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ON THE EDREI–GOLDBERG–OSTROVSKII THEOREM FOR MINIMAL SURFACES

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Abstract. This paper is devoted to the development of Beckenbach's theory of the meromorphic minimal surfaces. We consider the relationship between the number of separated maximum points of a meromorphic minimal surface and the Baernstein's T^*-function. The results of Edrei, Goldberg, Heins, Ostrovskii, Wiman are generalized. We also give examples showing that the obtained estimates are sharp.

1. Introduction

In the years 1960–1970 Beckenbach and collaborators generalized the original Nevanlinna's theory of value distribution of meromorphic functions by introducing the theory of meromorphic minimal surfaces [3–5]. In 1993 Fujimoto [11] generalized Nevanlinna's theory to minimal immersions from M to $\mathbb{R}^n$ where M is a Riemann surface of parabolic type.

A surface S is called *minimal* if the mean curvature of S vanishes at all points on the surface [6]. Let us remind the main definitions and results of Beckenbach's theory. We say that the surface

$$S = \left\{ \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_i = x_i(u, v),\ i = 1, 2, 3,\ (u, v) \in D \subset \mathbb{R}^2 \right\}$$

is given in terms of *isothermal parameters* u, v ([3]) if $E = G$, $F = 0$, where $x_i(u, v)$, $i = 1, 2, 3$, are twice continuously differentiable real-valued functions for $(u, v) \in D \subset \mathbb{R}^2$. Here E, F, G are the coefficients of the first fundamental form for the surface S

$$E = \|\mathbf{x}_u\|^2 = \sum_{j=1}^{3} \left(\frac{\partial x_j}{\partial u} \right)^2, \quad F = (\mathbf{x}_u, \mathbf{x}_v) = \sum_{j=1}^{3} \frac{\partial x_j}{\partial u} \frac{\partial x_j}{\partial v},$$

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$$G = \|\mathbf{x}_v\|^2 = \sum_{j=1}^{3} \Big(\frac{\partial x_j}{\partial v}\Big)^2,$$

where $\mathbf{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$. A necessary and sufficient condition for a regular surface S, given in terms of isothermal parameters, to be minimal is that the coordinate functions $x_i(u,v)$, $i = 1,2,3$, are harmonic on D ([6]).

Let us recall now some facts from the theory of harmonic functions.

The point $z_0 \in \mathbb{C}$ is an isolated singular point of a function $x(z) = x(u,v)$, if in a neighborhood of a point z_0 the function $x(z)$ is harmonic. If the point $z_0 \in \mathbb{C}$ is an isolated singular point of the harmonic function $x(z)$, then in the neighborhood of a point z_0 the function $x(z)$ can be presented by a series of the form

$$x(z) = c\log r + \sum_{k=-\infty}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta), \quad (b_0 = 0), \tag{1.1}$$

where $z - z_0 = re^{i\theta}$. Expansion (1.1) is an analogue of Laurent's series of a harmonic function. This expansion allows us to define poles, logarithmic poles and essential singular points [23]. Point $z_0 \in D$ is called *regular* for the function $x(z)$, if for the representation (1.1) in a neighborhood of the point z_0 we have $c = 0$ and $\min_{a_k^2+b_k^2\neq 0}\{k\} \geq 0$. If $\min_{a_k^2+b_k^2\neq 0}\{k\} = t \geq 1$ and $x(z_0) = a_0$, then the point z_0 is called an *a_0-point of order t* of the harmonic function. In particular, if $a_0 = 0$ then the point z_0 is called a *zero of order t* of the harmonic function.

We say that a point $z_0 \in D$ is *a pole of order $t = |l|$* of the function $x(z)$, if in the representation (1.1) we have $\min_{a_k^2+b_k^2\neq 0}\{k\} = l < 0$. On the other hand, if in (1.1) we have $c \neq 0$ and $\min_{a_k^2+b_k^2\neq 0}\{k\} \geq 0$ then the point z_0 is called a *logarithmic pole*. If in (1.1) there are infinitely many coefficients with negative indices, such that $a_k^2 + b_k^2 \neq 0$, then we say that z_0 is an essential singular point of the function $x(z)$.

We say that a harmonic function $x(z)$ is a *meromorphic harmonic* function in the domain D if, except for the poles, there are no more singular points of the function $x(z)$ in D ($z = u + iv$).

Definition [3]. The surface $S = \{x_1(u,v), x_2(u,v), x_3(u,v)\}$ is called a *meromorphic minimal surface* (m.m.s., for short) in a domain D if the parameters u, v are isothermal (i.e. $E = G$ and $F = 0$ for each $(u,v) \in D$) and the coordinate functions $x_1(u,v), x_2(u,v), x_3(u,v)$ are single-valued and harmonic in D, except for the poles.

In this paper we shall consider meromorphic minimal surfaces defined on the whole complex plane $\mathbb{C}$.

We say that a surface S is an *entire minimal surface* if the coordinate functions are harmonic in $\mathbb{C}$. A point $z_0 \in D$ is called a *pole* of m.m.s. in a domain D, if at least one of the coordinate functions $x_1(z), x_2(z), x_3(z)$ has a pole at z_0. Moreover if l_1, l_2, l_3 are the orders of the poles of functions $x_1(z)$, $x_2(z)$, $x_3(z)$ accordingly, then $l = \max\{l_1, l_2, l_3\}$ is called the *order of the pole of a m.m.s.* at z_0. A meromorphic minimal surface S cannot have a logarithmic poles [3]. A point $z_0 \in D$ is called an $\mathbf{a} = (a_1, a_2, a_3)$-point of a surface S, if z_0 is an a_i-point of the harmonic function $x_i(z)$, $i = 1, 2, 3$. Let l_i be the order of an a_i-point of the function $x_i(z)$. Then $l = \min\{l_1, l_2, l_3\}$ is the order of an $\mathbf{a}$-point of a surface S. The $\mathbf{a}$-points and the poles of a m.m.s. are isolated [3].

For m.m.s. S, Beckenbach and Hutchison defined the following three functions: $m(r, \mathbf{a}, S)$ – a *proximity* function of S; $N(r, \mathbf{a}, S)$ – an ***a**-points counting* function of S; and $H(r, \mathbf{a}, S)$ – a *visibility* function, which are defined in the following way:

$$m(r,\mathbf{a},S) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} \log^+ \|\mathbf{x}(re^{i\theta})\|\, d\theta & \text{for } \mathbf{a} = \infty, \\ \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\|\mathbf{x}(re^{i\theta}) - \mathbf{a}\|}\, d\theta & \text{for } \mathbf{a} \neq \infty, \end{cases}$$

where $\log^+ x = \max\{\log x, 0\}$ for $x \geq 0$, $\|\mathbf{x}(z)\| = \sqrt{x_1^2(z) + x_2^2(z) + x_3^2(z)}$ $(z = re^{i\theta})$;

$$N(r,\mathbf{a},S) = \begin{cases} \int_0^r \frac{n(\rho,\infty,S) - n(0,\infty,S)}{\rho}\, d\rho + n(0,\infty,S)\log r & \text{for } \mathbf{a} = \infty, \\ \int_0^r \frac{n(\rho,\mathbf{a},S) - n(0,\mathbf{a},S)}{\rho}\, d\rho + n(0,\mathbf{a},S)\log r & \text{for } \mathbf{a} \neq \infty, \end{cases}$$

where $n(r, \mathbf{a}, S)$ and $n(r, \infty, S)$ denote, respectively, the number of $\mathbf{a}$-points ($\mathbf{a} \in \mathbb{R}^3$) and the number of poles of meromorphic minimal surface S in the disc $\{z\colon |z| \leq r\}$, counted according to multiplicity;

$$H(r,\mathbf{a},S) = \begin{cases} 0 & \text{for } \mathbf{a} = \infty, \\ \int_0^r \frac{h(\rho,\mathbf{a};S)}{\rho}\, d\rho & \text{for } \mathbf{a} \neq \infty, \end{cases}$$

where

$$h(\rho,\mathbf{a},S) = \frac{1}{2\pi}\iint_{A_\rho(0)} \triangle \log\|\mathbf{x}(u,v) - \mathbf{a}\|\, du\, dv, \quad \triangle = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

is the Laplace operator and $A_\rho(0) = \{z \in \mathbb{C}\colon |z| \leq \rho\}$. Notice that $N(r, \mathbf{a}, S)$ vanishes almost everywhere in $\mathbb{R}^3$ so the most important function in Beckenbach's theory is $H(r, \mathbf{a}, S)$.

The function $T(r, S) = m(r, \infty, S) + N(r, \infty, S)$ is called the *characteristic* of a meromorphic minimal surface S.

In [3] Beckenbach and Hutchison get an analogue of Nevanlinna's first fundamental theorem for minimal surfaces. They prove that if S is a meromorphic minimal surface then for each $\mathbf{a} \in \mathbb{R}^3 \cup \{\infty\}$,

$$m(r, \mathbf{a}, S) + N(r, \mathbf{a}, S) + H(r, \mathbf{a}, S) = T(r, S) + O(1) \quad (r \to \infty).$$

Beckenbach and Cootz in [4] generalize Nevanlinna's second fundamental theorem to minimal surfaces. The theorem says that for a meromorphic minimal surface S and points $\mathbf{a}_k \in \mathbb{R}^3 \cup \{\infty\}$ $(k = 1, \ldots, q)$ we have the inequality

$$\sum_{k=1}^{q} m(r, \mathbf{a}_k, S) \le 2T(r, S) + O(\log(rT(r, S))), \quad r \notin E, \ r \to \infty,$$

where E is a set of finite measure.

If $f(z)$ is a meromorphic function then $S_f = (\operatorname{Re} f(z), \operatorname{Im} f(z), 0)$ is a plane surface, so it is also meromorphic minimal surface. It is clear that in this case for $a \in \mathbb{C}$ we have

$$m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} \, d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\|\mathbf{x}(re^{i\theta}) - \mathbf{a}\|} \, d\theta = m(r, \mathbf{a}, S_f),$$

where $\mathbf{a} = (\operatorname{Re} a, \operatorname{Im} a, 0)$. Moreover,

$$m(r, \infty, f) = m(r, \infty, S_f), \quad N(r, a, f) = N(r, \mathbf{a}, S_f),$$
$$H(r, \mathbf{a}, S_f) = 0, \quad T(r, f) = T(r, S_f).$$

Therefore Beckenbach's theory is a generalization of the classic Nevanlinna theory of distribution of values of meromorphic functions ([13,18]).

The number

$$\lambda = \liminf_{r \to \infty} \frac{\log T(r, S)}{\log r}$$

is called the *lower order* of S and

$$\delta(\mathbf{a}, S) = \liminf_{r \to \infty} \frac{m(r, \mathbf{a}, S)}{T(r, S)}$$

is called the *Nevanlinna defect* of S in the point $\mathbf{a}$.

For a m.m.s. S for any $\mathbf{a} \in \mathbb{R}^3$ we have $0 \le \delta(\mathbf{a}, S) \le 1$ and by the Second Fundamental Theorem we have $\sum_{\mathbf{a} \in \mathbb{R}^3 \cup \{\infty\}} \delta(\mathbf{a}, S) \le 2$. In 2004 Rhoads and

Weitsman generalized Nevanlinna's lemma of the logarithmic derivative for meromorphic minimal surfaces [28].

In 1979, Marchenko applied Petrenko's theory of the growth of meromorphic functions ([19], see also [27]) to the theory of meromorphic minimal surfaces. In [19] were defined the quantities

$$\mathcal{L}(r,\mathbf{a},S)=\begin{cases}\max_{|z|=r}\log^+\frac{1}{\|\mathbf{x}(z)-\mathbf{a}\|} & \text{for } \mathbf{a}\neq\infty,\\ \max_{|z|=r}\log^+\|\mathbf{x}(z)\| & \text{for } \mathbf{a}=\infty\end{cases}$$

and

$$\beta(\mathbf{a},S)=\liminf_{r\to\infty}\frac{\mathcal{L}(r,\mathbf{a},S)}{T(r,S)}.$$

The value $\beta(\mathbf{a},S)$ is called the *magnitude of deviation of the meromorphic minimal surface* S at the point $\mathbf{a}$. It is clear that $\delta(\mathbf{a},S)\le\beta(\mathbf{a},S)$ for each $\mathbf{a}\in\mathbb{R}^3\cup\{\infty\}$. A sharp upper estimate of $\beta(\mathbf{a},S)$ for surfaces of the finite lower order was presented in [19].

THEOREM A [19]. *If S is a meromorphic minimal surface of a finite lower order λ, then for each $\mathbf{a}\in\mathbb{R}^3\cup\{\infty\}$ we have*

$$\beta(\mathbf{a},S)\le B(\lambda):=\begin{cases}\frac{\pi\lambda}{\sin\pi\lambda} & \textit{for } \lambda\le\frac{1}{2},\\ \pi\lambda & \textit{for } \lambda>\frac{1}{2}.\end{cases}$$

The sharp upper estimate of the magnitude of deviation for meromorphic functions of finite lower order was obtained by Petrenko in 1969 [25]. Petrenko proved that for a meromorphic function $f(z)$ of finite lower order λ and any $a\in\overline{\mathbb{C}}$ we have $\beta(a,f)\le B(\lambda)$ (Paley's hypothesis, 1932).

The sharp upper estimate of the sum of deviations was given by Marchenko and Shcherba in 1990 as a solution of Petrenko's problem given in his monograph [26]. They proved that for a meromorphic function $f(z)$ of finite lower order λ the inequality $\sum_{a\in\overline{\mathbb{C}}}\beta(a,f)\le 2B(\lambda)$ holds [22].

In 2004 Ciechanowicz and Marchenko applied a quantity measuring the number of separated maximum modulus points of a meromorphic function to obtain an upper estimate of deviation for meromorphic functions ([7], see also [20]). We defined in [16] a similar quantity for meromorphic minimal surfaces.

Let $\phi(r)$ be a positive, nondecreasing convex function of $\log r$ for $r>0$, such that $\phi(r)=o(T(r,S))$ and $\widehat{p}_\phi(r,\infty,S)$ be the number of component intervals of the set

$$\{\theta:\log\|\mathbf{x}(re^{i\theta})\|>\phi(r)\}$$

possessing at lest one maximum modulus point of the function $\|\mathbf{x}(re^{i\theta})\|$.

Moreover, let us denote $\widehat{p}_\phi(\infty,S) = \liminf_{r\to\infty} \widehat{p}_\phi(r,\infty,S)$. We set

$$\widehat{p}(\infty,S) = \sup_{\{\phi\}} p_\phi(\infty,S).$$

In [16] we get an upper estimate of the magnitude of deviation for a meromorphic minimal surface of the finite lower order.

Theorem B. *For a meromorphic minimal surface S of the finite lower order λ, we have*

$$\beta(\infty,S) \le \begin{cases} \frac{\pi\lambda}{\widehat{p}(\infty,S)} & \text{if } \frac{\lambda}{\widehat{p}(\infty,S)} \ge \frac{1}{2}, \\ \frac{\pi\lambda}{\sin\pi\lambda} & \text{if } \widehat{p}(\infty,S) = 1 \text{ and } \lambda < \frac{1}{2}, \\ \frac{\pi\lambda}{\widehat{p}(\infty,S)} \sin\frac{\pi\lambda}{\widehat{p}(\infty,S)} & \text{if } \widehat{p}(\infty,S) > 1 \text{ and } \frac{\lambda}{\widehat{p}(\infty,S)} < \frac{1}{2}. \end{cases}$$

Corollary. *For a meromorphic minimal surface S of the finite lower order λ, we have*

$$\widehat{p}(\infty,S) \le \max\left(1, \left[\frac{\pi\lambda}{\beta(\infty,S)}\right]\right).$$

Moreover if $\beta(\infty,S) > 0$ then $1 \le \widehat{p}(\infty,S) < +\infty$.

2. Main results

Theorem 1. *Let S be a meromorphic minimal surface of the finite lower order $\lambda < \frac{\widehat{p}(\infty,S)}{2}$. Then*

$$\limsup_{r\to\infty} \frac{\log^+ \mu(r,S)}{T(r,S)} \ge \frac{\frac{\pi\lambda}{\widehat{p}(\infty,S)}}{\sin\frac{\pi\lambda}{\widehat{p}(\infty,S)}} \Big(\delta(\infty,S) - 1 + \cos\frac{\pi\lambda}{\widehat{p}(\infty,S)}\Big),$$

where $\mu(r,S) = \min_{|z|=r} \|\mathbf{x}(z)\|$.

In the case of the meromorphic functions the result of Theorem 1 was obtained by Marchenko [21].

Corollary 2. *Let S be a meromorphic minimal surface of the finite lower order $\lambda < \frac{1}{2}$. Then*

$$\limsup_{r\to\infty} \frac{\log^+ \mu(r,S)}{T(r,S)} \ge \frac{\pi\lambda}{\sin\pi\lambda} \big(\delta(\infty,S) - 1 + \cos\pi\lambda\big),$$

where $\mu(r,S) = \min_{|z|=r} \|\mathbf{x}(z)\|$.

In the case of meromorphic functions Corollary 2 was obtained earlier by Goldberg and Ostrovskii [13,24].

COROLLARY 3. *Suppose that S is a meromorphic minimal surface of finite lower order* $\lambda < \frac{\widehat{p}(\infty,S)}{2}$ *and* $\delta(\infty,S) > 1-\cos\frac{\pi\lambda}{\widehat{p}(\infty,S)}$. *Then there exists a sequence of circles* $\{z\colon |z| = r_k\}$, $r_k\to\infty$, *on which* $\|\mathbf{x}(z)\|$ *tends to* ∞ *uniformly with respect to* $\arg z$.

COROLLARY 4. *Suppose that S is a meromorphic minimal surface of finite lower order* $\lambda < \frac{1}{2}$ *and* $\delta(\infty,S) > 1-\cos\pi\lambda$. *Then there is a sequence* $r_n\to\infty$, *such that* $\|\mathbf{x}(r_n e^{i\theta})\|$ *tends uniformly to* ∞ *for* $\theta\in[0,2\pi]$.

In the case of meromorphic functions the result of Corollary 4 was obtained earlier by Goldberg and Ostrovskii ([13], see also [24]) and Edrei [9].

It is necessary to admit that in 1939 Teichmüller [30] proved that for the meromorphic function $f(z)$ of the order $\rho < \frac{1}{2}$ such that $\delta(\infty,f) > 1-\cos\pi\rho$ it holds for all $\theta\in[0,2\pi]$ that

$$\limsup_{r\to\infty}|f(re^{i\theta})| = \infty.$$

Therefore Teichmüller get the result of Corollary 4 in the case of meromorphic functions such that $\delta(\infty,f) > \frac{1-\cos\pi\rho}{1-\varepsilon\cos\pi\rho}$, where $\varepsilon > 0$ $(0<\varepsilon<1)$.

COROLLARY 5. *Let S be an entire minimal surface of lower order* $\lambda < \frac{\widehat{p}(\infty,S)}{2}$. *Then there exists a sequence of circles* $\{z\colon |z| = r_k\}$, $r_k\to\infty$, *on which* $\|\mathbf{x}(z)\|$ *tends to* ∞ *uniformly with respect to* $\arg z$.

COROLLARY 6. *Let S be an entire minimal surface of lower order* $\lambda < \frac{1}{2}$. *Then there exists a sequence of circles* $\{z\colon |z| = r_k\}$, $r_k\to\infty$, *on which* $\|\mathbf{x}(z)\|$ *tends to* ∞ *uniformly with respect to* $\arg z$.

In the case of entire function the result of Corollary 6 was obtained by Heins [15] in 1948 and in case when $f(z)$ is an entire function of order $\rho < \frac{1}{2}$ by Wiman [31] in 1905.

COROLLARY 7. *Let S be an entire minimal surface of lower order* $\lambda < \frac{\widehat{p}(\infty,S)}{2}$. *Then for any* $\mathbf{a}\in\mathbb{R}^3$ *we have* $\delta(\mathbf{a},S) = 0$.

COROLLARY 8. *Let S be an entire minimal surface of lower order* $\lambda < \frac{1}{2}$. *Then for any* $\mathbf{a}\in\mathbb{R}^3$ *we have* $\delta(\mathbf{a},S) = 0$.

Result of Corollary 8 was obtained by Tafel in 1970 ([29, Theorem 9.4, p. 118], see also [2, p. 40]).

3. Auxiliary results

Let $S = \{\mathbf{x}(z) = (x_1(z),x_2(z),x_3(z))\colon z\in\mathbb{C}\}$ be a meromorphic minimal surface and let $\phi(r)$ be a positive nondecreasing convex function of $\log r$ such that $\phi(r) = o(T(r,S))$. We consider the function given by

$$u_\phi(z) = \max\bigl\{\log\|\mathbf{x}(z)\|, \phi(|z|)\bigr\}.$$

We say that the function $u(z)$ is $\delta-$subharmonic function on D if $u(z)$ can be represented as the difference of two subharmonic functions on D.

In [16] we proved the following lemma.

LEMMA 9. *The function $u_\phi(z)$ is a δ-subharmonic function in $\mathbb{C}$, i.e.*

$$u_\phi(z) = u_1(z) - u_2(z),$$

where $u_1(z), u_2(z)$ are subharmonic functions in $\mathbb{C}$ and

$$\frac{1}{2\pi}\int_0^{2\pi} u_2(re^{i\theta})\,d\theta = N(r,\infty,S).$$

Let (see [1,16])

$$m^*(r,\theta,u_\phi) = \sup_{|E|=2\theta} \frac{1}{2\pi}\int_E u_\phi(re^{i\varphi})\,d\varphi,$$

$$T^*(r,\theta,u_\phi) = m^*(r,\theta,u_\phi) + N(r,\infty,S),$$

where $r \in (0,\infty)$, $\theta \in [0,\pi]$, E is a measurable set and $|E|$ is the Lebesgue measure of E. Now for each $t \in (0,+\infty)$, consider the set

$$F_t = \big\{re^{i\varphi} : u_\phi(re^{i\varphi}) > t\big\},$$

and let

$$\widetilde{u}_\phi(re^{i\varphi}) = \sup\big\{t : re^{i\varphi} \in F_t^*\big\},$$

where F_t^* is the symmetric rearrangement of the set F_t (see [14]). The function $\widetilde{u}_\phi(re^{i\varphi})$ is non-negative and non-increasing in the interval $[0,\pi]$, even with respect to φ and for each fixed r equimeasurable with $u_\phi(re^{i\varphi})$. Moreover, it satisfies the equalities:

$$\widetilde{u}_\phi(r) = \max\big\{\log\max_{|z|=r}\|\mathbf{x}(z)\|, \phi(r)\big\},$$

$$\widetilde{u}_\phi(re^{i\pi}) = \max\big\{\log\min_{|z|=r}\|\mathbf{x}(z)\|, \phi(r)\big\},$$

$$m^*(r,\theta,u_\phi) = \frac{1}{\pi}\int_0^\theta \widetilde{u}_\phi(re^{i\varphi})\,d\varphi.$$

From Baernstein's theorem ([1]), the function $T^*(r,\theta,u_\phi)$ is subharmonic in $D = \{re^{i\theta}\colon 0<r<\infty,\ 0<\theta<\pi\}$, continuous in $D \cup (-\infty,0) \cup (0,\infty)$ and logarithmically convex in $r>0$ for each fixed $\theta \in [0,\pi]$. Moreover,

$$T^*(r,0,u_\phi) = N(r,\infty,S),$$

$$T^*(r,\pi,u_\phi) = T(r,S) + o(T(r,S)) \quad (r\to\infty),$$

$$\frac{\partial}{\partial\theta}T^*(r,\theta,u_\phi) = \frac{\widetilde{u}_\phi(re^{i\theta})}{\pi} \quad \text{for } \theta\in[0,\pi] \text{ and } r\in(0,\infty)$$

such that there are no zeros nor poles of $\|\mathbf{x}(z)\|$ on the circle $\{z : |z| = r\}$.

Let $\alpha(r)$ be a real-valued function of a real variable r and define

$$L\alpha(r) = \liminf_{h\to 0}\frac{\alpha(re^h)+\alpha(re^{-h})-2\alpha(r)}{h^2}.$$

When $\alpha(r)$ is twice differentiable in r, then $L\alpha(r) = r\frac{d}{dr}\big(r\frac{d}{dr}\alpha(r)\big)$.

In [16] (see also [17]) we proved the following lemma.

Lemma 10. *Let $S = \{\mathbf{x}(z) = (x_1(z),x_2(z),x_3(z)) : z\in\mathbb{C}\}$ be a meromorphic minimal surface. For almost all $\theta\in[0,\pi]$ and for all $r>0$ such that the function $\|\mathbf{x}(z)\|$ has neither zeros nor poles in $\{z : |z| = r\}$, we have*

$$LT^*(r,\theta,u_\phi) \ge -\frac{\widehat{p}^2_\phi(r,\infty,S)}{\pi}\,\frac{\partial\widetilde{u}_\phi(r,\theta)}{\partial\theta}.$$

Lemma 11 [20]. *Let the function $f(x)$ be non decreasing on the interval $[a,b]$ and let $\varphi(x)$ be a non negative function having a bounded derivative of the interval $[a,b]$. Then*

$$\int_a^b f'(x)\varphi(x)\,dx \le f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b \varphi'(x)f(x)\,dx.$$

We will remind now definition of the Pólya peaks for a monotonic functions [26]. Let $T(r)$ be a increasing and continuous for $r\ge r_0$ function of a finite lower order λ.

The sequence $\{r_k\}$ is called a sequence of Pólya peaks of the function $T(r)$ if there are a sequences $\{a_k\}$, $\{A_k\}$ and $\{\varepsilon_k\}$ of nonnegative numbers such that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\varepsilon_k = 0, \quad \lim_{k\to\infty} A_k = \lim_{k\to\infty} a_k r_k = \infty,$$

and for all $r\in[a_k r_k, A_k r_k]$ holds the following inequality

$$T(r) \ge (1-\varepsilon_k)\Big(\frac{r}{r_k}\Big)^\lambda T(r_k). \tag{3.1}$$

for $k>k_0$. We recall the lemma, which allows to estimate growth of $\frac{T(r_k)}{r_k^\lambda}$.

Lemma A [26, p. 40]. *Let S_k and R_k be two sequences such that*

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} R_k = \lim_{k\to\infty}\frac{R_k}{S_k} = \infty,$$

and for each k the numbers $2S_k$ and $2R_k$ are Pólya peaks of the function $T(r)$. Then for each positive number ε there exists $k_0(\varepsilon)$ such that for each $k > k_0$ we have

$$\frac{T(2S_k)}{S_k^\lambda} + \frac{T(2R_k)}{R_k^\lambda} < \varepsilon \int_{2S_k}^{R_k} \frac{T(r)}{r^{\lambda+1}}\, dr$$

In our later considerations instead of the function $T(r)$ we will be using the Nevallina's characteristic function of the growth of m.m.s. S of the finite lower order λ and $\{r_k\}$ will be a sequence of Pólya peaks for this function. By term of a sequence of Pólya peaks it is possible to estimate the growth of $\frac{T(r_k,S)}{r_k^\lambda}$. From Lemma A we have

$$\frac{T(2S_k,S)}{S_k^\lambda} + \frac{T(2R_k,S)}{R_k^\lambda} < \varepsilon \int_{2S_k}^{R_k} \frac{T(r,S)}{r^{\lambda+1}}\, dr \quad (k \to \infty). \tag{3.2}$$

4. Proof of Theorem 1

If $\widehat{p}(\infty,S) = +\infty$ then by Theorem B we have $\beta(\infty,S) = 0$. Thus $\delta(\infty,S) = 0$, so the right-hand side of the inequality in the statement of Theorem 1 is equal to zero and left side is non-negative.

Let now $\widehat{p}(\infty,S) < \infty$. If $\delta(\infty,S) \le 1 - \cos\frac{\pi\lambda}{\widehat{p}(\infty,S)}$ then Theorem 1 is obviously. Let $\delta(\infty,S) > 1 - \cos\frac{\pi\lambda}{\widehat{p}(\infty,S)}$. Then $\delta(\infty,S) > 0$ and for every $\phi(r)$ we have $\widehat{p}_\phi(\infty,S) \ge 1$. We shall first consider the case $\lambda > 0$. We put (see [10,12,20])

$$\sigma(r) = \int_0^\pi T^*(r,\varphi,u_\phi) \sin\frac{\lambda\varphi}{\widehat{p}_\phi(\infty,S)}\, d\varphi,$$

where $T^*(r,\varphi,u_\phi) = T^*(re^{i\varphi},u_\phi)$. Since $T^*(re^{i\varphi},u_\phi)$ is a convex function of $\log r$, it follows that for all $r > 0$ and $h > 0$ we have

$$T^*(re^h,\varphi,u_\phi) + T^*(re^{-h},\varphi,u_\phi) - 2T^*(r,\varphi,u_\phi) \ge 0.$$

Thus by Fatou's lemma for all $r > 0$ we have

$$L\sigma(r) \ge \int_0^\pi LT^*(r,\theta,u_\phi) \sin\frac{\lambda\theta}{\widehat{p}_\phi(\infty,S)}\, d\theta \ge 0. \tag{4.1}$$

It follows from this inequality that $\sigma(r)$ is a convex function of $\log r$, and so $r\sigma'_-(r)$ is an increasing function on $(0,\infty)$. Therefore, for almost all $r > 0$

$$L\sigma(r) = r\frac{d}{dr}(r\sigma'_-(r)).$$

It follows from (4.1) and Lemma 10 that for almost all $r > 0$

$$r\frac{d}{dr}\left(r\sigma'_{-}(r)\right) \ge -\int_0^\pi \frac{\widehat{p}^2_\phi(r,\infty,S)}{\pi}\frac{\partial \widetilde{u}_\phi(r,\theta)}{\partial\theta}\sin\frac{\lambda\theta}{\widehat{p}_\phi(r,\infty,S)}\,d\theta. \tag{4.2}$$

By definition $\widehat{p}_\phi(r,\infty,S)$ takes only the integral values. Thus for $r \ge r_0$ we have $\widehat{p}_\phi(\infty,S) \le \widehat{p}_\phi(r,\infty,S)$. From this and (4.2) it follows that for almost all $r \ge r_0$

$$r\frac{d}{dr}\left(r\sigma'_{-}(r)\right) \ge -\int_0^\pi \frac{\widehat{p}^2_\phi(\infty,S)}{\pi}\frac{\partial \widetilde{u}_\phi(r,\theta)}{\partial\theta}\sin\frac{\lambda\theta}{\widehat{p}_\phi(\infty,S)}\,d\theta. \tag{4.3}$$

If there are neither zeros nor poles of $\|\mathbf{x}(z)\|$ on the circle $\{z : |z| = r\}$ for $r > 0$, the function $u_\phi(r,\theta) = \max\left(\log\|\mathbf{x}(re^{i\theta})\|, \phi(r)\right)$ fulfills the Lipschitz condition in θ. Therefore $\widetilde{u}_\phi(r,\theta)$ also fulfills the Lipschitz condition on $[0,\pi]$ (see [14]). It implies that the function $\widetilde{u}_\phi(r,\theta)$ is absolutely continuous on $[0,\pi]$. Integrating twice by parts, we have for almost all $r \ge r_0$

$$\begin{aligned} r\frac{d}{dr}\left(r\sigma'_{-}(r)\right) &\ge -\frac{\widehat{p}^2_\phi(\infty,S)}{\pi}\widetilde{u}_\phi(r,\pi)\sin\frac{\lambda\pi}{\widehat{p}_\phi(\infty,S)} \\ &\quad + \lambda\widehat{p}_\phi(\infty,S)T^*(r,\pi,u_\phi)\cos\frac{\lambda\pi}{\widehat{p}_\phi(\infty,S)} \\ &\quad - \lambda\widehat{p}_\phi(\infty,S)N(r,\infty,S) + \lambda^2\sigma(r) := h(r) + \lambda^2\sigma(r). \end{aligned} \tag{4.4}$$

Dividing both sides of (4.4) by $r^{\lambda+1}$ and integrating by parts over the interval $[2S_k, R_k]$, where S_k, R_k are the sequences described in (3.2) we have

$$\int_{2S_k}^{R_k}\frac{h(r)}{r^{\lambda+1}}\,dr + \lambda^2\int_{2S_k}^{R_k}\frac{\sigma(r)}{r^{\lambda+1}}\,dr \le \int_{2S_k}^{R_k}\frac{1}{r^\lambda}\frac{d}{dr}\left(r\sigma'_{-}(r)\right)dr = I. \tag{4.5}$$

Invoking Lemma 11 we get

$$I \le \frac{\sigma'_{-}(r)}{r^{\lambda+1}}\Big|_{2S_k}^{R_k} + \lambda\int_{2S_k}^{R_k}\frac{\sigma'_{-}(r)}{r^\lambda}\,dr. \tag{4.6}$$

The function $\sigma(r)$ is a convex function of $\log r$ on the interval $(0,+\infty)$, i.e. $g(t) = \sigma(e^t)$ is convex on $(-\infty,\infty)$.Thus the function $g(t)$ satisfies a Lipschitz condition on each interval $[a,b] \subset (0,+\infty)$, so is also absolutely continuous on each interval. Then the function $\sigma(r) = g(\log r)$ is also absolutely continuous on the intervals $[a,b] \subset (0,+\infty)$. Integrating by parts the integral in the inequality (4.6) we have

$$\int_{2S_k}^{R_k}\frac{\sigma'_{-}(r)}{r^\lambda}\,dr = \int_{2S_k}^{R_k}\frac{\sigma'(r)}{r^\lambda}\,dr = \frac{\sigma(R_k)}{R_k^\lambda} - \frac{\sigma(2S_k)}{(2S_k)^\lambda} + \lambda\int_{2S_k}^{R_k}\frac{\sigma(r)}{r^{\lambda+1}}\,dr. \tag{4.7}$$

By (4.5), (4.6) and (4.7) we have

$$\int_{2S_k}^{R_k} \frac{h(r)}{r^{\lambda+1}}\,dr \le \Big(\frac{\sigma'_-(r)}{r^{\lambda-1}} + \lambda\frac{\sigma(r)}{r^{\lambda}}\Big)\Big|_{2S_k}^{R_k}. \tag{4.8}$$

By definition of $\sigma(r)$ we get

$$0 \le \sigma(R) \le \pi(1+o(1))T(R,S) < 2\pi T(R,S) \quad (R\to\infty). \tag{4.9}$$

The function $r\sigma'_-(r)$ is non-decreasing on $(0,\infty)$, hence

$$\sigma(2R) \ge \sigma(2R)-\sigma(R) = \int_R^{2R} \sigma'(r)\,dr = \int_R^{2R} \frac{r\sigma'_-(r)}{r}\,dr$$
$$\ge R\sigma'_-(R)\int_R^{2R}\frac{dr}{r} = R\sigma'_-(R)\log 2.$$

Consequently, for $R > R_0$ we have

$$R\sigma'_-(R) \le \frac{1}{\log 2}\sigma(2R) \le \frac{2\pi}{\log 2}T(2R,S). \tag{4.10}$$

Moreover, in view of the monotonicity of $R\sigma'_-(R)$ we have for $R \ge 1$

$$R\sigma'_-(R) \ge \sigma'_-(1) = C. \tag{4.11}$$

By (4.8), (4.9), (4.10) and (4.11) we have

$$\int_{2S_k}^{R_k} \frac{h(r)}{r^{\lambda+1}}\,dr \le 2\pi\Big(\frac{1}{\log 2}+\lambda\Big)\frac{T(2R_k,S)}{R_k^{\lambda}} - \frac{C}{(2S_k)^{\lambda}} \quad (k\to\infty).$$

It follows from the (3.2) that for $k \ge k_0(\varepsilon)$

$$\int_{2S_k}^{R_k} \frac{h(r)}{r^{\lambda+1}}\,dr < \varepsilon\int_{2S_k}^{R_k}\frac{T(r,S)}{r^{\lambda+1}}\,dr\,.$$

Therefore there exists a sequence $r_k \in [2S_k, R_k]$ such that $h(r_k) < \varepsilon T(r_k,S)$. Since $S_k\to\infty$ it follows that $r_k\to\infty$ as $k\to\infty$.

Recalling the definition of $h(r)$ we have for $k \ge k_0$

$$\frac{\widehat{p}_\phi^2(\infty,S)}{\pi}\Big(\frac{\pi\lambda}{\widehat{p}_\phi(\infty,S)}T^*(r_k,\pi,u_\phi)\cos\frac{\lambda\pi}{\widehat{p}_\phi(\infty,S)} \tag{4.12}$$
$$-\frac{\pi\lambda}{\widehat{p}_\phi(\infty,S)}N(r_k,\infty,S) - \widetilde{u}_\phi(r_k,\pi)\sin\frac{\lambda\pi}{\widehat{p}_\phi(\infty,S)}\Big) < \varepsilon T(r_k,S).$$

The quantity $\widehat{p}_\phi(\infty, S)$ is an entire non-negative number. Since $\widehat{p}(\infty, S) = \sup_\phi \widehat{p}_\phi(\infty, S)$ there is the function $\phi(r)$, such that $\widehat{p}_\phi(\infty, S) = \widehat{p}(\infty, S)$. If we apply the inequality (4.12) to the function ϕ then we have

$$\frac{\pi\lambda}{\widehat{p}(\infty,S)} T^*(r_k,\pi,u_\phi)\cos\frac{\lambda\pi}{\widehat{p}(\infty,S)} - \frac{\pi\lambda}{\widehat{p}(\infty,S)} N(r_k,\infty,S) - \widetilde{u}_\phi(r_k,\pi)\sin\frac{\lambda\pi}{\widehat{p}(\infty,S)} < \varepsilon T(r_k,S) \quad (k > k_0). \tag{4.13}$$

Since

$$\begin{aligned} T^*(r,\pi,u_\phi) &= \frac{1}{\pi}\int_0^\pi \widetilde{u}_\phi(r,\theta)\,d\theta + N(r,\infty,S) \\ &= \frac{1}{2\pi}\int_0^{2\pi} u_\phi^+(r,\theta)\,d\theta + N(r,\infty,S) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \|\mathbf{x}(re^{i\theta})\|\,d\theta + o(T(r,S)) + N(r,\infty,S) \\ &= m(r,\infty,S) + N(r,\infty,S) + o(T(r,S)) = T(r,S) + o(T(r,S)), \end{aligned}$$

then by (4.13) we have

$$\frac{\pi\lambda}{\widehat{p}(\infty,S)} T(r_k,f)\cos\frac{\pi\lambda}{\widehat{p}(\infty,S)} - \frac{\pi\lambda}{\widehat{p}(\infty,S)} N(r_k,\infty,S) - \widetilde{u}_\phi(r_k,\pi)\sin\frac{\pi\lambda}{\widehat{p}(\infty,S)} < \varepsilon T(r_k,S) \quad (k\to\infty).$$

Since $\delta(\infty,S) = 1 - \limsup_{r\to\infty}\frac{N(r,\infty,S)}{T(r,S)}$ then

$$N(r,\infty,S) < (1-\delta(\infty,S)+\varepsilon)T(r,S) \quad (r > r_0).$$

Hence

$$\begin{aligned} \widetilde{u}_\phi(r,\pi) &= \max\Big(\min_{|z|=r}\log\|\mathbf{x}(z)\|,\phi(r)\Big) = \max\Big(\min_{|z|=r}\log^+\|\mathbf{x}(z)\|,\phi(r)\Big) \\ &\le \min_{|z|=r}\log^+\|\mathbf{x}(z)\| + \phi(r) = \log^+\mu(r,S) + o(T(r,S)) \quad (r\to\infty). \end{aligned}$$

Thus

$$\frac{\pi\lambda}{\widehat{p}(\infty,S)} T(r_k,S)\cos\frac{\pi\lambda}{\widehat{p}(\infty,S)} - \frac{\pi\lambda}{\widehat{p}(\infty,S)}\big(1-\delta(\infty,S)+\varepsilon\big)T(r_k,S) - \log^+\mu(r_k,S)\sin\frac{\pi\lambda}{\widehat{p}(\infty,S)} < \varepsilon T(r_k,S) \quad (k\to\infty).$$

Therefore

$$\sin \frac{\pi\lambda}{\widehat{p}(\infty, S)} \limsup_{r\to\infty} \frac{\log^+ \mu(r,S)}{T(r,S)}$$
$$\geq \frac{\pi\lambda}{\widehat{p}(\infty, S)}\Big(\delta(\infty, S) - 1 + \cos\frac{\pi\lambda}{\widehat{p}(\infty, S)} - \varepsilon\Big) - \varepsilon.$$

Taking $\varepsilon \to 0^+$ we get statement of Theorem 1 for $\lambda > 0$. The proof for $\lambda = 0$ can be obtained similarly (see [20]). □

5. Examples

We consider the surface $S(f)$ given by the relations

$$\text{(5.1)} \qquad \begin{cases} x_1(z) = \operatorname{Re}[3f(z) - f^3(z)], \\ x_2(z) = \operatorname{Re}[i(3f(z) + f^3(z))], \\ x_3(z) = \operatorname{Re}[3f^2(z)], \end{cases}$$

where $f(z)$ is a meromorphic function [19]. Then the coordinate functions are harmonic in $\mathbb{C}$. From [8, p. 94], to prove that $S(f)$ is a m.m.s. it is enough to show that

$$\sum_{i=1}^{3}\Big(\frac{dg_i(z)}{dz}\Big)^2 \equiv 0,$$

where

$$g_1(z) = 3f(z) - f^3(z), \quad g_2(z) = i(3f(z) + f^3(z)), \quad g_3(z) = 3f^2(z).$$

By basic computations we see that

$$\|\mathbf{x}(z)\|^2 = 9|f(z)|^2 + |f(z)|^6 + 6(\operatorname{Im}[f(z)]\operatorname{Im}[f^3(z)]$$
$$- \operatorname{Re}[f(z)]\operatorname{Re}[f^3(z)]) + 9(\operatorname{Re}[f^2(z)])^2.$$

We consider the set $E(r) = \big\{\theta \in [0, 2\pi] \colon |f(re^{i\theta})| > 4\big\}$. If $z = re^{i\theta}$, $\theta \in E(r)$ then we have

$$\|\mathbf{x}(z)\|^2 \geq |f(z)|^6 - 12|f(z)|^4 \geq \frac{1}{4}|f(z)|^6.$$

Then $\log^+ \|\mathbf{x}(z)\| \geq 3\log^+ |f(z)| + O(1)$ $(r \to \infty)$. On the other hand, for $z = re^{i\theta}$, $\theta \in E(r)$ we get

$$\text{(5.2)} \qquad \|\mathbf{x}(z)\|^2 \leq 9|f(z)|^2 + |f(z)|^6 + 21|f(z)|^4 \leq 31|f(z)|^6.$$

Then $\log^+ \|\mathbf{x}(z)\| \le 3\log^+ |f(z)| + O(1)$ $(r \to \infty)$. Thus we obtain

$$m(r, \infty, S(f)) = 3m(r, \infty, f) + O(1), \quad r \to \infty. \tag{5.3}$$

It is easy to see that $N(r, \infty, S(f)) = 3N(r, \infty, f)$, so by (5.3) we have

$$T(r, S(f)) = 3T(r, f) + O(1),$$

which implies that

$$\delta(\infty, S(f)) = \delta(\infty, f).$$

EXAMPLE 12. We consider the entire function

$$\varphi_1(z) = \cos\sqrt{z} = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(2k)!}.$$

The function $\varphi_1(z)$ is of order $\rho = \frac{1}{2}$ $(\lambda = \rho)$ and $|\varphi_1(x)| = |\cos\sqrt{x}| \le 1$ for $x \ge 0$. For each $n \in \mathbb{N}$ we consider now the function

$$F_1(z) = \varphi_1(z^n) = \cos\sqrt{z^n} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{nk}}{(2k)!}.$$

The entire function $F_1(z)$ is of order $\rho = \frac{n}{2}$ $(\lambda = \rho)$ and $|F_1(x)| \le 1$ for $x \ge 0$.

We consider now the surface $S(F_1)$ with the coordinate functions described in (5.1). By (5.2) we get $\|\mathbf{x}(x)\| \le \sqrt{31}$ for $x \ge 0$. For entire minimal surface $S(F_1)$ we have $\widehat{p}(\infty, S(F_1)) = n$ and the lower order is $\lambda = \frac{n}{2} = \frac{\widehat{p}(\infty, S(F_1))}{2}$.

The example of surface $S(F_1)$ proves that the condition $\lambda < \frac{\widehat{p}(\infty, S)}{2}$ in Corollary 5 can not be replaced by condition $\lambda \le \frac{\widehat{p}(\infty, S)}{2}$.

EXAMPLE 13. In the paper [30] (see also [13, p. 282]) Teichmüller created for each ρ, $0 < \rho < \frac{1}{2}$, the meromorphic function $f_\rho(z)$ of order ρ $(\lambda = \rho)$ such that $\delta(\infty, f_\rho) = 1 - \cos\pi\rho$ and $|f_\rho(-r)| \le 2$ for $r \ge 0$.

For each $n \in \mathbb{N}$ and $\lambda > 0$ such that $\frac{\lambda}{n} < \frac{1}{2}$ we consider the function

$$F_2(z) = f_{\frac{\lambda}{n}}(z^n).$$

It is easy to see that $F_2(z)$ is a meromorphic function of lower order λ $(\lambda = \rho)$, $\delta(\infty, F_2) = 1 - \cos\frac{\pi\lambda}{n}$ and $|F_2(\sqrt[n]{-r})| = |f_{\frac{\lambda}{n}}(-r)| \le 2$ for $r \ge 0$. We consider now the surface $S(F_2)$ with the coordinate functions described in (5.1). By (5.2) we get $\|\mathbf{x}(\sqrt[n]{-r})\| \le 8\sqrt{31}$ for $r \ge 0$. For meromorphic minimal surface $S(F_2)$ we have $\widehat{p}(\infty, S(F_2)) = n$, the lower order of $S(F_2)$ is λ, $\frac{1}{2} > \frac{\lambda}{n} = \frac{\lambda}{\widehat{p}(\infty, S(F_2))}$ and $\delta(\infty, S(F_2)) = 1 - \cos\frac{\pi\lambda}{n} = 1 - \cos\frac{\pi\lambda}{\widehat{p}(\infty, S(F_2))}$.

Therefore the example of surface $S(F_2)$ proves that the condition $\delta(\infty, S) > 1 - \cos\frac{\pi\lambda}{\widehat{p}(\infty,S)}$ in Corollary 3 can not be replaced by condition $\delta(\infty, S) \geq 1 - \cos\frac{\pi\lambda}{\widehat{p}(\infty,S)}$.

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