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ON THE COEXISTENCE OF CONVERGENCE AND DIVERGENCE PHENOMENA FOR INTEGRAL AVERAGES AND AN APPLICATION TO THE FOURIER–HAAR SERIES

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Abstract. Let $C, D \subset \mathbb{N}$ be disjoint sets, and $\mathcal{C} = \{1/2^c \colon c \in C\}, \mathcal{D} = \{1/2^d \colon d \in D\}$. We consider the associate bases of dyadic, axis-parallel rectangles $\mathcal{R}_{\mathcal{C}}$ and $\mathcal{R}_{\mathcal{D}}$. We give necessary and sufficient conditions on the sets $\mathcal{C}$ and $\mathcal{D}$ such that there is a positive function $f \in L^1([0,1)^2)$ so that the integral averages are convergent with respect to $\mathcal{R}_{\mathcal{C}}$ and divergent for $\mathcal{R}_{\mathcal{D}}$. We next apply our results to the two-dimensional Fourier–Haar series and characterize convergent and divergent sub-indices. The proof is based on some constructions from the theory of low-discrepancy sequences such as the van der Corput sequence and an associated tiling of the unit square.

1. Introduction

Let $\mathcal{R}$ be the family of half-closed axis-parallel rectangles in $\mathbb{R}^2$, i.e. $\mathcal{R} = \{[a,b) \times [c,d) : a < b,\ c < d\}$. For $R \in \mathcal{R}$ we denote by $\operatorname{diam} R$ the length of the diagonal of R.

Definition. A family of rectangles $\mathcal{F} \subset \mathcal{R}$ is said to be a basis of differentiation (or simply a basis), if for any point $z \in \mathbb{R}^2$ there exists a sequence of rectangles $R_k \in \mathcal{F}$ such that $z \in R_k$, $k \in \mathbb{N}$, and $\operatorname{diam} R_k \to 0$ as $k \to \infty$.

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Let $\mathcal{F} \subset \mathcal{R}$ be a differentiation basis. For any function $f \in L^1(\mathbb{R}^2)$ and $z \in \mathbb{R}^2$ we define

$$\delta_{\mathcal{F}}(z,f) = \limsup_{\substack{\operatorname{diam} R \to 0;\\ z \in R \in \mathcal{F}}} \left| \frac{1}{\mu(R)} \int_R f\,dxdy - f(z) \right|.$$

(Here and below, let μ denote the Lebesgue measure on $\mathbb{R}^2$.) The function $f \in L^1(\mathbb{R}^2)$ is said to be differentiable at a point $z \in \mathbb{R}^2$ with respect to the basis $\mathcal{F}$ provided $\delta_{\mathcal{F}}(z,f) = 0$.

Let $\mathcal{R}_{\text{dyadic}}$ be the family of all dyadic rectangles in $[0,1)^2$ of the form

$$R_{n,m}(i,j) = \Big[\frac{i-1}{2^n}, \frac{i}{2^n}\Big) \times \Big[\frac{j-1}{2^m}, \frac{j}{2^m}\Big),$$

where $n, m \in \mathbb{N}$, $1 \le i \le 2^n$, and $1 \le j \le 2^m$. For an infinite subset $C \subset \mathbb{N}$ one can generate a rare basis as follows:

(1.1)
$$\mathcal{F}_C = \big\{ R_{n,m}(i,j) \in \mathcal{R}_{\text{dyadic}} : n, m \in C,\ i \in \{1, \dots, 2^n\},\ j \in \{1, \dots, 2^m\} \big\}.$$

In this paper, we have the following theorem which shows the coexistence of convergence and divergence phenomena for integral averages of positive functions. For $a \in \mathbb{N}$ and $B \subset \mathbb{N}$, let $\operatorname{dist}(a,B) = \min_{b \in B} |a-b|$.

Theorem 1.1. *Let $C, D \subset \mathbb{N}$ be two disjoint infinite subsets and let $\mathcal{F}_C$ and $\mathcal{F}_D$ be the corresponding bases as in* (1.1). *Then there exists a function $f \in L^1([0,1)^2)$, with $f \ge 0$, such that for almost every $z \in [0,1)^2$ we have*

$$\delta_{\mathcal{F}_C}(z,f) = 0,$$

and

$$\delta_{\mathcal{F}_D}(z,f) = \infty$$

if and only if

$$\sup_{n \in D} \operatorname{dist}(n, C) = \infty. \tag{1.2}$$

We give a few comments on the result above. Let $\mathcal{F}$ be a differentiation basis and consider classes of functions

$$\mathcal{L}(\mathcal{F}) = \big\{ f \in L^1(\mathbb{R}^2) : \delta_{\mathcal{F}}(z,f) = 0 \text{ for almost every } z \big\},$$

$$\mathcal{L}^+(\mathcal{F}) = \big\{ f \in L^1(\mathbb{R}^2) : \delta_{\mathcal{F}}(z,f) = 0 \text{ and } f \ge 0 \text{ for almost every } z \big\}.$$

Note that $\mathcal{L}(\mathcal{F})$ is the family of functions having almost everywhere differentiable integrals with respect to the basis $\mathcal{F}$.

In [14], Zerekidze showed that

$$\mathcal{L}^{+}\left(\mathcal{R}_{\text{dyadic}}\right)=\mathcal{L}^{+}(\mathcal{R}).$$

This means that for positive functions the basis $\mathcal{R}$ is equivalent to the basis of all dyadic rectangles $\mathcal{R}_{\text{dyadic}}$. Remark, however, that we do not have $\mathcal{L}\left(\mathcal{R}_{\text{dyadic}}\right)=\mathcal{L}(\mathcal{R})$, i.e. unlike the class of all non-negative functions, there is no equivalence between the differential bases of all rectangles and the class of dyadic rectangles in the sense that convergence with respect to $\mathcal{R}_{\text{dyadic}}$ does not guarantee convergence with respect to $\mathcal{R}$ and the divergence with respect to $\mathcal{R}$ does not guarantee divergence with respect to $\mathcal{R}_{\text{dyadic}}$.

In [13], Stokolos proved the above theorem for the case $C=\emptyset$ and an arbitrary infinite subset $D\subset\mathbb{N}$. We remark that the function constructed in the paper is positive. In [6], the authors considered the case $C=\mathbb{N}\setminus D$, where D is an arbitrary infinite subset and give a necessary and sufficient condition for the existence of a function f. The function constructed in the paper is unbounded both from above and below, hence is not positive.

In [4], the authors studied the problem for the basis $\mathcal{R}$, i.e. for the class of all rectangles. They considered two sets $\mathcal{C},\mathcal{D}\subset[0,1]$ and give conditions on the sets, under which one can construct a functions f which is convergent with respect to rectangles with sides in $\mathcal{C}$ and divergent with respect rectangles with sides in $\mathcal{D}$. The function constructed in the paper is unbounded both from above and below, hence is not positive. Non-positivity of the function is crucial in the proof. As was mentioned above we have $\mathcal{L}\left(\mathcal{R}_{\text{dyadic}}\right)\neq\mathcal{L}(\mathcal{R})$. Due to the non-constructive nature of the argument, the convergence and divergence properties of the function on the rectangles from the bases $\mathcal{F}_C$ and $\mathcal{F}_D$ is not clear. To overcome this issue a new, constructive approach is needed to the problem. We provide such an approach in this paper.

1.1. Coexistence of convergence and divergence phenomena for Fourier–Haar series. We now discuss an application of our theorem to the Fourier–Haar series. Let $\Psi=\{\psi_k\}_{k\in\mathbb{Z}^d}$ with $\psi_k\in L^2(\mathbb{T}^d)$ be an orthonormal system (i.e., $\|\psi_k\|_{L^2}=1$ and $\langle\psi_k,\psi_\ell\rangle=0$, when $k\neq\ell$), and $f\in L^1(\mathbb{T}^d)$. We consider the rectangular partial sums of the Fourier series with respect to the system Ψ, i.e. for every $n=(n_1,\dots,n_d)\in\mathbb{N}^d$,

$$S_nf=\sum_{\substack{k=(k_1,\dots,k_d)\in\mathbb{Z}^d\\|k_i|\leq n_i}}\langle f,\psi_k\rangle\psi_k. \tag{1.3}$$

One can as well use other summation methods, however, in this paper we will consider only rectangular summation methods. It is well known that for certain orthonormal systems, there exists $f\in L^1(\mathbb{T})$ so that S_nf diverges almost everywhere. For instance the classical example by A. Kolmogorov [8] shows this for the one dimensional trigonometric system. In [2], Gosselin proved that for every increasing sequence of natural numbers (n_k) there exists a function $f \in L^1([0, 2\pi))$ such that $\sup_{k\in\mathbb{N}} |S_{n_k}(f)(x)| = \infty$. Similar functions can also be constructed for the Fourier–Walsh system. Another classical system, for which divergence phenomena occur, is the Haar wavelet which will be defined in detail in Appendix A.

We are interested in the following question:

QUESTION. *Let $\mathcal{N}, \mathcal{M} \subset \mathbb{N}^d$ be two infinite subsets of indices. Under which conditions on the sets $\mathcal{N}, \mathcal{M}$ there exists a function $f \in L^1(\mathbb{T}^d)$, with $f \geq 0$, such that*

$$\lim_{\substack{|n|\to\infty \\ n\in\mathcal{N}}} S_n f(z) = f(z) \quad \text{and} \quad \limsup_{\substack{|n|\to\infty \\ n\in\mathcal{M}}} |S_n f(z)| = \infty$$

for Lebesgue almost every $z \in \mathbb{T}^d$? Here, for $n = (n_1, \dots, n_d) \in \mathbb{N}^d$, we define $|n| = \min_i n_i$.

In this paper we give a complete answer to this question for the two dimensional univariate Haar system (see Theorem 1.2). The corresponding problem for spherical summation methods and for systems such as the trigonometric and Walsh systems appears to be open.

Denote the univariate Haar system by $\{H_m\}_{m\in\mathbb{N}^2}$. (See Appendix A for definition and properties). For $n, m \in \mathbb{N}$, consider the rectangular partial sums of the Fourier–Haar series as in (1.3)

$$S_{(n,m)} f = \sum_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} \langle f, H_{(i,j)} \rangle H_{(i,j)}.$$

It is well known that the correct Orlicz class of convergence for this sums is $L \ln^+ L$ (see [5], [12]). Hence, there exist a function $f \in L^1([0,1)^2)$ for which $S_{(n,m)} f$ diverges almost everywhere.

For $n \in \mathbb{N} \setminus \{1\}$, one can let $n = 2^k + i$, where $k \in \mathbb{N} \cup \{0\}$ and $i = 1, \dots, 2^k$. Given $\mathcal{N} \subset \mathbb{N}$, denote

$$B_{\mathcal{N}} = \left\{ k \in \mathbb{N} \cup \{0\} : 2^k + i \in \mathcal{N} \text{ for some } i \in \{1, \dots, 2^k\} \right\} \tag{1.4}$$

We have the following theorem:

THEOREM 1.2. *Let $\mathcal{N}, \mathcal{M} \subset \mathbb{N}$ be two disjoint infinite subsets and let $C = B_{\mathcal{N}}$, $D = B_{\mathcal{M}}$ be defined as above, respectively. For the Fourier–Haar series there exists a function $f \in L^1([0,1)^2)$, with $f \geq 0$, such that for almost every $z \in [0,1)^2$ we have*

$$\lim_{\substack{n,m\to\infty; \\ n,m\in\mathcal{N}}} S_{(n,m)} f(z) = f(z), \quad \text{and} \quad \limsup_{\substack{n,m\to\infty; \\ n,m\in\mathcal{M}}} |S_{(n,m)} f(z)| = \infty$$

if and only if

$$\sup_{k\in D} \operatorname{dist}(k, C) = \infty. \tag{1.5}$$

We will derive Theorem 1.2 from Theorem 1.1, the coexistence of convergence and divergence phenomena for integral averages of positive functions.

1.2. Idea of the proof of Theorem 1.2. The new ingredient is an application of some ideas from discrepancy theory, which is of independent interest. We now sketch the idea of the proof of Theorem 1.1. Suppose we are given $C, D \subset \mathbb{N}$ fulfilling (1.2), that is, they are far from each other. The idea is to construct an intermediate function satisfying properties in Proposition 3.1. In order to do so we choose rectangles from the basis $\mathcal{F}_D$ and distribute them in a way that they cover a substantial portion of the unit square, then we distribute the support of f in such a way that the integral averages with respect to each rectangle from $\mathcal{F}_D$ is larger than the prescribed number $M \gg 1$ thus full-filling condition (3.2). Hence, for any point that belongs to any of the rectangles the integral averages will be large.

However, at the same time the distribution of the support of f needs to be such that the integral averages with respect to rectangles from $\mathcal{F}_C$ are small (property (3.3)). If we think of the support of f as being concentrated at finite number of points and assume that each point has the same mass, then the question of estimating the expressions $(1/\mu(R)) \int_R f\, dxdy$ will boil down to computing the number of point-supports that fall inside R. This is nothing else but a discrepancy estimates for the rectangles in $\mathcal{F}_C$ and for the set $P = \{x_1, \dots, x_N\} \subset [0, 1)^2$ that carries the support of f. Namely, if $\mathcal{R}$ is a collection of axis-parallel rectangles, then one defines discrepancy, using Kuipers and Niederreiter's notation [9, p. 93] as follows:

$$D_N(P) = \sup_{R\in\mathcal{R}} \left| \frac{\#\{x \in P : x \in R\}}{N} - \mu(R) \right|. \tag{1.6}$$

It is known that for the van der Corput sequence the above expressions reaches the lowest possible asymptotic bound, i.e. there exists a constant $C > 0$ such that

$$D_N(P) \le C\, \frac{\log N}{N},$$

where $P \subset [0, 1)^2$ is the set of the N-points van der Corput sequence. Therefore, it is natural to use this sequence to minimize the discrepancy of the distribution of the support of f. We remark that the situation is in fact more complicated than the one described above, however the general idea is the same. A natural question arises whether the ideas in this paper can be used to construct a sequence for which its discrepancy with respect to one bases of rectangle is different from that of with respect to another bases, i.e. $\mathcal{F}_C$ and $\mathcal{F}_D$?

To find such an optimal distribution of the support, in Section 4 we introduce a tilling that follows the dichotomy of the van der Corput sequence and describes a way of distributing rectangles inside the unit square. In Section 4.3 we consider several van der Corput tilings and create pairings between them which eventually leads us to the definition of the function f in Section 5.4. The support of f is placed at the intersection of the rectangles that are paired with each other. The resulting function turns out to satisfy the desired properties of Proposition 3.1. Due to the constructive nature of the function we are also able to deduce all the necessary information for the rectangles in the bases $\mathcal{F}_C$ and $\mathcal{F}_D$.

To prove the necessity, we note that the maximal function

$$M_D f(z) = \sup_{z\in R\in\mathcal{F}_D} \frac{1}{\mu(R)} \int_R f\,dxdy$$

can essentially be estimated from above by the maximal function

$$M_C f(z) = \sup_{z\in R\in\mathcal{F}_C} \frac{1}{\mu(R)} \int_R f\,dxdy,$$

if the sets C and D are close.

1.3. Organization of the paper. To start with, we prove Theorem 1.2 assuming Theorem 1.1 and Theorem 1.1 assuming the main proposition (Proposition 3.1) in Section 2 and Section 3, respectively. Then Sections 4 to 6 will be devoted to proving Proposition 3.1. In Section 4, as was mentioned above, we define a tiling of the unit square by rectangles from $\mathcal{F}_D$, and create pairings between several tilings. Next, in Section 5, we will construct certain figures consisting of the tiles from a chain of tilings, and using the geometry of such figures, we define a positive function. We present two key estimates, Lemma 5.4 and Lemma 5.5: the former summarizes a consequence of the geometric features of the figures, and the latter shows the crucial properties of the function. In Section 6, we will make use of these two estimates to obtain the main lemma (Lemma 6.2), which describes a domain of convergence for integral averages over rectangles from $\mathcal{F}_C$. Then using Lemma 6.2, we prove Proposition 3.1 in Section 6.4, completing the proof of the main theorems.

1.4. Notations. Throughout the paper, let $\pi_x\colon \mathbb{R}^2 \ni (x,y) \mapsto x \in \mathbb{R}_x$ denote the projection onto x-axis and respectively, $\pi_y\colon \mathbb{R}^2 \ni (x,y) \mapsto y \in \mathbb{R}_y$ denote the projection onto y-axis. Let μ denote the Lebesgue measure on $\mathbb{R}^d$ for $d=1,2$. For a finite set A, let $\#A$ denote the cardinality.

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 assuming Theorem 1.1. We also use a result (Proposition A.1) connecting the convergence of the rectangular sums of the Fourier–Haar series with the differentiation of integrals with respect to the basis of dyadic rectangles. See Appendix A.

Proof of Theorem 1.2 (assuming Theorem 1.1). We first prove the sufficiency of (1.5). By Proposition A.1 and the discussion right after it we have for every $z \in [0,1)^2$ that

$$S_{(n,m)}f(z) = \frac{1}{\mu(I_{n,m}(z))} \int_{I_{n,m}(z)} f \, dxdy, \tag{2.1}$$

where $I_{n,m}(z)$ is a dyadic rectangle containing z with sides $1/2^k$ and $1/2^s$, or $1/2^{k+1}$ and $1/2^{s+1}$.

Consider the sets $\mathcal{N}$ and $\mathcal{M}$ and the associated bases $\mathcal{F}_C$ and $\mathcal{F}_D$ as in (1.1), where $C = B_{\mathcal{N}}$ and $D = B_{\mathcal{M}}$ are as in (1.4) respectively. If now $n, m \in \mathcal{N}$ or $n, m \in \mathcal{M}$, then the rectangle $I_{n,m}$ above will belong to either $\mathcal{F}_C$ or $\mathcal{F}_{C+1}$, or $\mathcal{F}_D$ and $\mathcal{F}_{D+1}$, where $C+1 = \{c+1 : c \in C\}$. Define new sets $C^+ = C \cup (C+1)$ and $D^+ = D \cup (D+1)$. Note that C^+ and D^+ satisfy the assumption (1.5) since it is also satisfied by C and D. Hence, by Theorem 1.1, there exists a non-negative function $f \in L^1([0,1)^2)$ such that for almost every z we have

$$\lim_{\substack{\operatorname{diam} R \to 0 \\ z \in R \in \mathcal{F}_{C^+}}} \frac{1}{\mu(R)} \int_R f \, dxdy = f(z) \quad \text{and} \quad \limsup_{\substack{\operatorname{diam} R \to 0 \\ z \in R \in \mathcal{F}_{D^+}}} \frac{1}{\mu(R)} \int_R f \, dxdy = \infty.$$

In view of (2.1), it follows from the first relation that for almost every z we have

$$\lim_{\substack{n,m \to \infty \\ n,m \in C}} S_{(n,m)}f(z) = \lim_{\substack{n,m \to \infty \\ n,m \in C^+}} S_{(n,m)}f(z) = f(z).$$

To see the second part, note that for every z the dyadic rectangle with sides $1/2^{k+1}, 1/2^{s+1}$ which contains z, is also contained in the rectangle with sides $1/2^k, 1/2^s$ containing z. Hence, since f is positive and we have divergence with respect to the dyadic rectangle with sides $1/2^{k+1}, 1/2^{s+1}$ $(k, s \in D)$ then we also have it for the dyadic rectangle with sides $1/2^k, 1/2^s$. This proves the divergence part of the theorem.

Next, we prove the necessity of (1.5) by contraposition. Hence, assume that (1.5) fails. Then there exists an integer $d > 0$ so that for every $s \in D^+$ we have

$$C^+ \cap [s-d, s+d] \neq \emptyset. \tag{2.2}$$

Suppose

$$\lim_{\substack{n,m\to\infty;\\ n,m\in\mathcal{N}}} S_{(n,m)}f(z) = f(z)$$

for almost every z. For a given $\delta > 0$ consider the set

$$E_\delta = \Big\{ z \in \mathbb{R}^2 : \sup_{n,m\geq \frac{1}{\delta},\, n,m\in\mathcal{N}} |S_{(n,m)}f(z) - f(z)| < 1 \Big\}.$$

We have $\mu(E_\delta) > 0$ for $\delta > 0$ small enough. Then for the indicator function $\mathbb{1}_{E_\delta}$ of E_δ, we will have by the Jessen–Marcinkiewicz–Zygmund theorem [5], that almost all points $z \in E_\delta$ are Lebesgue points, namely for almost every $z \in E_\delta$ we have

$$\lim_{\substack{\operatorname{diam} R\to 0\\ z\in R\in\mathcal{R}}} \frac{\mu(R\cap E_\delta)}{\mu(R)} = 1.$$

Let $z \in E_\delta$. Assume $n, m \in \mathcal{M}$ are so large that for the rectangle $B = I_{n,m}(z)$ from (A.1), i.e. $S_{(n,m)}f(z) = (1/\mu(I_{n,m}(z))) \int_{I_{n,m}(z)} f\,dxdy$, we have that

$$\frac{\mu(B\cap E_\delta)}{\mu(B)} > 1 - c$$

for small $c \in (0,1)$. Assume the sides of B are $1/2^s$ and $1/2^k$, where $s, k \in D^+$. We now represent B as a union of dyadic rectangles with sides $1/2^{s+d}$ and $1/2^{k+d}$, i.e. $B = \sqcup_q B_q$. Note that, since d is fixed and the constant c above can be taken arbitrarily small, then for an appropriate choice of c we can make sure that each of the rectangles B_q has a non-empty intersection with E_δ. Thus for each B_q we can choose a point $w = w_q \in E_\delta \cap B_q$. Then by (2.2) and Proposition A.1, there exists $\tilde{q} = (n_1, m_1)$, with $n_1, m_1 \in \mathcal{N}$, a rectangle $I_{\tilde{q}} = I_{\tilde{q}}(w)$ containing w, with sides $1/2^{\mu(w)}$ and $1/2^{\nu(w)}$, where $\mu(w), \nu(w) \in C^+$ so that $\mu(w) \in [s-d, s+d]$ and $\nu(w) \in [k-d, k+d]$, respectively, and

$$S_{(n_1,m_1)}f(w) = \frac{1}{\mu(I_{\tilde{q}})} \int_{I_{\tilde{q}}} f\,dxdy < 1 + f(w). \tag{2.3}$$

Thus, we will also have that $B_q \subset I_{\tilde{q}}$. Note that

$$\frac{\mu(B_q)}{\mu(I_{\tilde{q}})} \geq \frac{1/2^{s+d}\cdot 1/2^{k+d}}{1/2^{s-d}\cdot 1/2^{k-d}} = \frac{1}{2^{4d}}. \tag{2.4}$$

Repeating the same argument for all remaining dyadic rectangles B_q we can find a collection of rectangles $\{I_{\tilde{q}} \in \mathcal{F}_{C^+}\}_{\tilde{q}}$ such that $B \subset \bigcup_{\tilde{q}} I_{\tilde{q}}$.

Set $K = \max_q(1 + f(w_q))$. Since $f \geq 0$, we have

$$\int_B f\, dxdy \leq \sum_{\tilde{q}} \int_{I_{\tilde{q}}} f\, dxdy,$$

and by (2.3) and (2.4) we have

$$\sum_{\tilde{q}} \int_{I_{\tilde{q}}} f\, dxdy \leq K \sum_{\tilde{q}} \mu(I_{\tilde{q}}) \leq K \sum_q 2^{4d} \mu(B_q) = K 2^{4d} \mu(B).$$

It follows that

$$\frac{1}{\mu(B)} \int_B f\, dxdy \leq K 2^{4d}.$$

This implies that for almost every $z \in E_\delta$ we have

$$\lim_{\substack{n,m\to\infty;\\ n,m\in\mathcal{M}}} S_{(n,m)} f(z) \leq \limsup_{\substack{\operatorname{diam} B\to 0;\\ z\in B\in\mathcal{F}_{D^+}}} \frac{1}{\mu(B)} \int_B f\, dxdy < \infty.$$

This finishes the proof. $\square$

3. Main proposition and proof of Theorem 1.1

In this section, we will prove Theorem 1.1 assuming the following main proposition. Let $C, D \subset \mathbb{N}$ be two infinite disjoint subsets and let $\mathcal{F}_C$ and $\mathcal{F}_D$ be the corresponding bases as in (1.1).

Proposition 3.1. *Assume* (1.2). *There is a constant $C_0 > 0$, so that for every $\varepsilon > 0$ and every $M > 0$, there exists a function $f \in L^\infty([0,1)^2)$, with $f \geq 0$ and*

$$\|f\|_{L^1} < 2, \tag{3.1}$$

for which one can find a subset $E \subset [0,1)^2$, with $\mu(E) \geq 1 - \varepsilon$, such that for every $z \in E$ there exists a dyadic rectangle R from $\mathcal{F}_D$, such that $z \in R$ and

$$\frac{1}{\mu(R)} \int_R f\, dxdy \geq M, \tag{3.2}$$

and for every $z \in E$ and any dyadic rectangle $R \in \mathcal{F}_C$, with $z \in R$, we have that

$$\frac{1}{\mu(R)} \int_R f\, dxdy \leq C_0. \tag{3.3}$$

Sections 4 to 6 will be devoted to the proof of Proposition 3.1.

PROOF OF THEOREM 1.1 (ASSUMING PROPOSITION 3.1). Assume (1.2). Using Proposition 3.1 for $\varepsilon_n = 1/2^n$ and $M = n^3$, we will get a sequence of positive functions $f_n \in L^\infty$ and sets $E_n \subset [0,1)^2$ such that $\mu(E_n) > 1 - \varepsilon_n$. We then consider the function

$$f = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n.$$

Note that by Proposition 3.1(3.1), we have

$$\Big\| \sum_{n=1}^{\infty} \frac{1}{n^2} f_n \Big\|_{L^1} \le \sum_{n=1}^{\infty} \frac{1}{n^2} \|f_n\|_{L^1} < \infty.$$

Hence, f is well defined. It is positive and $f \in L^1$. By the Borel–Cantelli lemma, we also have

$$\mu\big(\liminf_{n\to\infty} E_n \big) = 1$$

Since $f_n \in L^\infty \subset L \ln^+ L$, then by the Jessen–Marcinkiewicz–Zygmund theorem [5], we have that for all $n \in \mathbb{N}$, there is $\Gamma_n \subset [0,1)^2$ with $\mu(\Gamma_n) = 1$ such that

$$\lim_{\substack{\operatorname{diam} R \to 0;\\ z \in R \in \mathcal{R}}} \frac{1}{\mu(R)} \int_R f_n \, dxdy = f_n(z) \tag{3.4}$$

for every $z \in \Gamma_n$. Define

$$\Gamma_\infty = \bigcap_{n=1}^{\infty} \Gamma_n \quad \text{and} \quad \Lambda = \big(\liminf_{n\to\infty} E_n \big) \cap \Gamma_\infty.$$

Clearly $\mu(\Lambda) = 1$.

First, we have that almost every $z \in [0,1)^2$ eventually belongs to all sets E_k, i.e. there exists $K = K(z) \in \mathbb{N}$ so that for all $k \ge K$ we have $z \in E_k \cap \Gamma_\infty$. If $z \in E_k$, for some k, then by (3.2) we can find $R \in \mathcal{F}_D$ with $R \ni z$ so that

$$\frac{1}{\mu(R)} \int_R f \, dxdy \ge \frac{1}{k^2} \frac{1}{\mu(R)} \int_R f_k \, dxdy \ge \frac{k^3}{k^2} = k.$$

Next, for $R \in \mathcal{F}_C$ write

$$\frac{1}{\mu(R)} \int_R \sum_{n=1}^{\infty} \frac{1}{n^2} f_n \, dxdy \tag{3.5}$$

$$= \sum_{n=1}^{N} \frac{1}{n^2\mu(R)} \int_R f_n\,dxdy + \sum_{n=N+1}^{\infty} \frac{1}{n^2\mu(R)} \int_R f_n\,dxdy.$$

Since almost every z eventually belongs to all sets E_n, then for large enough N, in view of property (3.3), we have

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2\mu(R)} \int_R f_n\,dxdy \le C_0 \sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

Hence this can be made small if N is large. While for the first term in (3.5) we have by (3.4)

$$\lim_{\substack{\operatorname{diam} R\to 0;\\ z\in R\in\mathcal{F}_C}} \sum_{n=1}^{N} \frac{1}{n^2\mu(R)} \int_R f_n\,dxdy = \sum_{n=1}^{N} \frac{1}{n^2} f_n(z)$$

for $z \in \Lambda$. Thus

$$\lim_{\substack{\operatorname{diam} R\to 0;\\ z\in R\in\mathcal{F}_C}} \frac{1}{\mu(R)} \int_R \sum_{n=1}^{\infty} \frac{1}{n^2} f_n\,dxdy = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(z) = f(z),$$

for almost every $z \in [0,1)^2$.

The proof of the opposite direction is analogous to the necessary part of Theorem 1.2, so we will skip it. □

4. The van der Corput sequence and a tiling of the unit square

In this section, we make some preparatory work for proving Proposition 3.1.

4.1. Van der Corput tiling. We now define a tiling of the unit square that is associated with the van der Corput sequence. For $i \in \mathbb{N} \cup \{0\}$, let $i = a_0 + 2a_1 + 2^2a_2 + \cdots$, where $a_j \in \{0,1\}$, be the binary expression. Set

$$v(i) = \frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + \cdots.$$

Then define

$$p_i = \Big(\frac{i}{N}, v(i)\Big) \in [0,1)^2$$

for $i = 0,1,\dots,N-1$. The set $P = \{p_0, p_1, \dots, p_{N-1}\} \subset [0,1)^2$ is called the N-points van der Corput set. As was already mentioned in the introduction, the van der Corput sequence is known to have a low discrepancy. See [9,10].

We are given a rectangle $R = [0,b) \times [0,a)$ and $(\xi,\eta) \in [0,1)^2$. Henceforth, to simplify the exposition, we will say that R *is placed at* (ξ,η) when we translate R by the vector (ξ,η):

$$R \mapsto R(\xi,\eta) = R + (\xi,\eta) = [\xi, b+\xi) \times [\eta, a+\eta).$$

Note that (ξ,η) specifies the lower left corner of $R(\xi,\eta)$.

Let $\mathcal{E} = \{1/2^k \in [0,1)\colon k \in \mathbb{N}\}$. Given $a,b \in \mathcal{E}$, let R be an axis-parallel rectangle with height a and width b. Let $N = 1/(ab)$, and $P = \{p_0, p_1, \dots, p_{N-1}\}$ be the van der Corput set. We will define a tiling on $[0,1)^2$ which is generated by R and associated with P. First, we place R at $p_0, p_1, \dots, p_{a^{-1}-1}$, respectively. Since $v(0), v(1), \dots, v(a^{-1}-1)$ are distributed equidistantly with intervals of a, the rectangles $R(p_0), R(p_1), \dots, R(p_{a^{-1}-1})$ are disjoint and

$$\pi_y\Bigg(\bigsqcup_{i=0}^{a^{-1}-1} R(p_i)\Bigg) = [0,1). \tag{4.1}$$

This finishes the first "column" of tiling.

To determine the second column we now translate the first column by the horizontal vector $(b,0)$ and subsequently by vectors $j(b,0)$, $j = 1,\dots,b^{-1}-1$. That is, we will have the collections $\{R(p_i) + (jb,0) : i \in \{0,1,\dots,a^{-1}-1\}\}$ for each $j \in \{0,1,\dots,b^{-1}-1\}$. One then can see that the resulting placement of figures will look like Figure 1. Identifying $\{0\} \times [0,1) \ni (0,y) \sim (1,y) \in \{1\} \times [0,1)$ will give a tiling of $[0,1)^2$ generated by R. We denote the collection of all tiles by $\mathcal{T}_{a,b}$, more specifically

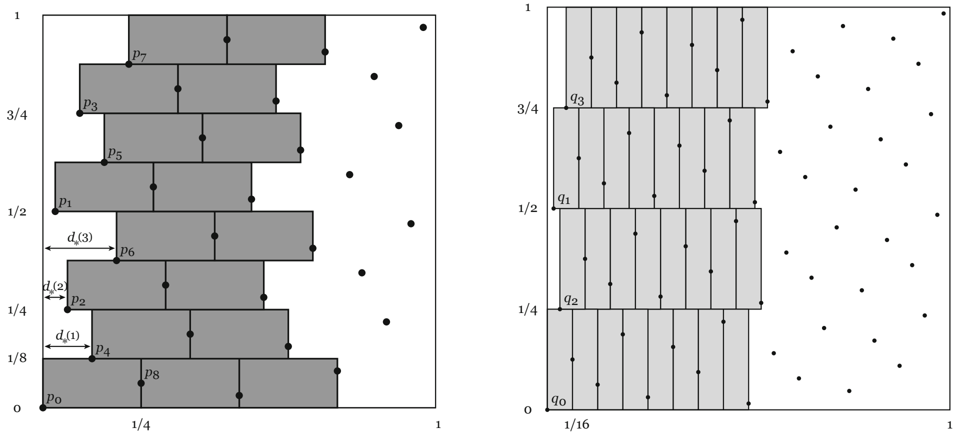
$$\mathcal{T}_{a,b} = \big\{R(p_i) + (jb,0) : i \in \{0,1,\dots,a^{-1}-1\},\ j \in \{0,1,\dots,b^{-1}-1\}\big\},$$

and thus $\#\mathcal{T}_{a,b} = a^{-1} \times b^{-1} = N = \#P$.

4.2. Horizontal translation. One can see from the figures in Figure 1 that each horizontal row of rectangles is a horizontal translation of other rows. Therefore, any row can be described by the amount of horizontal translation vector with respect to the bottom row. Given a tiling $\mathcal{T}_{a,b}$, the horizontal translation length will be denoted by $d_*(i)$ with $i = 0,\dots,a^{-1}-1$, starting from the bottom row. Thus the i'th row of $\mathcal{T}_{a,b}$ can be given as

$$R + j(b,0) + (d_*(i), ia), \quad j = 0,1,\dots,b^{-1}-1. \tag{4.2}$$

Note that $d_*(0) = 0$. One can see that the sequence $d_*(i)$ is similar to the y coordinate of the van der Corput sequence in the sense that $d_*(i) = v(i)b$.



(a) *The van der Corput sequence for $N = 32$ and the associated van der Corput tiling for the rectangle with sides $a = 1/8$ and $b = 1/4$. See Section 4.2 for d_*.*

(b) *The van der Corput sequence for $N = 64$ and the associated van der Corput tiling for the rectangle with sides $a = 1/4$ and $b = 1/16$.*

Figure 1: The N-points van der Corput sequence and the associated van der Corput tiling. Remark that each tile contains only one van der Corput point.

Finally, we introduce the following notation which will be used in Section 4.3 below. Let $\mathcal{T}_{a,b}$ be a tiling, and $x \in \mathcal{E}$ with $x > a$. For each $\ell \in \{0, 1, \ldots, x^{-1} - 1\}$, define

$$H_x^{(\ell)} = [0, 1) \times [\ell x, (\ell + 1)x). \tag{4.3}$$

It is the horizontal strip of height x at the ℓ'th row. We consider the first column of $\mathcal{T}_{a,b}$ in each $H_x^{(\ell)}$. More precisely, define

$$\mathcal{T}_{a,b}^{(\ell)}(x) = \Big\{R(p_i) \in \mathcal{T}_{a,b} : R(p_i) \subset H_x^{(\ell)},\ i \in \{0, 1, \ldots, a^{-1} - 1\}\Big\}. \tag{4.4}$$

Recall (4.1). Hence, for each $\ell \in \{0, 1, \ldots, x^{-1} - 1\}$, there exist sub-indices $i_u = i_u^{(\ell)}$ with $0 \le i_0 < i_1 < \cdots < i_{(x/a)-1} \le a^{-1} - 1$ such that $\mathcal{T}_{a,b}^{(\ell)}(x) = \{R(p_{i_u})\colon u \in \{0, 1, \ldots, (x/a) - 1\}\}$. See Example 4.1 below.

Now, sort the horizontal drifts for $R(p_{i_u}) \in \mathcal{T}_{a,b}^{(\ell)}(x)$ in an ascending order. To do this, let $\sigma\colon \{0, 1, \ldots, (x/a) - 1\} \ni u \mapsto \sigma(u) \in \{0, 1, \ldots, (x/a) - 1\}$ be a permutation such that

$$R(p_{i_u}) \subset [0, 1) \times \Big[\Big(\sigma(u) + \ell\frac{x}{a}\Big)a, \Big(\sigma(u) + 1 + \ell\frac{x}{a}\Big)a\Big) \tag{4.5}$$

for every $u \in \{0, 1, \dots, (x/a) - 1\}$. Namely, the row of $R(p_{i_u}) \in \mathcal{T}^{(\ell)}_{a,b}(x)$ is determined by $\sigma(u) \bmod \ell x/a$. Consequently, one has

$$0 \le d_*(\sigma(0)) < d_*(\sigma(1)) < \dots < d_*(\sigma((x/a) - 1)).$$

Note also that for every $u \in \{0, 1, \dots, (x/a) - 1\}$, we have

$$d_*(\sigma(u)) = \frac{u/x}{N} = \frac{uab}{x}$$

as $1/N = ab$, and thus it follows that

$$d_*(\sigma(u + 1)) - d_*(\sigma(u)) = \frac{(u + 1)ab}{x} - \frac{uab}{x} = \frac{ab}{x}. \tag{4.6}$$

EXAMPLE 4.1. See Figure 1(a). Consider $\mathcal{T}_{a,b}$ with $(a, b) = (1/8, 1/4)$. Let $x = 1/2$. Then we see

$$\mathcal{T}^{(0)}_{1/8,1/4}(1/2) = \big\{R(p_0), R(p_2), R(p_4), R(p_6)\big\},$$
$$\mathcal{T}^{(1)}_{1/8,1/4}(1/2) = \big\{R(p_1), R(p_3), R(p_5), R(p_7)\big\}.$$

For the former family, we have $0 = i^{(0)}_0 < 2 = i^{(0)}_1 < 4 = i^{(0)}_2 < 6 = i^{(0)}_3$, and thus

$$\sigma(0) = 0, \quad \sigma(1) = 2, \quad \sigma(2) = 1, \quad \sigma(3) = 3.$$

For the latter, we have $1 = i^{(1)}_0 < 3 = i^{(1)}_1 < 5 = i^{(1)}_2 < 7 = i^{(1)}_3$, and

$$\sigma(0) = 0\,(\equiv 4),\ \sigma(1) = 2\,(\equiv 6),\ \sigma(2) = 1\,(\equiv 5),\ \sigma(3) = 3\,(\equiv 7) \pmod 4.$$

Hence

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix}.$$

For $x = 1/4$, we have

$$\mathcal{T}^{(0)}_{1/8,1/4}(1/4) = \{R(p_0), R(p_4)\}, \quad \mathcal{T}^{(1)}_{1/8,1/4}(1/4) = \{R(p_2), R(p_6)\},$$
$$\mathcal{T}^{(2)}_{1/8,1/4}(1/4) = \{R(p_1), R(p_5)\}, \quad \mathcal{T}^{(3)}_{1/8,1/4}(1/4) = \{R(p_3), R(p_7)\}.$$

In this case, we have $\sigma = \mathrm{id}$.

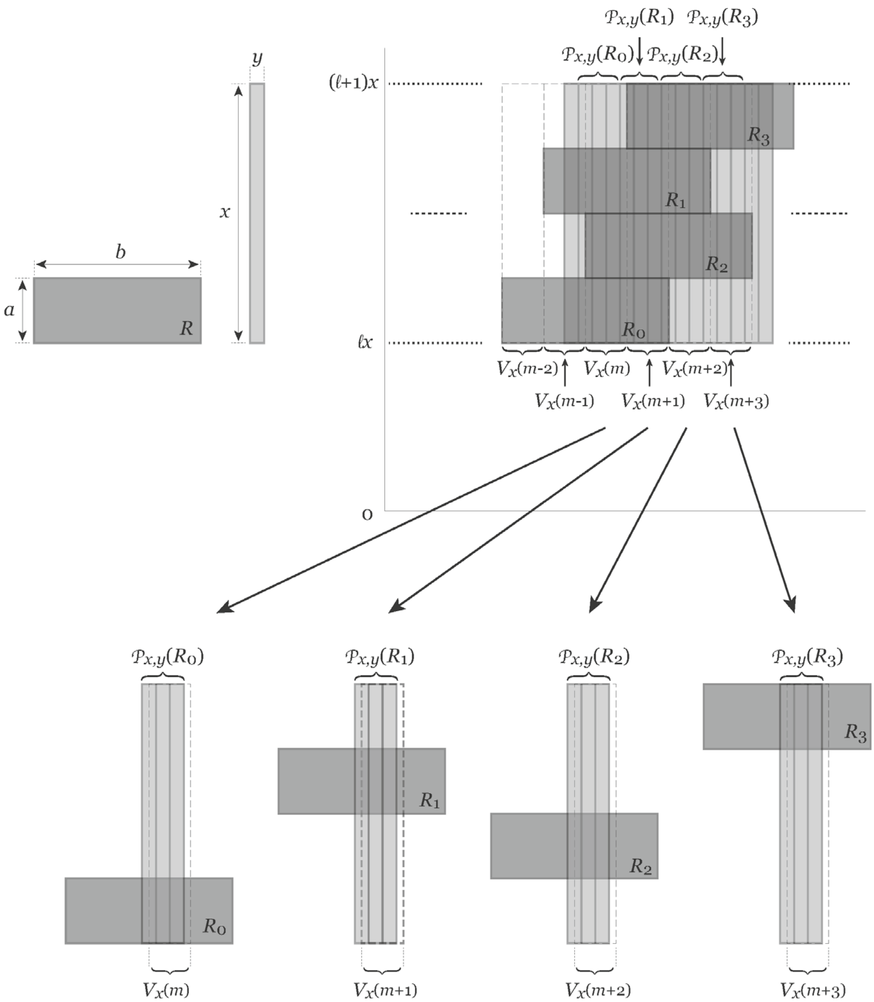


Figure 2: Pairing construction for $R \in \mathcal{T}^{(\ell)}_{a,b}(x)$. Here we abbreviate $R(p_{i_u})$ and $V_x^{(\ell)}$ as R_u and V_x, respectively. The light gray rectangles belong to $\mathcal{T}_{x,y}$. The dashed ones are $V_x^{(\ell)}$. The integer m is defined in (4.7): it is chosen in such a way that $V_x^{(\ell)}$, or $\mathcal{P}_{x,y}(R)$ will be placed (almost) at the horizontal center of $R \in \mathcal{T}^{(\ell)}_{a,b}(x)$.

4.3. Pairing. In this section, we consider two collections of van der Corput tilings $\mathcal{T}_{a,b}$ and $\mathcal{T}_{x,y}$, with $x > a$, $y < b$, and $xy < ab$. Then, in Lemma 4.2 below, we will define a pairing between the rectangles from each collection: this means we will create a correspondence between the rectangles from each collection. To begin with, we first comment on the pairing procedure described in Lemma 4.2. The procedure is slightly involved, but the idea is simple. (See Figure 2.)

Note that there cannot exist bijective pairing between $R \in \mathcal{T}_{a,b}$ and $Q \in \mathcal{T}_{x,y}$ since $\#\mathcal{T}_{a,b} = 1/(ab) < 1/(xy) = \#\mathcal{T}_{x,y}$. We will associate to each $R \in \mathcal{T}_{a,b}$ a collection of $ab/(xy)$-many adjacent tiles from $\mathcal{T}_{x,y}$. Hence, the union of such tiles defines a rectangle of height x and width ab/x. It remains to determine the position where to place the collection. For our purpose, constructing a function which fulfills the desired properties of Proposition 3.1, we need to proceed with this task a concrete way such that the resulting pairing will admit a nice geometric structure which eventually leads us to the definition of the function. To obtain such a geometric structure, roughly speaking, we will associate to $R \in \mathcal{T}_{a,b}$ the rectangles $Q \in \mathcal{T}_{x,y}$ intersecting with R at the horizontal center of it. Here we need to take the horizontal translations of tilings into account, and we will make use of the properties summarized in Section 4.2.

Now, we prove the following lemma.

Lemma 4.2 (pairing lemma). *Let $a, b, x, y \in \mathcal{E}$ so that $x > 5a$, $y < b$ and $xy < ab$, and $n > 5$. Then for any $R \in \mathcal{T}_{a,b}$ one can find a collection $\mathcal{P}_{x,y}(R) \subset \mathcal{T}_{x,y}$ consisting of adjacent tiles from $\mathcal{T}_{x,y}$ so that the following properties hold*:

(1) $\#\mathcal{P}_{x,y}(R) = ab/(xy)$.

(2) *The union $\bigcup \mathcal{P}_{x,y}(R) = \bigcup_{Q\in\mathcal{P}_{x,y}(R)} Q$ is a rectangle of height x and width ab/x such that $\pi_y\big(\bigcup \mathcal{P}_{x,y}(R)\big) \supset \pi_y(R)$ and $\pi_x\big(\bigcup \mathcal{P}_{x,y}(R)\big) \subset \pi_x(R)$. In particular, every $Q \in \mathcal{P}_{x,y}(R)$ intersects with R. Furthermore $\pi_x(R) \setminus \pi_x\big(\bigcup \mathcal{P}_{x,y}(R)\big)$ consists of two components* (*intervals*), *each of them has a length of at least $2b/n$.*

(3) *For different $R \in \mathcal{T}_{a,b}$ the corresponding unions* (*rectangles*) $\bigcup_{Q\in\mathcal{P}_{x,y}(R)} Q$ *have disjoint interiors*, *and*

$$[0,1)^2 = \bigcup_{R\in\mathcal{T}_{a,b}} \Bigg(\bigcup_{Q\in\mathcal{P}_{x,y}(R)} Q \Bigg).$$

Proof. Let $N = 1/(ab)$, and let $P = \{p_0, p_1, \dots, p_{N-1}\}$ be the van der Corput set. We will use the notation prepared in Section 4.2, for instance, (4.3) and (4.4). Note here that d_* will be used as the horizontal translation length of $\mathcal{T}_{a,b}$ throughout the proof.

We divide our argument into two steps. In the first step, we define pairing for $R \in \mathcal{T}_{a,b}$ which belongs to the bottom row of $\mathcal{T}_{x,y}$, that is, the horizontal strip $H_x^{(0)}$. Then in the second step, we define paring for $R \in \mathcal{T}_{a,b}$ which belongs to the other rows of $\mathcal{T}_{x,y}$.

Step 1. In this step, we will define $\mathcal{P}_{x,y}(R)$ for $R \in \mathcal{T}_{a,b}$ with $R \subset H_x^{(0)}$. To begin with, we consider $\mathcal{T}_{a,b}^{(0)}(x)$. As was observed above, there exist indices $0 = i_0 < i_1 < \cdots < i_{(x/a)-1} \le a^{-1} - 1$ such that $\mathcal{T}^{(0)}_{a,b}(x) = \{R(p_{i_u})\colon u \in \{0, 1, \ldots, (x/a) - 1\}\}$.

For $k = 0, 1, \ldots, (x/(ab)) - 1$, let

$$V_x(k) = \Big[k\frac{ab}{x}, (k+1)\frac{ab}{x}\Big) \times [0, x) \subset H^{(0)}_x.$$

Note that $\mu(V_x(k)) = ab = \mu(R)$. In view of (4.6), for every $u \in \{0, 1, \ldots, (x/a) - 1\}$, one sees that $V_x(u)$ is well-placed with respect to $R(p_{i_u}) \in \mathcal{T}^{(0)}_{a,b}(x)$ in the sense that $\partial_{\text{left}} R(p_{i_u}) \subset \partial_{\text{left}} V_x(u)$. Here and below, for a rectangle $R = [u_1, u_2) \times [v_1, v_2)$, let $\partial_{\text{left}} R = \{u_1\} \times [v_1, v_2)$. Take $m \in \mathbb{N}$ such that

$$\frac{2b}{n} < m\frac{ab}{x} < b - \frac{2b}{n},$$

or equivalently

$$\frac{2x}{an} < m < \frac{(n-2)x}{an}.$$

For instance, one can choose an integer m so that

$$m = \frac{x}{2a} \tag{4.7}$$

for $n > 5$ large. To distinguish we denote the m above by m_0. One has $m_0\frac{ab}{x} = b/2$, that is, the integrer m_0 determines the half of the width of $R \in \mathcal{T}^{(0)}_{a,b}(x)$. (The purpose of taking m_0 like this will be clear in Lemma 5.2 below.)

Now we pair $R(p_{i_u})$ with the tiles from $\mathcal{T}_{x,y}$ that fully fall inside $V_x(u+m_0)$ or intersect $\partial_{\text{left}} V_x(u + m_0)$, that is, define

$$\begin{aligned} \mathcal{P}_{x,y}(R(p_{i_u})) = \big\{Q \in \mathcal{T}_{x,y}\colon Q \subset V_x(u + m_0)\big\} \\ \cup \big\{Q \in \mathcal{T}_{x,y}\colon Q \cap \partial_{\text{left}} V_x(u + m_0) \ne \emptyset\big\}. \end{aligned} \tag{4.8}$$

(See Remark 4.3 below for the formulation of (4.8).) Thus for each $u \in \{0, 1, \ldots, (x/a) - 1\}$, one has $Q \cap R(p_{i_u}) \ne \emptyset$ for every $Q \in \mathcal{P}_{x,y}(R(p_{i_u}))$. Note that $\mathcal{P}_{x,y}(R(p_{i_u}))$ will be disjoint for distinct $u \in \{0, 1, \ldots, (x/a) - 1\}$. It follows from the construction above that for every $R = R(p_{i_u}) \in \mathcal{T}^{(0)}_{a,b}(x)$, $u \in \{0, 1, \ldots, (x/a) - 1\}$, we have $\#\mathcal{P}_{x,y}(R) = ab/(xy)$, and $\bigcup_{Q\in\mathcal{P}_{x,y}(R)} Q$ is a rectangle of height x and width ab/x. Further, due to the specific choice (4.7) of m_0, we see that $\pi_x(\bigcup_{Q\in\mathcal{P}_{x,y}(R)} Q)$ is placed at the middle of $\pi_x(R)$, thus, in particular, $\pi_x(R) \setminus \pi_x\big(\bigcup_{Q\in\mathcal{P}_{x,y}(R)} Q\big)$ consists of two intervals such that each of them has a length of at least $2b/n$. Hence, properties (1) and (2) are fulfilled.

Next, we define pairing for remaining tiles from $\mathcal{T}_{a,b}$ that belong to $H_x^{(0)}$. We recall that the tiling $\mathcal{T}_{a,b}$ in $H_x^{(0)}$ is just a translations of the rectangles in $\mathcal{T}_{a,b}^{(0)}(x)$ by $j(b,0)$, $j=1,\dots,b^{-1}-1$. Note also that for each $u=0,1,\dots,(x/a)-1$, one has

$$\partial_{\text{left}}\big(R(p_{i_u})+j(b,0)\big)\subset\partial_{\text{left}}\big(V_x(u)+j(b,0)\big)=\partial_{\text{left}}V_x\Big(u+j\frac{x}{a}\Big)$$

for $j=1,\dots,b^{-1}-1$. Hence repeating the argument above, we can associate to every $R\in\mathcal{T}_{a,b}$ in $H_x^{(0)}$ a tile from $\mathcal{T}_{x,y}$ that fully fall inside $V_x(k)$ for some $k\in\{0,1,\dots,(x/ab)-1\}$. We have defined $\mathcal{P}_{x,y}(R)$ for every $R\in\mathcal{T}_{a,b}$ with $R\subset H_x^{(0)}$.

Remark 4.3. Note that for $R(p_{i_u})\in\mathcal{T}_{a,b}^{(0)}(x)$ considered in Step 1 above, we have

$$\mathcal{P}_{x,y}(R(p_{i_u}))=\big\{Q\in\mathcal{T}_{x,y}\colon Q\subset V_x(u+m_0)\big\}$$

in fact. However, for $R\in\mathcal{T}_{a,b}$ with $R\subset H_x^{(\ell)}$, a tile $Q\in\mathcal{T}_{x,y}$ that intersecting $V_x^{(\ell)}$ partially will exists, where $V_x^{(\ell)}$ is the counterpart of V_x at $H_x^{(\ell)}$ (see (4.9) below), and hence there will be a missmatch between $\mathcal{P}_{x,y}(R)$ and $V_x^{(\ell)}$ as in Figure 2. (Essentially, this is caused by the miss-match of tiles: for instance, the tile $Q(q_2)$ in Figure 1(b) does not match $R(p_2)$ in Figure 1(a).) This justifies the definition of $\mathcal{P}_{x,y}(R)$ in (4.8).

Step 2, Recall that, each horizontal row of tiling $\mathcal{T}_{a,b}$ in $H_x^{(\ell)}=[0,1)\times[\ell x,(\ell+1)x)$, $\ell=1,\dots,x^{-1}-1$, is a horizontal translation of the tilling in $H_x^{(0)}$. To define $\mathcal{P}_{x,y}(R)$ for $R\in\mathcal{T}_{a,b}$ with $R\subset H_x^{(\ell)}$, consider

$$V_x^{(\ell)}(k)=\Big[k\frac{ab}{x},(k+1)\frac{ab}{x}\Big)\times[\ell x,(\ell+1)x)+(d_*(\ell x/a),0)\subset H_x^{(\ell)}\tag{4.9}$$

for $k=0,1,\dots,(x/(ab))-1$. Recall that d_* is the horizontal translation length for $\mathcal{T}_{a,b}$. In view of (4.2), note that each $R\in\mathcal{T}_{a,b}^{(\ell)}(x)$ can be given as

$$R=R_0+(d_*(\ell x/a),(\ell x/a)a)=R_0+(d_*(\ell x/a),\ell x)$$

for some $R_0\in\mathcal{T}_{a,b}^{(0)}(x)$. Hence, by the same argument for $R\in\mathcal{T}_{a,b}^{(0)}(x)$, one can associate to each $R\in\mathcal{T}_{a,b}^{(\ell)}(x)$ a collection $\mathcal{P}_{x,y}(R)$, the set defined as in (4.8) with replacing m_0 by $m_\ell=m_0+d_*(\ell x/a)$. Therefore, repeating the same construction as in Step 1, one can associate to each $R\in\mathcal{T}_{a,b}$ a collection $\mathcal{P}_{x,y}(R)$, and the properties (1) and (2) will follow. Since every $Q\in\mathcal{T}_{x,y}$ belongs to a (unique) family $\mathcal{P}_{x,y}(R)$ for some $R\in\mathcal{T}_{a,b}$, we have property (3). $\square$

5. Construction of a positive function

In this section, we construct a positive function that will play a crucial role in the proof of Proposition 3.1. To this end, we will make use of the pairing procedure developed in Section 4 for a sequence of van der Corput tilings, and define certain figures in Section 5.2. For the construction of figures, we use some ideas from [3, Chapter IV]. In Section 5.3, a key feature of such figures is summarized in Lemma 5.4. Next, in Section 5.4, we will define a positive function associated with each such a figure using the geometric structure, (5.6) below, and glue them to obtain an aimed function defined on $[0,1)^2$. We prove Lemma 5.5 summarizing key features of the function at the end of Section 5.4.

5.1. Notation and assumptions. Let $b_1 > b_2 > \cdots > b_{2n}$ with $b_i \in \mathcal{E}$, where $n > 5$ be a power of two, and $\lambda \in (0, 1/\{2n(n-1)\})$. (Recall that $\mathcal{E} = \{1/2^k \in [0,1) : k \in \mathbb{N}\}$.) Throughout this section, suppose that the sequence $\{b_k\}_{k=1}^{2n}$ decay fast enough so that the following three conditions are fulfilled:

$$b_n b_{n+1} > b_{n-1} b_{n+2} > \cdots > b_2 b_{2n-1} > b_1 b_{2n}, \tag{5.1}$$

$$\frac{b_{k+1}}{b_k} < \lambda \quad \text{for all } k \in \{n+1, \dots, 2n-1\}, \tag{5.2}$$

and

$$\frac{b_{k+1}}{b_k} < \lambda \quad \text{for all } k \in \{1, 2, \dots, n-1\}. \tag{5.3}$$

For each pair $(b_{n-k}, nb_{n+k+1}) \in \mathcal{E} \times \mathcal{E}$, consider the tiling $\mathcal{T}_{b_{n-k}, nb_{n+k+1}}$, where $k \in \{0, 1, \dots, n-1\}$. In view of (5.1) and (5.3), given a tile $R \in \mathcal{T}_{b_{n-k}, nb_{n+k+1}}$, one can associate to it a collection

$$\mathcal{P}_{b_{n-k-1}, nb_{n+k+2}}(R) \subset \mathcal{T}_{b_{n-k-1}, nb_{n+k+2}}$$

by Lemma 4.2. For the sake of simplicity, we may write

$$\mathcal{T}_k = \mathcal{T}_{b_{n-k}, nb_{n+k+1}},$$
$$\mathcal{P}_k(R) = \mathcal{P}_{b_{n-k}, nb_{n+k+1}}(R) \quad \text{for } R \in \mathcal{T}_{k-1} = \mathcal{T}_{b_{n-k-1}, nb_{n+k}}$$

for $k \in \{0, 1, \dots, n-1\}$. Note here that we use a convention

$$\mathcal{P}_0(R) = \{R\} \quad \text{for } R \in \mathcal{T}_{-1} = \mathcal{T}_0.$$

One has $\mathcal{P}_k(R) \subset \mathcal{T}_k$, where $R \in \mathcal{T}_{k-1}$ for $k \in \{0, 1, \dots, n-1\}$.

Given a rectangle $R = [x_1, x_2) \times [y_1, y_2)$, $n \in \mathbb{N}$, and

$$\omega \in \Big[0, x_2 - x_1 - \frac{x_2 - x_1}{n}\Big),$$

let

$$R^*(\omega) = \Big[x_1 + \omega, x_1 + \omega + \frac{x_2 - x_1}{n}\Big) \times [y_1, y_2). \tag{5.4}$$

Note that $R^*(\omega) \subset R$, with $\pi_x(R^*(\omega)) \subset \pi_x(R)$ and $\pi_y(R^*(\omega)) = \pi_y(R)$, and $\mu(R^*(\omega)) = (1/n)\mu(R)$.

5.2. A geometric construction. We now define chains of "admissible" tiles from the sequence $\{\mathcal{T}_k\}_{k=0}^{n-1}$, and then construct certain figures. First, consider a truncation of the form (5.4) for tiles from each $\mathcal{T}_k$. Then, for every $k \in \{0, 1, \dots, n-1\}$, one has

$$\mu\Big(\bigcup_{R \in \mathcal{T}_k} R^*(\omega_R)\Big) = \frac{1}{n}$$

for every $\omega_R \in [0, (n-1)b_{n+k+1})$. Note here that $\omega = \omega_R$ can vary with each $R \in \mathcal{T}_k$. We will fix certain choice of ω, with the aid of Lemma 4.2, in Lemma 5.2 below.

We can show the following lemma due to the big difference (5.2) between the sides of tiles in $\mathcal{T}_k$ for $k \in \{0, 1, \dots, n-1\}$, and the proof is independent of pairing procedure.

LEMMA 5.1. *Suppose* (5.2). *Then for arbitrary choices of translation vectors* $\{\omega_R\}$ *we have*

$$\mu\Big(\bigcup_{k=0}^{n-1} \bigcup_{R \in \mathcal{T}_k} R^*(\omega_R)\Big) \geq \frac{1}{2}.$$

PROOF. For each $k \in \{0, 1, \dots, n-1\}$, let

$$W_k = \bigcup_{R \in \mathcal{T}_k} R^*(\omega_R).$$

One has $\mu(W_k) = 1/n$ as mentioned above, and thus

$$\mu\Big([0,1)^2 \setminus \bigcup_{k=0}^{n-1} W_k\Big) = \int \prod_{k=0}^{n-1} (1 - \mathbb{1}_{W_k}(x,y))\, dxdy$$

$$\leq 1 - \sum_{k=0}^{n-1} \int \mathbb{1}_{W_k}\, dxdy + \sum_{0 \leq k < \ell < n} \int \mathbb{1}_{W_k} \mathbb{1}_{W_\ell}\, dxdy = \sum_{0 \leq k < \ell < n} \int \mathbb{1}_{W_k} \mathbb{1}_{W_\ell}\, dxdy.$$

For $k < \ell$, observe that for a given $R \in \mathcal{T}_k$, we have

$$\#\big\{A \in \mathcal{T}_\ell : A \cap R^*(\omega_R) \neq \emptyset\big\} \leq \frac{nb_{n+k+1}/n}{nb_{n+\ell+1}} + 2 = \frac{b_{n+k+1}}{nb_{n+\ell+1}} + 2$$

and

$$\mu\left(R^*(\omega_k) \cap A^*(\omega_A)\right) \leq b_{n-k}\frac{nb_{n+\ell+1}}{n} = b_{n-k}b_{n+\ell+1}.$$

Hence

$$\int \mathbb{1}_{W_k}\mathbb{1}_{W_\ell}\,dxdy \leq \sum_{R\in\mathcal{T}_k} \mu\Bigg(\bigsqcup_{\substack{A\in\mathcal{T}_\ell:\\ A\cap R^*(\omega_R)\neq\emptyset}} R^*(\omega_R)\cap A^*(\omega_A)\Bigg)$$

$$\leq \sum_{R\in\mathcal{T}_k}\Big(\frac{b_{n+k+1}}{nb_{n+\ell+1}} + 2\Big) \times b_{n-k}b_{n+\ell+1} = \sum_{R\in\mathcal{T}_k}\Big(\frac{b_{n-k}b_{n+k+1}}{n} + 2b_{n-k}b_{n+\ell+1}\Big)$$

$$\leq \frac{1}{nb_{n-k}b_{n+k+1}}\Big(\frac{b_{n-k}b_{n+k+1}}{n} + 2b_{n-k}b_{n+\ell+1}\Big) = \frac{1}{n^2} + \frac{2b_{n+\ell+1}}{nb_{n+k+1}}$$

as $\#\mathcal{T}_k = 1/(nb_{n-k}b_{n+k+1})$. In view of (5.2), it follows that

$$\mu\Bigg([0,1)^2 \setminus \bigcup_{k=0}^{n-1} W_k\Bigg) \leq \sum_{0\leq k<\ell<n}\int \mathbb{1}_{W_k}\mathbb{1}_{W_\ell}\,dxdy$$

$$\leq \frac{n(n-1)}{2}\Big(\frac{1}{n^2} + \frac{2b_{n+\ell+1}}{nb_{n+k+1}}\Big) \leq \frac{n(n-1)}{2}\Big(\frac{1}{n^2} + \frac{2}{n}\lambda\Big) < \frac{1}{2}$$

since $\lambda < \frac{1}{2n(n-1)}$. This implies the lemma. $\square$

Using Lemma 4.2, with the aid of (5.3), we next fix $\omega = \omega_R$ in (5.4) for each $R \in \mathcal{T}_k$, and construct chains of (truncated) tiles from $\{\mathcal{T}_k\}_{k=0}^{n-1}$. As was mentioned in Section 5.1, given a tile $R_0 \in \mathcal{T}_0$, one can associate to it a collection $\mathcal{P}_1(R_0) \subset \mathcal{T}_1$ by Lemma 4.2.

Lemma 5.2. *Suppose* (5.3). *Let* $R_0 \in \mathcal{T}_0$. *One can choose* $\omega = \omega_{R_0} \in [0,(n-1)b_{n+1})$ *so that every* $R_1 \in \mathcal{P}_1(R_0)$ *intersects with* $R_0^*(\omega)$ *in such a way that*

$$\pi_x(R_1) \subset \pi_x(R_0^*(\omega)), \quad \pi_y(R_1) \supset \pi_y(R_0^*(\omega)).$$

Proof. Write $[x, x+nb_{n+1}) \times [y, y+b_n)$ for $R_0 \in \mathcal{T}_0$. Taking $\omega = (n-1)b_{n+1}/2$ implies

$$x + \omega + \frac{b_{n+1}}{2} = x + \frac{nb_{n+1}}{2},$$

and hence $R_0^*(\omega) \subset R_0$ is placed at the middle of R_0. Since we have

$$x + \frac{nb_{n+1}}{2} \in \pi_x\Bigg(\bigcup_{Q \in \mathcal{P}_1(R_0)} Q \Bigg)$$

by Lemma 4.2(2), with the aid of (4.7), it follows that $\bigcup_{Q \in \mathcal{P}_1(R_0)} Q$ intersects with $R_0^*(\omega)$. To obtain the first half of the claim, it remains to compare the widths of $\bigcup_{Q \in \mathcal{P}_1(R_0)} Q$ and $R_0^*(\omega)$ as both are rectangles. By Lemma 4.2(2), one has

$$\mu\Bigg(\pi_x\Bigg(\bigcup_{Q \in \mathcal{P}_1(R_0)} Q \Bigg)\Bigg) = \frac{nb_n b_{n+1}}{b_{n-1}}.$$

Here, by (5.3) with $k = n-1$, we have

$$\frac{nb_n b_{n+1}}{b_{n-1}} < \lambda n b_{n+1} = \lambda n \times \mu\left(\pi_x(R_0^*(\omega))\right) < \frac{1}{2}\mu\left(\pi_x(R_0^*(\omega))\right)$$

as $\lambda < \frac{1}{2n(n-1)}$. Therefore, one has

$$\pi_x\Bigg(\bigcup_{Q \in \mathcal{P}_1(R_0)} Q \Bigg) \subset \pi_x(R_0^*(\omega)),$$

and which implies the first claim.

Since the truncation $R \mapsto R^*(\omega)$ preserves the height of R, one has

$$\pi_y\left(R_1\right) \supset \pi_y(R_0) = \pi_y(R_0^*(\omega))$$

by Lemma 4.2(2). The proof of Lemma 5.2 is obtained. $\square$

Using Lemma 4.2 recursively, given a tile $R_1 \in \mathcal{P}_1(R_0) \subset \mathcal{T}_1$, one can associate to it a collection $\mathcal{P}_2(R_1)$, and for every $R_2 \in \mathcal{P}_2(R_1) \subset \mathcal{T}_2$, one can associate to it $\mathcal{P}_3(R_2)$, and so forth. Here, by the same argument as in Lemma 5.2, for each $R_1 \in \mathcal{P}_1(R_0)$, one can take $\omega_{R_1} \in [0, (n-1)b_{n+2})$ so that every $R_2 \in \mathcal{P}_2\left(R_1\right)$ intersects with $R_1^*(\omega_{R_1})$ in such a way that

$$\pi_x(R_2) \subset \pi_x\left(R_1^*(\omega_{R_1})\right), \quad \pi_y(R_2) \supset \pi_y\left(R_1^*(\omega_{R_1})\right). \tag{5.5}$$

Note here that by Lemma 5.2 one has

$$\pi_x\left(R_1^*(\omega_{R_1})\right) \subset \pi_x\left(R_0^*(\omega_{R_0})\right)$$

as $\pi_x\left(R_1^*(\omega_{R_1})\right) \subset \pi_x\left(R_1\right)$, and

$$\pi_y\left(R_1^*(\omega_{R_1})\right) \supset \pi_y\left(R_0^*(\omega_{R_0})\right)$$

as $\pi_y\left(R_1^*(\omega_{R_1})\right) = \pi_y\left(R_1\right)$. By a recursive use of the argument above, for every chain of tiles $R_k \in \mathcal{P}_k\left(R_{k-1}\right)$, one can take $\omega_{R_k} \in [0,(n-1)b_{n+1+k})$ such that

(5.6)
$$\begin{cases} \pi_x(R^*_{n-1}(\omega_{R_{n-1}})) \subset \pi_x(R^*_{n-2}(\omega_{R_{n-2}})) \subset \cdots \subset \pi_x(R^*_1(\omega_{R_1})) \subset \pi_x\left(R^*_0(\omega_{R_0})\right), \\ \pi_y(R^*_{n-1}(\omega_{R_{n-1}})) \supset \pi_y(R^*_{n-2}(\omega_{R_{n-2}})) \supset \cdots \supset \pi_y(R^*_1(\omega_{R_1})) \supset \pi_y(R^*_0(\omega_{R_0})). \end{cases}$$

In particular (5.6) yields

$$R^*_{n-1}(\omega_{R_{n-1}}) \cap R^*_{n-2}(\omega_{R_{n-2}}) \cap \cdots \cap R^*_0(\omega_{R_0}) = R^*_{n-1}(\omega_{R_{n-1}}) \cap R^*_0(\omega_{R_0}). \tag{5.7}$$

We will call the sets of the form (5.7) *core* rectangles. One sees that each core rectangle has height b_n and width b_{2n}, where $b_n = \mu(\pi_y(R^*_0(\omega_{R_0})))$ and $b_{2n} = \mu(\pi_x(R^*_{n-1}(\omega_{R_{n-1}})))$. In Section 5.4, we will define a positive function such that its support is contained in these core rectangles determined by (5.7).

We proceed with the construction of chains of tiles and figures. Once the translation parameters ω_{R_k} are chosen so that (5.6) is fulfilled, they will remain unchanged in the sequel. Henceforth, the parameter ω_{R_k} will be omitted from $R^*_k(\omega_{R_k})$ and it is abbreviated as

$$R^*_k = R^*_k(\omega_{R_k})$$

for simplicity of notation.

Now, we define subsets of $[0,1)^2$ as follows. Given a tile $R_0 \in \mathcal{T}_0$, define

$$B_0(R_0) = R^*_0,$$

and for each $k \in \{1,\dots,n-1\}$, define $B_k(R_0) \subset [0,1)^2$ by

$$B_1(R_0) = \bigcup_{R_1 \in \mathcal{P}_1(R_0)} R^*_1, \quad B_2(R_0) = \bigcup_{R_1 \in \mathcal{P}_1(R_0)} \bigcup_{R_2 \in \mathcal{P}_2(R_1)} R^*_2\,,$$

$$\cdots$$

$$B_{n-1}(R_0) = \bigcup_{R_1 \in \mathcal{P}_1(R_0)} \bigcup_{R_2 \in \mathcal{P}_2(R_1)} \cdots \bigcup_{R_{n-2} \in \mathcal{P}_{n-2}(R_{n-3})} \bigcup_{R_{n-1} \in \mathcal{P}_{n-1}(R_{n-2})} R^*_{n-1}\,,$$

see Figure 3. Set

$$F(R_0) = B_0(R_0) \cup B_1(R_0) \cup B_2(R_0) \cup \cdots \cup B_{n-1}(R_0),$$

and define $\mathcal{A} = \left\{F(R_0) \subset [0,1)^2 \colon R_0 \in \mathcal{T}_0\right\}$. Note that we have

$$\#\mathcal{A} = \#\mathcal{T}_0 = \frac{1}{nb_nb_{n+1}} = \frac{1}{\mu(R_0)}, \tag{5.8}$$

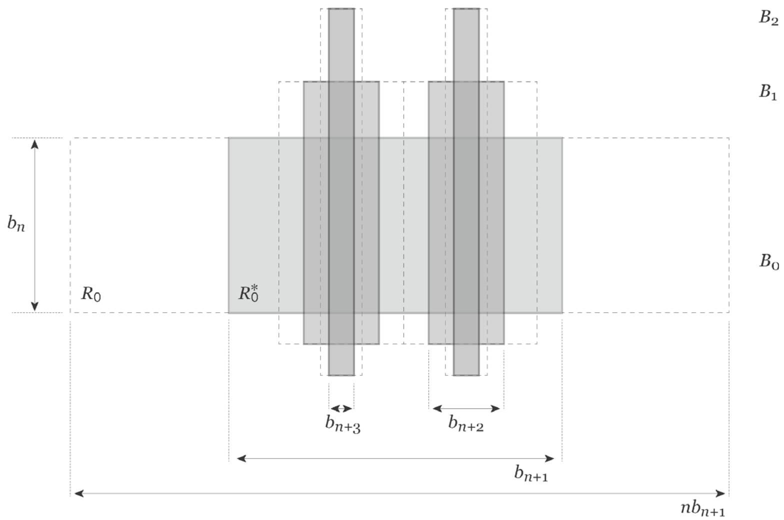


Figure 3: Examples of B_k for $k = 0, 1, 2$. The gray rectangles are R_0^*, R_1^*, and R_2^*. They are $1/n$'th part of the tiles (the rectangles with broken lines) R_0, R_1, and R_2, respectively. We see that B_1 consists of two R_1^*'s, and B_2 consists of two R_2^*'s in this figure, and hence there are two "core" rectangles determined by (5.7).

since each $F(R_0) \in \mathcal{A}$ is indexed by a unique $R_0 \in \mathcal{T}_0$ and every R_0 has the identical area. For simplicity of notation, we may write B_k and F instead of $B_k(R_0)$ and $F(R_0)$, respectively.

Remark 5.3. We can state Lemma 5.1 in terms of F as follows: one has

$$\mu\Big(\bigcup_{F\in\mathcal{A}} F\Big) \geq \frac{1}{2}. \tag{5.9}$$

Indeed, by Lemma 4.2(3), we see that every tile $R \in \mathcal{T}_k$, $k \in \{0, 1, \dots, n-1\}$, participates in the pairing procedure described above and each tile is paired with exactly one set $F = F(R_0) \in \mathcal{A}$. It follows that

$$\bigcup_{F\in\mathcal{A}} F = \bigcup_{k=0}^{n-1} \bigcup_{R\in\mathcal{T}_k} R^*,$$

and hence we have (5.9) by Lemma 5.1.

5.3. Small intersection property of figures. Let

$$q_k = \frac{\mu(R_{k-1}^*)}{\mu(R_k^*)} \tag{5.10}$$

for $k \in \{1, 2, \ldots, n-1\}$. One has

$$q_k = \frac{\mu(R_{k-1})}{\mu(R_k)} = \frac{b_{n-k+1}b_{n+k}}{b_{n-k}b_{n+k+1}}.$$

Note that $q_k > 1$ as $\mu(R_0^*) > \mu(R_1^*) > \cdots > \mu(R_{n-1}^*)$ by (5.1). Note also that

$$\text{(5.11)} \quad q_1 \cdots q_{k-1}q_k \times \mu(R_k^*) = q_1 \cdots q_{k-1} \times \mu(R_{k-1}^*) = \cdots = q_1 \times \mu(R_1^*) = \mu(R_0^*).$$

By Lemma 4.2(1), one sees that

$$\text{(5.12)} \qquad \#\mathcal{P}_k(R_{k-1}) = \#\mathcal{P}_k\left(R'_{k-1}\right) = \frac{b_{n-k+1}b_{n+k}}{b_{n-k}b_{n+k+1}} = q_k$$

for every $R_{k-1}, R'_{k-1} \in \mathcal{P}_{k-1}\left(R_{k-2}\right)$, $k = 2, \ldots, n-1$.

In the following lemma, we show that due to the big difference (5.3) between their sides, the collections $B_k(R_0)$, $k = 0, 1 \ldots, n-1$, have a very small intersection with each other. Hence, the total area of the figure $F(R_0)$ is almost the same as the sum of individual sets $B_k(R_0)$.

Lemma 5.4. *Assume* (5.3). *Let $R_0 \in \mathcal{T}_0$. Then we have*

$$1 - \lambda < \frac{\mu(F(R_0))}{\sum_{k=0}^{n-1} \mu(B_k(R_0))} \leq 1,$$

that is, the area of each $F \in \mathcal{A}$ is close to the sum of its components.

Proof. By definition and Lemma 4.2(1), with the aid of (5.11) and (5.12), we have

$$\mu(B_k(R_0)) = q_1q_2 \cdots q_k \times \mu(R_k^*) = \mu(R_0^*) = b_n b_{n+1}$$

for every $k \in \{0, 1, \ldots, n-1\}$. It follows that

$$\text{(5.13)} \qquad \sum_{k=0}^{n-1} \mu(B_k(R_0)) = nb_nb_{n+1} = \mu(R_0) = n\mu(R_0^*).$$

By the construction of $F = F(R_0)$, one has

$$\begin{aligned}\mu(F) &= \mu(B_0) + \mu(B_1 \setminus B_0) + \mu(B_2 \setminus B_1) + \cdots + \mu(B_{n-1} \setminus B_{n-2}) \\ &= \mu(R_0^*) + (b_{n-1} - b_n)q_1b_{n+2} + (b_{n-2} - b_{n-1})q_1q_2b_{n+3} + \cdots \\ &\qquad + (b_1 - b_2)q_1q_2 \ldots q_{n-1}b_{2n},\end{aligned}$$

and thus

$$\frac{\mu(F)}{\sum_{k=0}^{n-1} \mu(B_k(R_0))} = \frac{\mu(F)}{n\mu(R_0^*)}$$

$$= \frac{1}{n}\Big\{1 + \frac{(b_{n-1} - b_n)q_1 b_{n+2}}{\mu(R_0^*)} + \frac{(b_{n-2} - b_{n-1})q_1 q_2 b_{n+3}}{\mu(R_0^*)} + \cdots$$
$$+ \frac{(b_1 - b_2)q_1 q_2 \cdots q_{n-1} b_{2n}}{\mu(R_0^*)}\Big\}.$$

Here, for every $k \in \{1, \dots, n-1\}$, it follows from (5.11) and (5.3) that

$$\frac{(b_{n-k} - b_{n-k+1})q_1 \cdots q_k b_{n+k+1}}{\mu(R_0^*)} = \frac{(b_{n-k} - b_{n-k+1})q_1 \cdots q_k b_{n+k+1}}{q_1 \cdots q_k \times \mu(R_k^*)}$$
$$= \frac{b_{n-k} b_{n+k+1}}{\mu(R_k^*)} - \frac{b_{n-k+1} b_{n+k+1}}{\mu(R_k^*)} = 1 - \frac{b_{n-k+1}}{b_{n-k}} > 1 - \lambda.$$

Hence

$$\frac{\mu(F)}{\sum_{k=0}^{n-1} \mu(B_k(R_0))} = \frac{\mu(F)}{n\mu(R_0^*)} > \frac{n - (n-1)\lambda}{n} > 1 - \lambda.$$

The opposite estimate $\mu(F) \le \sum_{k=0}^{n-1} \mu(B_k(R_0))$ is clear. The proof of Lemma 5.4 is obtained. $\square$

5.4. Geometric construction of a function. As we have seen in Section 5.2, each $F(R_0) \in \mathcal{A}$, $R_0 \in \mathcal{T}_0$, contains core rectangles determined by the property (5.7). In view of (5.12), one sees that for each $F(R_0)$ there are $q_1 q_2 \cdots q_{n-1}$ many such rectangles. To obtain an aimed function associated with $F(R_0) \in \mathcal{A}$, for each $R_0 \in \mathcal{T}_0$, we will place a mass at each core rectangle in such a way that they are distributed uniformly along the vertical direction. More specifically, we do as follows.

Let $R_0 \in \mathcal{T}_0$, and let $c_n = q_1 q_2 \cdots q_{n-1}$. See (5.10) for q_k. One has

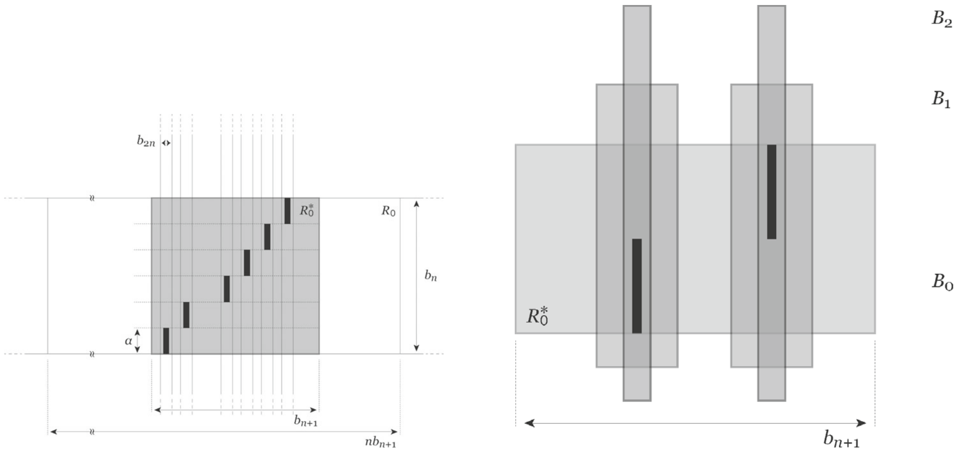
$$c_n = \frac{\mu(R_0^*)}{\mu(R_1^*)} \frac{\mu(R_1^*)}{\mu(R_2^*)} \cdots \frac{\mu(R_{n-2}^*)}{\mu(R_{n-1}^*)} = \frac{\mu(R_0^*)}{\mu(R_{n-1}^*)} = \frac{b_n b_{n+1}}{b_1 b_{2n}}. \tag{5.14}$$

As was mentioned above c_n is the number of core rectangles contained in $F(R_0)$. In fact, due to (5.7), such rectangles are contained in R_0, and thus c_n is the total number of truncated rectangles R_{n-1}^* that intersect with R_0^*. We enumerate core rectangles as

$$\big\{\gamma_j^* \subset R_0^* \colon j = 0, 1, \dots, c_n - 1\big\}.$$

For $\rho \in (0, b_{2n})$, define the set s_ρ as the rectangle with side length ρ (width) and $\alpha = b_1 b_{2n}/b_{n+1}$ (height). For each γ_j^*, we place a single s_ρ inside it in such way that

$$\pi_x(s_\rho) \subset \pi_x(\gamma_j^*), \quad \pi_y(s_\rho) = \pi_y\bigg(\Big[j\frac{b_n}{c_n}, (j+1)\frac{b_n}{c_n}\Big)\bigg). \tag{5.15}$$



(a) *The gray rectangle is* R_0^*. *The vertical, long rectangles are* R_{n-1}^**: each one has sides* b_1 *(height) and* b_{2n} *(width). Each rectangle* s_ρ *has height* $\alpha = b_1 b_{2n}/b_{n+1}$ *and width* $\rho(< b_{2n})$.

(b) *Example with* 3 *generations*

Figure 4: Placement of s_ρ*'s (black rectangles)*

The exact x coordinate of s_ρ inside γ_j^* is not important. Since

$$\frac{b_n}{c_n} = \frac{b_1 b_{2n}}{b_{n+1}} = \alpha$$

by (5.14), one can achieve (5.15), and hence the rectangles are distributed uniformly along vertical direction such that

$$\mu\left(\pi_y\left(\bigcup_{j=1}^{c_n-1} s_\rho\right)\right) = b_n = \mu(\pi_y(R_0)).$$

See Figure 4. Now, for each $R_0 \in \mathcal{T}_0$ and the associated $F(R_0) \in \mathcal{A}$, we define a positive function $h = h_{R_0}$ by

$$h_{R_0}(z) = \begin{cases} \frac{nb_n b_{n+1}}{\mu(s_\rho)c_n}, & \text{if } z \in s_\rho \subset \gamma_j^* \text{ for some } \gamma_j^*, \\ 0, & \text{otherwise,} \end{cases} \tag{5.16}$$

where $\mu(s_\rho) = \rho\alpha$, the area of s_ρ. Hence one has $\operatorname{supp} h \subset R_0^* \subset R_0$. The purpose of distributing the support of h_{R_0} like this will be clear in Lemma 5.5(3) below.

Since each $F(R_0) \in \mathcal{A}$ is indexed by $R_0 \in \mathcal{T}_0$, the definition of function h_{R_0} in (5.16) can be extended to a positive function f on the unit square as follows:

$$f(z) = \sum_{R_0 \in \mathcal{T}_0} h_{R_0}(z). \tag{5.17}$$

LEMMA 5.5. *Let f be defined as in* (5.17).

(1) *We have $f \geq 0$ and $\|f\|_{L^1} = 1$.*

(2) *For every $R_0 \in \mathcal{T}_0$, $k \in \{0, 1, \ldots, n-1\}$ and $R_k \in \mathcal{P}_k\,(R_{k-1})$, with a convention that $R = R_0$ for $k = 0$, we have*

$$\frac{1}{\mu(R_k^*)} \int_{R_k^*} f \, dxdy = n \quad \textit{and} \quad \frac{1}{\mu(R_k)} \int_{R_k} f \, dxdy = 1.$$

(3) *For every $R_0 \in \mathcal{T}_0$ and a dyadic rectangle $A \subset R_0$ with height x and width y, so that $x \leq b_n$, $y = nb_{n+1}$ we have that*

$$\frac{1}{\mu(A)} \int_A f \, dxdy = 1.$$

PROOF. By (5.17), we have

$$\|f\|_{L^1} = \int f \, dxdy = \sum_{R_0 \in \mathcal{T}_0} \int h_{R_0} \, dxdy.$$

Here, it follows from the construction of $h = h_{R_0}$ that

$$\int h_{R_0} \, dxdy = \int_{R_0} h_{R_0} \, dxdy = \int_{R_0^*} h_{R_0} \, dxdy$$
$$= \frac{nb_n b_{n+1}}{\mu(s_\rho) c_n} \times \mu(\operatorname{supp} h \cap R_0^*) = \frac{nb_n b_{n+1}}{\mu(s_\rho) c_n} \times \mu(s_\rho) c_n = nb_n b_{n+1} = \mu(R_0).$$

Thus

$$\|f\|_{L^1} = \sum_{R_0 \in \mathcal{T}_0} \mu(R_0) = 1.$$

Next, we show (2). By construction, one has

$$\int_{R_k^*} f \, dxdy = \int_{R_k^*} h \, dxdy = \frac{nb_n b_{n+1}}{\mu(s_\rho) c_n} \times \mu(\operatorname{supp} h \cap R_k^*)$$
$$= \frac{nb_n b_{n+1}}{\mu(s_\rho) q_1 \cdots q_{n-1}} \times \mu(s_\rho) q_{k+1} \cdots q_{n-1} = \frac{nb_n b_{n+1}}{q_1 \cdots q_k}.$$

Thus we have

$$\frac{1}{\mu(R_k^*)}\int_{R_k^*} f\,dxdy = \frac{1}{\mu(R_k^*)}\frac{nb_nb_{n+1}}{q_1\cdots q_k} = \frac{nb_nb_{n+1}}{b_nb_{n+1}} = n$$

since $q_1\cdots q_{k-1}q_k \times \mu(R_k^*) = \mu(R_0^*) = b_nb_{n+1}$ from (5.11).

For the latter half of (2), we have

$$\int_{R_k} f\,dxdy = \int_{R_k} h\,dxdy = \int_{R_k^*} h\,dxdy = \int_{R_k^*} f\,dxdy$$

since $\operatorname{supp} h$ is contained in the core rectangles $s_\rho \subset R_{n-1}^* \cap R_0^* \subset R_k^*$ by (5.15) and (5.7). It then follows from the first assertion that

$$\int_{R_k} f\,dxdy = \int_{R_k^*} f\,dxdy = n\mu(R_k^*) = \mu(R_k).$$

Now, we show (3). Since the support of f is (or the rectangles s_ρ) distributed uniformly along the vertical direction, it follows that

$$\int_A f\,dxdy = \frac{x}{b_n}\int_{R_0} f\,dxdy.$$

Thus, in view of (2) of this lemma, one has

$$\frac{1}{\mu(A)}\int_A f\,dxdy = \frac{1}{\mu(A)}\frac{x}{b_n}\int_{R_0} f\,dxdy$$

$$= \frac{1}{\mu(A)}\frac{x}{b_n}\mu(R_0) = \frac{xnb_nb_{n+1}}{xyb_n} = \frac{nb_{n+1}}{y} = 1.$$

The proof of Lemma 5.5 is obtained. □

Remark. Note that for every $F(R_0) \in \mathcal{A}$ one has

$$\int_{F(R_0)} h_{R_0}\,dxdy = \int_{R_0} h_{R_0}\,dxdy.$$

Hence it follows that

$$\frac{1}{\mu(F(R_0))}\int_{F(R_0)} h_{R_0}\,dxdy = \frac{1}{\mu(F(R_0))}\int_{R_0} h_{R_0}\,dxdy = \frac{\mu(R_0)}{\mu(F(R_0))}.$$

Here, in view of Lemma 5.4 with the aid of (5.13), we have

$$\mu(R_0) \geq \mu(F(R_0)) \geq (1-\lambda)\mu(R_0).$$

Thus we obtain

$$1 \leq \frac{1}{\mu(F(R_0))}\int_{F(R_0)} h_{R_0}\,dxdy \leq \frac{1}{1-\lambda}.$$

6. Proof of Proposition 3.1

In this section, we conclude the proof of Proposition 3.1. Recall that we are given two infinite disjoint subsets $C, D \subset \mathbb{N}$, then $\mathcal{F}_C$ and $\mathcal{F}_D$ be the corresponding bases as in (1.1). Proposition 3.1 is deduced from the following result:

Proposition 6.1. *Assume* (1.2). *For every* $\varepsilon > 0$ *and every* $n > 5$ *which is a power of two, there exists a function* $f \in L^\infty([0,1)^2)$, *with* $f \geq 0$ *and* $\|f\|_{L^1} = 1$, *for which there exist two disjoint subsets* $E, N \subset [0,1)^2$, *with* $\mu(E) \geq 1/3$ *and* $\mu(N) \leq \varepsilon$, *such that for every* $z \in E$ *there exists a dyadic rectangle* R *from* $\mathcal{F}_D$, *with* $z \in R$, *such that*

$$\frac{1}{\mu(R)} \int_R f \, dxdy \geq \frac{n}{2}, \tag{6.1}$$

and for every $z \in [0,1)^2 \setminus N$ *and any dyadic rectangle* $R \in \mathcal{F}_C$, *with* $z \in R$, *we have*

$$\frac{1}{\mu(R)} \int_R f \, dxdy \leq 3. \tag{6.2}$$

In Section 6.2, we will prove a rectangle removal lemma (Lemma 6.2) to determine the set $N \subset [0,1)^2$ in Proposition 6.1, and then prove Proposition 6.1 in Section 6.3. In Section 6.4, we prove Proposition 3.1.

6.1. Notation and assumptions. We will continue to use some notation from Section 5.

Define the collections

$$\mathcal{C} = \Big\{ \frac{1}{2^k} \in [0,1] : k \in C \Big\} \quad \text{and} \quad \mathcal{D} = \Big\{ \frac{1}{2^s} \in [0,1] : s \in D \Big\}.$$

Note that $\mathcal{C}$ is the set of lengths of dyadic rectangles $R \in \mathcal{F}_C$, and the same is true for $\mathcal{D}$. Below, we write $\mathcal{D} = \{b_n\}_{n \in \mathbb{N}}$.

For each $b_n \in \mathcal{D}$, we define

$$\overline{a}_n = \sup \big\{ a \in \mathcal{C} : a < b_n \big\}, \quad \text{and} \quad \underline{a}_n = \inf \big\{ a \in \mathcal{C} : a > b_n \big\}.$$

Then note that condition (1.2) is equivalent to the following condition

$$\liminf_{n \to \infty} \left(\max \Big\{ \frac{\bar{a}_n}{b_n}, \frac{b_n}{\underline{a}_n} \Big\} \right) = 0. \tag{6.3}$$

In view of (6.3), we can obtain the following assertions. Let $n > 5$ be a power of two, and $\lambda \in (0, 1/\{2n(n-1)\})$. Given $b_i \in \mathcal{D}$, $i = 1, \ldots, n$, with (5.3), we can choose $b_i \in \mathcal{D}$, $i = n+1, \dots, 2n$, with (5.1) and (5.2), such that for every $k = 0, 1, \dots, n-1$

$$\frac{\overline{a}_{n+1}}{b_{n+1}} \le \lambda b_n \quad \text{and} \quad \frac{b_{n+k}}{\underline{a}_{n+k}} \le \frac{1}{n}, \tag{6.4}$$

$$\frac{\overline{a}_{n+k+1}}{b_{n+k+1}} \le \lambda \frac{b_{n-k}}{\overline{a}_{n-k-1}} \quad \text{and} \quad \frac{b_{n+k+1}}{\underline{a}_{n+k+1}} \le \lambda \frac{\overline{a}_{n-k}}{b_{n-k}}, \tag{6.5}$$

where $\overline{a}_0 = 1$ as a convention.

6.2. Removal lemma. We denote the axis-parallel *dyadic* rectangles with side lengths x and y by $A_{x,y}$, where x is the length of the vertical side and y is that of the horizontal one. Let

$$\mathcal{W}_C = \big\{ A_{x,y} \in \mathcal{F}_C : A_{x,y} \cap \operatorname{supp} f \neq \emptyset \big\},$$

where f is the positive function defined in (5.17).

Lemma 6.2. *Assume we have* (6.3). *Let f be a positive function defined as in* (5.17). *Given $\varepsilon \in (0,1)$, one can choose a sufficiently small $\lambda \in (0,1)$ and a set $N \subset [0,1)^2$ in such a way that the following properties hold*:

(1) *For every $z \in [0,1)^2 \setminus N$ and $A \in \mathcal{W}_C$ with $A \ni z$ one has*

$$\frac{1}{\mu(A)} \int_A f \, dxdy \le 3.$$

(2) $\mu(N) < \varepsilon$.

Proof. We divide our argument into the following two cases with respect to the range of x, the height of $A = A_{x,y}$: 1) $x \in (b_n, 1]$ and 2) $x \le b_n$. Depending on the case we will define sets denoted by $N_1, N_2 \subset [0,1)^2$. For these sets, the integral averages are expected to be large so they will constitute N and will need to be removed.

Case 1: $x \in (b_n, 1]$. Then there is $k \in \{1, 2, \dots, n-1, n\}$ such that $x \in (b_{n-k+1}, b_{n-k}]$, with a convention $b_0 = 1$.

1-i): Assume $y > n b_{n+k}$. By Lemma 5.5(2), for each rectangle $A = A_{x,y}$, one has

$$\int_A f(z)\, dz \le \sum_{\substack{R \in \mathcal{T}_{k-1}: \\ R \cap A \neq \emptyset}} \int_R f(z)\, dz \le \sum_{\substack{R \in \mathcal{T}_{k-1}: \\ R \cap A \neq \emptyset}} \mu(R).$$

Here observe that

$$\#\big\{ R \in \mathcal{T}_{k-1} \colon R \cap A \neq \emptyset \big\} \le 3 \, \frac{\mu(A)}{\mu(R)}. \tag{6.6}$$

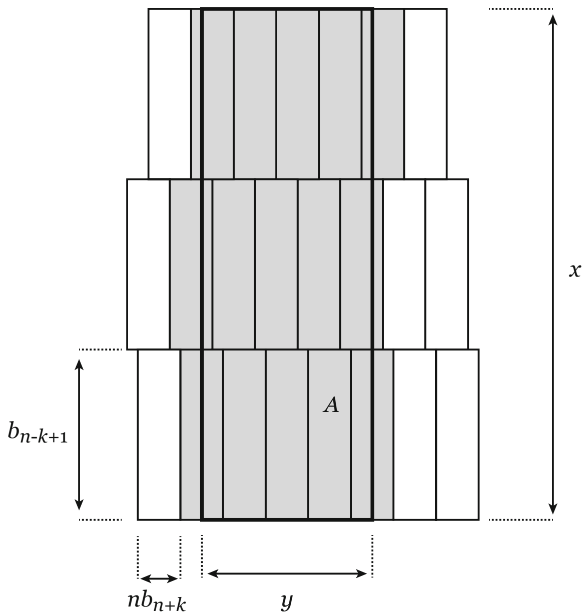


Figure 5: Intuition behind formula (6.6): one needs to estimate the number of gray rectangles that intersect $A = A_{x,y}$ partially. Since $b_{n-k+1} < x$ and $nb_{n+k} < y$ the total measure of such rectangles is small.

Indeed, since $\pi_y(A)$ is dyadic, one has

$$\#\{R \in \mathcal{T}_{k-1} \colon R \cap A \neq \emptyset\} \leq \frac{x}{b_{n-k+1}}\Big(\frac{y}{nb_{n+k}} + 2\Big)$$

$$= \frac{xy}{nb_{n-k+1}b_{n+k}} + 2\frac{x}{b_{n-k+1}} < \frac{xy}{nb_{n-k}b_{n+k+1}} + 2\,\frac{x}{b_{n-k+1}}\,\frac{y}{nb_{n+k}} = 3\,\frac{\mu(A)}{\mu(R)}$$

with the aid of (5.1). (See Figure 5.) Thus, it follows that

$$\frac{1}{\mu(A)}\int_A f(z)\,dz \leq \frac{1}{\mu(A)} \times 3\frac{\mu(A)}{\mu(R)} \times \mu(R) = 3.$$

1-ii) Assume $y \leq nb_{n+k}$. Then, by the second inequality of (6.4), we cannot have $b_{n+k} < y \leq nb_{n+k}$ (recall that $y \in \mathcal{C}$). Hence, we can assume that $y < b_{n+k}$: For $k \in \{1, 2, \ldots, n\}$, let $\mathcal{N}_{1,k} = \{A_{x,y} \in \mathcal{W}_C \colon x \in (b_{n-k+1}, b_{n-k}],\ y \in (0, b_{n+k}]\}$. Define

$$N_{1,k} = \bigcup_{A \in \mathcal{N}_{1,k}} A \quad \text{and} \quad N_1 = \bigcup_{k=1}^{n} N_{1,k}$$

(recall that we let $b_0 = 1$).

Case 2): $x \leq b_n$.

2-i): $y > nb_{n+1}$. For each rectangle $A = A_{x,y}$ due to the dyadicity of all rectangles involved

$$\frac{1}{\mu(A)} \int_A f(z)\,dz = \frac{1}{\mu(A)} \sum_{\substack{R_0 \in \mathcal{T}_0; \\ R_0 \cap A \neq \emptyset}} \int_{A \cap R_0} f(z)\,dz = \frac{1}{\mu(A)} \frac{y}{nb_{n+1}} \int_{R_0 \cap A} f(z)\,dz$$

as $\#\{R_0 \in \mathcal{T}_0 : R_0 \cap A \neq \emptyset\} = y/(nb_{n+1})$. For $R_0 \cap A (\neq \emptyset)$, by Lemma 5.5(3) one has

$$\int_{R_0 \cap A} f(z)\,dz = \mu(R_0 \cap A) = \frac{x}{b_n} \mu(R_0)$$

since the support of f (or the rectangles s_ρ) is distributed uniformly along the vertical direction. It follows that

$$\frac{1}{\mu(A)} \int_A f(z)\,dz = \frac{1}{\mu(A)} \frac{y}{nb_{n+1}} \frac{x}{b_n} \mu(R_0) = \frac{\mu(R_0)}{nb_n b_{n+1}} = \frac{nb_n b_{n+1}}{nb_n b_{n+1}} = 1.$$

2-ii): $y \leq nb_{n+1}$. Similar to above, by the second inequality of (6.4), we can assume that $y < b_{n+1}$. Define

$$N_2 = \bigcup_{\substack{A_{x,y} \in \mathcal{W}_C : \\ x \in (0,b_n],\ y \in (0,b_{n+1}]}} A_{x,y}.$$

Note in fact that we have $N_2 \subset N_1$. Indeed, let $x \in (0, b_n]$ $y \in (0, b_{n+1}]$. Then there is $k \in \{1, \dots, n-1\}$ such that $y \in (b_{n+k+1}, b_{n+k}]$, with the convention that $b_{2n+1} = 0$. For each $y \in (b_{n+k+1}, b_{n+k}]$, one has $A_{x,y} \subset A_{\tilde{x},y}$ for any $\tilde{x} = \tilde{x}_k \in (b_{n-k+1}, b_{n-k}]$ since $x \leq b_n \leq b_{n-k+1} < \tilde{x}_k$. Since $A_{\tilde{x}_k,y}$ belongs to $N_{1,k}$ defined in case 1-ii), one has $N_2 \subset N_1$.

It follows that letting

$$N = N_1 \cup N_2$$

will imply the first assertion. Note that $N = N_1$ since $N_2 \subset N_1$ as observed.

Next, we will show (2) which claims that the Lebesgue measure of $N = N_1$ can be made arbitrarily small. To see it, given $F = F(R_0) \in \mathcal{A}$, set $\mathcal{N} = \bigcup_{k=1}^n \mathcal{N}_{1,k}$, and $N(F) = \bigcup_{\{A \in \mathcal{N} : A \cap F \neq \emptyset\}} A$. Then note that $N = \bigcup_{F \in \mathcal{A}} N(F)$. In what follows, we will show $\mu(N(F)) < \varepsilon \mu(R_0)$. Once it is shown, we have (2). Indeed, with the aid of (5.8), one has

$$\mu(N) \leq \sum_{F = F(R_0) \in \mathcal{A}} \mu(N(F)) < \varepsilon \sum_{F = F(R_0) \in \mathcal{A}} \mu(R_0) = \varepsilon \times \frac{\mu(R_0)}{\mu(R_0)} = \varepsilon.$$

Below, for $k \in \{1, \dots, n\}$, we let

$$N_{1,k}(F) = \bigcup_{\{A \in \mathcal{N}_{1,k} : A \cap F \neq \emptyset\}} A.$$

Thus $N(F) = \bigcup_{k=1}^{n} N_{1,k}(F)$ since $N = N_1$. For each $k \in \{1, \dots, n\}$, the rectangle $A = A_{x,y}$, with $x \in (b_{n-k+1}, b_{n-k}]$, $y \in (0, b_{n+k}]$ satisfies

$$\mu(A_{x,y}) = xy \leq \overline{a}_{n-k}\overline{a}_{n+k} \leq \lambda b_{n-k+1} b_{n+k} \tag{6.7}$$

by the first inequality of (6.5). Note here that $b_{n-k+1}b_{n+k}$ is the area of R^*_{k-1}. Hence, repeating the same argument as in the proof of Lemma 5.4, by replacing B_{k-1} with $N_{1,k}(F)$, we will get

$$\mu(N(F)) = \mu\Big(\bigcup_{k=1}^{n} N_{1,k}(F)\Big) \leq \sum_{k=1}^{n} \mu(N_{1,k}(F))$$

with the aid of (6.4) and (6.5). Here for each $k \in \{1, \dots, n\}$ one has

$$\mu(N_{1,k}(F)) \leq q_1 \cdots q_{k-1} \overline{a}_{n-k} \overline{a}_{n+k} \leq \lambda q_1 \cdots q_{k-1} b_{n-k+1} b_{n+k} = \lambda b_n b_{n+1}$$

by (6.7) and (5.11). It follows that

$$\mu(N(F)) \leq \sum_{k=1}^{n} \mu(N_{1,k}(F)) \leq \lambda \sum_{k=1}^{n} b_n b_{n+1} = \lambda n b_n b_{n+1} = \lambda \mu(R_0).$$

The proof of Lemma 6.2 is obtained. $\square$

6.3. Proof of Proposition 6.1. Let f be the positive function defined in (5.17). Hence we have $\|f\|_{L^1} = 1$ by Lemma 5.5(1).

By construction and Lemma 5.5(2) it follows that for every R^* we have

$$\frac{1}{\mu(R^*)} \int_{R^*} f \, dxdy = n.$$

(Recall here that R^* is an abbreviation of $R^*(\omega_R)$ for $R \in \mathcal{T}_k$.) Note that $\pi_y(R^*)$ is dyadic. However, the rectangle R^* is not dyadic since neither is $\pi_x(R^*)$ in general. In order to fix this issue we now consider two dyadic, adjacent rectangles that are horizontal translations of R^* and that cover R^* (see Figure 6).

Denote the dyadic rectangles by R^*_{dl} and R^*_{dr}. Note that $R^*_{\mathrm{dl}}, R^*_{\mathrm{dr}} \in \mathcal{F}_D$. Note also that $\mu(R^*_{\mathrm{dl}}) = \mu(R^*_{\mathrm{dr}}) = \mu(R^*)$ and thus

$$\frac{1}{\mu(R^*)} \int_{R^*} f \, dxdy \leq \frac{1}{\mu(R^*_{\mathrm{dl}})} \int_{R^*_{\mathrm{dl}}} f \, dxdy + \frac{1}{\mu(R^*_{\mathrm{dr}})} \int_{R^*_{\mathrm{dr}}} f \, dxdy.$$

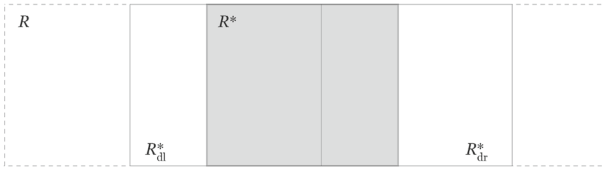


Figure 6: The gray rectangle in the middle is R^* while the other two are its dyadic translations R^*_{dl} and R^*_{dr}. They are inside of R, the dashed rectangle.

Hence for at least one rectangle $K(R^*) \in \{R^*_{\mathrm{dl}}, R^*_{\mathrm{dr}}\}$ we will have

$$\frac{1}{\mu(K(R^*))}\int_{K(R^*)} f\,dxdy \geq \frac{n}{2} \tag{6.8}$$

by Lemma 5.5(2). Here note that $K(R^*) \in \mathcal{F}_D$ as mentioned above. According to the definition, with the aid of Lemma 4.2(2), we see that both R^*_{dl} and R^*_{dr} are inside of R, and thus $K(R^*) \subset R$.

Now, for each R^* we consider the union of all such rectangles $K(R^*)$ for which (6.8) holds. Specifically, let

$$E' = \bigcup_{k=0}^{n-1} \bigcup_{R\in\mathcal{T}_k} K(R^*).$$

Then, due to Lemma 5.1 (or Remark 5.3), we have $\mu(E') \geq 1/2$. Define

$$E = E' \setminus N,$$

where $N \subset [0,1)^2$ is defined in Lemma 6.2. Since $\mu(N) < \varepsilon$ by Lemma 6.2(2), one has $\mu(E) \geq 1/3$. We also automatically have the first assertion for every point in E by (6.8).

Next, we will prove the second statement. Since $E \cap N = \emptyset$ and N is defined through the cases 1-ii) and 2-ii) in the proof of Lemma 6.2, it is enough to show the second assertion for $R \in \mathcal{F}_C$ belonging to the cases 1-i) and 2-i) there. Note here that the estimate (6.6) is verified for every $R \in \mathcal{F}_C$ with sides considered in the other cases 1-i) and 2-i). Hence one can obtain the desired estimate

$$\frac{1}{\mu(R)}\int_R f\,dxdy \leq 3$$

for every $z \in [0,1)^2 \setminus N$ and every $R \in \mathcal{F}_C$ with $R \ni z$. Proposition 6.1 is proven. □

Remark 6.3. We remark that instead of the unit square, we could do the same constructions inside any dyadic square. For our purposes in Proposition 3.1 it will be more convenient to consider a partition of the unit square into smaller, dyadic squares and carry out the same constructions inside each tiny square. Then, inside each square Q we will find the corresponding sets E, N satisfying the bounds $\mu(N) \le \mu(Q)\varepsilon$ and $\mu(E) \ge (1/3)\mu(Q)$. The estimates (6.1) and (6.2) will hold for rectangles that are strictly inside the partition squares. So to prove the analog of Proposition 6.1 in this case, it will remain to take care of rectangles from $\mathcal{F}_C$ that is not entirely contained inside a partition square. For this, we can write $R = \bigcup_{j=1}^{J} R_j$, where R_j entirely belongs to a partition square. Note that since all the rectangles are dyadic then the rectangles R_j will have identical sizes. Then the property (6.2) can be achieved as follows:

$$\frac{1}{\mu(R)} \int_R f\, dxdy = \frac{1}{N} \sum_{j=1}^{J} \frac{1}{\mu(R_j)} \int_{R_j} f\, dxdy \le C_0.$$

6.4. Proof of Proposition 3.1. For every $k \in \mathbb{N}$, let $\varepsilon_k = \varepsilon/2^{k+1}$. Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive integers. For each $k \in \mathbb{N}$, consider a partition $\mathcal{P}_k$ of the unit square into dyadic squares $Q_{k,j}$ of size $1/2^{n_k}$. Inside each square $Q_{k,j}$, we repeat the same procedure as in Proposition 6.1 as described in Remark 6.3. Hence we will find positive functions $f_{k,j} \in L^\infty$ with $\|f_{k,j}\|_{L^1} = \mu(Q_{k,j})$, and $E_{k,j}, N_{k,j} \subset Q_{k,j}$ with $\mu(E_{k,j}) \ge (1/3)\mu(Q_{k,j})$ and $\mu(N_{k,j}) \le \varepsilon_k \mu(Q_{k,j})$ such that (6.1) and (6.2) holds with $n \ge 2\max\{k^3, M^3\}$ which is a power of two. Namely, for every $z \in E_{k,j}$ there exists a dyadic rectangle $R \in \mathcal{F}_D$ with $R \ni z$ such that

$$\frac{1}{\mu(R)} \int_R f_{k,j}\, dxdy \ge \frac{n}{2} \ge \max\{k^3, M^3\}, \tag{6.9}$$

and for every $z \in Q_{k,j} \setminus N_{k,j}$ and any dyadic rectangle $R \in \{R \in \mathcal{F}_C : R \subset Q_{k,j}\}$ with $R \ni z$, one has

$$\frac{1}{\mu(R)} \int_R f_{k,j}\, dxdy \le 3. \tag{6.10}$$

Define

$$E_k = \bigsqcup_{j=1}^{2^{2n_k}} E_{k,j}, \quad N_k = \bigsqcup_{j=1}^{2^{2n_k}} N_{k,j}, \quad \text{and} \quad f_k = \sum_{j=1}^{2^{2n_k}} f_{k,j}.$$

Then for each $k \in \mathbb{N}$ one has $\mu(E_k) \ge 1/3$, $\mu(N_k) \le \varepsilon_k$ and $\|f_k\|_{L^1} \le 1$.

Now, given $L \in \mathbb{N}$ to be determined later, define

$$E = \left(\bigcup_{k=1}^{L} E_k\right) \setminus \left(\bigcup_{k=1}^{L} N_k\right), \quad \text{and} \quad f = \sum_{k=1}^{L} \frac{1}{k^2} f_k.$$

Then one has

$$\|f\|_{L^1} \leq \sum_{k=1}^{L} \frac{\|f_k\|_{L^1}}{k^2} \leq \sum_{k=1}^{L} \frac{1}{k^2} \leq C$$

for some $C \in (0,2)$, and for every $z \in E$ there exists a dyadic rectangle $R \in \mathcal{F}_D$ with $R \ni z$ such that

$$\begin{aligned} \frac{1}{\mu(R)} \int_R f \, dxdy &\geq \frac{1}{k^2} \frac{1}{\mu(R)} \int_R f_k \, dxdy \\ \geq \frac{1}{k^2} \frac{1}{\mu(R)} \int_R f_{k,j} \, dxdy &\geq \frac{1}{k^2} \frac{n}{2} \geq \frac{\max\{k^3, M^3\}}{k^2} \geq M \end{aligned}$$

by (6.9). We also have for every $z \in E$ and any dyadic rectangle $R \in \mathcal{F}_C$, with $z \in R$,

$$\frac{1}{\mu(R)} \int_R f \, dxdy \leq 3 \sum_{k=1}^{L} \frac{1}{k^2}$$

by (6.10) with the aid of Remark 6.3.

It remains to show $\mu(E) > 1 - \varepsilon$. Note first that

$$\mu\Big(\bigcup_{k=1}^{L} N_k\Big) \leq \sum_{k=1}^{L} \mu(N_k) \leq \sum_{k=1}^{L} \varepsilon_k = \sum_{k=1}^{L} \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{2}.$$

Thus it is enough to show $\mu(\bigcup_{k=1}^{L} E_k) \geq 1 - (\varepsilon/2)$, and this will be achieved by making n_k grow sufficiently fast and taking L sufficiently large. For $k \in \mathbb{N}$, we let $E(k) = E_1 \cup \cdots \cup E_k$. Note that if n_k grows fast then the partition at step $k+1$ can be made so small that $E_{k+1} \setminus E(k)$ will fill up almost 1/3rd of the complement of $E(k)$. Hence, by taking L large enough we can achieve the bound $\mu(E(L)) \geq 1 - (\varepsilon/2)$. It then follows that

$$\mu(E) \geq \mu(E(L)) - \mu\Big(\bigcup_{k=1}^{L} N_k\Big) > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon.$$

The proof of Proposition 3.1 is obtained. $\square$

Appendix A. The Haar wavelet and its properties

For simplicity, we will formulate the multivariate Haar system only in dimension 2. More general formulations can be found in [1,11]. Our presentation follows the notations of [11].

We recall the definition of the one dimensional Haar system $\{h_m\}_{m\in\mathbb{N}}$:

$$h_1(x) = 1 \quad (x \in \mathbb{T}^1),$$

and if

$$m = 2^k + i \quad \big(k \in \mathbb{N} \cup \{0\},\ i = 1, \dots, 2^k\big)$$

then

$$h_m(x) = \begin{cases} 2^{k/2} & \text{if } x \in \big(\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}\big) \\ -2^{k/2} & \text{if } x \in \big(\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}\big) \\ 0 & \text{if } x \notin \big[\frac{i-1}{2^k}, \frac{i}{2^k}\big]. \end{cases}$$

At inner points of discontinuity, h_m is defined as the mean value of the limits from the right and from the left, and at the endpoints of $\mathbb{T}^1$ as the limits from inside of the interval. The two dimensional Haar system $\big\{H_{(m,n)}\big\}_{(m,n)\in\mathbb{N}^2}$ is defined as follows:

$$H_{(m,n)}(x, y) = h_m(x) \times h_n(y) \quad \big((x, y) \in \mathbb{T}^2\big).$$

For $z \in \mathbb{T}^2$, let $H(z)$ be the spectrum of the Haar system at z, i.e.,

$$H(z) = \big\{(m, n) \in \mathbb{N}^2 : H_{(m,n)}(z) \neq 0\big\}.$$

We denote by $\Delta_{k,s}(z)$ the dyadic rectangle with sides $1/2^k$ and $1/2^s$ that contains z.

The next property connects the convergence of the rectangular Fourier–Haar sums with the differentiation of integrals with respect to the basis of dyadic rectangles. (See, e.g., [7, Ch. 3, §1] or [1, Ch. 1, §6].) Below, by $[\![a, b]\!]$ with $a, b \in \mathbb{N}$ and $a < b$ we mean the set $\{a, a + 1, \dots, b\}$.

Proposition A.1. *Let $f \in L^1(\mathbb{T}^2)$, $z \in \mathbb{T}^2$ and $(m, n) \in \mathbb{N}^2$. Then the following assertions hold: let $m = 2^k + i$ and $n = 2^s + j$, with $i = 1, \dots, 2^k$ and $j = 1, \dots, 2^s$;*

(1) *If $H(z) \cap ([\![2^k + 1, m]\!] \times [\![2^s + 1, n]\!]) \neq \emptyset$, then*

$$S_{(m,n)}f(z) = S_{(2^{k+1},2^{s+1})}f(z) = \frac{1}{\mu(\Delta_{k+1,s+1}(z))} \int_{\Delta_{k+1,s+1}(z)} f\,dxdy.$$

(2) *If $H(z) \cap ([\![2^k + 1, m]\!] \times [\![2^s + 1, n]\!]) = \emptyset$, then*

$$S_{(m,n)}f(z) = S_{(2^k,2^s)}f(z) = \frac{1}{\mu(\Delta_{k,s}(z))} \int_{\Delta_{k,s}(z)} f\,dxdy.$$

In other words for each $z \in \mathbb{T}^2$ and every $m, n \in \mathbb{N}$, we have

$$S_{(m,n)}f(z) = \frac{1}{\mu(I_{m,n}(z))} \int_{I_{m,n}(z)} f \, dxdy, \tag{A.1}$$

where $I_{m,n}(z)$ is a dyadic rectangle with sides $1/2^k$ and $1/2^s$ or $1/2^{k+1}$ and $1/2^{s+1}$ containing z.

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