguments as in Theorem 1.2, the desired result is obtained by Lemma 2.4. We omit the details. $\square$

Acknowledgement. I would like to thank the referee for the helpful comment which improved the presentation of this paper.

References

[1] J. Bastero, M. Milman and F.J. Ruiz, Commutators for the maximal and sharp functions, *Proc. Amer. Math. Soc.*, **128** (2000), 3329–3334.

[2] L. Chaffee, Characterizations of bounded mean oscillation through commutators of bilinear singular integral operators, *Proc. Royal Soc. Edinburgh A. 1*, **46** (2016), 1159–1166.

[3] L. Chaffee and D. Cruz-Uribe, Necessary conditions for the boundedness of linear and bilinear commutators on Banach function spaces, *Math. Inequal. Appl.*, **21** (2018), 1–16.

[4] R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. of Math.*, **103** (1976), 611–635.

[5] F. Chiarenza, M. Frasca and P. Longo, $W^{2,p}$-solvability of Dirichlet problem for nondivergence elliptic equations with VMO coefficients, *Trans. Amer. Math. Soc.*, **336** (1993), 841–853.

[6] R. A. Devore and R. C. Sharpley, Maximal functions measuring smoothness, *Mem. Amer. Math. Soc.*, **47** (1984).

[7] Y. Ding, A characterization of BMO via commutators for some operators, *Northeast. Math. J.*, **13** (1997), 422–432.

[8] G. Di Fazio and M. A. Ragusa, Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, *J. Funct. Anal.*, **112** (1993), 241–256.

[9] J. García-Cuerva, E. Harboure, C. Segovia and J. L. Torrea, Weighted norm inequalities for commutators of strongly singular integrals, *Indiana Univ. Math. J.*, **40** (1991), 1397–1420.

[10] J. Hart and R. H. Torres, John–Nirenberg inequalities and weight invariant BMO spaces, *J. Geom. Anal.*, **29** (2019), 1608–1648.

[11] M. Hu and D. H. Wang, The John–Nirenberg inequality for functions of bounded mean oscillation with bounded negative part, *Czechoslovak Math. J.*, **72** (2022), 1121–1131.

[12] S. Janson, Mean oscillation and commutators of singular integral operators, *Ark. Mat.*, **16** (1978), 263–270.

[13] S. Janson, M. Taibleson and G. Weiss, *Elementary Characterization of the Morrey–Campanato Spaces*, Lect. Notes in Math., vol. 992, Springer (1983), pp. 101–114.

[14] F. John and L. Nirenberg, On functions of bounded mean oscillation, *Comm. Pure Appl. Math.*, **2** (1961), 415–426.

[15] M. Lacey, The bilinear maximal functions map into L^p for $2/3 < p \leq 1$, *Ann. of Math. (2)*, **151** (2000), 35–57.

[16] A. Lerner, S. Ombrosi, C. Pérez, R. Torres and R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón–Zygmund theory, *Adv. Math.*, **220** (2009), 1222–1264.

[17] C. Segovia and J. L. Torrea, Weighted inequalities for commutators of fractional and singular integrals, *Publ. Mat.*, **35** (1991), 209–235.

[18] C. Segovia and J. L. Torrea, Higher order commutators for vector-valued Calderón–Zygmund operators, *Trans. Amer. Math. Soc.*, **336** (1993), 537–556.

[19] J.-O. Strömberg, Bounded mean oscillation with Orlicz norms and duality of Hardy spaces, *Indiana Univ. Math. J.*, **28** (1979), 511–544.

[20] D. H. Wang, A note on the boundedness of iterated commutators of Multilinear operators, *Math. Notes*, **112** (2022), 59–64.

[21] D. H. Wang and L. S. Shu, New function classes of Morrey-Campanato type and their applications, *Banach J. Math. Anal.*, **16** (2022), Paper No. 39, 33 pp.

[22] D. H. Wang and J. Zhou, Some new classes of BMO spaces, *Acta Math. Sin, (Chin. Ser.)*, **60** (2017), 833–846.

[23] D. H. Wang, J. Zhou and Z. D. Teng, A note on Campanato spaces and their applications, *Math. Notes*, **103** (2018), 483–489.

[24] D. H. Wang, J. Zhou and Z. D. Teng, Characterizations of weighted BMO space and its application, *Acta Math. Sin. (Engl. Ser.)*, **37** (2021), 1278–1292.

[25] P. Zhang, Characterization of boundedness of some commutators of maximal functions in terms of Lipschitz spaces, *Anal. Math. Phys.*, **9** (2019), 1411–1427.

[26] P. Zhang, Characterization of Lipschitz spaces via commutators of the Hardy–Littlewood maximal function, *C. R. Math. Acad. Sci. Paris*, **355** (2017), 336–344.

[27] P. Zhang, Multiple weighted estimates for commutators of multilinear maximal function, *Acta Math. Sin. (Engl. Ser.)*, **31** (2015), 973–994.

[28] P. Zhang and J. L. Wu, Commutators for the maximal functions on Lebesgue spaces with variable exponent, *Math. Inequal. Appl.*, **17** (2014), 1375–1386.

[29] P. Zhang and J. L. Wu, Commutators of the fractional maximal function on variable exponent Lebesgue spaces, *Czechoslovak Math. J.*, **64** (2014), 183–197.

[30] P. Zhang, J. L. Wu and J. Sun, Commutators of some maximal functions with Lipschitz function on Orlicz spaces, *Mediterr. J. Math.*, **15** (2018), Paper No. 216, 13 pp.