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DUALITY FOR VECTOR-VALUED BERGMAN–ORLICZ SPACES AND LITTLE HANKEL OPERATORS BETWEEN VECTOR-VALUED BERGMAN–ORLICZ SPACES ON THE UNIT BALL

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Abstract. In this paper, we consider vector-valued Bergman–Orlicz spaces which are generalization of classical vector-valued Bergman spaces. We characterize the dual space of vector-valued Bergman–Orlicz space, and study the boundedness of the little Hankel operators, h_b, with operator-valued symbols b, between different weighted vector-valued Bergman–Orlicz spaces on the unit ball $\mathbb{B}_n$. More precisely, given two complex Banach spaces X, Y, we characterize those operator-valued symbols $b\colon \mathbb{B}_n \to \mathcal{L}(\overline{X}, Y)$ for which the little Hankel operator $h_b : A^{\Phi_1}_\alpha(\mathbb{B}_n, X) \to A^{\Phi_2}_\alpha(\mathbb{B}_n, Y)$, extends into a bounded operator, where Φ_1 and Φ_2 are either convex or concave growth functions.

1. Introduction and statement of results

Throughout this paper, we fix a positive integer n and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the n-dimensional Euclidean space. For

$$z = (z_1, \dots, z_n), \quad w = (w_1, \dots, w_n),$$

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in $\mathbb{C}^n$, we define the inner product of z and w by

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n},$$

where $\overline{w_k}$ is the complex conjugate of w_k. The resulting norm is then

$$|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

We know that $\mathbb{C}^n$ is a Hilbert space whose canonical basis consists of the following vectors:

$$e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \quad e_n = (0, \ldots, 0, 1).$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

When $\alpha > -1$, the weighted Lebesgue measure $d\nu_\alpha$ in $\mathbb{B}_n$ is defined by

$$d\nu_\alpha = c_\alpha(1 - |z|^2)^\alpha \, d\nu(z), \tag{1.1}$$

where $d\nu$ is the Lebesgue measure in $\mathbb{C}^n$ and

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\,\Gamma(\alpha + 1)} \tag{1.2}$$

is the normalizing constant so that $d\nu_\alpha$ becomes a probability measure on $\mathbb{B}_n$. A function defined on the unit ball $\mathbb{B}_n$ will be called a vector-valued function when it takes its values in some vector space. If X is a complex Banach space, a vector-valued function $f\colon \mathbb{B}_n \to X$ (a X-valued function) is said to be strongly holomorphic in $\mathbb{B}_n$ if for every $z \in \mathbb{B}_n$ and for every $k \in \{1, \ldots, n\}$, the limit

$$\lim_{\lambda \to 0} \frac{f(z + \lambda e_k) - f(z)}{\lambda}$$

exists in X, where $\lambda \in \mathbb{C}$ is different to zero. The space of all X-valued strongly holomorphic functions on $\mathbb{B}_n$ will be denoted by $\mathcal{H}(\mathbb{B}_n, X)$. We will also denote by $\mathcal{H}^\infty(\mathbb{B}_n, X)$ the space of all bounded X-valued holomorphic functions. Let $X^\star$ denote the space of all bounded linear functionals $x^\star\colon X \to \mathbb{C}$ (the topological dual space of X). We say that $f\colon \mathbb{B}_n \to X$ is weakly holomorphic if for every $x^\star \in X^\star$, the scalar-valued function $x^\star(f)\colon \mathbb{B}_n \to \mathbb{C}$ is holomorphic in the usual sense. An important result by N. Dunford [9] shows that a vector-valued function is strongly holomorphic if and only if it is weakly holomorphic. So, vector-valued holomorphic functions are vector-valued weakly or strongly holomorphic functions.

1.1. The conjugate $\overline{X}$ of the complex Banach space X. In the sequel, we will need the "conjugate" notion of a complex Banach space X.

We will use the following definition and notation which can be found in [12]. Let $x \in X$, $x^\star \in X^\star$ and $\lambda \in \mathbb{C}$, we define

$$(\lambda x^\star)(x) := \overline{\lambda} x^\star(x).$$

We also use the notation

$$\langle x, x^\star \rangle_{X,X^\star} = x^\star(x)$$

which will represent the 'inner product' in the complex Banach space X. We remark that

$$\langle \lambda x, x^\star \rangle_{X,X^\star} = \lambda \langle x, x^\star \rangle_{X,X^\star} = \langle x, \overline{\lambda} x^\star \rangle_{X,X^\star},$$

so that we have a regular rule of an inner product.

The complex conjugate $\overline{x}$ of $x \in X$ is the functional on $X^\star$ defined by

$$\overline{x}(x^\star) = \overline{\langle x, x^\star \rangle}_{X,X^\star},$$

for every $x^\star \in X^\star$. Let

$$\overline{X} = \{\overline{x} : x \in X\}$$

the complex conjugate of the Banach space X. With the norm

$$\|\overline{x}\|_{\overline{X}} := \sup_{\|x^\star\|_{X^\star}=1} |\overline{x}(x^\star)|,$$

$\overline{X}$ becomes a Banach space. Moreover, we have $\|x\|_X = \|\overline{x}\|_{\overline{X}}$ for any $x \in X$, so that X and $\overline{X}$ are isometrically anti-isomorphic.

1.2. Vector-valued Bergman spaces. In the sequel, we will integrate vector-valued measurable functions in the sense of Bochner. Let X be a complex Banach space.

Definition 1.1. A function $s\colon \mathbb{B}_n \to X$ is called a "vector-valued" simple function if it has the form

$$s = \sum_{i=1}^{k} x_i \chi_{A_i},$$

where k is a nonnegative integer, $(A_i)_{i=1}^k$ is a finite sequence of pairwise disjoint members of the σ-algebra of $\mathbb{B}_n$ such that $\bigcup_{i=1}^k A_i = \mathbb{B}_n$ and x_1, x_2, $\dots$, x_k are finite distinct values of X. We have $A_i = s^{-1}(\{x_i\})$ and $\chi_A\colon \mathbb{B}_n \to \{0,1\}$ denotes the characteristic function of A. We denote by $\mathcal{S}(\mathbb{B}_n, X)$ the set of simple functions.

Definition 1.2 (Borel σ-algebra on X). A function $f\colon \mathbb{B}_n \to X$ is a Borel σ-algebra on X with respect to the measure ν_α in the unit ball $\mathbb{B}_n$ if there exists a sequence of simple functions $(s_k)_{k\geq 1} \subset \mathcal{S}(\mathbb{B}_n, X)$ such that

$$s_k \to f \quad \nu_\alpha\text{-a.e. as } k \to \infty. \tag{1.3}$$

Definition 1.3 (Bochner–Lebesgue space). A function $f\colon \mathbb{B}_n \to X$ is Bochner integrable with respect to the measure ν_α in the unit ball $\mathbb{B}_n$ if there exists a sequence of simple functions $(s_k)_{k\geq 1} \subset \mathcal{S}(\mathbb{B}_n, X)$ such that

(i) $s_k \to f$ ν_α-a.e. as $k \to \infty$,

(ii) $\int_{\mathbb{B}_n} \|s_k(z) - f(z)\|_X \, d\nu_\alpha(z) \to 0$ as $n \to \infty$.

A sequence of simple functions that satisfy assertion (i) and (ii) is called an approximant sequence for f. For $0 < p < \infty$, the Bochner–Lebesgue space $L^p_\alpha(\mathbb{B}_n, X)$ consists of all vector-valued measurable functions $f\colon \mathbb{B}_n \to X$ such that

$$\|f\|^p_{p,\alpha,X} := \int_{\mathbb{B}_n} \|f(z)\|^p_X \, d\nu_\alpha(z) < \infty.$$

The vector-valued Bergman space $A^p_\alpha(\mathbb{B}_n, X)$ is the closed subspace of $L^p_\alpha(\mathbb{B}_n, X)$ consisting of holomorphic functions.

It is known (see [4]) that, for any $1 < p < \infty$, the dual of the Bergman space $A^p_\alpha(\mathbb{B}_n, X)$, $(A^p_\alpha(\mathbb{B}_n, X))^\star$, identifies with $A^{p'}_\alpha(\mathbb{B}_n, X^\star)$ where p' is the conjugate exponent of p, under the duality pairing

$$\langle f, g\rangle_{\alpha,X} := \int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_{X,X^\star} \, d\nu_\alpha(z),$$

with $f \in A^p_\alpha(\mathbb{B}_n, X)$ and $g \in A^{p'}_\alpha(\mathbb{B}_n, X^\star)$.

For $0 < p < \infty$, the weak Bochner–Lebesgue space $L^{p,\infty}_\alpha(\mathbb{B}_n, X)$ consists of all vector-valued measurable functions $f\colon \mathbb{B}_n \to X$ for which

$$\|f\|_{L^{p,\infty}_\alpha(\mathbb{B}_n,X)} := \Big(\sup_{\lambda>0} \lambda^p \nu_\alpha\big(\{z \in \mathbb{B}_n : \|f(z)\|_X > \lambda\}\big)\Big)^{1/p} < \infty.$$

The weak vector-valued Bergman space $A^{p,\infty}_\alpha(\mathbb{B}_n, X)$ is defined by

$$A^{p,\infty}_\alpha(\mathbb{B}_n, X) = \mathcal{H}(\mathbb{B}_n, X) \cap L^{p,\infty}_\alpha(\mathbb{B}_n, X).$$

For $f \in L^1_\alpha(\mathbb{B}_n, X)$ and $z \in \mathbb{B}_n$, the Bergman projection $P_\alpha f$ of f is the integral operator defined by

$$P_\alpha f(z) = \int_{\mathbb{B}_n} K_\alpha(z, w) f(w) \, d\nu_\alpha(w),$$

where $K_\alpha(z,w)=\frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}$ is the Bergman reproducing kernel. In this case, $P_\alpha f$ is also a X-valued holomorphic function. We recall the following well-known result that will be used later [3].

Proposition 1.4. *Let $\alpha>-1$, there exists a constant $C>0$ such that for any $f\in L^1_\alpha(\mathbb{B}_n,X)$,*

$$\nu_\alpha\big(\{z\in\mathbb{B}_n:\|P_\alpha f(z)\|_X>\lambda\}\big)\le C\,\frac{\|f\|_{1,\alpha,X}}{\lambda}.$$

Let X, Y be two complex Banach spaces and $\alpha>-1$. We have the following two lemmas whose proof can be found in [12].

Lemma 1.5. *Let $T\colon X\to Y$ be a bounded linear operator. If $f\colon\mathbb{B}_n\to X$ is ν_α-Bochner integrable in the unit ball, then $Tf\colon\mathbb{B}_n\to Y$ is ν_α-Bochner integrable in the unit ball and we have*

$$\int_{\mathbb{B}_n}Tf(z)\,d\nu_\alpha(z)=T\bigg(\int_{\mathbb{B}_n}f(z)\,d\nu_\alpha(z)\bigg).$$

Lemma 1.6. *If $f\colon\mathbb{B}_n\to X$ is a ν_α-Bochner integrable vector-valued function in the unit ball, then we have the inequality*

$$\bigg\|\int_{\mathbb{B}_n}f(z)\,d\nu_\alpha(z)\bigg\|_X\le\int_{\mathbb{B}_n}\|f(z)\|_X\,d\nu_\alpha(z).$$

1.3. Vector-valued Bergman–Orlicz spaces. We say that a function Φ is a growth function if it is a continuous and non-decreasing function from $[0,\infty)$ onto itself. We say that a growth function Φ is of lower type $0<p\le1$ if there exists $C>0$ such that, for $s>0$ and $0<t\le1$,

$$\Phi(st)\le Ct^p\Phi(s). \tag{1.4}$$

We say that a growth function Φ is of upper type $q\ge1$ if there exists $C>0$ such that, for $s>0$ and $t\ge1$,

$$\Phi(st)\le Ct^q\Phi(s). \tag{1.5}$$

We say that Φ satisfies the Δ_2-condition if there exists a constant $K>1$ such that, for any $s\ge0$,

$$\Phi(2s)\le K\,\Phi(s). \tag{1.6}$$

Observe the equivalence between the properties (1.5) and (1.6).

Let X be a complex Banach space. For Φ a growth function, the vector-valued Orlicz space $L^{\Phi}_{\alpha}(\mathbb{B}_n, X)$ is the space of all vector-valued measurable functions $f\colon \mathbb{B}_n \to X$ such that for some $\lambda > 0$,

$$\|f\|_{\Phi,\alpha,X,\lambda} := \int_{\mathbb{B}_n} \Phi\left(\frac{\|f(z)\|_X}{\lambda}\right) d\nu_\alpha(z) < \infty.$$

The vector-valued Bergman–Orlicz space $A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$ is defined by

$$A^{\Phi}_{\alpha}(\mathbb{B}_n, X) = \mathcal{H}(\mathbb{B}_n, X) \cap L^{\Phi}_{\alpha}(\mathbb{B}_n, X).$$

For $0 < p < \infty$, when $\Phi(t) = t^p$, we recover the classical vector-valued Bergman space $A^p_\alpha(\mathbb{B}_n, X)$ defined in Subsection 1.2.

Recall that two growth functions Φ_1 and Φ_2 are said equivalent if there exists some constant c such that

$$c\Phi_1(ct) \le \Phi_2(t) \le c^{-1}\Phi_1(c^{-1}t) \quad \text{for all } t > 0.$$

Such equivalent growth functions define the same Orlicz space. Let us define two classes of growth functions of our interest in this paper.

Definition 1.7. We call $\mathscr{L}$ the set of growth function Φ of lower type p, for some $0 < p \le 1$, such that the function $t \mapsto \frac{\Phi(t)}{t}$ is non-increasing. For $\Phi \in \mathscr{L}$ of lower type p, we say that $\Phi \in \mathscr{L}_p$.

Definition 1.8. We call $\mathscr{U}$ the set of growth function Φ of upper type q, for some $q \ge 1$, such that the function $t \mapsto \frac{\Phi(t)}{t}$ is non-decreasing. For $\Phi \in \mathscr{U}$ of upper type q, we say that $\Phi \in \mathscr{U}^q$.

Examples of growth functions of particular interest include

$$\Phi_1(t) = \Big(\frac{t}{\log(e+t)}\Big)^p,\ 0 < p \le 1 \ \text{ and } \ \Phi_2(t) = \big(t\log(e+t)\big)^q,\ 0 < q < \infty.$$

We have $\Phi_1 \in \mathscr{L}_p$ and $\Phi_2 \in \mathscr{U}^q$ for $q > 1$. Clearly, the functions in $\mathscr{L}$ or $\mathscr{U}$ satisfy the Δ_2-condition. Note that if $\Phi \in \mathscr{U}$, then Φ is of lower type 1. For $\Phi \in \mathscr{L}$ (resp. $\mathscr{U}$), without loss of generality, possibly replacing Φ by the equivalent growth function $\int_0^t \frac{\Phi(s)}{s}\,ds$, we may always suppose that Φ is concave (resp. convex) and Φ is a $\mathscr{C}^1$ function with derivative $\Phi' \backsimeq \frac{\Phi(t)}{t}$ (see [7] for the proof for the case $\Phi \in \mathscr{L}$).

For $\Phi \in \mathscr{U} \cup \mathscr{L}$, we define on $L^{\Phi}_{\alpha}(\mathbb{B}_n, X)$ the Luxemburg (quasi)-norm

$$\|f\|^{\mathrm{lux}}_{\Phi,\alpha,X} := \inf\left\{\lambda > 0 : \int_{\mathbb{B}_n} \Phi\Big(\frac{\|f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) \le 1\right\} \tag{1.7}$$

which is finite for $f \in L^{\Phi}_{\alpha}(\mathbb{B}_n, X)$ (see [15]).

In order to give the dual of $A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$ when $\Phi \in \mathscr{U}$, we need to recall the notion of complementary function of a growth function [14]. For $\Phi \in \mathscr{U}$, we recall that the complementary function of Φ, is the function $\Psi \colon \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$\Psi(s) = \sup_{t \in \mathbb{R}_+} \{ts - \Phi(t)\}.$$

One easily checks that if $\Phi \in \mathscr{U}$, then Ψ is also a growth function of lower type such that $t \mapsto \frac{\Psi(t)}{t}$ is non-decreasing but which may not satisfy the Δ_2-condition. The fact that Ψ also satisfies the Δ_2-condition is relevant in our results here. We thus introduce another class of growth functions.

Definition 1.9. For $\Phi \in \mathscr{U}$, we say that Φ satisfies the ∇_2-condition whenever both Φ and its complementary satisfy the Δ_2-condition.

Several characterizations that guarantee that a growth function has a complementary function satisfying the Δ_2-condition are known. One of these characterizations is the Dini condition which we recall here. We say that $\Phi \in \mathscr{U}$ satisfies the Dini condition if there exists a constant $C > 0$ such that, for $t > 0$,

$$\int_0^t \frac{\Phi(s)}{s^2}\, ds \le C \frac{\Phi(t)}{t}. \tag{1.8}$$

So when Φ satisfies (1.8), then Φ satisfies the ∇_2-condition [5, Proposition 3]. Under this condition, we will show that the space $A^{\Psi}_{\alpha}(\mathbb{B}_n, X^{\star})$ is the topological dual space of $A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$.

If Φ and Ψ are a pair of complementary growth functions, then using the fact that $st \le \Psi(s) + \Phi(t)$, for $s, t \ge 0$, we obtain the following Hölder's type inequality

$$\int_{\mathbb{B}_n} \|f(z)\|_X \|g(z)\|_Y \, d\nu_\alpha(z) \le 2 \|f\|^{\mathrm{lux}}_{\Phi,\alpha,X} \|g\|^{\mathrm{lux}}_{\Psi,\alpha,Y}, \tag{1.9}$$

where X and Y are complex Banach spaces and $f \in A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$, $g \in A^{\Psi}_{\alpha}(\mathbb{B}_n, Y)$.

The following proposition extends [17, Proposition 4.4]. The proof is similar.

Lemma 1.10. *Let X a complex Banach space, $\Phi \in \mathscr{L}$ and $\alpha > -1$. Suppose that Φ satisfies the Dini condition*

$$\int_1^{\infty} \frac{\Phi(s)}{s^2}\, ds < \infty. \tag{1.10}$$

Then $A^{1,\infty}_{\alpha}(\mathbb{B}_n, X)$ embeds continuously into $A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$.

1.4. Vector-valued Lipschitz spaces. The radial derivative of a vector-valued holomorphic function $f\colon \mathbb{B}_n \to X$ denoted Nf is defined for $z \in \mathbb{B}_n$ by

$$Nf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z). \tag{1.11}$$

For $\beta \geq 0$, we denote by $\Gamma_\beta(\mathbb{B}_n, X)$ the space of vector-valued holomorphic functions $f\colon \mathbb{B}_n \to X$ for which there exists an integer $k > \beta$ such that

$$\|f\|_{\beta,X} = \|f(0)\|_X + \sup_{z\in\mathbb{B}_n} \big(1-|z|^2\big)^{k-\beta} \|N^k f(z)\|_X < \infty,$$

where $N^k = N \circ N \circ \cdots \circ N$ $k-$times. As in the scalar case, the definition of the space $\Gamma_\beta(\mathbb{B}_n, X)$ is independent of the integer k used. The space $\Gamma_\beta(\mathbb{B}_n, X)$ will be called the vector-valued holomorphic Lipschitz space and for $\beta = 0$, the class $\Gamma_\beta(\mathbb{B}_n, X)$ coincides with the usual vector valued Bloch space $\mathcal{B}(\mathbb{B}_n, X)$. We recall that $f \in \mathcal{B}(\mathbb{B}_n, X)$ if

$$\|f\|_{\mathcal{B}(\mathbb{B}_n,X)} := \|f(0)\|_X + \sup_{z\in\mathbb{B}_n} (1-|z|^2)\|Nf(z)\|_X < \infty.$$

It is known (see [3]) that, for $0 < p < 1$, the dual of the Bergman space $A^p_\alpha(\mathbb{B}_n, X)$, coincides with $\Gamma_\beta(\mathbb{B}_n, X^\star)$ with $\beta = (n+1+\alpha)\big(\frac{1}{p} - 1\big)$ under the duality pairing

$$\langle f, g\rangle_{\alpha,X} = c_k \int_{\mathbb{B}_n} \langle f(z), M^\alpha_k g(z)\rangle_{X,X^\star} (1-|z|^2)^k \, d\nu_\alpha(z),$$

$k > \beta$, is an integer, $g \in \Gamma_\beta(\mathbb{B}_n, X^\star)$, $f \in A^p_\alpha(\mathbb{B}_n, X)$ and M^α_k is the differential operator of order k defined by (2.22).

For Φ a growth function, we associate the function

$$\rho_\Phi(t) = \frac{1}{t\,\Phi^{-1}\big(\frac{1}{t}\big)}.$$

The function ρ_Φ is quite relevant in the study of Orlicz space of analytic functions (see [7], [11], and the references therein). Note in particular that in the case of $A^p_\alpha(\mathbb{B}_n, X)$, $\Phi(t) = t^p$ and $\rho(t) = t^{\frac{1}{p}-1}$, hence $f \in \Gamma_\beta(\mathbb{B}_n, X)$, where $\beta = (n+1+\alpha)\big(\frac{1}{p} - 1\big)$ if

$$\|N^k f(z)\|_X \leq C\big(1-|z|^2\big)^{-k} \rho\Big((1-|z|^2)^{n+1+\alpha}\Big).$$

From this observation, we will make the following generalization. Let ρ be a positive continuous increasing function from $[0,\infty)$ onto itself and let $\gamma > 0$.

We say that ρ is of upper type γ on $[0,1]$ if there exists a constant C such that

$$\rho(st) \leq C s^{\gamma} \rho(t),$$

for $s > 1$ and $st \leq 1$. We will call a weight, a function ρ which is a continuous nondecreasing function from $[0,\infty)$ onto itself, which is of upper type γ, for some $\gamma > 0$. We recall the following fact from [19, Proposition 3.10]: if $\Phi \in \mathscr{L}_p$ and $\rho(t) = \frac{1}{t\,\Phi^{-1}(\frac{1}{t})}$, then ρ is a weight of upper type $\frac{1}{p} - 1$.

Now, for $\alpha > -1$ and a weight ρ (of upper type γ), we define the weighted Lipschitz space $\Gamma_{\alpha,\rho}(\mathbb{B}_n, X)$ as the space of holomorphic functions $f \in \mathcal{H}(\mathbb{B}_n, X)$ such that, for some integer $k > \gamma(n+1+\alpha)$ and a positive constant $C > 0$, we have

$$\big\|N^k f(z)\big\|_X \leq C\big(1-|z|^2\big)^{-k} \rho\big((1-|z|^2)^{n+1+\alpha}\big), \quad z \in \mathbb{B}_n.$$

We will also need a logarithmic version of the above space, $L\Gamma_{\alpha,\rho}(\mathbb{B}_n, X)$, defined as the space of holomorphic functions f in $\mathbb{B}_n$ such that, for some $k > \gamma(n+1+\alpha)$ and a positive constant $C > 0$, we have

$$\big\|N^k f(z)\big\|_X \leq C\big(1-|z|^2\big)^{-k} \rho\big((1-|z|^2)^{n+1+\alpha}\big)\Big(\log \frac{1}{1-|z|^2}\Big)^{-1}, \quad z \in \mathbb{B}_n.$$

One can show that, as in the weighed classical Lipschitz spaces in Part 2 of [17, Proposition 2.11], these spaces are independent of k. As a consequence, the spaces $\Gamma_{\alpha,\rho}(\mathbb{B}_n, X)$ and $L\Gamma_{\alpha,\rho}(\mathbb{B}_n, X)$ become Banach spaces under the norms

$$\|f\|_{\Gamma_{\alpha,\rho}(\mathbb{B}_n,X)} = \|f(0)\|_X + \sup_{z\in\mathbb{B}_n} \frac{(1-|z|^2)^k \|N^k f(z)\|_X}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)},$$

$$\|f\|_{L\Gamma_{\alpha,\rho}(\mathbb{B}_n,X)} = \|f(0)\|_X + \sup_{z\in\mathbb{B}_n} \frac{(1-|z|^2)^k \|N^k f(z)\|_X}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)} \big|\log(1-|z|^2)\big|,$$

where k is a fixed integer strictly greater than $\gamma(n+1+\alpha)$.

1.5. Little Hankel operators with operator-valued symbols. Given two complex Banach spaces X and Y, we denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators $T\colon X \to Y$ endowed with the norm

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1} \|Tx\|_Y = \sup_{\|x\|_X=1, \|y^\star\|_{Y^\star}=1} |\langle Tx, y^\star\rangle_{Y,Y^\star}|,$$

where $T \in \mathcal{L}(X,Y)$. Then $\mathcal{L}(X,Y)$ is a Banach space. We consider an operator-valued function $b\colon \mathbb{B}_n \to \mathcal{L}\big(\overline{X}, Y\big)$ and we suppose that $b \in$

$\mathcal{H}\big(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)\big)$. The little Hankel operator with operator-valued symbol b, denoted h_b, is defined for $z \in \mathbb{B}_n$ by

$$h_b f(z) = \int_{\mathbb{B}_n} \frac{b(w)\big(\overline{f(w)}\big)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}\, d\nu_\alpha(w), \quad f \in \mathcal{H}^\infty(\mathbb{B}_n, X).$$

In the sequel, we will assume that the symbol b satisfies the following condition:

$$\int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{X},Y)}}{|1 - \langle z, w\rangle|^{n+1+\alpha}}\, d\nu_\alpha(w) < \infty \quad \text{for every } z \in \mathbb{B}_n. \tag{1.12}$$

It is easy to check that if b satisfies (1.12), then the little Hankel operator h_b is well defined on $\mathcal{H}^\infty(\mathbb{B}_n, X)$.

The boundedness properties of the little Hankel operator in the classical case (that is, when $X = Y = \mathbb{C}$) have been extensively studied and many results are now well known [6,13,22]. In recent years, the study of the little Hankel operator h_b, with operator-valued holomorphic symbols b, between vector-valued Bergman spaces, $A^p_\alpha(\mathbb{B}_n, X)$ and $A^q_\alpha(\mathbb{B}_n, Y)$, have gained some interest and the boundedness properties are now well-established [1,3,12].

We are here concerned with the question of characterizing the operator-valued holomorphic symbols b for which the little Hankel operator h_b extends into a bounded operator from $A^{\Phi_1}_\alpha(\mathbb{B}_n, X)$ into $A^{\Phi_2}_\alpha(\mathbb{B}_n, Y)$ where Φ_i, $i = 1, 2$, is a concave or convex growth function. Our results here extend the results in [17] to the vector-valued setting with the same parameters.

1.6. Statement of results. Our first interest in this paper is the characterization of the dual space of the vector-valued Bergman–Orlicz spaces when Φ is in $\mathscr{U}$ (resp. $\mathscr{L}$). We obtain the following results, which extend the duality results for classical vector-valued Bergman spaces [3,12].

THEOREM 1.11. *Let X be a complex Banach space, $\alpha > -1$ and $\Phi \in \mathscr{U}$. Let Ψ be the complementary function of Φ and suppose that Φ satisfies the ∇_2-condition. Then the topological dual space $\big(A^\Phi_\alpha(\mathbb{B}_n, X)\big)^\star$ of $A^\Phi_\alpha(\mathbb{B}_n, X)$ can be identified with $A^\Psi_\alpha\big(\mathbb{B}_n, X^\star\big)$ (with equivalent norms) under the duality pairing*

$$\langle f, g\rangle_{\alpha,X} = \int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_{X,X^\star}\, d\nu_\alpha(z),$$

where $f \in A^\Phi_\alpha(\mathbb{B}_n, X)$ and $g \in A^\Psi_\alpha(\mathbb{B}_n, X^\star)$. Moreover,

$$\|g\|^{\text{lux}}_{\Psi,\alpha,X^\star} \simeq \sup_{\|f\|^{\text{lux}}_{\Phi,\alpha,X}=1} |\langle f, g\rangle_{\alpha,X}|.$$

Theorem 1.12. *Let X be a complex Banach space, $\alpha > -1$, $\Phi \in \mathscr{L}_p$ and $\rho(t) = \frac{1}{t\,\Phi^{-1}(\frac{1}{t})}$. Then the topological dual space $\big(A_\alpha^\Phi(\mathbb{B}_n, X)\big)^\star$ of $A_\alpha^\Phi(\mathbb{B}_n, X)$ can be identified with $\Gamma_{\alpha,\rho}\big(\mathbb{B}_n, X^\star\big)$ under the duality pairing*

$$\langle f, g\rangle_{\alpha,X} = C_{k,\alpha}^{-1} \int_{\mathbb{B}_n} \langle f(z), M_k^\alpha g(z)\rangle_{X,X^\star}\, d\nu_{\alpha+k}(z),$$

$k > (n+1+\alpha)\big(\frac{1}{p} - 1\big)$ is an integer, $f \in A_\alpha^\Phi(\mathbb{B}_n, X)$, $g \in \Gamma_{\alpha,\rho}(\mathbb{B}_n, X^\star)$, $C_{k,\alpha}$ and M_k^α are respectively the constant and the differential operator defined by (2.22) *and* (2.23). *Moreover,*

$$\|g\|_{\Gamma_{\alpha,\rho}(\mathbb{B}_n,X^\star)} \simeq \sup_{\|f\|_{\Phi,\alpha,X}^{\text{lux}}=1} \big|\langle f, g\rangle_{\alpha,X}\big|.$$

Our second interest in this paper is the study of the boundedness of the little Hankel operators, h_b, from $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ to $A_\alpha^{\Phi_2}(\mathbb{B}_n, Y)$. We do not use a specific method but combine several techniques some of them appearing in [17] or used in the case of vector valued Bergman spaces in [3].

Before stating the results on Hankel operators, we need to make another assumption on the operator-valued symbol b. More precisely, we assume that the operator-valued symbol b satisfies the following condition:

$$\int_{\mathbb{B}_n} \|b(z)\|_{\mathcal{L}(\overline{X},Y)} \log\Big(\frac{1}{1-|z|^2}\Big)\, d\nu_\alpha(z) < \infty. \tag{1.13}$$

Let X and Y be two complex Banach spaces. Our contribution to the boundedness problem of the little Hankel operator with operator-valued symbol is the following.

Theorem 1.13. *Let $\Phi \in \mathscr{U}$ such that Φ satisfies the ∇_2-condition, and $\alpha > -1$. Then the Hankel operator h_b extends into a bounded operator from $A_\alpha^\Phi(\mathbb{B}_n, X)$ into $A_\alpha^\Phi(\mathbb{B}_n, Y)$ if and only if its symbol b belongs to $\mathcal{B}\big(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)\big)$.*

Theorem 1.14. *Let $\Phi_1 \in \mathscr{L}$, $\alpha > -1$ and $\Phi_2 \in \mathscr{L}$. If the Hankel operator h_b extends into a bounded operator from $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ into $A_\alpha^{\Phi_2}(\mathbb{B}_n, Y)$, then its symbol b belongs to $\Gamma_{\alpha,\rho_1}\big(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)\big)$ with $\rho_1(t) = \frac{1}{t\,\Phi_1^{-1}(\frac{1}{t})}$. Conversely, if $b \in \Gamma_{\alpha,\rho_1}\big(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)\big)$, then there exists a bounded operator T_b from $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ into $L_\alpha^1(\mathbb{B}_n, Y)$ such that $h_b = P_\alpha T_b$.*

As a direct consequence, we have the following result.

Corollary 1.15. *Let $\Phi_1, \Phi_2 \in \mathscr{L}$, $\alpha > -1$. Suppose that Φ_2 satisfies the Dini condition* (1.10). *Then the Hankel operator h_b extends into a bounded operator from $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ into $A_\alpha^{\Phi_2}(\mathbb{B}_n, Y)$ if and only if its symbol b belongs to $\Gamma_{\alpha,\rho_1}\big(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)\big)$ with $\rho_1(t) = \frac{1}{t\,\Phi_1^{-1}(\frac{1}{t})}$.*

Theorem 1.16. *Let $\Phi \in \mathscr{L}$, $\alpha > -1$. Then the Hankel operator h_b extends into a bounded operator from $\underline{A^{\Phi}_{\alpha}(\mathbb{B}_n, X)}$ into $A^{1}_{\alpha}(\mathbb{B}_n, Y)$ if and only if its symbol b belongs to $L\Gamma_{\alpha,\rho}\big(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)\big)$ with $\rho(t) = \frac{1}{t\,\Phi^{-1}(\frac{1}{t})}$.*

Theorem 1.17. *Let $\Phi_1 \in \mathscr{L}$ and $\Phi_2 \in \mathscr{U}$, $\rho_i(t) = \frac{1}{t\,\Phi_i^{-1}(\frac{1}{t})}$ and assume that Φ_2 satisfies the ∇_2-condition. Then the Hankel operator h_b extends into a bounded operator from $\underline{A^{\Phi_1}_{\alpha}(\mathbb{B}_n, X)}$ into $A^{\Phi_2}_{\alpha}(\mathbb{B}_n, Y)$ if and only if its symbol b belongs to $\Gamma_{\alpha,\rho}\big(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)\big)$, where*

$$\rho := \frac{\rho_1}{\rho_2}.$$

Theorem 1.18. *Let Φ_1 and $\Phi_2 \in \mathscr{U}$, and $\rho_i(t) = \frac{1}{t\,\Phi_i^{-1}(\frac{1}{t})}$. Let Ψ_2 be the complementary function of Φ_2. We suppose that*

(i) *Φ_2 satisfies the ∇_2-condition,*

(ii) *$\frac{\Phi_1^{-1}(t)\Psi_2^{-1}(t)}{t}$ is nondecreasing or $\frac{\Phi_2^{-1}\circ\Phi_1(t)}{t}$ is nonincreasing.*

Then the Hankel operator h_b extends into a bounded operator from $\underline{A^{\Phi_1}_{\alpha}(\mathbb{B}_n, X)}$ into $A^{\Phi_2}_{\alpha}(\mathbb{B}_n, Y)$ if and only if its symbol b belongs to $\Gamma_{\alpha,\rho_\Phi}\big(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)\big)$, where

$$\rho_\Phi := \frac{\rho_1}{\rho_2}.$$

1.7. Plan of the paper. The paper is divided into four sections. In Section 2, we recall some preliminary notions on vector-valued holomorphic functions and we also give the proofs of some important results. Section 3 contains the proof of Theorem 1.11 and 1.12 on the dual of the vector-valued Bergman–Orlicz spaces $A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$ for $\Phi \in \mathscr{U}$ and for $\Phi \in \mathscr{L}$. In the last section, each subsection is devoted to the study, in each case, of the boundedness of the little Hankel operator, h_b, from $A^{\Phi_1}_{\alpha}(\mathbb{B}_n, X)$ into $A^{\Phi_2}_{\alpha}(\mathbb{B}_n, Y)$, with $\Phi_i \in \mathscr{L}$ or $\mathscr{U}$, $i = 1, 2$.

Finally, all over the text, X and Y will denote two complex Banach spaces, the real parameter α will be chosen such that $\alpha > -1$ and C will be a positive constant not necessary the same at each occurrence. We will also use the notation C_k to express the fact that the constant depends on the underlined parameters k. Given two quantities A and B, the notation $A \lesssim B$ means that $A \leq CB$ for some positive uniform constant C. When $A \lesssim B$ and $B \lesssim A$, we write $A \backsimeq B$.

2. Preliminaries

In this section, we recall some known results and establish some auxiliary results that are needed in our study, we extend to Orlicz setting some classical results known for the vector-valued Bergman spaces.

2.1. Some properties of growth functions. We collect in this subsection some properties of growth functions we shall use later. We start with these two useful lemmas which give a relation between functions in the classes $\mathscr{L}$ and $\mathscr{U}$.

LEMMA 2.1 (see [16, Section 2]). *The following assertion holds*:

$$\Phi \in \mathscr{L} \quad \textit{if and only if} \quad \Phi^{-1} \in \mathscr{U}.$$

LEMMA 2.2 (see [16, Section 2]). *Let $\Phi \in \mathscr{U}$, and Ψ the complementary function of Φ. We have for any $t \geq 0$,*

$$t \leq \Phi^{-1}(t)\Psi^{-1}(t) \leq 2t.$$

This lemma implies the following three lemmas.

LEMMA 2.3. *Let A be a Borel set of $\mathbb{B}_n$. Let Φ and Ψ two complementary functions. We have the following estimate of the Luxemburg norm of χ_A in $L^\Psi_\alpha(\mathbb{B}_n, d\nu_\alpha)$.*

$$\|\chi_A\|^{\mathrm{lux}}_{\Psi,\alpha} \simeq \nu_\alpha(A)\Phi^{-1}\Big(\frac{1}{\nu_\alpha(A)}\Big). \tag{2.1}$$

PROOF. By the definition of the Luxemburg norm (1.7), we have

$$\|\chi_A\|^{\mathrm{lux}}_{\Psi,\alpha} = \inf\left\{\lambda > 0 : \int_A \Psi\Big(\frac{1}{\lambda}\Big)\, d\nu_\alpha(z) \leq 1\right\}$$

$$= \inf\left\{\lambda > 0 : \Psi\Big(\frac{1}{\lambda}\Big)\nu_\alpha(A) \leq 1\right\} = \inf\left\{\lambda > 0 : \lambda \geq \frac{1}{\Psi^{-1}\big(\frac{1}{\nu_\alpha(A)}\big)}\right\}.$$

Then $\|\chi_A\|^{\mathrm{lux}}_{\Psi,\alpha} = \frac{1}{\Psi^{-1}(\frac{1}{\nu_\alpha(A)})}$. By Lemma 2.2, we obtain (2.1). $\square$

LEMMA 2.4. *Let $\Phi_1 \in \mathscr{L}$ and $\Phi_2 \in \mathscr{U}$, and Ψ_2 the complementary function of Φ_2. Let Φ be such that*

$$\Phi^{-1}(t) = \Phi_1^{-1}(t)\Psi_2^{-1}(t).$$

Then $\Phi \in \mathscr{L}$.

LEMMA 2.5. *Let Φ_1 be a growth function and $\Phi_2 \in \mathscr{U}$, $\rho_i(t) = \frac{1}{t\,\Phi_i^{-1}(\frac{1}{t})}$ and Ψ_2 the complementary function of Φ_2. Then, if*

$$\rho_\Phi = \frac{\rho_1}{\rho_2},$$

we also have

$$\Phi^{-1}(t) \backsimeq \Phi_1^{-1}(t)\Psi_2^{-1}(t) \tag{2.2}$$

and vice-versa.

The following lemma whose proof is in [16] will be also useful.

Lemma 2.6. *Let Φ_1 and Φ_2 be in $\mathscr{U}$, and Ψ_2 the complementary function of Φ_2. Let Φ be such that $\Phi^{-1}(t) = \Phi_1^{-1}(t)\Psi_2^{-1}(t)$. We suppose that Φ_2 satisfies the ∇_2-condition, and that*

$$\frac{\Phi_2^{-1} \circ \Phi_1^{-1}(t)}{t}$$

is nonincreasing. Then $\Phi \in \mathscr{L}$.

The following can be adapted from [20].

Proposition 2.7 (Volberg, Tolokonnikov). *For Φ_1 and Φ_2 two growth functions of lower type and $\alpha > -1$, the bilinear map $(f, g) \mapsto fg$ sends $L_\alpha^{\Phi_1}(\mathbb{B}_n) \times L_\alpha^{\Phi_2}(\mathbb{B}_n)$ onto $L_\alpha^{\Phi}(\mathbb{B}_n)$, with the inverse mappings of Φ_1, Φ_2 and Φ related by*

$$\Phi^{-1} = \Phi_1^{-1} \times \Phi_2^{-1}.$$

Moreover, there exists some constant c such that

$$\|fg\|_{\Phi,\alpha}^{\text{lux}} \leq c\|f\|_{\Phi_1,\alpha}^{\text{lux}}\|g\|_{\Phi_2,\alpha}^{\text{lux}}.$$

2.2. Density in $A_\alpha^\Phi(\mathbb{B}_n, X)$ where $\Phi \in \mathscr{U}$ or $\mathscr{L}$. In this subsection, X is a complex Banach space and $\alpha > -1$.

Lemma 2.8. *The space of simple functions $\mathcal{S}(\mathbb{B}_n, X)$ forms a dense subspace of the vector-valued Orlicz space $L_\alpha^\Phi(\mathbb{B}_n, X)$, where $\Phi \in \mathscr{U}$ or $\mathscr{L}$.*

Proof. Let $f \in L_\alpha^\Phi(\mathbb{B}_n, X)$, then there exists $\lambda > 0$ such that

$$\int_{\mathbb{B}_n} \Phi\left(\lambda\|f(z)\|_X\right) d\nu_\alpha(z) < \infty.$$

Since $f\colon \mathbb{B}_n \to X$ is a Borel σ-algebra on X with respect to the measure ν_α in the unit ball $\mathbb{B}_n$, let a sequence of simple functions $(t_k)_{k\geq 1} \subset \mathcal{S}(\mathbb{B}_n, X)$ such that

$$t_k \to f \quad \nu_\alpha\text{-a.e. as } k \to \infty. \tag{2.3}$$

Let $s_k = t_k\chi_{\{z\in\mathbb{B}_n : \|t_k(z)\|_X \leq 2\|f(z)\|_X\}}$ is a sequence of simple functions from $\mathbb{B}_n$ to X which converges to f ν_α-a.e. and satisfies $\|s_k(z) - f(z)\|_X \leq 3\|f(z)\|_X$. Using the increasing of Φ, we have

$$\Phi\Big(\frac{\lambda}{3}\|s_k(z) - f(z)\|_X\Big) \leq \Phi\left(\lambda\|f(z)\|_X\right) \in L_\alpha^1(\mathbb{B}_n).$$

So by the dominated convergence theorem for real-valued functions, the continuity of Φ and the fact that $\Phi(0) = 0$, it follows that

$$\int_{\mathbb{B}_n} \Phi\Big(\frac{\lambda}{3}\|s_k(z) - f(z)\|_X\Big)\, d\nu_\alpha(z) \to 0,$$

that is

$$\forall \varepsilon > 0, \exists k_0 \in \mathbb{N} : k > k_0 \implies \int_{\mathbb{B}_n} \Phi\Big(\frac{\lambda}{3}\|s_k(z) - f(z)\|_X\Big)\, d\nu_\alpha(z) < \varepsilon.$$

From this, using the fact that Φ is of lower type p for $\Phi \in \mathscr{L}$ or of lower type 1 for $\Phi \in \mathscr{U}$, we have

$$\forall \varepsilon > 0, \exists k_0 \in \mathbb{N} : k > k_0 \implies \int_{\mathbb{B}_n} \Phi\Big(\frac{\lambda}{3K}\|s_k(z) - f(z)\|_X\Big)\, d\nu_\alpha(z) \leq 1,$$

with $K^p = 2C\varepsilon$ where C is the constant in the definition of lower type (1.4). That is

$$k > k_0 \implies \|s_k - f\|_{\Phi,\alpha,X} \leq \frac{3K}{\lambda},$$

hence $s_k \to f$ in $L^\Phi_\alpha(\mathbb{B}_n, X)$. $\square$

The proof of the following result can be adapted from [21, Proposition 2.6].

Lemma 2.9. *Suppose $\Phi \in \mathscr{U}$ or $\mathscr{L}$. Given a function $f \in A^\Phi_\alpha(\mathbb{B}_n, X)$, let f_r defined for $z \in \mathbb{B}_n$ by $f_r(z) := f(rz)$, where $0 < r < 1$. Then*

$$\lim_{r\to 1^-} \|f_r - f\|^{\mathrm{lux}}_{\Phi,\alpha,X} = 0.$$

In particular, the space of all bounded vector-valued holomorphic functions, $\mathcal{H}^\infty(\mathbb{B}_n, X)$, is dense in $A^\Phi_\alpha(\mathbb{B}_n, X)$.

Proof. Given a function $f \in A^\Phi_\alpha(\mathbb{B}_n, X)$, let f_ρ defined for $z \in \mathbb{B}_n$ by $f_\rho(z) := f(\rho z)$, where $0 < \rho < 1$. Then f_ρ is holomorphic in the set $\{z \in \mathbb{B}_n : |z| < 1/\rho\}$ hence is bounded on $\mathbb{B}_n$. We first recall that there exists $\lambda > 0$ such that

$$\int_{\mathbb{B}_n} \Phi\Big(2\frac{\|f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) < \infty.$$

For $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\int_{\{z\in\mathbb{B}_n : 1-\delta<|z|<1\}} \Phi\Big(2\frac{\|f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) < \varepsilon. \tag{2.4}$$

Let $\eta \in]0,\delta[$, we have $\mathbb{B}_n = \{z \in \mathbb{B}_n : |z| \leq 1-\eta\} \cup \{z \in \mathbb{B}_n : 1-\eta < |z| < 1\}$. The function $z \mapsto f_r(z) - f(z)$ is uniformly continuous on any compact subset of $\{z \in \mathbb{B}_n : |z| \leq 1-\eta\}$ then, there exists $R = R(\eta, \varepsilon) > 0$ such that for any $r \in (R, 1)$ and $z \in \mathbb{B}_n : |z| \leq 1-\eta$,

$$\|f_r(z) - f(z)\|_X < \varepsilon. \tag{2.5}$$

For $r \in \big(\max\big(R, \frac{1-\delta}{1-\eta}\big), 1\big)$,

$$\int_{\{z\in\mathbb{B}_n:|z|\leq 1-\eta\}} \Phi\Big(\frac{\|f_r(z) - f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) < \Phi\Big(\frac{\varepsilon}{\lambda}\Big). \tag{2.6}$$

Using the fact that Φ is increasing, we have

$$\begin{aligned} &\int_{\{z\in\mathbb{B}_n:1-\eta<|z|<1\}} \Phi\Big(\frac{\|f_r(z) - f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) \\ &\quad \leq \int_{\{z\in\mathbb{B}_n:1-\eta<|z|<1\}} \Phi\Big(\frac{2\|f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) \\ &\quad + \int_{\{z\in\mathbb{B}_n:1-\eta<|z|<1\}} \Phi\Big(\frac{2\|f_r(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z). \end{aligned} \tag{2.7}$$

Since $\eta < \delta$, we have $\{z \in \mathbb{B}_n : 1-\eta < |z| < 1\} \subset \{z \in \mathbb{B}_n : 1-\delta < |z| < 1\}$.

From (2.4), we have

$$\begin{aligned} &\int_{\{z\in\mathbb{B}_n:1-\eta<|z|<1\}} \Phi\Big(\frac{2\|f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) \\ &\quad \leq \int_{\{z\in\mathbb{B}_n:1-\delta<|z|<1\}} \Phi\Big(\frac{2\|f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) < \varepsilon. \end{aligned} \tag{2.8}$$

Since $r \in \big(\max\big(R, \frac{1-\delta}{1-\eta}\big), 1\big)$, we have $\{w \in \mathbb{B}_n : r(1-\eta) < |w| < r\} \subset \{w \in \mathbb{B}_n : 1-\delta < |w| < 1\}$, and taking $w = rz$, and by (2.4), we have

$$\begin{aligned} &\int_{\{z\in\mathbb{B}_n:1-\eta<|z|<1\}} \Phi\Big(\frac{2\|f_r(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) \\ &\quad \leq \frac{1}{r^{2n}} \int_{\{w\in\mathbb{B}_n:1-\delta<|w|<1\}} \Phi\Big(\frac{2\|f(w)\|_X}{\lambda}\Big)\, d\nu_\alpha(w) < \frac{1}{r^{2n}}\varepsilon. \end{aligned} \tag{2.9}$$

Using (2.9) and (2.8) in (2.7), we have

$$\int_{\{z\in\mathbb{B}_n:1-\eta<|z|<1\}} \Phi\Big(\frac{\|f_r(z) - f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) < \Big(\frac{1}{r^{2n}} + 1\Big)\varepsilon. \tag{2.10}$$

From (2.10) and (2.6), we obtain

$$\int_{\mathbb{B}_n} \Phi\Big(\frac{\|f_r(z) - f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) < \Phi\Big(\frac{\varepsilon}{\lambda}\Big) + \Big(\frac{1}{r^{2n}} + 1\Big)\varepsilon. \tag{2.11}$$

Taking r such that $2r^{2n} \geq 1$ in (2.11), we have

$$\int_{\mathbb{B}_n} \Phi\Big(\frac{\|f_r(z) - f(z)\|_X}{\lambda}\Big)\, d\nu_\alpha(z) < \Phi\Big(\frac{\varepsilon}{\lambda}\Big) + 3\varepsilon. \tag{2.12}$$

From this, using the fact that Φ is of lower type p for $\Phi \in \mathscr{L}$ or of lower type 1 for $\Phi \in \mathscr{U}$, we have

$$\int_{\mathbb{B}_n} \Phi\Big(\frac{\|f_r(z) - f(z)\|_X}{\lambda B}\Big)\, d\nu_\alpha(z) \leq 1, \tag{2.13}$$

with $B^p = 2C\big(\Phi\big(\frac{\varepsilon}{\lambda}\big) + 3\varepsilon\big)$, where C is the constant in the definition of lower type (1.4). Hence,

$$\|f_r - f\|^{\mathrm{lux}}_{\Phi,\alpha,X} \leq \lambda B. \tag{2.14}$$

Since ε is arbitrary, letting $\varepsilon \to 0$ in (2.14), we obtain

$$\lim_{r\to 1^-} \|f_r - f\|^{\mathrm{lux}}_{\Phi,\alpha,X} = 0.$$

This finishes the proof. $\square$

Corollary 2.10. *Suppose $\Phi \in \mathscr{U}$ or $\mathscr{L}$. Then the space of all vector-valued holomorphic polynomials $\mathcal{P}(\mathbb{B}_n, X)$ is dense in $A^\Phi_\alpha(\mathbb{B}_n, X)$.*

2.3. Some results on vector-valued Bergman spaces. In this subsection, X and Y are complex Banach spaces and $\alpha > -1$.

The following reproducing kernel formula also holds for vector-valued Bergman spaces. The proof can be found in [12, Proposition 2.1.2].

Proposition 2.11. *Let $f \in A^1_\alpha(\mathbb{B}_n, X)$. We have, for any $z \in \mathbb{B}_n$*

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}\, d\nu_\alpha(w).$$

We recall the following facts whose details can be found in [12]. Let $f \in \mathcal{H}(\mathbb{B}_n, X)$ and $f(z) = \sum_{k=0}^\infty f_k(z), \quad z \in \mathbb{B}_n$ the homogeneous expansion of the function f, where f_k is an homogeneous holomorphic polynomial of degree k with coefficients in X. For any two real parameters α and t such that neither $n+\alpha$ nor $n+\alpha+t$ is a negative integer, we define an invertible operator $R^{\alpha,t}:\mathcal{H}(\mathbb{B}_n,X)\to\mathcal{H}(\mathbb{B}_n,X)$ as

$$R^{\alpha,t}f(z):=\sum_{k=0}^{\infty}\frac{\Gamma(n+1+\alpha)\Gamma(n+1+\alpha+k+t)}{\Gamma(n+1+\alpha+t)\Gamma(n+1+\alpha+k)}f_k(z). \tag{2.15}$$

where $z\in\mathbb{B}_n$ and Γ is the classical Euler Gamma function.

PROPOSITION 2.12. *Let $\alpha>-1$, $f\in A^1_\alpha(\mathbb{B}_n,X)$ and $t>0$. Then*

$$R^{\alpha,t}f(z)=\int_{\mathbb{B}_n}\frac{f(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+t}}\,d\nu_\alpha(w),$$

for any $z\in\mathbb{B}_n$.

COROLLARY 2.13. *Suppose $t>0$ and $1<r<\infty$. If $b\in A^r_\alpha\big(\mathbb{B}_n,\mathcal{L}(\overline{X},Y)\big)$ then the equality*

$$\int_{\mathbb{B}_n}\langle b(z)\big(\overline{f(z)}\big),g(z)\rangle_{Y,Y^\star}\,d\nu_\alpha(z)=\int_{\mathbb{B}_n}\langle\big(R^{\alpha,t}b(z)\big)\,\big(\overline{f(z)}\big),g(z)\rangle_{Y,Y^\star}d\nu_{\alpha+t}(z)$$

holds whenever $f\in\mathcal{H}^\infty(\mathbb{B}_n,X)$ and $g\in\mathcal{H}^\infty(\mathbb{B}_n,Y^\star)$.

The following theorem gives us several conditions that are equivalent to be in the vector-valued Bloch space.

THEOREM 2.14 [12]. *Suppose $t>0$ and $\alpha>-1$. If $f\in\mathcal{H}(\mathbb{B}_n,X)$, then the following conditions are equivalent.*

(i) *The function f is in $\mathcal{B}(\mathbb{B}_n,X)$.*

(ii) *We have $f=P_\alpha g$ for some $g\in L^\infty(\mathbb{B}_n,X)$.*

(iii) *The function $f_{\alpha,t}(z):=(1-|z|^2)^tR^{\alpha,t}f(z)$ is in $L^\infty(\mathbb{B}_n,X)$.*

Moreover,

$$\|f\|_{\mathcal{B}(\mathbb{B}_n;X)}\simeq\|g\|_{\infty,X}\simeq\|f_{\alpha,t}\|_{\infty,X}.$$

We recall the following property of vector-valued Bloch spaces that can be found in [12].

LEMMA 2.15. *Let $\alpha>-1$. Then $\mathcal{B}(\mathbb{B}_n,X)\subset A^p_\alpha(\mathbb{B}_n,X)$, for any $1\le p<\infty$.*

To finish this subsection, we recall the duality theorem for the vector-valued Bergman space $A^1_\alpha(\mathbb{B}_n,X)$ (see [2]). The result is stated as follows.

THEOREM 2.16 (duality). *The dual space $(A^1_\alpha(\mathbb{B}_n,X))^\star$ of $A^1_\alpha(\mathbb{B}_n,X)$ can be identified with $\mathcal{B}(\mathbb{B}_n,X^\star)$ under the pairing defined by*

$$\langle f,g\rangle_{\alpha,X}=\int_{\mathbb{B}_n}\langle f(z),g(z)\rangle_{X,X^\star}\,d\nu_\alpha(z),$$

for any $f\in H^\infty(\mathbb{B}_n,X)$ and $g\in\mathcal{B}\,(\mathbb{B}_n,X^\star)$, with equivalent norms.

2.4. Vector-valued measures. (See [9,10,18].) In this subsection, we concentrate on vector-valued measures and, more specifically, on the theory of integration with respect to such measures. Throughout this subsection, Σ is a σ-algebra of $\mathbb{B}_n$, and ν_α the normalized measure defined in (1.1).

Definition 2.17. Let X a Banach space. A function $m\colon \Sigma \to X$ is called a finitely additive vector measure if $m\big(\bigcup_{n=1}^k E_n\big) = \sum_{n=1}^k m(E_n)$ for all finite collections $\{E_n\}_{n=1}^k$ of pairwise disjoint sets. If, in addition, m satisfies

$$m\bigg(\bigcup_{n=1}^{\infty} E_n\bigg) = \sum_{n=1}^{\infty} m(E_n), \tag{2.16}$$

for all sequence of pairwise disjoint sets $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$, where the series is convergent in the norm topology of X, then m is simply called a vector-valued measure. We also say that m is $\sigma-$ additive on Σ. If $m\colon \Sigma \to \mathbb{R}^+$ and satisfies (2.16) then m is called a positive measure.

Example 2.18 [9]. Let $T\colon L^1(\mathbb{B}_n) \to X$ be a continuous linear operator. For each Lebesgue measurable set $E \subseteq \mathbb{B}_n$, define $m(E)$ to be $T(\chi_E)$ (χ_E denotes the characteristic or indicator function of E). Then m is a vector-valued measure.

Definition 2.19. Let $m\colon \Sigma \to X$ be a vector-valued measure. The variation of m is the extended nonnegative measure denoted by $|m|$ and defined for $E \in \Sigma$ by

$$|m|(E) := \sup_{\Pi} \sum_{A\in\Pi} \|m(A)\|_X$$

where the supremum is taken over all finite partitions Π of E. If $|m|(\mathbb{B}_n)<\infty$, then m is of bounded variation.

Example 2.20. Let λ be the Lebesgue measure in $\mathbb{B}_n$ and m a measure of the type discussed in Example 2.18. Since $\|m(E)\|_X \le \lambda(E)\|T\|$, it is clear that $|m|(\mathbb{B}_n) \le \|T\|$, so that m is of bounded variation.

Proposition 2.21. *A vector-valued measure of bounded variation is countably additive if and only if its variation is also countably additive.*

Definition 2.22. Given a Banach space X, if m is an $X^\star$-valued measure of bounded variation, and $s = \sum_{i=1}^k a_i \chi_{A_i}$ is a simple X-valued function, then we define the integral of s by

$$\int_{\mathbb{B}_n} s(z)\, dm(z) = \sum_{i=1}^k \langle a_i, m(A_i)\rangle_{X,X^\star}.$$

Since $\left|\int_{\mathbb{B}_n} s(z)\,dm(z)\right| \le \|s\|_\infty |m|(\mathbb{B}_n)$, using the density of simple functions in $L^\infty(\mathbb{B}_n, X)$, we extend the continuous linear functional $\int_{\mathbb{B}_n} s(z)\,dm(z)$ to any X-valued bounded function.

Definition 2.23. Let $m\colon \Sigma \to X$ be a vector-valued measure. We say that m is absolutely continuous with respect to ν_α (i.e. ν_α-continuous) also denoted by $m \ll \nu_\alpha$, if and only if $m(E) = 0$ whenever $E \in \Sigma$ satisfies $\nu_\alpha(E) = 0$.

Definition 2.24. Let X be a complex Banach space, $\alpha > -1$ and $\Phi \in \mathscr{U}$. Let Ψ be the complementary function of Φ. We define the space $V_\alpha^\Phi(\mathbb{B}_n, X)$ as the space of all countably additive vector-valued measure $m\colon \Sigma \to X$ (ν_α-continuous) for which the supremum

$$\sup\left\{\sum_{A\in\Pi} |\lambda_A| \|m(A)\|_X : \left\|\sum_{A\in\Pi} \lambda_A \chi_A\right\|_{\Psi,\alpha}^{\mathrm{lux}} \le 1\right\}$$

is finite. Here the supremum is taken over all finite partitions Π of $\mathbb{B}_n$. This supremun is called the Φ-variation of m and it is denoted by $\|m\|_{\Phi,\alpha,X}$.

Remark 2.25. A simple argument allows us to replace $\|m(A)\|_X$ by $|m|(A)$ in the previous definition, we thus obtain

$$\|m\|_{\Phi,\alpha,X} = \sup\left\{\sum_{A\in\Pi} |\lambda_A| |m|(A) : \left\|\sum_{A\in\Pi} \lambda_A \chi_A\right\|_{\Psi,\alpha}^{\mathrm{lux}} \le 1\right\}.$$

As in the theory of Orlicz spaces, the structure of $V_\alpha^\Phi(\mathbb{B}_n, X)$ will be analyzed with the introduction of the norm

$$\|m\|_{V_\alpha^\Phi(\mathbb{B}_n,X)} = \inf\left\{\lambda > 0 : \sup_{\Pi} \sum_{A\in\Pi} \nu_\alpha(A)\Phi\left(\frac{\|m(A)\|_X}{\lambda\,\nu_\alpha(A)}\right) \le 1\right\},$$

the supremum is taken over all finite partitions Π and the convention $\frac{0}{0} = 0$ is used.

Proposition 2.26 [18, Theorem 19]. *Let X be a complex Banach space, $\alpha > -1$ and $\Phi \in \mathscr{U}$. For any vector-valued measure $m \in V_\alpha^\Phi(\mathbb{B}_n, X)$, we have*

$$\|m\|_{V_\alpha^\Phi(\mathbb{B}_n,X)} \le \|m\|_{\Phi,\alpha,X} \le 2\|m\|_{V_\alpha^\Phi(\mathbb{B}_n,X)}.$$

Theorem 2.27 [18, Theorem 11]. *Let X be a complex Banach space, $\alpha > -1$ and $\Phi \in \mathscr{U}$. Then $V_\alpha^\Phi(\mathbb{B}_n, X)$ is a complex Banach space.*

Now we can give this characterization of vector-valued measures of bounded Φ-variation in $\mathbb{B}_n$ who extends to the Orlicz setting what is known for vector-valued Lebesgue spaces.

LEMMA 2.28 [10]. *Let X be a complex Banach space, $\alpha > -1$ and $\Phi \in \mathscr{U}$. Let Ψ be the complementary function of Φ and suppose that Φ satisfies the ∇_2-condition. For any vector-measure $m \in V_\alpha^\Phi(\mathbb{B}_n, X)$, there exists $h \in L_\alpha^\Phi(\mathbb{B}_n, \mathbb{R}_+)$ such that*

$$d|m| = h\,d\nu_\alpha \quad \textit{and} \quad \|h\|_{\Phi,\alpha}^{\mathrm{lux}} \simeq \|m\|_{V_\alpha^\Phi(\mathbb{B}_n,X)}.$$

PROOF. For $m \in V_\alpha^\Phi(\mathbb{B}_n, X)$, the set function $|m|$ is ν_α-continuous positive measure. The Radon-Nikodym theorem provides a non negative function $h \in L_\alpha^1(\mathbb{B}_n, \mathbb{R}_+)$ such that

$$|m|(A) = \int_A h(z)\,d\nu_\alpha(z), \quad A \in \Sigma. \tag{2.17}$$

Using the duality of $L_\alpha^\Phi(\mathbb{B}_n)$ (see [8, Theorem 2.5]), we have

$$\|h\|_{\Phi,\alpha}^{\mathrm{lux}} = \sup\left\{\left|\int_{\mathbb{B}_n} h(z)s(z)\,d\nu_\alpha(z)\right| : s \in L_\alpha^\Psi(\mathbb{B}_n),\ \|s\|_{\Psi,\alpha}^{\mathrm{lux}} \le 1\right\}.$$

Approximating the supremum with the use of simple functions in $L_\alpha^\Psi(\mathbb{B}_n)$ and using (2.17) we obtain that

$$\|h\|_{\Phi,\alpha}^{\mathrm{lux}} = \sup\left\{\sum_{A\in\Pi} |\lambda_A||m|(A) : \left\|\sum_{A\in\Pi} \lambda_A \chi_A\right\|_{\Psi,\alpha}^{\mathrm{lux}} \le 1\right\} = \|m\|_{\Phi,\alpha,X}.$$

The desired result follows at once using Proposition 2.26. $\square$

This allows, as in the scalar valued case, to get the duality result for vector-valued Orlicz-spaces. The proof is adapted from the proof in the classical weighted vector-valued Lebesgue spaces [10].

THEOREM 2.29. *Let $\Phi \in \mathscr{U}$, $\alpha > -1$ and Ψ the complementary function of Φ. Suppose that Φ satisfies the ∇_2-condition. Let X be a Banach space. Then the topological dual space $(L_\alpha^\Phi(\mathbb{B}_n, X))^\star$ of $L_\alpha^\Phi(\mathbb{B}_n, X)$ can be identified with $V_\alpha^\Psi(\mathbb{B}_n, X^\star)$ under the duality pairing*

$$\langle f, m\rangle_{\alpha,X} = \int_{\mathbb{B}_n} f(z)\,dm(z), \tag{2.18}$$

where $f \in L_\alpha^\Phi(\mathbb{B}_n, X)$ and $m \in V_\alpha^\Psi(\mathbb{B}_n, X^\star)$. Moreover,

$$\|m\|_{V_\alpha^\Psi(\mathbb{B}_n,X^\star)} \simeq \sup_{\|f\|_{\Phi,\alpha,X}^{\mathrm{lux}}=1} |\langle f, m\rangle_{\alpha,X}|.$$

PROOF. We first suppose that $m \in V_\alpha^\Psi(\mathbb{B}_n, X^\star)$ and define the functional

$$\bigwedge_m : L_\alpha^\Phi(\mathbb{B}_n, X) \to \mathbb{C} f \mapsto \bigwedge_m(f) = \int_{\mathbb{B}_n} f(z)\, dm(z).$$

It is clear that $\bigwedge_m$ is linear and is well defined on $L_\alpha^\Phi(\mathbb{B}_n, X)$. Indeed, by Lemma 2.28, there exists $h \in L_\alpha^\Psi(\mathbb{B}_n, \mathbb{R}_+)$ such that

$$d|m|(z) = h(z)\, d\nu_\alpha(z) \quad \text{and} \quad \|h\|_{\Psi,\alpha}^{\mathrm{lux}} \lesssim \|m\|_{V_\alpha^\Psi(\mathbb{B}_n, X^\star)}. \tag{2.19}$$

Let $f \in L_\alpha^\Phi(\mathbb{B}_n, X)$, using (2.19) and Hölder's inequality (1.9), we have

$$\begin{aligned}\left|\bigwedge_m(f)\right| &\le \int_{\mathbb{B}_n} \|f(z)\|_X d|m|(z) = \int_{\mathbb{B}_n} \|f(z)\|_X h(z)\, d\nu_\alpha(z)\\ &\le 2\|f\|_{\Phi,\alpha,X}^{\mathrm{lux}} \|h\|_{\Psi,\alpha}^{\mathrm{lux}} \lesssim \|f\|_{\Phi,\alpha,X}^{\mathrm{lux}} \|m\|_{V_\alpha^\Psi(\mathbb{B}_n, X^*)}.\end{aligned}$$

We conclude that $\bigwedge_m$ is bounded on $L_\alpha^\Phi(\mathbb{B}_n, X)$ and

$$\left\|\bigwedge_m\right\|_{(L_\alpha^\Phi(\mathbb{B}_n, X))^\star} := \sup_{\|f\|_{\Phi,\alpha,X}^{\mathrm{lux}}=1} \left|\bigwedge_m(f)\right| \lesssim \|m\|_{V_\alpha^\Psi(\mathbb{B}_n, X^\star)}.$$

Conversely, let G be a bounded linear functional on $L_\alpha^\Phi(\mathbb{B}_n, X)$. Let us show that there exists $m \in V_\alpha^\Psi(\mathbb{B}_n, X^\star)$ such that $G = \bigwedge_m$. For every set $A \in \Sigma$, the mapping

$$m(A) : X \to \mathbb{C} \quad x \longmapsto \langle x, m(A)\rangle_{X,X^\star} = G(x\,\chi_A)$$

is linear and continuous. Indeed, by the continuity of G, using (2.1) we have

$$\begin{aligned}|m(A)\,x| &= |G(x\,\chi_A)| \le \|G\|_{(L_\alpha^\Phi(\mathbb{B}_n, X))^\star} \|x\,\chi_A\|_{\Phi,\alpha,X}^{\mathrm{lux}}\\ &\lesssim \|G\|_{(L_\alpha^\Phi(\mathbb{B}_n, X))^\star} \nu_\alpha(A) \Psi^{-1}\Big(\frac{1}{\nu_\alpha(A)}\Big)\|x\|_X.\end{aligned}$$

Then,

$$\|m(A)\|_{X^*} = \sup_{\|x\|_X=1} |m(A)\,x| \lesssim \|G\|_{(L_\alpha^\Phi(\mathbb{B}_n, X))^\star} \nu_\alpha(A) \Psi^{-1}\Big(\frac{1}{\nu_\alpha(A)}\Big). \tag{2.20}$$

The mapping $m : \Sigma \to X^\star$ is countably additive and ν_α-continuous.

$$\|m\|_{\Psi,\alpha,X^\star} = \sup\Big\{ \sum_{A\in\Pi} |\lambda_A| \|m(A)\|_{X^\star} : \Big\|\sum_{A\in\Pi} \lambda_A \chi_A\Big\|_{\Phi,\alpha}^{\mathrm{lux}} \le 1\Big\}$$

$$= \sup\Big\{ \sum_{A\in\Pi} \left|\langle |\lambda_A| x, m(A)\rangle_{X,X^\star}\right| : \Big\|\sum_{A\in\Pi} \lambda_A \chi_A\Big\|_{\Phi,\alpha}^{\mathrm{lux}} \le 1,\ \|x\|_X = 1\Big\}$$

$$\leq \sup\left\{ \sum_{A\in\Pi} \left|\langle y_A, m(A)\rangle_{X,X^\star}\right| : \left\|\sum_{A\in\Pi} y_A\chi_A\right\|^{\text{lux}}_{\Phi,\alpha,X} \leq 1\right\}$$

Now, using the fact that $\langle y_A, m(A)\rangle_{X,X^\star} = |\langle y_A, m(A)\rangle_{X,X^\star}|e^{i\theta_A}, \theta_A \in \mathbb{R}$, we have

$$\|m\|_{\Psi,\alpha,X^\star} = \sup\left\{ \sum_{A\in\Pi} \langle e^{-i\theta_A} y_A, m(A)\rangle_{X,X^\star} : \left\|\sum_{A\in\Pi} y_A\chi_A\right\|^{\text{lux}}_{\Phi,\alpha,X} \leq 1\right\}$$

$$\leq \sup\left\{ \sum_{A\in\Pi} \langle z_A, m(A)\rangle_{X,X^\star} : \left\|\sum_{A\in\Pi} z_A\chi_A\right\|^{\text{lux}}_{\Phi,\alpha,X} \leq 1,\ \langle z_A, m(A)\rangle_{X,X^\star} \geq 0\right\}$$

$$= \sup\left\{ \sum_{A\in\Pi} G(z_A\chi_A) : \left\|\sum_{A\in\Pi} z_A\chi_A\right\|^{\text{lux}}_{\Phi,\alpha,X} \leq 1\right\}$$

$$= \sup\left\{ G\left(\sum_{A\in\Pi} z_A\chi_A\right) : \left\|\sum_{A\in\Pi} z_A\chi_A\right\|^{\text{lux}}_{\Phi,\alpha,X} \leq 1\right\} \leq \|G\|_{(L^\Phi_\alpha(\mathbb{B}_n,X))^\star}.$$

From Definition 2.24 and Proposition 2.26, we have $m \in V^\Psi_\alpha(\mathbb{B}_n, X^\star)$ and

$$\|m\|_{V^\Psi_\alpha(\mathbb{B}_n,X^\star)} \leq \|G\|_{(L^\Phi_\alpha(\mathbb{B}_n,X))^\star}.$$

For every σ-step function $s = \sum_{i=1}^k x_i\chi_{A_i}$, we have

$$G(s) = G\left(\sum_{i=1}^k x_i\chi_{A_i}\right) = \sum_{i=1}^k G(x_i\chi_{A_i}) = \sum_{i=1}^k m(A_i)x_i = \int_{\mathbb{B}_n} s(z)\,dm(z).$$

To finish the proof, it remains to show that (2.18) remains true for functions in $L^\Phi_\alpha(\mathbb{B}_n, X)$ which is a direct consequence of the density in Lemma 2.8. $\square$

Definition 2.30. Let X be a complex Banach space and $m\colon \Sigma \to X$ be a measure of bounded variation. We define the Bergman projection of the vector measure m as the analytic function in the unit ball $\mathbb{B}_n$ given by

$$P_\alpha m(z) = \int_{\mathbb{B}_n} \frac{dm(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}, \quad z \in \mathbb{B}_n.$$

Since $\sup_{z\in\mathbb{B}_n} \frac{1}{|1-\langle z,w\rangle|^{n+1+\alpha}} \leq \frac{1}{(1-|z|)^{n+1+\alpha}}$, then $P_\alpha m$ is well defined. We are now ready to obtain the boundedness of the Bergman projection on the vector-valued Lebesgue Orlicz space $L^\Phi_\alpha(\mathbb{B}_n, X)$. We will use the following result from [8, Theorem 6.2] proved for $n = 1$ and the same proof works for any n.

Theorem 2.31. *Let $\Phi \in \mathscr{U}$ and $\alpha > -1$. Suppose Φ satisfies the ∇_2-condition. Then the positive Bergman projection P_α^+ is bounded on the Orlicz space $L_\alpha^\Phi(\mathbb{B}_n)$, where $P_\alpha^+ h(z) = \int_{\mathbb{B}_n} \frac{h(w)\, d\nu_\alpha(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}$.*

We deduce the following result which is important in the proof of the duality result in the case of convex Orlicz functions.

Theorem 2.32. *Let $\Phi \in \mathscr{U}$ and $\alpha > -1$. Suppose Φ satisfies the ∇_2-condition. Then the Bergman projection P_α is bounded from $V_\alpha^\Phi(\mathbb{B}_n, X)$ onto $A_\alpha^\Phi(\mathbb{B}_n, X)$.*

Proof. Since $m \in V_\alpha^\Phi(\mathbb{B}_n, X)$, from Lemma 2.28, there exists a nonnegative function h in $L_\alpha^\Phi(\mathbb{B}_n, \mathbb{R}_+)$ such that $d|m|(z) = h(z)\, d\nu_\alpha(z)$ and $\|h\|_{\Phi,\alpha}^{\mathrm{lux}} \simeq \|m\|_{V_\alpha^\Phi(\mathbb{B}_n,X)}$. Now, for each $z \in \mathbb{B}_n$ we have

$$\|P_\alpha m(z)\|_X = \left\| \int_{\mathbb{B}_n} \frac{dm(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}} \right\|_X$$
$$\le \int_{\mathbb{B}_n} \frac{d|m|(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}} = \int_{\mathbb{B}_n} \frac{h(w)\, d\nu_\alpha(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}} = P_\alpha^+ h(z).$$

To finish the proof, we now make use of Theorem 2.31 to obtain that $\|P_\alpha^+ h\|_{\Phi,\alpha}^{\mathrm{lux}} \lesssim \|h\|_{\Phi,\alpha}^{\mathrm{lux}} \simeq \|m\|_{V_\alpha^\Phi(\mathbb{B}_n,X)}$. Then

$$\|P_\alpha m\|_{\Phi,\alpha,X}^{\mathrm{lux}} \lesssim \|m\|_{V_\alpha^\Phi(\mathbb{B}_n,X)}. \quad \square$$

2.5. Some useful estimates. The following result will be very useful in many situations.

Theorem 2.33 [21, Theorem 1.12]. *Let $\alpha > -1$. For $\beta \in \mathbb{R}$, let*

$$I_{\alpha,\beta}(z) := \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha\, d\nu_\alpha(w)}{|1-\langle z,w\rangle|^{n+1+\alpha+\beta}}, \quad z \in \mathbb{B}_n.$$

(i) *If $\beta = 0$, there exists a constant $C > 0$ such that*

$$I_{\alpha,\beta}(z) \le C \log \frac{1}{1-|z|^2}.$$

(ii) *If $\beta > 0$, there exists a constant $C > 0$ such that*

$$I_{\alpha,\beta}(z) \le C \frac{1}{(1-|z|^2)^\beta}.$$

(iii) *If $\beta < 0$, there exists a constant $C > 0$ such that*

$$I_{\alpha,\beta}(z) \le C.$$

The next lemma is a generalization of the inequality

$$\int_{\mathbb{B}_n} \|f(z)\|_X (1-|z|^2)^{(\frac{1}{p}-1)(n+1+\alpha)}\, d\nu_\alpha(z) \lesssim \|f\|_{p,\alpha,X},$$

known for vector-valued Bergman spaces with exponent $0<p<1$ (see [3, Lemma 15]). As in the classical weighted vector-valued Bergman spaces, this lemma is crucial to characterize the dual space $\big(A^\Phi_\alpha(\mathbb{B}_n,X)\big)^\star$ of $A^\Phi_\alpha(\mathbb{B}_n,X)$ for $\Phi\in\mathscr{L}$. The proof follows exactly as in [17, Lemma 3.1].

Lemma 2.34. *Let $\alpha>-1$, $\Phi\in\mathscr{L}$ and $\rho(t)=\frac{1}{t\,\Phi^{-1}(\frac{1}{t})}$. There is a constant $C>1$ such that for any $f\in A^\Phi_\alpha(\mathbb{B}_n,X)$,*

$$\int_{\mathbb{B}_n} \|f(z)\|_X \rho\big((1-|z|^2)^{n+1+\alpha}\big)\, d\nu_\alpha(z) \le C\|f\|^{\mathrm{lux}}_{\Phi,\alpha,X}. \tag{2.21}$$

This lemma will be crucial to characterize the boundedness of Hankel operators from $A^{\Phi_1}_\alpha(\mathbb{B}_n,X)$ into $A^{\Phi_2}_\alpha(\mathbb{B}_n,Y)$, $\Phi_1,\Phi_2\in\mathscr{L}$. The next technical results are easy extension to the vector-valued case of the results in the scalar case proved in [17]. We will use them in Section 4.

Lemma 2.35. *Let $\alpha>-1$, $\Phi\in\mathscr{U}$. Let $x\in X$, $w\in\mathbb{B}_n$ and $t>0$. Then the function*

$$f^x_w(z)=\frac{x}{(1-\langle z,w\rangle)^t},\quad z\in\mathbb{B}_n,$$

belongs to $\mathcal{H}^\infty(\mathbb{B}_n,X)$ and to $A^\Phi_\alpha(\mathbb{B}_n,X)$ and we have

$$\|f^x_w\|^{\mathrm{lux}}_{\Phi,\alpha,X} \lesssim \frac{\|x\|_X}{(1-|w|^2)^t\Phi^{-1}\big(\frac{1}{(1-|w|^2)^{n+1+\alpha}}\big)},\quad \textit{for } t>n+1+\alpha.$$

Lemma 2.36. *Let $\alpha>-1$ and $\Phi\in\mathscr{L}_p$. Let $x\in X$, $w\in\mathbb{B}_n$ and $k>(n+1+\alpha)\big(\frac{1}{p}-1\big)$. Then the function*

$$f^x_w(z)=\Phi^{-1}\Big(\frac{1}{(1-|w|^2)^{n+1+\alpha}}\Big)\frac{(1-|w|^2)^{n+1+\alpha+k}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}\,x,\quad z\in\mathbb{B}_n,$$

is uniformly in $A^\Phi_\alpha(\mathbb{B}_n,X)$. Precisely, there exists $C>0$ independent of w such that

$$\|f^x_w\|^{\mathrm{lux}}_{\Phi,\alpha,X} \le C\|x\|_X.$$

Lemma 2.37. *Let $\alpha>-1$ and $\Phi\in\mathscr{L}_p$. Let $x\in X$, an integer*

$$k>(n+1+\alpha)\Big(\frac{1}{p}-1\Big)$$

and fix $w \in \mathbb{B}_n$. Then the function

$$\varphi_w^x(z) = \log\left(\frac{1-\langle z,w\rangle}{1-|w|^2}\right)\Phi^{-1}\Big(\frac{1}{(1-|w|^2)^{n+1+\alpha}}\Big)\frac{(1-|w|^2)^{n+1+\alpha+k}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}\,x,$$

$z \in \mathbb{B}_n$, belongs to $\mathcal{H}^\infty(\mathbb{B}_n;X)$ and to $A_\alpha^\Phi(\mathbb{B}_n,X)$; and there exists $C>0$, independent of w and x such that

$$\|\varphi_w^x\|_{\Phi,\alpha,X}^{\mathrm{lux}} \le C\|x\|_X.$$

Lemma 2.38. *Let $\alpha > -1$ and $\Phi \in \mathscr{U}^q$. Let Ψ be the complementary function of Φ. Let $x \in X$ and $w \in \mathbb{B}_n$. Then the function*

$$f_w^x(z) = \Psi^{-1}\Big(\frac{1}{(1-|w|^2)^{n+1+\alpha}}\Big)\frac{(1-|w|^2)^{n+2+\alpha}}{(1-\langle z,w\rangle)^{n+2+\alpha}}\,x, \quad z \in \mathbb{B}_n,$$

is uniformly in $A_\alpha^\Phi(\mathbb{B}_n,X)$. More precisely, there exists $C>0$ independent of w such that

$$\|f_w^x\|_{\Phi,\alpha,X}^{\mathrm{lux}} \le C\|x\|_X.$$

2.6. Differential operators and equivalent norms for $\boldsymbol{\Gamma_{\alpha,\rho}(\mathbb{B}_n;X)}$. Given a positive integer k, we define the differential operator M_k^α by

$$M_k^\alpha := [(n+\alpha+k)I+N]\circ\cdots\circ[(n+\alpha+2)I+N]\circ[(n+\alpha+1)I+N], \tag{2.22}$$

where I is the identity operator and N is the differential operator given in (1.11). There exists $C_{k,\alpha}>0$ such that for any $x \in X$,

$$M_k^\alpha\Big(\frac{x}{(1-\langle z,w\rangle)^{n+1+\alpha}}\Big) = \frac{C_{k,\alpha}x}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}, \tag{2.23}$$

It is easy to see that for every $f \in A_\alpha^1(\mathbb{B}_n;X)$,

$$M_k^\alpha f(z) = C_{k,\alpha}\int_{\mathbb{B}_n}\frac{f(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}\,d\nu_\alpha(w), \quad z \in \mathbb{B}_n. \tag{2.24}$$

By using Proposition 2.11, Lemma 1.5, the Fubini Theorem and (2.24), we easily obtain the following result.

Lemma 2.39. *For all $f \in \mathcal{H}^\infty(\mathbb{B}_n,X)$ and $g \in \mathcal{H}^\infty(\mathbb{B}_n,X^\star)$, the following equality holds*:

$$\int_{\mathbb{B}_n}\langle f(z),g(z)\rangle_{X,X^\star}\,d\nu_\alpha(z) = C_{k,\alpha}^{-1}\int_{\mathbb{B}_n}\langle f(z),M_k^\alpha g(z)\rangle_{X,X^\star}\,d\nu_{\alpha+k}(z),$$

where $C_{k,\alpha}$ is defined by (2.23). The above identities are valid for vector-valued holomophic functions when both sides make sense.

The following lemma will be very useful in the sequel. The proof is similar as in the classical vector-valued Bergman space in [3, Lemma 20].

LEMMA 2.40. *Let $\{a_k\}$ a sequence of positive numbers. For any positive integer k, let M_k the differential operator of order k defined by*

$$M_k = (a_0 I + N) \circ (a_1 I + N) \circ \cdots \circ (a_{k-1} I + N).$$

For a weight ρ of upper type γ, f belongs to $\Gamma_{\alpha,\rho}(\mathbb{B}_n, X)$ if and only if there exists an integer $k > \gamma(n+1+\alpha)$ and a positive constant $C > 0$ such that

$$\|M_k f(z)\|_X \le C(1-|z|^2)^{-k} \rho\big((1-|z|^2)^{n+1+\alpha}\big).$$

The same is true for $L\Gamma_{\alpha,\rho}(\mathbb{B}_n, X)$.

As a consequence of this fact, we will write the equivalent norms of f in terms of M_k^α. More precisely, we have the following result:

COROLLARY 2.41. *Let M_k^α the differential operator of order k defined in* (2.22). *For a weight ρ of upper type γ, for vector-valued holomorphic functions, we have*

(i) $f \in \Gamma_{\alpha,\rho}(\mathbb{B}_n, X)$ *if only if there exists an integer $k > \gamma(n+1+\alpha)$ such that*

$$\sup_{z \in \mathbb{B}_n} \frac{(1-|z|^2)^k \|M_k^\alpha f(z)\|_X}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)} < \infty.$$

(ii) $f \in L\Gamma_{\alpha,\rho}(\mathbb{B}_n, X)$ *if only if there exists an integer $k > \gamma(n+1+\alpha)$ such that*

$$\sup_{z \in \mathbb{B}_n} \frac{(1-|z|^2)^k \|M_k^\alpha f(z)\|_X}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)} \big|\log(1-|z|^2)\big| < \infty.$$

Moreover, the following hold:

$$\|f\|_{\Gamma_{\alpha,\rho}(\mathbb{B}_n,X)} \simeq \|f(0)\|_X + \sup_{z \in \mathbb{B}_n} \frac{(1-|z|^2)^k \|M_k^\alpha f(z)\|_X}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)},$$

$$\|f\|_{L\Gamma_{\alpha,\rho}(\mathbb{B}_n,X)} \simeq \|f(0)\|_X + \sup_{z \in \mathbb{B}_n} \frac{(1-|z|^2)^k \|M_k^\alpha f(z)\|_X}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)} \big|\log(1-|z|^2)\big|.$$

The proof of some of our results obtained here will be based on the following lemmas. See [3] for their proofs.

LEMMA 2.42. *Let $f \in \mathcal{H}^\infty(\mathbb{B}_n, X)$ and $g \in \mathcal{H}^\infty(\mathbb{B}_n, Y^\star)$. Suppose that $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ is such that* (1.12) *and* (1.13) *hold. Then*

$$\langle h_b(f), g\rangle_{\alpha,Y} = \int_{\mathbb{B}_n} \big\langle b(z)\big(\overline{f(z)}\big), g(z)\big\rangle_{Y,Y^\star} \, d\nu_\alpha(z).$$

Lemma 2.43. *Let $f \in \mathcal{H}^\infty(\mathbb{B}_n, X)$ and $z \in \mathbb{B}_n$. For $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ satisfying* (1.12) *and* (1.13), *the function*

$$g_z(w) := \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}}, \quad w \in \mathbb{B}_n,$$

belongs to $\mathcal{H}^\infty(\mathbb{B}_n, X)$ and the following identity holds:

$$h_b(f)(z) = C_k \int_{\mathbb{B}_n} M_k^\alpha\big(b(w)\big(\overline{g_z(w)}\big)\big)\, d\nu_{\alpha+k}(w),$$

where k is any positive integer and C_k is a positive constant depending only on k.

Lemma 2.44. *Let $f \in \mathcal{H}^\infty(\mathbb{B}_n, X)$ and $g \in \mathcal{H}^\infty(\mathbb{B}_n, Y^\star)$. Suppose that $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ is such that* (1.12) *and* (1.13) *hold. Then we have*

$$\begin{aligned} \langle h_b(f), g \rangle_{\alpha,Y} &= \int_{\mathbb{B}_n} \big\langle b(z)\big(\overline{f(z)}\big), g(z) \big\rangle_{Y,Y^\star}\, d\nu_\alpha(z) \\ &= C_k \int_{\mathbb{B}_n} \langle M_k^\alpha\big(b(z)\big(\overline{f(z)}\big)\big), g(z) \rangle_{Y,Y^\star}\, d\nu_{\alpha+k}(z). \end{aligned} \tag{2.25}$$

3. Duality for vector-valued Bergman–Orlicz spaces

In this section, we prove Theorem 1.11 and Theorem 1.12 about the characterisation of the topological dual space of $A_\alpha^\Phi(\mathbb{B}_n, X)$ for $\Phi \in \mathscr{L} \cup \mathscr{U}$.

3.1. Duality of $A_\alpha^\Phi(\mathbb{B}_n, X)$, for $\Phi \in \mathscr{U}$.

We will first prove the following embedding result which is of independent interest.

Lemma 3.1. *For $\Phi \in \mathscr{U}$, the following inclusion holds*:

$$A_\alpha^\Phi(\mathbb{B}_n, X) \subset A_\alpha^1(\mathbb{B}_n, X).$$

Proof. Let $f \in A_\alpha^\Phi(\mathbb{B}_n, X)$ such that $\|f\|_{\Phi,\alpha,X}^{\mathrm{lux}} > 0$. Since $\Phi \in \mathscr{U}$, using the fact that $t \mapsto \frac{\Phi(t)}{t}$ is nondecreasing, we have for all $z \in \mathbb{B}_n$,

$$\begin{aligned} \frac{\|f(z)\|_X}{\|f\|_{\Phi,\alpha,X}^{\mathrm{lux}}} \geq 1 \implies \frac{\Phi\Big(\frac{\|f(z)\|_X}{\|f\|_{\Phi,\alpha,X}^{\mathrm{lux}}}\Big)}{\frac{\|f(z)\|_X}{\|f\|_{\Phi,\alpha,X}^{\mathrm{lux}}}} \geq \Phi(1) \\ \iff \|f(z)\|_X \leq \frac{\|f\|_{\Phi,\alpha,X}^{\mathrm{lux}}}{\Phi(1)} \Phi\bigg(\frac{\|f(z)\|_X}{\|f\|_{\Phi,\alpha,X}^{\mathrm{lux}}}\bigg). \end{aligned} \tag{3.1}$$

From the relation (3.1) and the definition of $\|.\|^{\text{lux}}_{\Phi,\alpha,X}$ we have

$$\int_{\mathbb{B}_n} \|f(z)\|_X \, d\nu_\alpha(z) \tag{3.2}$$

$$= \int_{\left\{z : \frac{\|f(z)\|_X}{\|f\|^{\text{lux}}_{\Phi,\alpha,X}} < 1\right\}} \|f(z)\|_X \, d\nu_\alpha(z) + \int_{\left\{z : \frac{\|f(z)\|_X}{\|f\|^{\text{lux}}_{\alpha,\Phi,X}} \geq 1\right\}} \|f(z)\|_X \, d\nu_\alpha(z)$$

$$\leq \nu_\alpha(\mathbb{B}_n)\|f\|^{\text{lux}}_{\Phi,\alpha,X} + \frac{\|f\|^{\text{lux}}_{\Phi,\alpha,X}}{\Phi(1)} \int_{\left\{z\in\mathbb{B}_n : \frac{\|f(z)\|_X}{\|f\|^{\text{lux}}_{\Phi,\alpha,X}} \geq 1\right\}} \Phi\left(\frac{\|f(z)\|_X}{\|f\|^{\text{lux}}_{\Phi,\alpha,X}}\right) d\nu_\alpha(z)$$

$$\leq \|f\|^{\text{lux}}_{\Phi,\alpha,X} + \frac{\|f\|^{\text{lux}}_{\Phi,\alpha,X}}{\Phi(1)} \int_{\mathbb{B}_n} \Phi\left(\frac{\|f(z)\|_X}{\|f\|^{\text{lux}}_{\Phi,\alpha,X}}\right) d\nu_\alpha(z) \leq \|f\|^{\text{lux}}_{\Phi,\alpha,X} + \frac{\|f\|^{\text{lux}}_{\Phi,\alpha,X}}{\Phi(1)}.$$

Then $f \in A^1_\alpha(\mathbb{B}_n, X)$ and $\|f\|_{1,\alpha,X} \leq \left(1 + \frac{1}{\Phi(1)}\right)\|f\|^{\text{lux}}_{\Phi,\alpha,X}$. The proof is complete. □

In a similar manner, we could also obtain the following embedding result.

LEMMA 3.2. *For $\Phi \in \mathscr{L}$, the following inclusion holds*:

$$A^1_\alpha(\mathbb{B}_n, X) \subset A^\Phi_\alpha(\mathbb{B}_n, X).$$

We will now prove the duality result in Theorem 1.11 which extends the classical duality result for vector-valued Bergman space, $A^p_\alpha(\mathbb{B}_n, X)$, for $p > 1$. We recall it here for the reader's convenience.

THEOREM 3.3. *Let X be a complex Banach space, $\alpha > -1$ and $\Phi \in \mathscr{U}$. Let Ψ be the complementary function of Φ and suppose that Φ satisfies the ∇_2-condition. Then the topological dual space $\left(A^\Phi_\alpha(\mathbb{B}_n, X)\right)^\star$ of $A^\Phi_\alpha(\mathbb{B}_n, X)$ can be identified with $A^\Psi_\alpha(\mathbb{B}_n, X^\star)$ (with equivalent norms) under the duality pairing*

$$\langle f, g\rangle_{\alpha,X} = \int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_{X,X^\star} \, d\nu_\alpha(z),$$

where $f \in A^\Phi_\alpha(\mathbb{B}_n, X)$ and $g \in A^\Psi_\alpha(\mathbb{B}_n, X^\star)$. Moreover,

$$\|g\|^{\text{lux}}_{\Psi,\alpha,X^\star} \simeq \sup_{\|f\|^{\text{lux}}_{\Phi,\alpha,X}=1} |\langle f, g\rangle_{\alpha,X}|.$$

PROOF. We first suppose that $g \in A^\Psi_\alpha(\mathbb{B}_n, X^\star)$ and define the functional

$$J_g \colon A^\Phi_\alpha(\mathbb{B}_n, X) \to \mathbb{C}, \quad f \longmapsto J_g(f) = \int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_{X,X^\star} \, d\nu_\alpha(z).$$

It is clear that J_g is linear and well defined on $A^\Phi_\alpha(\mathbb{B}_n, X)$. Indeed, it follows from Hölder's inequality (1.9) that J_g is bounded on $A^\Phi_\alpha(\mathbb{B}_n, X)$ and

$$\|J_g\|_{(A^\Phi_\alpha(\mathbb{B}_n,X))^\star} \le 2\|g\|^{\text{lux}}_{\Psi,\alpha,X^\star}.$$

Conversely, given $J \in (A^\Phi_\alpha(\mathbb{B}_n, X))^\star$, let us show that $J = J_g$ for some $g \in A^\Psi_\alpha(\mathbb{B}_n, X)$. The Hahn–Banach Theorem gives an extension $\tilde{J} \in (L^\Phi_\alpha(\mathbb{B}_n, X))^\star$ with the same norm. Using Theorem 2.29 there exists a vector-valued measure $m \in V^\Psi_\alpha(\mathbb{B}_n, X^\star)$, with Ψ-variation equals to $\|J\|$, for which

$$\tilde{J}(f) = \int_{\mathbb{B}_n} f(z)\, dm(z),$$

for every $f \in L^\Phi_\alpha(\mathbb{B}_n, X)$. Define $g = P_\alpha m$, by Theorem 2.32 we get $g \in A^\Psi_\alpha(\mathbb{B}_n, X^*)$. Let us show that $J = J_g$. For $f \in \mathcal{P}(\mathbb{B}_n, X)$, we can write $f(z) = \sum_{|\beta|\le N} z^\beta x_\beta$, where each $x_\beta \in X, N \in \mathbb{N}$ and $z \in \mathbb{B}_n$. By using Fubini's Theorem and the reproducing kernel property, we have

$$\begin{aligned}
\int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_{X,X^\star}\, d\nu_\alpha(z) &= \int_{\mathbb{B}_n} \Big\langle \sum_{|\beta|\le N} z^\beta x_\beta, P_\alpha m(z) \Big\rangle_{X,X^\star} d\nu_\alpha(z) \\
&= \sum_{|\beta|\le N} \int_{\mathbb{B}_n} z^\beta\, \overline{P_\alpha m(z)}(x_\beta)\, d\nu_\alpha(z) \\
&= \sum_{|\beta|\le N} \int_{\mathbb{B}_n} z^\beta\, \overline{\left(\int_{\mathbb{B}_n} \frac{dm(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\right)}(x_\beta)\, d\nu_\alpha(z) \\
&= \sum_{|\beta|\le N} \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{z^\beta dm(w)(x_\beta)}{(1-\langle w,z\rangle)^{n+1+\alpha}}\, d\nu_\alpha(z) \\
&= \sum_{|\beta|\le N} \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{z^\beta\, d\nu_\alpha(z)}{(1-\langle w,z\rangle)^{n+1+\alpha}}\right) dm(w)(x_\beta) \\
&= \sum_{|\beta|\le N} \int_{\mathbb{B}_n} w^\beta dm(w)(x_\beta) = \int_{\mathbb{B}_n} \sum_{|\beta|\le N} w^\beta x_\beta\, dm(w) \\
&= \int_{\mathbb{B}_n} f(w)\, dm(w) = \tilde{J}(f) = J(f).
\end{aligned}$$

Hence, $J = J_g$ for functions in $A^\Phi_\alpha(\mathbb{B}_n, X)$ since the set of vector-valued holomorphic polynomials is dense in $A^\Phi_\alpha(\mathbb{B}_n, X)$ by Corollary 2.10. This completes the proof of the theorem. □

3.2. Duality of $A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$, for $\Phi \in \mathscr{L}$. Here we give the proof of Theorem 1.12 about the characterization of the topological dual space of $A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$ in the case $\Phi \in \mathscr{L}$. Our proof is adapted from the proof in the classical weighted Bergman–Orlicz spaces in [17]. We recall Theorem 1.12 here for the reader's convenience.

THEOREM 3.4. *Let X be a complex Banach space, $\alpha > -1$, $\Phi \in \mathscr{L}_p$ and $\rho(t) = \frac{1}{t\,\Phi^{-1}(\frac{1}{t})}$. Then the topological dual space $(A^{\Phi}_{\alpha}(\mathbb{B}_n, X))^{\star}$ of $A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$ can be identified with $\Gamma_{\alpha,\rho}(\mathbb{B}_n, X^{\star})$ under the duality pairing*

$$\langle f, g\rangle_{\alpha,X} = C^{-1}_{k,\alpha} \int_{\mathbb{B}_n} \langle f(z), M^{\alpha}_{k} g(z)\rangle_{X,X^{\star}}\, d\nu_{\alpha+k}(z),$$

$k > (n+1+\alpha)\big(\frac{1}{p} - 1\big)$ is an integer, $f \in A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$, $g \in \Gamma_{\alpha,\rho}(\mathbb{B}_n, X^{\star})$, $C_{k,\alpha}$ and M^{α}_{k} are the constant and the differential operator defined by (2.22) *and* (2.23). *Moreover,*

$$\|g\|_{\Gamma_{\alpha,\rho}(\mathbb{B}_n,X^{\star})} \simeq \sup_{\|f\|^{\mathrm{lux}}_{\Phi,\alpha,X}=1} |\langle f, g\rangle_{\alpha,X}|.$$

PROOF. We first suppose that $g \in \Gamma_{\alpha,\rho}(\mathbb{B}_n; X^{\star})$. Given a positive integer $k > (n+1+\alpha)\big(\frac{1}{p} - 1\big)$, we define the functional, with $A_{k,\alpha} := C^{-1}_{k,\alpha}$,

$$\bigwedge\nolimits_g : A^{\Phi}_{\alpha}(\mathbb{B}_n, X) \to \mathbb{C},$$
$$f \mapsto \bigwedge\nolimits_g(f) = A_{k,\alpha} \int_{\mathbb{B}_n} \langle f(z), M^{\alpha}_{k} g(z)\rangle_{X,X^{\star}} (1-|z|^2)^k\, d\nu_{\alpha}(z).$$

It is clear that $\bigwedge_g$ is linear and is well defined on $A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$. Indeed, let $f \in A^{\Phi}_{\alpha}(\mathbb{B}_n, X)$, by Corollary 2.41 and Lemma 2.34, we have

$$\begin{aligned}
\big|\bigwedge\nolimits_g(f)\big| &= A_{k,\alpha}\left| \int_{\mathbb{B}_n} \langle f(z), M^{\alpha}_{k} g(z)\rangle_{X,X^{*}} (1-|z|^2)^k\, d\nu_{\alpha}(z)\right| \\
&\le A_{k,\alpha} \int_{\mathbb{B}_n} \|f(z)\|_X \|M^{\alpha}_{k} g(z)\|_{X^{*}} (1-|z|^2)^k\, d\nu_{\alpha}(z) \\
&= A_{k,\alpha} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^k \|M^{\alpha}_{k} g(z)\|_{X^{*}}}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)} \rho\big((1-|z|^2)^{n+1+\alpha}\big) \|f(z)\|_X\, d\nu_{\alpha}(z) \\
&\le A_{k,\alpha} \sup_{z\in\mathbb{B}_n} \left\{ \frac{(1-|z|^2)^k \|M^{\alpha}_{k} g(z)\|_{X^{*}}}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)} \right\} \int_{\mathbb{B}_n} \|f(z)\|_X \rho\big((1-|z|^2)^{n+1+\alpha}\big)\, d\nu_{\alpha}(z) \\
&\lesssim \|g\|_{\Gamma_{\alpha,\rho}(\mathbb{B}_n;X^{*})} \|f\|^{\mathrm{lux}}_{\Phi,\alpha,X},
\end{aligned}$$

we conclude that $\bigwedge_g$ is bounded on $A^\Phi_\alpha(\mathbb{B}_n, X)$ and $\|\bigwedge_g\| \lesssim \|g\|_{\Gamma_{\alpha,\rho}(\mathbb{B}_n;X^\star)}$.

Conversely, let $\bigwedge$ be a bounded linear functional on $A^\Phi_\alpha(\mathbb{B}_n, X)$. Let us show that there exists $g \in \Gamma_{\alpha,\rho}(\mathbb{B}_n, X^*)$ such that $\bigwedge = \bigwedge_g$. Since, by Lemma 3.2, $A^1_\alpha(\mathbb{B}_n, X) \subset\hookrightarrow A^\Phi_\alpha(\mathbb{B}_n, X)$ and $\bigwedge$ is bounded from $A^\Phi_\alpha(\mathbb{B}_n, X)$ into $\mathbb{C}$, then $\bigwedge$ is also in $(A^1_\alpha(\mathbb{B}_n, X))^*$. By Theorem 2.16, there exists $g \in \mathcal{B}(\mathbb{B}_n, X^\star)$ such that

$$\bigwedge(f) = \int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_{X,X^\star}\, d\nu_\alpha(z), \tag{3.3}$$

for all $f \in A^1_\alpha(\mathbb{B}_n, X)$. Since $\mathcal{B}(\mathbb{B}_n, X^\star) \subset A^1_\alpha(\mathbb{B}_n, X^\star)$ (see Lemma 2.15), we have $g \in A^1_\alpha(\mathbb{B}_n, X^\star)$. For any positive integer k, we have $M^\alpha_k g \in A^1_{\alpha+k}(\mathbb{B}_n, X^\star)$, (see [3]). Applying Lemma 2.39 in (3.3), we obtain that

$$\bigwedge(f) = \frac{1}{C_{k,\alpha}} \int_{\mathbb{B}_n} \langle f(z), M^\alpha_k g(z)\rangle_{X,X^\star}\, d\nu_{\alpha+k}(z) \tag{3.4}$$

for all $f \in A^1_\alpha(\mathbb{B}_n, X)$. Now let $x \in X$ and $w \in \mathbb{B}_n$. Let

$$\begin{aligned} f^x_w(z) &= \Phi^{-1}\Big[\frac{1}{(1-|w|^2)^{n+1+\alpha}}\Big]\frac{(1-|w|^2)^{n+1+\alpha+k}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}\, x \\ &= \frac{(1-|w|^2)^k}{(1-\langle z,w\rangle)^{n+1+\alpha+k}\rho\big((1-|w|^2)^{n+1+\alpha}\big)}\, x, \quad z \in \mathbb{B}_n, \end{aligned}$$

where k is a fixed integer satisfying $k > (n+1+\alpha)\big(\frac{1}{p} - 1\big)$. Using Lemma 2.36, we see that f^x_w is uniformly in $A^\Phi_\alpha(\mathbb{B}_n, X)$ and there exists $C > 0$ independent of w such that $\|f^x_w\|^{\mathrm{lux}}_{\Phi,\alpha,X} \le C\|x\|_X$. Since $\bigwedge \in \big(A^\Phi_\alpha(\mathbb{B}_n, X)\big)^\star$, then

$$|\bigwedge(f^x_w)| \le \|\bigwedge\| \|f^x_w\|^{\mathrm{lux}}_{\Phi,\alpha,X} \le C\|\bigwedge\|\|x\|_X. \tag{3.5}$$

By the identity (3.4) and the reproducing kernel property, we have

$$\begin{aligned} \bigwedge(f^x_w) &= \frac{1}{C_{k,\alpha}} \int_{\mathbb{B}_n} \big\langle f^x_w(z), M^\alpha_k g(z)\big\rangle_{X,X^*} (1-|z|^2)^k\, d\nu_\alpha(z) \\ &= \frac{1}{C_{k,\alpha}} \int_{\mathbb{B}_n} \Big\langle \frac{(1-|w|^2)^k}{(1-\langle z,w\rangle)^{n+1+\alpha+k}\rho\big((1-|w|^2)^{n+1+\alpha}\big)}\, x, M^\alpha_k g(z)\Big\rangle_{X,X^*} \\ &\qquad \times (1-|z|^2)^k\, d\nu_\alpha(z) \\ &= \frac{1}{C_{k,\alpha}} \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \int_{\mathbb{B}_n} \Big\langle \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}\, x, M^\alpha_k g(z)\Big\rangle_{X,X^*} \\ &\qquad \times (1-|z|^2)^k\, d\nu_\alpha(z) \end{aligned}$$

$$= \frac{1}{C_{k,\alpha}} \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \int_{\mathbb{B}_n} \left\langle x, \frac{(1-|z|^2)^k}{(1-\langle w,z\rangle)^{n+1+\alpha+k}} M_k^\alpha g(z) \right\rangle_{X,X^*} d\nu_\alpha(z)$$

$$= \frac{1}{C_{k,\alpha}} \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \left\langle x, \int_{\mathbb{B}_n} \frac{(1-|z|^2)^k}{(1-\langle w,z\rangle)^{n+1+\alpha+k}} M_k^\alpha g(z)\, d\nu_\alpha(z) \right\rangle_{X,X^\star}$$

$$= \frac{c_\alpha}{c_{k+\alpha} C_{k,\alpha}} \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \left\langle x, \int_{\mathbb{B}_n} \frac{M_k^\alpha g(z)}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}\, d\nu_{\alpha+k}(z) \right\rangle_{X,X^\star}$$

$$= \frac{1}{A_{k,\alpha}} \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \big\langle x, M_k^\alpha g(w)\big\rangle_{X,X^\star},$$

where $A_{k,\alpha} = \frac{c_{k+\alpha} C_{k,\alpha}}{c_\alpha}$ and c_α is defined in (1.2). Using (3.5), we have

$$\|M_k^\alpha g(w)\|_{X^\star} = \sup_{\|x\|_X=1} \big|\langle x, M_k^\alpha g(w)\rangle_{X,X^\star}\big|$$

$$= A_{k,\alpha} \frac{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}{(1-|w|^2)^k} \sup_{\|x\|_X=1} \big|\bigwedge(f_w^x)\big| \le C\, A_{k,\alpha} \frac{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}{(1-|w|^2)^k} \|\bigwedge\|.$$

According to Corollary 2.41, we conclude that $g \in \Gamma_{\alpha,\rho}(\mathbb{B}_n, X^*)$ and $\|g\|_{\Gamma_{\alpha,\rho}(\mathbb{B}_n,X^*)} \lesssim \|\bigwedge\|$.

To finish the proof, it remains to show that (3.3) remains true for functions in $A_\alpha^\Phi(\mathbb{B}_n, X)$ which is a direct consequence of the density of $\mathcal{H}^\infty(\mathbb{B}_n, X)$ in $A_\alpha^\Phi(\mathbb{B}_n, X)$ by Lemma 2.9. $\square$

4. Hankel operators between vector-valued Bergman–Orlicz spaces

This section is devoted to the study of the boundedness of the little Hankel operator, h_b, between two Bergman–Orlicz spaces $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ and $A_\alpha^{\Phi_2}(\mathbb{B}_n, Y)$ with $\Phi_i \in \mathscr{U}$ or $\mathscr{L}$, $i = 1, 2$.

4.1. Hankel operators from $A_\alpha^\Phi(\mathbb{B}_n, X)$ into $A_\alpha^\Phi(\mathbb{B}_n, Y)$, $\Phi \in \mathscr{U}$. We consider in this subsection the boundedness of Hankel operators, h_b, from $A_\alpha^\Phi(\mathbb{B}_n, X)$ into $A_\alpha^\Phi(\mathbb{B}_n, Y)$ for Φ a convex growth function. Our result is contained in Theorem 1.13 that we give a proof here.

Proof. Suppose that h_b is a bounded linear operator from $A_\alpha^\Phi(\mathbb{B}_n, X)$ into $A_\alpha^\Phi(\mathbb{B}_n, Y)$ with norm $\|h_b\| = \|h_b\|_{A_\alpha^\Phi(\mathbb{B}_n,X)\to A_\alpha^\Phi(\mathbb{B}_n,Y)}$. Let $x \in X$ and $t > n+1+\alpha$. For $z \in \mathbb{B}_n$, let

$$f_z^x(w) = \frac{x}{(1-\langle w,z\rangle)^t}, \quad w \in \mathbb{B}_n.$$

By Lemma 2.35, $f_z^x \in \mathcal{H}^\infty(\mathbb{B}_n, X)$ and

$$\|f_z^x\|_{\Phi,\alpha,X}^{\mathrm{lux}} \lesssim \frac{\|x\|_X}{(1-|z|^2)^t \Phi^{-1}\big(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\big)}. \tag{4.1}$$

Therefore, with this particular f_z^x and using Proposition 2.12, we have

$$\begin{aligned} h_b(f_z^x)(z) &= \int_{\mathbb{B}_n} \frac{b(w)\big(\overline{f_z^x(w)}\big)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\, d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{b(w)(\overline{x})}{(1-\langle z,w\rangle)^{n+1+\alpha+t}}\, d\nu_\alpha(w) = (R^{\alpha,t} b(z))(\overline{x}). \end{aligned}$$

Then, applying the mean value property, Hölder's inequality (1.9), (4.1) and (2.1) we have

$$\begin{aligned} \big\|(R^{\alpha,t} b(z))(\overline{x})\big\|_Y &= \|h_b f_z^x(z)\|_Y \\ &\le \frac{1}{\nu_\alpha\big(B\big(z, \frac{1-|z|}{2}\big)\big)} \int_{\mathbb{B}_n} \|h_b f_z^x(w)\|_Y \chi_{B(z,1-|z|)}(w)\, d\nu_\alpha(w) \\ &\le \frac{2}{\nu_\alpha\big(B\big(z, \frac{1-|z|}{2}\big)\big)} \|h_b f_z^x\|_{\Phi,\alpha,Y}^{\mathrm{lux}} \|\chi_{B(z,1-|z|)}\|_{\Psi,\alpha}^{\mathrm{lux}} \\ &\lesssim \frac{2\|h_b\|\|x\|_X}{(1-|z|^2)^t \Phi^{-1}\big(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\big)} \Phi^{-1}\Big(\frac{1}{\nu_\alpha(B(z,1-|z|))}\Big) \lesssim \frac{2\|h_b\|\|x\|_X}{(1-|z|^2)^t}. \end{aligned}$$

Since $x \in X$ is arbitrary and $\|x\|_X = \|\overline{x}\|_{\overline{X}}$, we get that

$$\big\|R^{\alpha,t} b(z)\big\|_{\mathcal{L}(\overline{X},Y)} = \sup_{\|\overline{x}\|_{\overline{X}}=1} \big\|(R^{\alpha,t} b(z))(\overline{x})\big\|_Y \lesssim \frac{\|h_b\|}{(1-|z|^2)^t}.$$

By Theorem 2.14, this implies that b is in the Bloch space $\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ with

$$\|b\|_{\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{X},Y))} \lesssim \|h_b\|.$$

Conversely, suppose that $b \in \mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. We recall that Φ satisfies ∇_2, then by Theorem 1.11, the topological dual space of $A_\alpha^\Phi(\mathbb{B}_n, X)$ coincides with $A_\alpha^\Psi(\mathbb{B}_n, X^\star)$. Let $f \in H^\infty(\mathbb{B}_n, X)$, $g \in H^\infty(\mathbb{B}_n, Y^\star)$ and $t > 0$. By Lemma 2.42, we have

$$\langle h_b(f), g\rangle_{\alpha,Y} = \int_{\mathbb{B}_n} \Big\langle b(z)\big(\overline{f(z)}\big), g(z)\Big\rangle_{Y,Y^*} d\nu_\alpha(z).$$

Then, by Lemma 2.15, Corollary 2.13, Theorem 2.14, and Hölder's inequality (1.9), we obtain that

$$\begin{aligned}
\left|\langle h_b(f), g\rangle_{\alpha,Y}\right| &= \left|\int_{\mathbb{B}_n} \left\langle b(z)\big(\overline{f(z)}\big), g(z)\right\rangle_{Y,Y^*} d\nu_\alpha(z)\right| \\
&= \left|\int_{\mathbb{B}_n} \left\langle (R^{\alpha,t}b(z))\big(\overline{f(z)}\big), g(z)\right\rangle_{Y,Y^*} d\nu_{\alpha+t}(z)\right| \\
&\leq \int_{\mathbb{B}_n} \left\|R^{\alpha,t}b(z)\big(\overline{f(z)}\big)\right\|_Y \|g(z)\|_{Y^*}\, d\nu_{\alpha+t}(z) \\
&\leq \int_{\mathbb{B}_n} \left\|R^{\alpha,t}b(z)\right\|_{\mathcal{L}(\overline{X},Y)} \left\|\overline{f(z)}\right\|_{\overline{X}} \|g(z)\|_{Y^*}\, d\nu_{\alpha+t}(z) \\
&\simeq \int_{\mathbb{B}_n} \left\|(1-|z|^2)^t R^{\alpha,t}b(z)\right\|_{\mathcal{L}(\overline{X},Y)} \|f(z)\|_X \|g(z)\|_{Y^*}\, d\nu_\alpha(z) \\
&\lesssim \|b\|_{\mathcal{B}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \int_{\mathbb{B}_n} \|f(z)\|_X \|g(z)\|_{Y^*}\, d\nu_\alpha(z) \\
&\leq 2\|b\|_{\mathcal{B}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \|f\|^{\mathrm{lux}}_{\Phi,\alpha,X} \|g\|^{\mathrm{lux}}_{\Psi,\alpha,Y^\star}.
\end{aligned}$$

Therefore, by duality, we obtain

$$\|h_b\| = \sup_{\substack{\|f\|^{\mathrm{lux}}_{\Phi,\alpha,X}=1 \\ \|g\|^{\mathrm{lux}}_{\Psi,\alpha,Y^\star}=1}} \left|\langle h_b(f), g\rangle_{\alpha,Y}\right| \lesssim \|b\|_{\mathcal{B}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}.$$

The proof of the Theorem is complete. □

4.2. Hankel operators from $A^{\Phi_1}_{\alpha}(\mathbb{B}_n, X)$ into $A^{\Phi_2}_{\alpha}(\mathbb{B}_n, Y)$, $\Phi_1, \Phi_2 \in \mathscr{L}$. In this subsection, we will give the proofs of Theorem 1.14 and Corollary 1.15. We recall Theorem 1.14 here for the reader's convenience.

THEOREM 4.1. *Let $\Phi_1 \in \mathscr{L}$, $\alpha > -1$ and $\Phi_2 \in \mathscr{L}$. If the Hankel operator h_b extends into a bounded operator from $A^{\Phi_1}_{\alpha}(\mathbb{B}_n, X)$ into $A^{\Phi_2}_{\alpha}(\mathbb{B}_n, Y)$, then its symbol b belongs to $\Gamma_{\alpha,\rho_1}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ with $\rho_1(t) = \frac{1}{t\,\Phi_1^{-1}(\frac{1}{t})}$. Conversely, if $b \in \Gamma_{\alpha,\rho_1}\big(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)\big)$, then there exists a bounded operator from $A^{\Phi_1}_{\alpha}(\mathbb{B}_n, X)$ into $L^1_\alpha(\mathbb{B}_n, Y)$, which we denote by T_b, such that $h_b = P_\alpha T_b$.*

PROOF. First assume that h_b extends into a bounded operator from $A^{\Phi_1}_{\alpha}(\mathbb{B}_n, X)$ into $A^{\Phi_2}_{\alpha}(\mathbb{B}_n, Y)$, with norm $\|h_b\| = \|h_b\|_{A^{\Phi_1}_{\alpha}(\mathbb{B}_n,X)\to A^{\Phi_2}_{\alpha}(\mathbb{B}_n,Y)}$. We want to show that $b \in \Gamma_{\alpha,\rho_1}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Since $h_b \colon A^{\Phi_1}_{\alpha}(\mathbb{B}_n, X) \to A^{\Phi_2}_{\alpha}(\mathbb{B}_n, Y)$ is a bounded operator, by Theorem 1.12, we have

$$|\langle h_b(f), g\rangle_{\alpha,Y}| \leq \|h_b\| \|f\|^{\mathrm{lux}}_{\Phi_1,\alpha,X} \|g\|_{\Gamma_{\alpha,\rho_2}(\mathbb{B}_n,Y^\star)}, \tag{4.2}$$

for every $f \in A^{\Phi_1}_\alpha(\mathbb{B}_n, X)$ and $g \in (A^{\Phi_2}_\alpha(\mathbb{B}_n, Y))^* = \Gamma_{\alpha,\rho_2}(\mathbb{B}_n, Y^\star)$. Let $x \in X$, $y^\star \in Y^\star$, and an integer k such that $k > \gamma = (n+1+\alpha)(1/p-1)$, where p is such that $\Phi_1 \in \mathscr{L}_p$. Let $w \in \mathbb{B}_n$ and let $g(z) = y^\star$ and

$$\begin{aligned} f^x_w(z) &= \Phi_1^{-1}\Big[\frac{1}{(1-|w|^2)^{n+1+\alpha}}\Big]\frac{(1-|w|^2)^{n+1+\alpha+k}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}\,x \\ &= \frac{(1-|w|^2)^k x}{(1-\langle z,w\rangle)^{n+1+\alpha+k}\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}. \end{aligned}$$

It is clear that $g \in \Gamma_{\alpha,\rho_2}(\mathbb{B}_n, Y^\star)$ with $\|g\|_{\Gamma_{\alpha,\rho_2}(\mathbb{B}_n,Y^\star)} = \|y^\star\|_{Y^\star}$. We also have, by Lemma 2.36, that the function f^x_w is uniformly in $A^{\Phi_1}_\alpha(\mathbb{B}_n, X)$. More precisely, there exists $C > 0$, independent of w, such that $\|f^x_w\|^{\mathrm{lux}}_{\Phi_1,\alpha,X} \le C\|x\|_X$. Hence by (4.2),

$$\big|\langle h_b(f^x_w), g\rangle_{\alpha,Y}\big| \le C\|h_b\|\|x\|_X\|y^\star\|_{Y^*}. \tag{4.3}$$

Applying Lemma 2.42, (2.23), Lemma 1.5 and the reproducing kernel property, we have that

$$\begin{aligned} \langle h_b(f^x_w), g\rangle_{\alpha,Y} &= \int_{\mathbb{B}_n}\big\langle h_b(f^x_w)(z), g(z)\big\rangle_{Y,Y^*}\,d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n}\big\langle b(z)\big(\overline{f^x_w(z)}\big), g(z)\big\rangle_{Y,Y^\star}\,d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n}\Big\langle b(z)\Big(\overline{\frac{(1-|w|^2)^k}{(1-\langle z,w\rangle)^{n+1+\alpha+k}\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}x}\Big), y^\star\Big\rangle_{Y,Y^\star}\,d\nu_\alpha(z) \\ &= \frac{(1-|w|^2)^k}{\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}\int_{\mathbb{B}_n}\Big\langle \frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}, y^\star\Big\rangle_{Y,Y^\star}\,d\nu_\alpha(z) \\ &= \frac{(1-|w|^2)^k}{\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}\Big\langle \int_{\mathbb{B}_n}\frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}\,d\nu_\alpha(z), y^\star\Big\rangle_{Y,Y^\star} \\ &= \frac{(1-|w|^2)^k}{\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}\Big\langle \int_{\mathbb{B}_n}\frac{1}{C_{k,\alpha}}M^\alpha_k\Big(\frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha}}\Big)\,d\nu_\alpha(z), y^\star\Big\rangle_{Y,Y^\star} \\ &= \frac{(1-|w|^2)^k}{\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}\frac{1}{C_{k,\alpha}}\Big\langle M^\alpha_k\Big(\int_{\mathbb{B}_n}\frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha}}\,d\nu_\alpha(z)\Big), y^\star\Big\rangle_{Y,Y^\star} \\ &= \frac{(1-|w|^2)^k}{\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}\frac{1}{C_{k,\alpha}}\big\langle M^\alpha_k\left(b(w)(\overline{x})\right), y^\star\big\rangle_{Y,Y^\star}. \end{aligned}$$

Thus,

$$\left\langle M_k^\alpha(b(w)(\overline{x})), y^*\right\rangle_{Y,Y^\star} = C_{k,\alpha}\frac{\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}{(1-|w|^2)^k}\left\langle h_b(f_w^x), g\right\rangle_{\alpha,Y}.$$

Since $x \in X$ and $y^* \in Y$ are arbitrary, and $\|x\|_X = \|\overline{x}\|_{\overline{X}}$, from (4.3), we deduce that

$$\begin{aligned}\|M_k^\alpha b(w)\|_{\mathcal{L}(\overline{X},Y)} &= \sup_{\substack{\|x\|_X=1\\ \|y^*\|_{Y^*}=1}} \left|\left\langle M_k^\alpha(b(w)(\overline{x})), y^*\right\rangle_{Y,Y^\star}\right|\\ &= C_{k,\alpha}\frac{\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}{(1-|w|^2)^k}\sup_{\substack{\|x\|_X=1\\ \|y^*\|_{Y^*}=1}} \left|\langle h_b(f_w^x), g\rangle_{\alpha,Y}\right|\\ &\leq CC_{k,\alpha}\frac{\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}{(1-|w|^2)^k}\|h_b\|.\end{aligned}$$

Then

$$\sup_{w\in\mathbb{B}_n}\left\{\frac{(1-|w|^2)^k\|M_k^\alpha b(w)\|_{\mathcal{L}(\overline{X},Y)}}{\rho_1\big((1-|w|^2)^{n+1+\alpha}\big)}\right\} \lesssim \|h_b\|.$$

By Corollary 2.41, we have that $b \in \Gamma_{\alpha,\rho_1}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and $\|b\|_{\Gamma_{\alpha,\rho_1}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \lesssim \|h_b\|$.

Conversely, assume that $b \in \Gamma_{\alpha,\rho_1}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and let us prove that h_b extends to a bounded operator from $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ into $A_\alpha^{1,\infty}(\mathbb{B}_n, Y)$. Let $T_b\colon A_\alpha^{\Phi_1}(\mathbb{B}_n, X) \to L_\alpha^1(\mathbb{B}_n, Y)$ be defined by

$$T_bf(z) = \frac{1}{C_{k,\alpha}}(1-|z|^2)^k M_k^\alpha\big(b(z)\big(\overline{f(z)}\big)\big), \quad f \in A_\alpha^{\Phi_1}(\mathbb{B}_n, X),\ z \in \mathbb{B}_n,$$

where $k > (n+1+\alpha)\big(\frac{1}{p}-1\big)$ and M_k^α is the differential operator given in (2.22). It is clear that T_b is linear and is well defined on $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$. Indeed, let $f \in A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$, by Corollary 2.41 and Lemma 2.34, we have

$$\begin{aligned}\int_{\mathbb{B}_n}\|T_bf(z)\|_Y\,d\nu_\alpha(z) &= \int_{\mathbb{B}_n}\left\|\frac{1}{C_{k,\alpha}}(1-|z|^2)^k M_k^\alpha\big(b(z)\big(\overline{f(z)}\big)\big)\right\|_Y d\nu_\alpha(z)\\ &\leq \frac{1}{C_{k,\alpha}}\int_{\mathbb{B}_n}(1-|z|^2)^k\|M_k^\alpha b(z)\|_{\mathcal{L}(\overline{X},Y)}\left\|\overline{f(z)}\right\|_{\overline{X}}d\nu_\alpha(z)\\ &= \frac{1}{C_{k,\alpha}}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^k\|M_k^\alpha b(z)\|_{\mathcal{L}(\overline{X},Y)}}{\rho_1\big((1-|z|^2)^{n+1+\alpha}\big)}\rho_1\big((1-|z|^2)^{n+1+\alpha}\big)\left\|\overline{f(z)}\right\|_{\overline{X}}d\nu_\alpha(z)\end{aligned}$$

$$\lesssim \sup_{z\in\mathbb{B}_n}\left\{\frac{(1-|z|^2)^k\|M_k^\alpha b(z)\|_{\mathcal{L}(\overline{X},Y)}}{\rho_1\big((1-|z|^2)^{n+1+\alpha}\big)}\right\}\int_{\mathbb{B}_n}\rho_1\big((1-|z|^2)^{n+1+\alpha}\big)\big\|\overline{f(z)}\big\|_{\overline{X}}\,d\nu_\alpha(z)$$

$$\lesssim \|b\|_{\Gamma_{\alpha,\rho_1}\left(\mathbb{B}_n,\mathcal{L}(\overline{X},Y)\right)}\|f\|^{\mathrm{lux}}_{\Phi_1,\alpha,X}.$$

We conclude that T_b is bounded from $A^{\Phi_1}_\alpha(\mathbb{B}_n,X)$ into $L^1_\alpha(\mathbb{B}_n,Y)$ and

$$\|T_bf\|_{1,\alpha,Y}\lesssim \|b\|_{\Gamma_{\alpha,\rho_1}\left(\mathbb{B}_n,\mathcal{L}(\overline{X},Y)\right)}\|f\|^{\mathrm{lux}}_{\Phi_1,\alpha,X}. \tag{4.4}$$

It is easy to check that $h_b=P_\alpha T_b$. Indeed, by Lemma 2.43, we have

$$\begin{aligned}
P_\alpha T_bf(z)&=\int_{\mathbb{B}_n}\frac{T_bf(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\,d\nu_\alpha(w)\\
&=\int_{\mathbb{B}_n}\frac{\frac{1}{C_{k,\alpha}}(1-|w|^2)^kM_k^\alpha\big(b(w)\big(\overline{f(w)}\big)\big)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\,d\nu_\alpha(w)\\
&=\frac{1}{C_{k,\alpha}}\int_{\mathbb{B}_n}M_k^\alpha\left(b(w)\left(\overline{\frac{f(w)}{(1-\langle w,z\rangle)^{n+1+\alpha}}}\right)\right)d\nu_{\alpha+k}(w)\\
&=\frac{1}{C_{k,\alpha}}\int_{\mathbb{B}_n}M_k^\alpha\big(b(w)\big(\overline{g_z(w)}\big)\big)\,d\nu_{\alpha+k}(w)=h_b(f)(z),
\end{aligned}$$

where $g_z(w)=\frac{f(w)}{(1-\langle w,z\rangle)^{n+1+\alpha}}$. Moreover, by Proposition 1.4 and (4.4) we have

$$\begin{aligned}
\|h_bf\|_{A^{1,\infty}_\alpha(\mathbb{B}_n,Y)}&=\|P_\alpha T_bf\|_{A^{1,\infty}_\alpha(\mathbb{B}_n,Y)}\\
&\lesssim \|T_bf\|_{1,\alpha,Y}\lesssim \|b\|_{\Gamma_{\alpha,\rho_1}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}\|f\|^{\mathrm{lux}}_{\Phi_1,\alpha,X}.
\end{aligned}$$

So h_b extends into a bounded operator from $A^{\Phi_1}_\alpha(\mathbb{B}_n,X)$ into $A^{1,\infty}_\alpha(\mathbb{B}_n,Y)$, and

$$\|h_b\|=\sup_{\|f\|^{\mathrm{lux}}_{\Phi_1,\alpha,X}=1}\|h_b(f)\|_{A^{1,\infty}_\alpha(\mathbb{B}_n,Y)}\lesssim\|b\|_{\Gamma_{\alpha,\rho_1}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}. \quad \square$$

Proof of Corollary 1.15. We observe that the necessity is given by Theorem 1.14. The sufficiency condition follows from Lemma 1.10 and the second part of Theorem 1.14. $\square$

4.3. Hankel operators from $A^{\Phi}_{\alpha}(\mathbb{B}_n,X)$ into $A^1_\alpha(\mathbb{B}_n,Y)$, $\Phi\in\mathscr{L}$. In this subsection, we will give the proof of Theorem 1.16 which generalizes the case $h_b\colon A^p_\alpha(\mathbb{B}_n,X)\to A^1_\alpha(\mathbb{B}_n,Y)$ with $0<p<1$ (see [3]).

Proof. We first prove the sufficiency of the theorem. We suppose that $b \in L\Gamma_{\alpha,\rho}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Applying Lemma 2.43, for $f \in \mathcal{H}^\infty(\mathbb{B}_n, X)$, we get

$$h_b(f)(z) = \frac{1}{C_{k,\alpha}} \int_{\mathbb{B}_n} \frac{M_k^\alpha\big(b(w)\big(\overline{f(w)}\big)\big)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\, d\nu_{\alpha+k}(w).$$

Thus, by Lemma 1.6, Theorem 2.33, Corollary 2.41 and Lemma 2.34, we have

$$\begin{aligned}
\|h_b(f)\|_{1,\alpha,Y} &= \frac{1}{C_{k,\alpha}} \int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} \frac{M_k^\alpha\big(b(w)\big(\overline{f(w)}\big)\big)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\, d\nu_{\alpha+k}(w) \right\|_Y d\nu_\alpha(z) \\
&\le \frac{1}{C_{k,\alpha}} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \left\| \frac{M_k^\alpha\big(b(w)\big(\overline{f(w)}\big)\big)}{(1-\langle z,w\rangle)^{n+1+\alpha}} \right\|_Y d\nu_{\alpha+k}(w)\, d\nu_\alpha(z) \\
&\le \frac{1}{C_{k,\alpha}} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|M_k^\alpha b(w)\|_{\mathcal{L}(\overline{X},Y)} \|\overline{f(w)}\|_{\overline{X}}}{|1-\langle z,w\rangle|^{n+1+\alpha}}\, d\nu_{\alpha+k}(w)\, d\nu_\alpha(z) \\
&= \frac{c_{\alpha+k}}{c_\alpha C_{k,\alpha}} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^k \|M_k^\alpha b(w)\|_{\mathcal{L}(\overline{X},Y)} \|f(w)\|_X}{|1-\langle z,w\rangle|^{n+1+\alpha}}\, d\nu_\alpha(w)\, d\nu_\alpha(z) \\
&= \frac{c_{\alpha+k}}{c_\alpha C_{k,\alpha}} \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{d\nu_\alpha(z)}{|1-\langle z,w\rangle|^{n+1+\alpha}} \right) \\
&\quad \times (1-|w|^2)^k \|M_k^\alpha b(w)\|_{\mathcal{L}(\overline{X},Y)} \|f(w)\|_X\, d\nu_\alpha(w) \\
&= \frac{c_{\alpha+k}}{C_{k,\alpha}} \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{(1-|z|^2)^\alpha\, d\nu(z)}{|1-\langle z,w\rangle|^{n+1+\alpha}} \right) \\
&\quad \times (1-|w|^2)^k \|M_k^\alpha b(w)\|_{\mathcal{L}(\overline{X},Y)} \|f(w)\|_X\, d\nu_\alpha(w) \\
&\lesssim \int_{\mathbb{B}_n} \log\left(\frac{1}{1-|w|^2}\right) (1-|w|^2)^k \|M_k^\alpha b(w)\|_{\mathcal{L}(\overline{X},Y)} \|f(w)\|_X\, d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \frac{(1-|w|^2)^k \|M_k^\alpha b(w)\|_{\mathcal{L}(\overline{X},Y)}}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \log\left(\frac{1}{1-|w|^2}\right) \\
&\quad \times \rho\big((1-|w|^2)^{n+1+\alpha}\big) \|f(w)\|_X\, d\nu_\alpha(w) \\
&\le \sup_{z\in\mathbb{B}_n} \left\{ \frac{(1-|z|^2)^k \|M_k^\alpha b(z)\|_{\mathcal{L}(\overline{X},Y)}}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)} \log\left(\frac{1}{1-|z|^2}\right) \right\} \\
&\quad \times \int_{\mathbb{B}_n} \rho\big((1-|w|^2)^{n+1+\alpha}\big) \|f(w)\|_X\, d\nu_\alpha(w) \\
&\lesssim \|b\|_{L\Gamma_{\alpha,\rho}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \|f\|_{\Phi,\alpha,X}^{\mathrm{lux}}.
\end{aligned}$$

So h_b extends into a bounded operator from $A^\Phi_\alpha(\mathbb{B}_n, X)$ into $A^1_\alpha(\mathbb{B}_n, Y)$, and

$$\|h_b\| = \sup_{\|f\|^{\text{lux}}_{\Phi,\alpha,X}=1} \|h_b f\|_{A^1_\alpha(\mathbb{B}_n,Y)} \lesssim \|b\|_{L\Gamma_{\alpha,\rho}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}.$$

Conversely, we assume that h_b extends into a bounded operator from $A^\Phi_\alpha(\mathbb{B}_n, X)$ into $A^1_\alpha(\mathbb{B}_n, Y)$, with $\|h_b\| := \|h_b\|_{A^\Phi_\alpha(\mathbb{B}_n,X)\to A^1_\alpha(\mathbb{B}_n,Y)}$. Then we have, by Theorem 2.16, for every $f \in A^\Phi_\alpha(\mathbb{B}_n, X)$ and $g \in (A^1_\alpha(\mathbb{B}_n, Y))^\star = \mathcal{B}(\mathbb{B}_n, Y^\star)$,

$$|\langle h_b(f), g\rangle_{\alpha,Y}| \le \|h_b\| \|f\|^{\text{lux}}_{\Phi,\alpha,X} \|g\|_{\mathcal{B}(\mathbb{B}_n,Y^\star)}. \tag{4.5}$$

Now, take $x \in X$, $y^\star \in Y^\star$ and an integer k such that $k > (n+1+\alpha)(\frac{1}{p} - 1)$, Fix $w \in \mathbb{B}_n$ and let $g_w(z) = [\log(1 - \langle z, w\rangle)]\, y^\star$, and

$$\begin{aligned} f^x_w(z) &= \Phi^{-1}\Big(\frac{1}{(1-|w|^2)^{n+1+\alpha}}\Big) \frac{(1-|w|^2)^{n+1+\alpha+k}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}\, x \\ &= \frac{(1-|w|^2)^k}{(1-\langle z,w\rangle)^{n+1+\alpha+k}\rho\big((1-|w|^2)^{n+1+\alpha}\big)}\, x, \end{aligned}$$

where log is the principal branch of the logarithm. We have seen, by Lemma 2.36, that the function f^x_w is uniformly in $A^\Phi_\alpha(\mathbb{B}_n, X)$ and it is well known that g_w is uniformly in $\mathcal{B}\,(\mathbb{B}_n, Y^\star)$. Thus, by relation (4.5), we obtain that

$$|\langle h_b(f^x_w), g_w\rangle|_{\alpha,Y} \lesssim \|h_b\| \|x\|_X \|y^\star\|_{Y^\star}. \tag{4.6}$$

Applying Lemma 2.42 for those particular vector-valued holomorphic functions f^x_w and g_w, we obtain that

$$\begin{aligned} \langle h_b(f^x_w), g_w\rangle_{\alpha,Y} &= \int_{\mathbb{B}_n} \big\langle h_b(f^x_w)(z), g_w(z)\big\rangle_{Y,Y^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \big\langle b(z)\big(\overline{f^x_w(z)}\big), g_w(z)\big\rangle_{Y,Y^\star} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \Big\langle b(z)\Big(\overline{\frac{(1-|w|^2)^k}{(1-\langle z,w\rangle)^{n+1+\alpha+k}\rho\big((1-|w|^2)^{n+1+\alpha}\big)}\, x}\Big), g_w(z)\Big\rangle_{Y,Y^\star} d\nu_\alpha(z) \\ &= \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \\ &\quad\times \int_{\mathbb{B}_n} \Big\langle \frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}, \big[\log(1-\langle z,w\rangle)\big] y^\star\Big\rangle_{Y,Y^\star} d\nu_\alpha(z) \end{aligned}$$

$$= \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}$$

$$\times \Big\langle \int_{\mathbb{B}_n} \frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}} \log(1-\langle w,z\rangle)\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star}.$$

Now, using the fact that $\log(1-\langle w,z\rangle) = \log(1-|w|^2) + \log\big(\frac{1-\langle w,z\rangle}{1-|w|^2}\big)$, we have

$$\big\langle h_b(f_w^x), g_w\big\rangle_{\alpha,Y} = \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}$$

$$\times \Big\langle \int_{\mathbb{B}_n} \frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}} \log(1-|w|^2)\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star}$$

$$+ \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}$$

$$\times \Big\langle \int_{\mathbb{B}_n} \frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}} \log(\frac{1-\langle w,z\rangle}{1-|w|^2})\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star} = I_1 + I_2,$$

where

$$I_1 = \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}$$

$$\times \Big\langle \int_{\mathbb{B}_n} \frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}} \log(1-|w|^2)\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star},$$

and

$$I_2 = \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}$$

$$\times \Big\langle \int_{\mathbb{B}_n} \frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}} \log\Big(\frac{1-\langle w,z\rangle}{1-|w|^2}\Big)\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star}.$$

To estimate I_1, using (2.23), Lemma 1.5 and the reproducing kernel property gives

$$I_1 = \frac{(1-|w|^2)^k \log(1-|w|^2)}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \Big\langle \int_{\mathbb{B}_n} \frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star}$$

$$= \frac{(1-|w|^2)^k \log(1-|w|^2)}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \Big\langle \int_{\mathbb{B}_n} \frac{1}{C_{k,\alpha}} M_k^\alpha\Big(\frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha}}\Big)\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star}$$

$$
\begin{aligned}
&= \frac{1}{C_{k,\alpha}} \frac{(1-|w|^2)^k \log(1-|w|^2)}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \\
&\quad \times \Big\langle M_k^\alpha\Big(\int_{\mathbb{B}_n} \frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha}}\, d\nu_\alpha(z)\Big), y^\star \Big\rangle_{Y,Y^\star} \\
&= \frac{1}{C_{k,\alpha}} \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \log(1-|w|^2) \big\langle M_k^\alpha(b(w)(\overline{x})), y^\star \big\rangle_{Y,Y^\star}.
\end{aligned}
$$

For I_2, we observe that

$$
\begin{aligned}
I_2 &= \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \\
&\quad \times \Big\langle \int_{\mathbb{B}_n} \frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}} \log\Big(\frac{1-\langle w,z\rangle}{1-|w|^2}\Big)\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star} \\
&= \Big\langle \int_{\mathbb{B}_n} b(z)\Bigg(\overline{\frac{(1-|w|^2)^k \log\big(\frac{1-\langle z,w\rangle}{1-|w|^2}\big) x}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)(1-\langle z,w\rangle)^{n+1+\alpha+k}}}\Bigg)\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star} \\
&= \Big\langle \int_{\mathbb{B}_n} b(z)\Big(\overline{\log\Big(\frac{1-\langle z,w\rangle}{1-|w|^2}\Big) f_w^x(z)}\Big)\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star}.
\end{aligned}
$$

Then,

$$
I_2 = \Big\langle \int_{\mathbb{B}_n} b(z)\big(\overline{\varphi_w^x(z)}\big)\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star}, \tag{4.7}
$$

where $\varphi_w^x(z) = \log\big(\frac{1-\langle z,w\rangle}{1-|w|^2}\big) f_w^x(z)$. From Lemma 2.37, we have that $\varphi_w^x \in \mathcal{H}^\infty(\mathbb{B}_n, X)$ and there exists a constant $C>0$, independent of w and x, such that

$$
\|\varphi_w^x\|_{\Phi,\alpha,X}^{\mathrm{lux}} \le C\|x\|_X.
$$

From (4.7), Lemma 2.42 and (4.5), we obtain the following estimate of I_2:

$$
\begin{aligned}
|I_2| &= \Big|\Big\langle \int_{\mathbb{B}_n} b(z)\big(\overline{\varphi_w^x(z)}\big)\, d\nu_\alpha(z), y^\star \Big\rangle_{Y,Y^\star}\Big| = \big|\langle h_b(\varphi_w^x), y^\star\rangle_{\alpha,Y}\big| \\
&\lesssim \|h_b\| \|\varphi_w^x\|_{\Phi,\alpha,X}^{\mathrm{lux}} \|y^\star\|_{\mathcal{B}(\mathbb{B}_n,Y^\star)} \le C\|h_b\|\, \|x\|_X \|y^\star\|_{Y^\star}.
\end{aligned} \tag{4.8}
$$

Since $I_1 = \langle h_b(f_w^x), g_w\rangle_{\alpha,Y} - I_2$, by (4.6) and (4.8), we have

$$
|I_1| \le |\langle h_b(f_w^x), g_w\rangle_{\alpha,Y}| + |I_2| \lesssim \|h_b\|\|x\|_X\|y^\star\|_{Y^\star}.
$$

Since $\|x\|_X = \|\overline{x}\|_{\overline{X}}$, we get that

$$\frac{1}{C_{k,\alpha}} \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)} \big|\log(1-|w|^2)\big| \big|\big\langle M_k^\alpha\left(b(z)(\overline{x})\right), y^\star\big\rangle_{Y,Y^\star}\big|$$
$$= |I_1| \lesssim \|h_b\|\, \|\overline{x}\|_{\overline{X}} \|y^\star\|_{Y^\star}.$$

Since $x \in X$, $y^\star \in Y^\star$ are arbitrary, we deduce that

$$\|M_k^\alpha b(w)\|_{\mathcal{L}(\overline{X},Y)} = \sup_{\substack{\|\overline{x}\|_{\overline{X}}=1\\ \|y^\star\|_{Y^\star}=1}} \big|\big\langle M_k^\alpha\left(b(w)(\overline{x})\right), y^*\big\rangle_{Y,Y^\star}\big|$$
$$\lesssim \frac{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}{(1-|w|^2)^k \big|\log(1-|w|^2)\big|} \|h_b\|.$$

The desired result follows immediately by Corollary 2.41. □

4.4. Hankel operators from $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ into $A_\alpha^{\Phi_2}(\mathbb{B}_n, Y)$, $\Phi_1 \in \mathscr{L} \cup \mathscr{U}$, $\Phi_2 \in \mathscr{U}$. In this subsection, we give the proof of Theorem 1.17 which generalizes the case $h_b\colon A_\alpha^p(\mathbb{B}_n, X) \to A_\alpha^q(\mathbb{B}_n, Y)$ with $0 < p < 1$ and $q > 1$ and also the proof of Theorem 1.18, which generalizes the case $h_b\colon A_\alpha^p(\mathbb{B}_n, X) \to A_\alpha^q(\mathbb{B}_n, Y)$ with $1 < p \le q < \infty$, whose proofs can be found in [3,12]. We start with the proof of Theorem 1.17 that we recall here for the reader's convenience.

THEOREM 4.2. *Let* $\Phi_1 \in \mathscr{L}$ *and* $\Phi_2 \in \mathscr{U}$, $\rho_i(t) = \frac{1}{t\,\Phi_i^{-1}(\frac{1}{t})}$ *and assume that* Φ_2 *satisfies the* ∇_2*-condition. Then the Hankel operator* h_b *extends into a bounded operator from* $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ *into* $A_\alpha^{\Phi_2}(\mathbb{B}_n, Y)$ *if and only if its symbol* b *belongs to* $\Gamma_{\alpha,\rho}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, *where*

$$\rho = \rho_\Phi := \frac{\rho_1}{\rho_2}.$$

PROOF. We assume that h_b extends into a bounded operator from $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ into $A_\alpha^{\Phi_2}(\mathbb{B}_n, Y)$, with $\|h_b\| := \|h_b\|_{A_\alpha^{\Phi_1}(\mathbb{B}_n,X)\to A_\alpha^{\Phi_2}(\mathbb{B}_n,Y)}$. Then by Theorem 1.12, we have, for every $f \in A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ and $g \in (A_\alpha^{\Phi_2}(\mathbb{B}_n, Y))^\star \equiv A_\alpha^{\Psi_2}(\mathbb{B}_n, Y^\star)$,

$$|\langle h_b(f), g\rangle_{\alpha,Y}| \le \|h_b\| \|f\|^{\mathrm{lux}}_{\Phi_1,\alpha,X} \|g\|^{\mathrm{lux}}_{\Psi_2,\alpha,Y^*}. \tag{4.9}$$

Now, take $x \in X, y^\star \in Y^\star$ and an integer k with $k > (n+1+\alpha)\big(\frac{1}{p} - 1\big) + 1$. Fix $w \in \mathbb{B}_n$ and let

$$f_w^x(z) = \Phi_1^{-1}\Big(\frac{1}{(1-|w|^2)^{n+1+\alpha}}\Big) \frac{(1-|w|^2)^{k-1}}{(1-\langle z, w\rangle)^{k-1}}\, x$$

and

$$g_w^{y^*}(z) = \Psi_2^{-1}\Big(\frac{1}{(1-|w|^2)^{n+1+\alpha}}\Big)\frac{(1-|w|^2)^{n+2+\alpha}}{(1-\langle z,w\rangle)^{n+2+\alpha}}\ y^*.$$

By Lemma 2.36 (resp. Lemma 2.38), we have $f_w^x \in \mathcal{H}^\infty(\mathbb{B}_n;X)$ (resp. $g_w^{y^*} \in \mathcal{H}^\infty(\mathbb{B}_n,Y^*)$) and there exist C_1, C_2 independent of w, x and y^* such that

$$\|f_w^x\|^{\mathrm{lux}}_{\Phi_1,\alpha,X} \le C_1\|x\|_X \quad \text{and} \quad \|g_w^{y^*}\|^{\mathrm{lux}}_{\Psi_2,\alpha,Y*} \le C_2\|y^*\|_{Y^*}.$$

Applying the inequality (4.9) to f_w^x and $g_w^{y^*}$, we have

$$\begin{aligned}\big|\langle h_b(f_w^x), g_w^{y^*}\rangle_{\alpha,Y}\big| &\le \|h_b\|\|f_w^x\|^{\mathrm{lux}}_{\Phi_1,\alpha,X}\|g_w^{y^*}\|^{\mathrm{lux}}_{\Psi_2,\alpha,Y^*} \\ &\le C_1C_2\|h_b\|\|x\|_X\|y^*\|_{Y^*}.\end{aligned} \tag{4.10}$$

Applying Lemma 2.42 for those particular vector-valued holomorphic functions f_w^x and $g_w^{y^*}$, and using Lemma 2.5, (2.23), Lemma 1.5 and the reproducing kernel property, we have that

$$\begin{aligned}\langle h_b(f_w^x), g_w^{y^*}\rangle_{\alpha,Y} &= \int_{\mathbb{B}_n}\big\langle h_b(f_w^x)(z), g_w^{y^*}(z)\big\rangle_{Y,Y^*}\,d\nu_\alpha(z)\\
&= \int_{\mathbb{B}_n}\big\langle b(z)\big(\overline{f_w^x(z)}\big), g_w^{y^*}(z)\big\rangle_{Y,Y^*}\,d\nu_\alpha(z)\\
&= (1-|w|^2)^{n+1+\alpha+k}\Phi^{-1}\Big(\frac{1}{(1-|w|^2)^{n+1+\alpha}}\Big)\\
&\quad\times\int_{\mathbb{B}_n}\Big\langle\frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}, y^*\Big\rangle_{Y,Y^*}\,d\nu_\alpha(z)\\
&= \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}\Big\langle\int_{\mathbb{B}_n}\frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}\,d\nu_\alpha(z), y^*\Big\rangle_{Y,Y^*}\\
&= \frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}\Big\langle\int_{\mathbb{B}_n}\frac{1}{C_{k,\alpha}}M_k^\alpha\Big(\frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha}}\Big)\,d\nu_\alpha(z), y^*\Big\rangle_{Y,Y^*}\\
&= \frac{1}{C_{k,\alpha}}\frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}\Big\langle M_k^\alpha\Big(\int_{\mathbb{B}_n}\frac{b(z)(\overline{x})\,d\nu_\alpha(z)}{(1-\langle w,z\rangle)^{n+1+\alpha}}\Big), y^*\Big\rangle_{Y,Y^*}\\
&= \frac{1}{C_{k,\alpha}}\frac{(1-|w|^2)^k}{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}\big\langle M_k^\alpha(b(w)(\overline{x})), y^*\big\rangle_{Y,Y^*}.\end{aligned}$$

Then

$$\big\langle M_k^\alpha(b(w)(\overline{x})), y^*\big\rangle_{Y,Y^*} = C_{k,\alpha}\frac{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}{(1-|w|^2)^k}\,\langle h_b(f_w^x), g_w^{y^*}\rangle_{\alpha,Y}.$$

Since $x \in X$, $y^\star \in Y^\star$ are arbitrary and $\|x\|_X = \|\overline{x}\|_{\overline{X}}$, by (4.10), we have

$$\begin{aligned}
\|M_k^\alpha b(w)\|_{\mathcal{L}(\overline{X},Y)} &= \sup_{\substack{\|\overline{x}\|_{\overline{X}}=1 \\ \|y^*\|_{Y^*}=1}} \big|\big\langle (M_k^\alpha b(w))(\overline{x}), y^* \big\rangle_{Y,Y^\star}\big| \\
&= C_{k,\alpha} \frac{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}{(1-|w|^2)^k} \sup_{\substack{\|\overline{x}\|_{\overline{X}}=1 \\ \|y^*\|_{Y^*}=1}} \big|\langle h_b f_w^x, g_w\rangle_{\alpha,Y}\big| \\
&\le C_1 C_2 C_{k,\alpha} \frac{\rho\big((1-|w|^2)^{n+1+\alpha}\big)}{(1-|w|^2)^k} \|h_b\|.
\end{aligned}$$

The desired result follows immediately by Corollary 2.41.

Conversely, assume that $b \in \Gamma_{\alpha,\rho}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and let us prove that h_b extends to a bounded operator from $A_\alpha^{\Phi_1}(\mathbb{B}_n, X)$ into $A_\alpha^{\Phi_2}(\mathbb{B}_n, Y)$. For every $f \in \mathcal{H}^\infty(\mathbb{B}_n, X)$ and $g \in \mathcal{H}^\infty(\mathbb{B}_n, Y^*)$, by Lemma 2.42 and Lemma 2.44 we have

$$\begin{aligned}
\big|\langle h_b(f), g\rangle_{\alpha,Y}\big| &= \left| \int_{\mathbb{B}_n} \big\langle h_b(f)(z), g(z)\big\rangle_{Y,Y^*} \, d\nu_\alpha(z)\right| \\
&= \left| \int_{\mathbb{B}_n} \big\langle b(z)\big(\overline{f(z)}\big), g(z)\big\rangle_{Y,Y^\star} \, d\nu_\alpha(z)\right| \\
&= C_k \left| \int_{\mathbb{B}_n} \big\langle M_k^\alpha\big(b(z)\big(\overline{f(z)}\big)\big), g(z)\big\rangle_{Y,Y^\star} \, d\nu_{\alpha+k}(z)\right| \\
&\le C_k \int_{\mathbb{B}_n} \big|\langle M_k^\alpha\big(b(z)\big(\overline{f(z)}\big)\big), g(z)\rangle_{Y,Y^\star}\big| \, d\nu_{\alpha+k}(z) \\
&\le C_k \int_{\mathbb{B}_n} \big\|M_k^\alpha\big(b(z)\big(\overline{f(z)}\big)\big)\big\|_Y \|g(z)\|_{Y^\star} \, d\nu_{\alpha+k}(z) \\
&\le C_k \int_{\mathbb{B}_n} \big\|M_k^\alpha b(z)\big\|_{\mathcal{L}(\overline{X},Y)} \big\|\overline{f(z)}\big\|_{\overline{X}} \|g(z)\|_{Y^\star} \, d\nu_{\alpha+k}(z) \\
&= C_k \frac{c_{\alpha+k}}{c_\alpha} \int_{\mathbb{B}_n} (1-|z|^2)^k \big\|M_k^\alpha b(z)\big\|_{\mathcal{L}(\overline{X},Y)} \|f(z)\|_X \|g(z)\|_{Y^\star} \, d\nu_\alpha(z) \\
&= C_k \frac{c_{\alpha+k}}{c_\alpha} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^k \|M_k^\alpha b(z)\|_{\mathcal{L}(\overline{X},Y)}}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)} \\
&\quad \times \rho\big((1-|z|^2)^{n+1+\alpha}\big) \|f(z)\|_X \|g(z)\|_{Y^\star} \, d\nu_\alpha(z).
\end{aligned}$$

Thus,

$$|\langle h_b(f), g\rangle_{\alpha,Y}| \leq C_k \frac{c_{\alpha+k}}{c_\alpha} \int_{\mathbb{B}_n} b^{\alpha,k}(z) h(z)\, d\nu_\alpha(z), \tag{4.11}$$

where

$$b^{\alpha,k}(z) = \frac{(1-|z|^2)^k \|M_k^\alpha(b(z))\|_{\mathcal{L}(\overline{X},Y)}}{\rho\big((1-|z|^2)^{n+1+\alpha}\big)}$$

and

$$h(z) = \rho\big((1-|z|^2)^{n+1+\alpha}\big)\|f(z)\|_X \|g(z)\|_{Y^\star}.$$

Since $b \in \Gamma_{\alpha,\rho}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ then $b^{\alpha,k} \in L^\infty(\mathbb{B}_n)$ and by Corollary 2.41, we have

$$\|b^{\alpha,k}\|_{L^\infty(\mathbb{B}_n)} \leq \|b\|_{\Gamma_{\alpha,\rho}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \tag{4.12}$$

By Proposition 2.7, the function $z \mapsto h(z) = h_1(z)h_2(z) \in L^\Phi_\alpha(\mathbb{B}_n)$ with

$$h_1(z) = \rho_1\big((1-|z|^2)^{n+1+\alpha}\big)\|f(z)\|_X \in L^{\Phi_1}_\alpha(\mathbb{B}_n)$$

and

$$h_2(z) = \frac{\|g(z)\|_{Y^*}}{\rho_2\big((1-|z|^2)^{n+1+\alpha}\big)} \in L^{\Psi_2}_\alpha(\mathbb{B}_n).$$

Moreover, we have

$$\|h\|^{\text{lux}}_{\Phi,\alpha,\mathbb{C}} \lesssim \|f\|^{\text{lux}}_{\Phi_1,\alpha,\,X}\|g\|^{\text{lux}}_{\Psi_2,\alpha,Y^\star}. \tag{4.13}$$

From the relation (4.11), using (4.12) and (4.13), we have

$$\big|\langle h_b(f), g\rangle_{\alpha,Y}\big| \lesssim \left|\int_{\mathbb{B}_n} b^{\alpha,k}(z)h(z)\, d\nu_\alpha(z)\right|$$

$$\lesssim \|b^{\alpha,k}\|_{L^\infty(\mathbb{B}_n)}\|h\|^{\text{lux}}_{\Phi,\alpha,\mathbb{C}} \lesssim \|b\|_{\Gamma_{\alpha,\rho}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}\|f\|^{\text{lux}}_{\Phi_1,\alpha,X}\|g\|^{\text{lux}}_{\Psi_2,\alpha,Y^\star}.$$

Then h_b extends into a bounded operator from $A^{\Phi_1}_\alpha(\mathbb{B}_n, X)$ into $A^{\Phi_2}_\alpha(\mathbb{B}_n, Y)$ and

$$\|h_b\| = \sup_{\substack{\|f\|^{\text{lux}}_{\Phi_1,\alpha,X}=1 \\ \|g\|_{\Psi_2,\alpha,Y^\star}}} \big|\big\langle h_b(f), g\big\rangle_{\alpha,Y}\big| \lesssim \|b\|_{\Gamma_{\alpha,\rho}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}. \quad \square$$

We can adapt the proof of Theorem 1.17 to prove Theorem 1.18 and we outline the main argument here.

Proof. From Theorem 1.11 we see that condition (i) implies that the dual space $(A_\alpha^{\Phi_2}(\mathbb{B}_n, Y))^*$ of $A_\alpha^{\Phi_2}(\mathbb{B}_n, Y)$ identified with $A_\alpha^{\Psi_2}(\mathbb{B}_n, Y^*)$. From Lemma 2.6, condition (ii) implies that $\Phi \in \mathscr{L}_p$ for some $0 < p \leq 1$. The whole proof follows the lines of the proof of Theorem 1.17. We omit the details. $\square$

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