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NON-ARCHIMEDEAN BANACH SPACES OF UNIVERSAL DISPOSITION

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Abstract. A space of universal disposition is a Banach space which has certain natural extension properties for isometric embeddings of Banach spaces belonging to a specific class. We study spaces of universal disposition for non-archimedean Banach spaces. In particular, we introduce the classification of non-archimedean Banach spaces depending on the cardinality of maximal orthogonal sets, which can be viewed as a kind of special density and characterize spaces of universal disposition for each distinguished class.

1. Introduction

We say that a Banach space E is a *space of universal* (*almost universal*) *disposition* for a given class of Banach spaces $\mathcal{U}$ such that $\{0\} \in \mathcal{U}$ if for every linear isometric embedding $g\colon X \to Y$, where $X, Y \in \mathcal{U}$ and X is a linear subspace of E, there exists a linear isometric embedding $f\colon Y \to E$ (for every $\varepsilon > 0$ there is a linear ε-isometric embedding $f\colon Y \to E$, i.e. for every $x \in Y$ one has $(1+\varepsilon)^{-1}\|x\| \le \|f(x)\| \le (1+\varepsilon)\|x\|$) such that $f(g(x)) = x$ for all $x \in X$.

The concept of Banach spaces of (almost) universal disposition was introduced by Gurariĭ in [9], who constructed a separable (real) Banach space $\mathbb{G}$ of almost universal disposition for the class of finite-dimensional normed

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spaces. Banach spaces of universal disposition, especially for the class of finite-dimensional normed spaces $\mathcal{F}$ and the class of separable Banach spaces $\mathcal{G}$, were recently extensively developed by several authors, see [2–4,6, 8] among others. Briefly characterizing the known results, there is no separable Banach space of universal disposition for the class $\mathcal{F}$, there are spaces of universal disposition for the class $\mathcal{F}$ that are not of universal disposition for the class $\mathcal{G}$. A Banach space of universal disposition for the class $\mathcal{G}$ must contain an isometric copy of each Banach space of density $\aleph_1$ or less. Under (CH) there is only one space (up to isometrics) of universal disposition for the class of separable Banach spaces and density character $\aleph_1$ (Kubiś space $\mathcal{K}$).

This paper deals with non-archimedean normed spaces, i.e. linear spaces over a non-archimedean complete valued field equipped with the norm satisfying the strong triangle inequality. Non-archimedean Banach spaces of (almost) universal disposition were studied previously in [11]. Among others, for the class of non-archimedean finite-dimensional normed spaces $\mathcal{U}_{FNA}$ two non-isometrically isomorphic spaces of universal disposition were constructed (see [11, Theorems 4.6 and 4.7]). The paper continues this line of research.

Let E be a non-archimedean Banach space. By [16, Theorem 5.4], all maximal orthogonal sets of elements of E have the same cardinality. Hence, the cardinality of maximal orthogonal set can be viewed as a kind of special density and seems to be a good feature to classify non-archimedean Banach spaces, see Section 2.2. Let $\operatorname{ort}(E)$ be a cardinal number defined as $\operatorname{ort}(E) := \operatorname{card}(X)$, where X is a maximal orthogonal set in E. For a given infinite cardinal number k we denote by $\mathcal{U}_{kNA}$ the class of non-archimedean Banach spaces defined as follows: $E \in \mathcal{U}_{kNA}$ if and only if $\operatorname{ort}(E) < k$. Then, the class $\mathcal{U}_{kNA}$ contains all non-archimedean Banach spaces with an orthogonal base with cardinality not bigger than k and all immediate extensions of such spaces, see Section 2.2.1, in particular spherical completions of such spaces. Clearly, $k_1 < k_2$ follows $\mathcal{U}_{k_1NA} \subset \mathcal{U}_{k_2NA}$.

In this paper we characterize spaces of universal disposition for the class $\mathcal{U}_{kNA}$ for a given infinite cardinal number k, demonstrating the necessary and sufficient conditions for the non-archimedean Banach space to be a space of universal disposition for a specific class $\mathcal{U}_{kNA}$ (Theorem 3.4). Next, in Theorem 3.10 we construct a space of universal disposition for the class $\mathcal{U}_{\aleph_0 NA}$. We compare spaces of universal disposition for the class $\mathcal{U}_{kNA}$, where $k = \aleph_0$ or $k = \aleph_1$, with spaces of universal disposition for the class $\mathcal{U}_{FNA}$ and the class of all non-archimedean Banach spaces of countable type $\mathcal{U}_{CNA}$, respectively. We show that $\mathcal{U}_{FNA} = \mathcal{U}_{\aleph_0 NA}$ if and only if $\mathbb{K}$ is spherically complete (Remark 3.12) and $\mathcal{U}_{CNA} \neq \mathcal{U}_{\aleph_1 NA}$ in general, even if $\mathbb{K}$ is spherically complete (Remark 3.15). However, under (CH) spaces of universal disposition for classes $\mathcal{U}_{CNA}$ and $\mathcal{U}_{\aleph_1 NA}$ coincide with each other (Theorem 3.17).

These results are supplemented by characterization of injectivity, universality and transitivity of non-archimedean Banach spaces of universal disposition (see Sections 4 and 5).

2. Preliminaries and notations

2.1. Non-archimedean valued fields. A *valuation* on a field $\mathbb{K}$ is a map $|.|\colon \mathbb{K} \to [0,\infty)$, satisfying the following properties:

(1) $|\lambda| = 0 \iff \lambda = 0$, $\lambda \in \mathbb{K}$;

(2) $|\lambda\mu| = |\lambda| \cdot |\mu|$ for all $\lambda, \mu \in \mathbb{K}$;

(3) $|\lambda + \mu| \leq |\lambda| + |\mu|$ for all $\lambda, \mu \in \mathbb{K}$.

The pair $(\mathbb{K}, |.|)$ is called a *valued field*. We often will write $\mathbb{K}$ instead of $(\mathbb{K}, |.|)$. Then the map $(\lambda, \mu) \longmapsto |\lambda - \mu|$ is a metric on $\mathbb{K}$ which induces the topology for which $\mathbb{K}$ is a topological field.

The valuation $|.|$ is called *non-archimedean* and $\mathbb{K}$ is called a *non-archimedean valued field* if $|.|$ satisfies *the strong triangle inequality*, i.e.

$$(3') \qquad |\lambda + \mu| \leq \max\{|\lambda|, |\mu|\} \quad \text{for all } \lambda, \mu \in \mathbb{K}.$$

We say that a valuation $|.|$ is *trivial* if $|\lambda| = 1$ for $\lambda \neq 0$; otherwise, we will say that the valuation $|.|$ is *non-trivial*.

Note that any complete valued field is either non-archimedean or isometrically isomorphic to the field of real or complex numbers, see [15, Theorem 1.2.18].

From now on, by $\mathbb{K}$ we will denote a non-archimedean non-trivially valued field, which is commutative and complete under the metric generating by a non-archimedean valuation.

2.1.1. The value group. Recall that $|\mathbb{K}^\times| := \{|\lambda| : \lambda \in \mathbb{K}\backslash\{0\}\}$ is the *value group* of $\mathbb{K}$. Set $|\mathbb{K}| := |\mathbb{K}^\times| \cup \{0\}$. A valued field $\mathbb{K}$ is said to be *discretely valued* if 0 is the only accumulation point of $|\mathbb{K}^\times|$; then, there exists an *uniformizing element*: $\rho \in \mathbb{K}$ with $0 < |\rho| < 1$ such that $|\mathbb{K}^\times| = \{|\rho|^n : n \in \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the set of integers. Otherwise, we say that $\mathbb{K}$ is *densely valued*, then, $|\mathbb{K}^\times|$ is a dense subset of $(0, \infty)$.

The field of p-adic numbers $\mathbb{Q}_p$ is an example of non-archimedean valued field, which is discretely valued (see [15, Example 1.2.5]), whereas the field of p-adic complex numbers $\mathbb{C}_p$ (the completion of the algebraic closure of $\mathbb{Q}_p$) is densely valued (see [15, Example 1.2.11]).

Recall that $\mathbb{R}^+$, the set of positive real numbers, is a multiplicative group and $|\mathbb{K}^\times|$ is its normal subgroup. Let

$$(2.1) \qquad \pi_G\colon \mathbb{R}^+ \to G := \mathbb{R}^+/\left|\mathbb{K}^\times\right|$$

be the natural quotient map and let $S := \{s_g : g \in G\}$ be the set of representatives of elements of G in $(r, 1]$, i.e. $\pi_G(s_g) = g$, where $r = |\rho|$ if $\mathbb{K}$

is discretely valued (then $\rho \in \mathbb{K}$ is an uniformizing element of $|\mathbb{K}^{\times}|$ with $0 < |\rho| < 1$) and $r = \frac{1}{2}$ if $\mathbb{K}$ is densely valued. Let $g_0 \in G$ be the identity element of G. Then, we additionally assume that $s_{g_0} = 1$.

2.2. Non-archimedean normed spaces. Let E be a linear space over $\mathbb{K}$. A norm on E is defined as usual, i.e. it is a map $\|.\| \colon E \to [0, \infty)$ such that the following conditions are satisfied:

(1) $\|x\| = 0 \iff x = 0$;

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and $x \in E$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

We say that a norm on E is *non-archimedean* if it satisfies the strong triangle inequality, i.e.

$$\|x + y\| \leq \max\big\{\|x\|, \|y\|\big\} \quad \text{for all } x, y \in E. \tag{3'}$$

Remark 2.1. Observe that if $\|.\|$ is non-archimedean, then $\|x\| < \|y\|$, $x, y \in E$ implies $\|x + y\| = \|y\|$ (see [14, Lemma 1.1.1]).

By a *non-archimedean normed space* we mean a normed space E over $\mathbb{K}$ equipped with a non-archimedean norm $\|.\|$. Note that not every normed space over $\mathbb{K}$ is non-archimedean, for example $l_1(\mathbb{K})$, the space of all sequences $x = (x_n)$ of members of $\mathbb{K}$ such that $\|x\|_1 := \sum_n |x_n|$ is finite, is not non-archimedean and does not even have an equivalent non-archimedean norm, see [5].

By $\mathbb{K}_s$ $(s > 0)$ we will denote the normed space whose underlying linear space is $\mathbb{K}$ itself, normed by the norm $\|x\|_s := s \cdot |x|$, $x \in \mathbb{K}$.

Let $\|E^{\times}\| := \big\{\|x\| : x \in E \backslash \{0\}\big\}$ and $\|E\| := \|E^{\times}\| \cup \{0\}$. Let X be a subset of E; by $[X]$ we will denote the linear span generated by elements of X. Let I be a nonempty set. Let $t \in (0, 1]$. A subset $\{x_i : i \in I\} \subset E$ is called *t-orthogonal* (*orthogonal* if $t = 1$) if for each finite subset $J \subset I$ and all $\{\lambda_i\}_{i \in J} \subset \mathbb{K}$ we have

$$\Big\| \sum_{i \in J} \lambda_i x_i \Big\| \geq t \cdot \max_{i \in J} \|\lambda_i x_i\| .$$

An orthogonal set $\{x_i\}_{i \in I}$ in E is said to be an *orthogonal base* of E if $\overline{\big[\{x_i\}_{i \in I}\big]} = E$. Then every $x \in E$ has an unequivocal expansion

$$x = \sum_{i \in I} \lambda_i x_i \quad (\lambda_i \in \mathbb{K},\ i \in I).$$

Every orthogonal set of elements of E can be extended to a maximal orthogonal one (see [16, Chapter 5]). By [16, Theorem 5.4], all maximal orthogonal sets of elements of E have the same cardinality. Every orthogonal base of E is a maximal orthogonal set in E, however not every maximal orthogonal set in E has to be an orthogonal base (see [16, Remark following Theorem 3.16]).

Recall that every closed linear subspace of a non-archimedean Banach space with an orthogonal base has an orthogonal base (see [16, Theorem 5.9]).

We say that a normed space E over $\mathbb{K}$ is of *countable type* if it contains a countable set whose linear hull is dense in E. If $\mathbb{K}$ is separable, then a normed space is of countable type if and only if it is separable.

Linear subspaces D_1, D_2 of a non-archimedean normed space E are called *orthogonal* if $\|x+y\| = \max\{\|x\|, \|y\|\}$ for all $x \in D_1$ and $y \in D_2$; then, we write $D_1 \perp D_2$.

For $x, y \in E\backslash\{0\}$ and a linear subspace $D \subset E$ we will write shortly $x \perp y$ instead of $[x] \perp [y]$ and $x \perp D$ instead of $[x] \perp D$. Let D_1 be a closed linear subspace of E. Then, we say that D_1 is *orthocomplemented* in E if there is a linear subspace D_2 of E such that $D_1 + D_2 = E$ and $D_1 \perp D_2$. Consequently, there exists a surjective projection (called an *orthoprojection*) $P\colon E \to D$ with $\|P\| \le 1$. Observe that $D_1 \perp D_2$ implies $D_1 \cap D_2 = \{0\}$; hence, the sum $D_1 + D_2$ is direct.

2.2.1. The spherical completion, an immediate extension of a non-archimedean normed space. Let E be a non-archimedean normed space. Let $r > 0$. The *closed ball* of E is the set $B(x,r) := \{z \in E : \|x - z\| \le r\}$. Note that topologically $B(x,r)$ is both closed and open.

A sequence of balls $(B_n)_n$ in E is called *nested* if $B_{n+1} \subset B_n$ for all $n \in \mathbb{N}$. A non-archimedean Banach space E is *spherically complete* (in particular a non-archimedean valued field $\mathbb{K}$) if every nested sequence of closed balls in E has a non-empty intersection; otherwise, we say that E is *non-spherically complete*. If $\mathbb{K}$ is spherically complete, then every non-archimedean Banach space over $\mathbb{K}$ which is of countable type has an orthogonal base (see [15, Theorem 2.3.25]).

We say that a spherically complete Banach space $\widehat{E}$ is a *spherical completion* of a non-archimedean normed space E, if there exists an isometric embedding $i\colon E \to \widehat{E}$ and $\widehat{E}$ has no proper spherically complete linear subspace containing $i(E)$. Applying the natural identification, we will usually identify E with $i(E)$. Every Banach space (in particular a non-archimedean valued field $\mathbb{K}$) has a spherical completion and any two spherical completions of E are isometrically isomorphic, see [16, Theorem 4.43]. Let E, F be non-archimedean normed spaces such that $E \subseteq F$. If F is spherically complete, then F contains a spherical completion of E, see [16, Corollary 4.45].

Let D be a linear subspace of E. E is called an *immediate extension* of D (see [16, Chapter 4]) if there is no nonzero element of E that is orthogonal to D; in other words, for every $x \in E\backslash D$ we have $\operatorname{dist}(x, D) < \|x - d\|$ for all $d \in D$, where $\operatorname{dist}(x, D) := \inf_{d\in D}\{\|x-d\|\}$. A spherical completion $\widehat{E}$ of E is a maximal immediate extension of E (see [16, Corollary 4.44]). The set $X \subset E$ is a maximal orthogonal set in E if and only if E is an immediate extension of $[X]$, see [16, page 167].

The spherical completion $\widehat{\mathbb{K}}$ of $\mathbb{K}$ can be made into a valued field. Then, $\mathbb{K}$ can be viewed as a subfield of $\widehat{\mathbb{K}}$ and the valuation of $\widehat{\mathbb{K}}$ extends the valuation from $\mathbb{K}$, see [16, Theorem 3.19]. Hence, every Banach space over $\widehat{\mathbb{K}}$ can be viewed also as a Banach space over $\mathbb{K}$ (see [16, p. 162]). Usually we will consider $\widehat{\mathbb{K}}$ as a Banach space over $\mathbb{K}$. Then, $\widehat{\mathbb{K}}$ is infinite-dimensional (see Remark 3.12) and every one-dimensional linear subspace of $\widehat{\mathbb{K}}$ is isomorphic with $\mathbb{K}$; by $\mathbb{K}$ we will denote the one-dimensional linear subspace of $\widehat{\mathbb{K}}$, generated by the element $1 \in \widehat{\mathbb{K}}$.

2.2.2. The spaces $l^\infty(I\colon s,\mathbb{K})$, $c_0(I\colon s,\mathbb{K})$, $\ell^\infty(I_{uk},\widehat{\mathbb{K}})$, $(c_0(I_{uk},\widehat{\mathbb{K}})$. Using some ideas of [16, Chapter 3] we define specific non-archimedean Banach spaces which will be used in the sequel.

Let I be a nonempty set and let $s\colon I \to (0,\infty)$, $h\colon I \to \mathbb{K}$ be maps. Set

$$\|h\|_s := \sup\{|h(i)| \cdot s(i) : i \in I\}. \tag{2.2}$$

The maps $h\colon I \to \mathbb{K}$ for which $\|h\|_s$ is finite form the linear space $l^\infty(I\colon s,\mathbb{K})$, which is a non-archimedean Banach space under the norm $\|.\|_s$. By $c_0(I\colon s,\mathbb{K})$ we will denote the closed linear subspace of $l^\infty(I : s,\mathbb{K})$, which consists of all $h \in l^\infty(I : s,\mathbb{K})$ such that for every $\varepsilon > 0$ the set $\{i \in I : |h(i)| \cdot s(i) \geq \varepsilon\}$ is finite. If $s(i) = 1$ for all $i \in I$, we will write shortly $l^\infty(I,\mathbb{K})$ and $c_0(I,\mathbb{K})$, respectively.

Usually, when there is no risk of confusion, the ground field will be omitted; then we will write $l^\infty(I : s)$ and $c_0(I : s)$ instead of $l^\infty(I : s,\mathbb{K})$ and $c_0(I : s,\mathbb{K})$ or $l^\infty(I)$ and $c_0(I)$ instead of $l^\infty(I,\mathbb{K})$ and $c_0(I,\mathbb{K})$.

Let k be an infinite cardinal number and let J be a set such that $\operatorname{card}(J) = k$. Denote $I_{uk} := G \times J$, where G is explained in Section 2.1.1. Define the function $s_{uk}\colon I_{uk} \to (r, 1]$ by $s_{uk}((g, i)) := s_g$ and the norm on $l^\infty(I_{uk})$ by

$$\|x\|_{uk} := \sup_{p=(g,i_p)\in I_{uk}} \{s_{uk}((g, i_p)) \cdot |x_p|\}, \quad x = (x_p)_{p\in I_{uk}} \in \ell^\infty(I_{uk}). \tag{2.3}$$

Let $E_{uk} := (c_0(I_{uk}), |.|_{uk})$ be the closed linear subspace of $(\ell^\infty(I_{uk}), \|.\|_{uk})$ consisting of all $(x_p)_{p\in I_{uk}}$, $(x_p \in \mathbb{K},\ p \in I_{uk})$ such that for each $\varepsilon > 0$, $s_{uk}(p)\cdot(x_p) > \varepsilon$ only for finitely many indices. We will write shortly I_u, E_u, s_u and $\|.\|_u$ instead of $I_{u\aleph_0}$, $E_{u\aleph_0}$, $s_{u\aleph_0}$ and $\|.\|_{u\aleph_0}$.

This way the defined space E_{uk} has the useful property: for every $g \in G$, E_{uk} contains an orthogonal set X_g such that $\operatorname{card}(X_g) = k$ and $\pi_G(\|x\|) = g$

for all $x \in X_g$ (values of norms of elements of X_g are in the same coset of $|\mathbb{K}^{\times}|$ in $\mathbb{R}^{+}$).

Let $\widehat{\mathbb{K}}$ be the non-archimedean valued field which is a spherical completion of $\mathbb{K}$. Similarly as above we define the space $(\ell^{\infty}(I_{uk}, \widehat{\mathbb{K}}), \|.\|_{uk})$, the Banach space over $\mathbb{K}$ of all bounded maps $I_{uk} \to \widehat{\mathbb{K}}$ equipped with the norm $\|.\|_{uk}$ (where $\|.\|_{uk}$ is defined in the same way as in (2.3) and $|.|$ denotes the valuation defined on $\widehat{\mathbb{K}}$ which extends the valuation of $\mathbb{K}$. In the same spirit we define the space $(c_0(I_{uk}, \widehat{\mathbb{K}}), \|.\|_{uk})$, a Banach space over $\mathbb{K}$.

We refer the reader to the monographs [7], [14] [15] and [16] for more background on non-archimedean normed spaces.

3. Spaces of universal disposition

We start this section with a few preparing lemmas.

Lemma 3.1. *Let E be a non-archimedean Banach space. Then, E is spherically complete if and only if for every linear subspace of countable type $D \subset E$ there exists a linear subspace $\widehat{D} \subset E$ which is a spherical completion of D.*

Proof. "$\Rightarrow$" If E is spherically complete, then, by [16, Corollary 4.45], it contains a spherical completion of each of its subspaces, in particular every linear subspace of countable type.

"$\Leftarrow$" Assume that E contains a spherical completion of each of its linear subspace of countable type. Let $(B(x_n, r_n))_n$ be a nested sequence of closed balls in E. Then $D := \overline{[(x_n)_n]}$ is a linear subspace of E which is of countable type. Let $\widehat{D} \subset E$ be a spherical completion of D. Then, $V_n := \{x \in \widehat{D} : \|x_n - x\| \le r_n\}$, $n \in \mathbb{N}$ form a nested sequence of closed balls in $\widehat{D}$. Since, by assumption, $\widehat{D}$ is spherically complete, there exists $x_0 \in \bigcap_n V_n$. Clearly $x_0 \in \bigcap_n B(x_n, r_n)$; hence, E is spherically complete. □

Lemma 3.2. *Let E, F be non-archimedean normed spaces, D be a spherically complete linear subspace of E. If $T\colon E \to F$ is an isometric embedding, then $T(D) \subseteq F$ is spherically complete.*

Proof. Straightforward. □

Lemma 3.3. *Let E, F be non-archimedean normed spaces, D be a linear subspace of E such that E is an immediate extension of D, F be spherically complete and $T\colon D \to F$ be an isometric embedding. Then T can be extended to a linear isometry $T'\colon E \to F$.*

Proof. See [11, Lemma 4.2] and [16, Lemma 4.42]. □

Theorem 3.4. *Let k be an infinite cardinal number and E_k be an infinite-dimensional non-archimedean Banach space. Consider the following conditions.*

(A) *E_k satisfies the following properties*:

(i) *for every $g \in G$, where G is explained in Section 2.1.1, E_k contains an orthogonal set X_g such that* $\operatorname{card}(X_g) = k$ *and $\pi_G(\|x\|) = g$ for all $x \in X_g$;*

(ii) *for every linear subspace $F \subset E_k$ such that* $\operatorname{ort}(F) < k$ *there is a linear subspace F_0 of E_k such that $F \subset F_0$ and F_0 is a spherical completion of F.*

(B) *E_k is a space of universal disposition for the class $\mathcal{U}_{kNA}$.*

(C) *E_k is spherically complete.*

Then, (A) $\Leftrightarrow$ (B), (A) $\Rightarrow$ (C) *if $k > \aleph_0$.*

Proof. (A) $\Rightarrow$ (B). Let $F, H \in \mathcal{U}_{kNA}$, $F \subset E_k$ and let $j : F \to H$ be an isometric embedding. We prove, that there exists an isometric embedding $f : H \to E_k$ such that $f(j(x)) = x$ for all $x \in F$.

First, choose a maximal orthogonal set X_F in $j(F) \subset H$. Next, we extend X_F to a maximal orthogonal set X_H in H. Denote $Y_H := X_H \backslash X_F$. Clearly, $Y_H \cap j(F) = \varnothing$, as X_F is a maximal orthogonal set in $j(F)$. Thus, $[X_F] \perp [Y_H]$.

We show that $j(F) \perp [Y_H]$. Indeed, take any $x \in j(F)$ and $y \in [Y_H]$. If $x \in [X_F]$, the conclusion is obvious since $[X_F] \perp [Y_H]$. So, assume that $x \notin [X_F]$. Then, since $j(F)$ is an immediate extension of $[X_F]$ (see [16, page 167]), there exists $x_0 \in [X_F]$ such that $\|x\| = \|x_0\|$ and $\|x - x_0\| < \|x\|$. Thus, as $x_0 \perp y$, we obtain

$$\|x_0 + y\| = \max\{\|x_0\|, \|y\|\} = \max\{\|x\|, \|y\|\}.$$

Hence,

$$\|x - x_0\| < \|x\| \leq \max\{\|x\|, \|y\|\} = \|x_0 + y\|.$$

Finally, since $\|x - x_0\| < \|x_0 + y\|$, using Remark 2.1 we get

$$\|x + y\| = \|(x - x_0) + (x_0 + y)\| = \|x_0 + y\| = \max\{\|x\|, \|y\|\},$$

concluding $x \perp y$.

For every $g \in G$ denote

$$X_{F,g} := \{x \in X_F : \pi_G(\|x\|) = g\} \quad \text{and} \quad Y_{H,g} := \{x \in Y_H : \pi_G(\|x\|) = g\},$$

respectively. Then, $Y_H = \cup_{g \in G} Y_{H,g}$ and the components of the union are disjoint.

Fix $g \in G$. Since $H, F \in \mathcal{U}_{kNA}$, $\operatorname{card}(Y_{H,g}) < k$ and $\operatorname{card}(X_{F,g}) < k$. Thus, if $X'_{F,g}$ is a maximal orthogonal set in $\{x \in F : \pi_G(\|x\|) = g\}$ then $\operatorname{card}(X'_{F,g}) < k$. Now, we extend $X'_{F,g}$ to X'_g, a maximal orthogonal set in $\{x \in E_k : \pi_G(\|x\|) = g\}$. Since, by assumption, E_k contains an orthogonal set X_g with $\operatorname{card}(X_g) = k$ and $\pi_G(\|x\}) = g$ for all $x \in X_g$, thus $\operatorname{card}(X'_g)$ $\geq k$ (note that by [16, Remark following Theorem 5.2] for every fixed $s > 0$ all maximal orthogonal subsets of $\{x \in E_k : \|x\| \in s \cdot |\mathbb{K}|\}$ have the same cardinality). Therefore, since $Y_{H,g} \subset Y_H$ and $[Y_H] \perp j(F)$ we can select an orthogonal set $Y'_g = \{x_y : y \in Y_{H,g}\} \subset X'_g \backslash X'_{F,g}$, such that for every $z \in Y'_g$, $z \perp F$.

Since for every $y \in Y_{H,g}$ we have $\pi_G(y) = g$ and for every $x \in Y'_g$, $\pi_G(x) = g$, for every $y \in Y_{H,g}$ there is $\lambda_y \in \mathbb{K}$ such that $\|\lambda_y x_y\| = \|y\|$. Since the same procedure we can carry out for every $g \in G$, this way we define the map $f\colon j(F) + [Y_H] \to E_k$, setting

$$f\Big(x + \sum_{y \in Y_H} \mu_y y\Big) := j^{-1}(x) + \sum_{y \in Y_H} \mu_y \lambda_y x_y,$$

where $x \in j(F)$, $x_y \in \bigcup_{g \in G} Y'_g$ and $\mu_y \in \mathbb{K}$. Since $j(F) \perp [Y_H]$ and, as we showed above, Y_H and $\bigcup_{g \in G} Y'_g$ are both orthogonal sets, we obtain

$$\Big\|x + \sum_{y \in Y_H} \mu_y y\Big\| = \max\Big\{\|x\|, \Big\|\sum_{y \in Y_H} \mu_y y\Big\|\Big\} = \max\Big\{\|x\|, \max_{y \in Y_H} \|\mu_y y\|\Big\}.$$

On the other hand, we get

$$\Big\|j^{-1}(x) + \sum_{y \in Y_H} \lambda_y \mu_y x_y\Big\| = \max\Big\{\|j^{-1}(x)\|, \Big\|\sum_{y \in Y_H} \lambda_y \mu_y x_y\Big\|\Big\}$$

$$= \max\Big\{\|j^{-1}(x)\|, \max_{y \in Y_H} \|\lambda_y \mu_y x_y\|\Big\} = \max\Big\{\|x\|, \max_{y \in Y_H} \|\mu_y y\|\Big\}.$$

Hence, we conclude that f is isometric.

If $j(F) + [Y_H] = H$, we are done, f is a required isometry defined on H; otherwise, if $j(X) + [Y_H] \neq H$ then, by [13, Proposition 2.1], H is an immediate extension of $j(F) + [Y_H]$. By assumption, E_k contains a spherically complete linear subspace E_0 which is a spherical completion of $f(j(F) + [Y_H])$ such that $f(H) \subset E_0$.Thus, applying Lemma 3.3, we can extend f to the isometry: $H \to E_0 \subset E_k$.

(B) $\Rightarrow$ (A). Assume that E_k is a space of universal disposition for the class $\mathcal{U}_{kNA}$. We show that E_k satisfies the conditions (i) and (ii) of Theorem 3.4. First, suppose that there is $g \in G$ for which the maximal orthogonal set X_g of $\{x \in E_k : \pi_G(\|x\|) = g\}$ has a cardinality less then k.

Let $F := \overline{[X_g]}$. Set $H := F \oplus \mathbb{K}_{s_g}$, where $\mathbb{K}_{s_g}$is explained in Section 2.2, and $j\colon F \to H$, $j(x) = (x, 0)$ be the inclusion map. Then, since by assumption E_k is a space of universal disposition for the class $\mathcal{U}_{kNA}$ and $F, H \in \mathcal{U}_{kNA}$, there exists an isometric embedding $f\colon H \to E_k$ such that $f(j(x)) = x$ for all $x \in F$. Hence, the set $X_g \cup \{z_0\}$, where $z_0 := f((0, 1))$, is orthogonal and $\pi_G(\|z_0\|) = g$. This conclusion contradicts with the maximality of X_g. So, the assertion (i) is satisfied.

Now, take any linear subspace $F \subset E_k$ such that $\operatorname{ort}(F) < k$. Let $\widehat{F}$ be a spherical completion of F and $i\colon F \to \widehat{F}$ be the isometric embedding. Clearly, $F, \widehat{F} \in \mathcal{U}_{kNA}$. Then, arguing as above, there is an isometric embedding $h\colon \widehat{F} \to E_k$ such that $h(i(x)) = x$ for all $x \in F$. But, then, by Lemma 3.2, the linear subspace $h(\widehat{F})$ of E_k is spherically complete and by [16, Corollary 4.45] it contains a spherical completion of F. Thus, we get (ii).

Finally assume that $k > \aleph_0$. Then, it follows from (A) that every linear subspace of countable type has a spherical completion contained in E_k. By Lemma 3.1, we conclude that E_k is spherically complete. □

Remark 3.5. The condition (i) of Theorem 3.4 is equivalent to

(i′) E_k contains an isometric copy of the space E_{uk}, where the space E_{uk} is explained in Section 2.2.2.

Proof. Assume that E_k satisfies the condition (i), i.e., for every $g \in G$, E_k contains an orthogonal set X_g such that $\operatorname{card}(X_g) = k$ and $\pi_G(\|x\|) = g$ for all $x \in X_g$. Let $X := \bigcup_{g \in G} X_g$. Then $\overline{[X]}$ is isometrically isomorphic with E_{uk}. The implication (i′) ⇒ (i) is obvious. □

Corollary 3.6. (1) *The space* $\widehat{E_{uk}}$ *is a space of universal disposition for the class* $\mathcal{U}_{kNA}$.

(2) *If* E_k *is a space of universal disposition for the class* $\mathcal{U}_{kNA}$ *then* E_k *contains an isometric copy of the space* E_{uk}.

(3) *If* $k > \aleph_0$, *then any space of universal disposition for the class* $\mathcal{U}_{kNA}$ *contains an isometric copy of* $\widehat{E_{uk}}$.

(4) *Let* $\mathcal{U}_1, \mathcal{U}_2$ *be classes of infinite-dimensional non-archimedean Banach spaces such that* $\mathcal{U}_1 \subseteq \mathcal{U}_2$. *If* E *is a space of universal disposition for the class* $\mathcal{U}_2$ *then* E *is a space of universal disposition for the class* $\mathcal{U}_1$. *In particular, if* k_1, k_2 *are infinite cardinal numbers such that* $k_1 \le k_2$, *then, if* E *is a space of universal disposition for the class* $\mathcal{U}_{k_2NA}$, *then* E *is a space of universal disposition for the class* $\mathcal{U}_{k_1NA}$.

Proof. (1) By Remark 3.5, $\widehat{E_{uk}}$ satisfies the condition (i) of Theorem 3.4. Since $\widehat{E_{uk}}$ is spherically complete, by [16, Corollary 4.45] it contains a spherical completion of its every linear subspace; hence, it satisfies the condition (ii) of Theorem 3.4 and the conclusion follows.

(2) For the class $\mathcal{U}_{kNA}$ the condition (i) of Theorem 3.4 satisfies. Thus, using Remark 3.5, we conclude that E_k contains an isometric copy of E_{uk}.

(3) Let E_k be a space of universal disposition for the class $\mathcal{U}_{kNA}$. If $k > \aleph_0$, then by Theorem 3.4, E_k is spherically complete. As we proved above, E_k contains an isometric copy of the space E_{uk}. Let's denote it by D. By [16, Corollary 4.45] E_k contains a linear subspace D_0 which is a spherical completion of D. But, by [16, Theorem 4.43], D_0 is isometrically isomorphic with $\widehat{E_{uk}}$.

(4) Let E be a space of universal disposition for the class $\mathcal{U}_2$. Take $X, Y \in \mathcal{U}_1$ such that X is a linear subspace of E Let $g\colon X \to Y$ be an linear isometric embedding. Since $X, Y \in \mathcal{U}_2$ it follows directly from definition and assumption that there exists a linear isometric embedding $f\colon Y \to E$. So, E is a space of universal disposition for the class $\mathcal{U}_1$. $\square$

Remark 3.7. Note that the third assertion of Corollary 3.6 is not true if $k = \aleph_0$. Theorem 3.10 shows that there exists a space of universal disposition for the class $\mathcal{U}_{\aleph_0 NA}$ which is non-spherically complete and it does not contain any isometric copy of $\widehat{E_u}$.

First, we prove two lemmas (see also [16, 4.B]).

Lemma 3.8. *Let $\mathbb{K}$ be densely valued, I be an infinite set and $s\colon I \to (0,\infty)$ be a map. If there exists a countable $J \subset I$, say $J = \{k_1, k_2, \ldots\}$, such that $s(k_n)\cdot|\lambda_n| \geq s(k_{n+1})\cdot|\lambda_{n+1}|$ and $\lim_n s(k_n)\cdot|\lambda_n| > 0$ for some $\lambda_1, \lambda_2, \ldots \in \mathbb{K}$ then the space $c_0(I:s)$ is non-spherically complete.*

Proof. Recall that the space $c_0(I:s)$ is normed by the norm $\|.\|_s$ defined in Section 2.2. Let $r = \lim_n s(k_n)\cdot|\lambda_n|$. Set $x_n := \sum_{i=1}^n \lambda_i e_{k_i}$ and $r_n := s(k_{n+1})\cdot|\lambda_{n+1}|$ $(n \in \mathbb{N})$. Then, the sequence of balls $(B_{r_n}(x_n))_n$ in $c_0(I:s)$ is nested. Indeed, for every $n \in \mathbb{N}$ we have

$$\|x_{n+1} - x_n\|_s = s(k_{n+1})\cdot|\lambda_{n+1}| = r_n,$$

thus $x_{n+1} \in B_{r_n}(x_n)$, $n \in \mathbb{N}$ and by assumption for every $n \in \mathbb{N}$

$$s(k_{n+1})\cdot|\lambda_{n+1}| \geq s(k_{n+2})\cdot|\lambda_{n+2}|,$$

hence $r_n \geq r_{n+1}$.

We prove that the considered sequence of balls has an empty intersection. Assume the contrary and suppose that there is $z \in \bigcap_n B_{r_n}(x_n)$. Then, we can write $z = \sum_{i\in I} \mu_i e_i$ for some $\mu_i \in \mathbb{K}$ $(i \in I)$. On the other hand, since for each $\varepsilon > 0$, $s(i)\cdot|\mu_i| > \varepsilon$ only for finitely many indices, there is $p \in \mathbb{N}$ such that $s(k_j)\cdot|\mu_{k_j}| < r$ if $j > p$. Thus, $|\mu_{k_{p+1}}| < \frac{r}{s(k_{p+1})}$. Since $|\lambda_{p+1}| = \frac{r_p}{s(k_{p+1})}$, using Remark 2.1 for $n > p$ we have

$$\|z - x_n\|_s \geq s(k_{p+1})\cdot|\mu_{k_{p+1}} - \lambda_{p+1}| = s(k_{p+1})\cdot|\lambda_{p+1}| = r_p.$$

But, by assumption, $\|z - x_n\|_s = \max_{i\in I}\{s(i)\cdot|\mu_i - \lambda_i|\} \le r_n$ for all $n \in \mathbb{N}$; a contradiction. □

LEMMA 3.9. *Every finite-dimensional linear subspace of* $(c_0(I_u, \widehat{\mathbb{K}}), \|.\|_u)$ *is spherically complete.*

PROOF. If $\mathbb{K}$ is spherically complete, then $\widehat{\mathbb{K}} = \mathbb{K}$ and the conclusion is straightforward, as every finite-dimensional normed space over such $\mathbb{K}$ is spherically complete (see [16, Corollary 4.6]). So, assume that $\mathbb{K}$ is non-spherically complete and let F be a finite-dimensional linear subspace of $(c_0(I_u, \widehat{\mathbb{K}}), \|.\|_u)$. Let $\{v^1, \dots, v^m\}$, be an orthogonal base of F, $m \in \mathbb{N}$. Let $v^i = (x_1^i, x_2^i, \dots)$, $x_p^i \in \widehat{\mathbb{K}}$, $p \in I_u$, $i \in \{1, \dots, m\}$. Then, for every $i \in \{1, \dots, m\}$ there exists p_i and $s_{g_i} \in S$ such that $\|v^i\|_u = s_{g_i} \cdot |x_{p_i}|$. For each $i \in \{1, \dots, m\}$ define a one-dimensional normed space $H_i = (\widehat{\mathbb{K}}, \|.\|_i)$, where $\|.\|_i := s_{g_i} \cdot |.|$ and denote $H := \prod_{i=1}^n H_i$. Then, setting

$$T\colon \lambda_1 v^1 + \dots + \lambda_m v^m \longmapsto (\lambda_1, \dots, \lambda_m) \in H$$

where $\lambda_1, \dots, \lambda_m \in \mathbb{K}$, we define the isomorphism $T\colon F \to H$. Since $\widehat{\mathbb{K}}$ is spherically complete, for every $i \in \{1, \dots, n\}$ the space H_i is spherically complete. Hence, by [16, 4.A] the product space $H = \prod_{i=1}^n H_i$ is spherically complete. Hence, by Lemma 3.2 F is spherically complete. □

THEOREM 3.10. *For any* $\mathbb{K}$ *the space* $(c_0(I_u, \widehat{\mathbb{K}}), \|.\|_u)$ *is a space of universal disposition for the class* $\mathcal{U}_{\aleph_0 NA}$. *Furthermore, if* $\mathbb{K}$ *is densely valued then* $(c_0(I_u, \widehat{\mathbb{K}}), \|.\|_u)$ *is not spherically complete and it does not contain an isometric copy of* $\widehat{E_u}$.

PROOF. Recall, as we note in Section 2, that $(c_0(I_u, \widehat{\mathbb{K}}), \|.\|_u)$ denotes the space of all bounded maps $h\colon I_u \to \widehat{\mathbb{K}}$ such that for every $\varepsilon > 0$ the set $\{i \in I : |h(i)| \cdot s_u(i) \ge \varepsilon\}$ is finite, considered as a Banach space over $\mathbb{K}$, normed by the norm (2.3).

To prove that $(c_0(I_u, \widehat{\mathbb{K}}), \|.\|_u)$ is a space of universal disposition for the class $\mathcal{U}_{\aleph_0 NA}$ it is enough to show that it satisfies the conditions (i)-(ii) of Theorem 3.4. The condition (i) is clear, by the construction of I_u, for every $g \in G$ the set $\{e_i : i \in I_u\} \subset c_0(I_u, \widehat{\mathbb{K}})$ contains an orthogonal countable subset X_g such that $\pi_G(\|x\|) = g$ for all $x \in X_g$.

By Lemma 3.9, every finite-dimensional linear subspace of $(c_0(I_u, \widehat{\mathbb{K}}), \|.\|_u)$ is spherical complete, hence the condition (ii) is satisfied and this part of the proof is finished.

Since $\|(c_0(I_u, \widehat{\mathbb{K}}), \|.\|_u)\|_u = [0, \infty)$, defining the suitable map s we can apply Lemma 3.8 and conclude that the space $(c_0(I_u, \widehat{\mathbb{K}}), \|.\|_u)$ is non-spherically complete.

Assume now that there is a linear subspace H of $(c_0(I_u,\widehat{\mathbb{K}}),\|.\|_u)$ such that H is an isometric copy of $\widehat{E_u}$. Therefore, H is spherically complete. Since $(c_0(I_u,\widehat{\mathbb{K}}),\|.\|_u)$ has an orthogonal base, by [16, Theorem 5.9] H, as a closed linear subspace of $(c_0(I_u,\widehat{\mathbb{K}}),\|.\|_u)$ has an orthogonal base. Hence, H is isometrically isomorphic with $c_0(I:s)$ for some infinite I and a map $s\colon I\to(0,\infty)$ (see [16, page 171]). But $\|H\|_u\subseteq\|\widehat{E_u}\|_u=\|E_u\|_u=[0,\infty)$. Therefore, we can apply Lemma 3.8 again and imply that H is finite-dimensional, a contradiction. □

Lemma 3.11. *If $\mathbb{K}$ is non-spherically complete then a spherical completion $\widehat{\mathbb{K}}$ considered as a Banach space over $\mathbb{K}$ is infinite-dimensional.*

Proof. Assume that $\widehat{\mathbb{K}}$ is finite-dimensional. Then, by [16, Theorem 3.15] we imply that all linear functionals defined on $\widehat{\mathbb{K}}$ are continuous, a contradiction with [16, Corollary 4.3]. □

Remark 3.12. 1) Note that $\mathcal{U}_{FNA}\subset\mathcal{U}_{\aleph_0 NA}$ and all known examples of spaces of universal disposition for $\mathcal{U}_{FNA}$ are spaces of universal disposition for $\mathcal{U}_{\aleph_0 NA}$ (see [11, Theorems 4.6 and 4.7]and Theorem 3.10).

2) If $\mathbb{K}$ is spherically complete, then $\mathcal{U}_{FNA}=\mathcal{U}_{\aleph_0 NA}$ since every finite-dimensional normed space over such $\mathbb{K}$ is spherically complete by [16, Corollary 4.6]. Hence, in this case all known examples of spaces of universal disposition for $\mathcal{U}_{FNA}$ are spaces of universal disposition for $\mathcal{U}_{\aleph_0 NA}$ (see [11, Theorems 4.6 and 4.7], Corollary 3.6 and Theorem 3.10).

3) If $\mathbb{K}$ is non-spherically complete, then by Lemma 3.11 a spherical completion $\widehat{\mathbb{K}}$ considered as a Banach space over $\mathbb{K}$ is infinite-dimensional. Hence, $\widehat{\mathbb{K}}\notin\mathcal{U}_{FNA}$. On the other hand, $\widehat{\mathbb{K}}$ is an immediate extension of its one-dimensional linear subspace $\mathbb{K}$, thus it is a member of $\mathcal{U}_{\aleph_0 NA}$. Hence, for non-spherically complete $\mathbb{K}$ we have $\mathcal{U}_{FNA}\neq\mathcal{U}_{\aleph_0 NA}$.

However, in this context, it is natural to formulate the following question.

Problem 3.13. *Let $\mathbb{K}$ be non-spherically complete. Is every space of universal disposition for the class $\mathcal{U}_{FNA}$, a space of universal disposition for $\mathcal{U}_{\aleph_0 NA}$?*

Proposition 3.14. *A spherical completion of $c_0(\mathbb{N}:s)$ is not of countable type if one of the following conditions satisfies:*

- *$\mathbb{K}$ is discretely valued, $s(n)>s(n+1)$ for all $n\in\mathbb{N}$ and $\lim_n s(n)>0$;*
- *$\mathbb{K}$ is densely valued.*

Proof. When the first condition is satisfied, the conclusion follows from [16, Theorem 4.57]; then $\ell^\infty(\mathbb{N}:s)$ is a spherical completion of $c_0(\mathbb{N}:s)$. Recall that $\ell^\infty(\mathbb{N}:s)$ is not of countable type (see [15, Theorem 2.5.15, Corollary 2.3.14 and Remark 2.5.16]).

Assume that $\mathbb{K}$ is densely valued.

Let $\mathbb{K}$ be non-spherically complete. The conclusion is straightforward, as for every spherically complete Banach space E over non-spherically complete $\mathbb{K}$, $E^* = \{0\}$ (its topological dual is trivial, see [16, Corollary 4.3]). Hence, by [15, Theorem 4.2.4] E cannot be of countable type.

Let $\mathbb{K}$ be spherically complete. Then, $\ell^\infty(\mathbb{N} : s)$ is spherically complete (see [16, Remark below Theorem 4.56]) and by [16, Corollary 4.45] there exists a spherical completion E_0 of $c_0(\mathbb{N} : s)$ contained in $\ell^\infty(\mathbb{N} : s)$. We prove that E_0 is not of countable type.

First, we show that if $J = \{j_1, j_2, \dots\}$ is a countable subset of $\mathbb{N}$, $(p_n)_n$ is a strictly decreasing sequence of reals such that $\frac{p_n}{s(n)} \in |\mathbb{K}^\times|$ and $p_n \in [1, 2]$ for every $n \in \mathbb{N}$, then there exists $y = (y_1, y_2, \dots) \in E_0$ such that $s(j_n) \cdot |y_{j_n}| = p_n$ for all $n \in \mathbb{N}$ and $s(k) \cdot |y_k| \le \inf_{n\in\mathbb{N}} p_n$ if $k \notin J$.

Let $x = (x_1, x_2, \dots) \in \ell^\infty(\mathbb{N} : s)$ be such that $s(j_n) \cdot |x_{j_n}| = p_n$ for all $n \in \mathbb{N}$ and $x_k = 0$ if $k \notin J$.

If $x \in E_0$, then we set $y := x$ and we are done. So, assume that $x \notin E_0$. Let $z^n = \sum_{i=1}^n x_i e_i$, $n \in \mathbb{N}$. Then,

$$\operatorname{dist}(x, c_0(\mathbb{N} : s)) \le \inf_{n\in\mathbb{N}} \|x - z^n\| = \inf_{n\in\mathbb{N}} s(j_n) \cdot |x_{j_n}| \le \inf_{n\in\mathbb{N}} p_n.$$

Hence, $\operatorname{dist}(x, E_0) \le \operatorname{dist}(x, c_0(\mathbb{N} : s)) \le \inf_{n\in\mathbb{N}} p_n$.

By maximality of E_0 (see [16, Corollary 4.44]), $E_0 + [x]$ is not an immediate extension of $c_0(\mathbb{N} : s)$. Hence, by [13, Proposition 2.1], $E_0 + [x]$ is not an immediate extension of E_0, therefore, there is $y = (y_1, y_2, \dots) \in E_0$ for which $\operatorname{dist}(x, E_0) = \|x - y\|$. Thus,

$$\|x - y\| \le \inf_{n\in\mathbb{N}} p_n.$$

But, on the other hand $\|x - y\| \ge \sup_{n\in\mathbb{N}} s(j_n) \cdot |x_{j_n} - y_{j_n}|$; hence,

$$s(j_n) \cdot |y_{j_n}| = s(j_n) \cdot |x_{j_n}| = p_n$$

for all $n \in \mathbb{N}$. Since $x_k = 0$ if $k \notin J$, we imply that for such k we have $s(k) \cdot |y_k| \le \inf_{n\in\mathbb{N}} p_n$.

This shows that y, an element of E_0, satisfies the required conditions.

Let $\{J_i\}_{i\in I}$ be an uncountable almost disjoint family of infinite subsets of $\mathbb{N}$ (it is well known fact (see [10] or [1, Lemma 2.5.3]), that there is an uncountable almost disjoint family of infinite subsets of $\mathbb{N}$, i.e. the intersection of two members of this family is finite) and let $(p_n)_n$ be a strictly decreasing sequence of reals such that $\frac{p_n}{s(n)} \in |\mathbb{K}^\times|$ and $p_n \in [1, 2]$ for every $n \in \mathbb{N}$. Thus,

$$\inf_{n\in\mathbb{N}} p_n \ge 1. \tag{3.1}$$

Now, using the conclusion of the previous part of the proof, for each $i \in I$ we choose a corresponding $y_i \in E_0$ as follows. Fix $i \in I$ and write $J_i = \{j_1, j_2, \dots\}$. Then, as we proved above, there is $y^i = (y_1^i, y_2^i, \dots) \in E_0$ such that $s(j_n) \cdot |y_{j_n}^i| = p_n$ for all $n \in \mathbb{N}$ and $s(k) \cdot |y_k^i| \leq \inf_{n\in\mathbb{N}} p_n$ if $k \notin J_i$. Denote $X = \{y^i : i \in I\}$. We show that an uncountable set X is $\frac{1}{2}$-orthogonal. Take a finite $I_0 \subset I$, say $I_0 = \{i_1, \dots, i_m\}$. Set $M := \{1, \dots, m\}$. Take $\lambda_k \in \mathbb{K}\backslash\{0\}$, $k \in M$.

Since for every $k \in M$

$$\|y^{i_k}\| = \sup_{n\in\mathbb{N}}\{s(n) \cdot |y_n^{i_k}|\} = \sup_{n\in\mathbb{N}} p_n \leq 2,$$

we note $\|\lambda_k y^{i_k}\| = |\lambda_k| \cdot \|y^{i_k}\| \leq 2 \cdot |\lambda_k|$ and conclude

$$\max_{k\in M}\|\lambda_k y^{i_k}\| \leq 2 \cdot \max_{k\in M} |\lambda_k|. \tag{3.2}$$

Let $k_0 \in M$ be such that $|\lambda_{k_0}| = \max_{k\in M} |\lambda_k|$. Then for each $k \in M\backslash\{k_0\}$ we have $\left|\frac{\lambda_k}{\lambda_{k_0}}\right| \leq 1$.

Then, as J_{i_1}, ..., J_{i_m} are pairwise almost disjoint, for every $k \in \{1, \dots, m\}$ there is $n_k \in \mathbb{N}$ such that $s(n_k) \cdot |y_{n_k}^{i_k}| \geq \inf_{n\in\mathbb{N}} p_n$ and $s(n_k) \cdot |y_{n_k}^{i_l}| \leq \inf_{n\in\mathbb{N}} p_n$ if $l \neq k$. Therefore,

$$\begin{aligned}\Big\|\sum_{k=1}^{m} \lambda_k y^{i_k}\Big\| &= |\lambda_{k_0}| \cdot \Big\|y^{i_{k_0}} + \sum_{k\in M\backslash\{k_0\}} \frac{\lambda_k}{\lambda_{k_0}} y^{i_k}\Big\| \\ &\geq |\lambda_{k_0}| \cdot s(n_k) \cdot \Big|y_{n_k}^{i_{k_0}} + \sum_{k\in M\backslash\{k_0\}} \frac{\lambda_k}{\lambda_{k_0}} y_{n_k}^{i_k}\Big|.\end{aligned}$$

Applying Remark 2.1 we have

$$s(n_k) \cdot \Big|y_{n_k}^{i_{k_0}} + \sum_{k\in M\backslash\{k_0\}} \frac{\lambda_k}{\lambda_{k_0}} y_{n_k}^{i_k}\Big| \geq s(n_k) \cdot |y_{n_k}^{i_{k_0}}| \geq \inf_{n\in\mathbb{N}} p_n.$$

Thus, using (3.1) and (3.2) we obtain

$$\Big\|\sum_{k=1}^{m} \lambda_k y^{i_k}\Big\| \geq \Big(\frac{1}{2} \cdot 2\Big) \cdot \max_{k=1,\dots,m} |\lambda_k| \cdot \inf_{n\in\mathbb{N}} p_n \geq \frac{1}{2} \max_{k=1,\dots,m} \|\lambda_k y^{i_k}\|.$$

Hence, we get

$$\Big\|\sum_{k=1}^{m} \lambda_k y^{i_k}\Big\| \geq \frac{1}{2} \max_{k=1,\dots,m} \|\lambda_k y^{i_k}\|$$

and finally conclude that X is $\frac{1}{2}$-orthogonal. Now, assume the contrary and suppose that E_0 is of countable type. But then, by [15, Theorem 2.3.18] every $\frac{1}{2}$-orthogonal set of E_0 is countable, a contradiction. □

Let $\mathcal{U}_{CNA}$ be the class of non-archimedean Banach spaces of countable type.

Remark 3.15. Clearly, $\mathcal{U}_{CNA} \subset \mathcal{U}_{\aleph_1 NA}$. But, $\mathcal{U}_{CNA} \neq \mathcal{U}_{\aleph_1 NA}$ in general. Indeed, let $c_0(\mathbb{N} : s)$, be such that one of the conditions of Proposition 3.14 is satisfied and E_0 be its spherical completion. Then, E_0 is a member of $\mathcal{U}_{\aleph_1 NA}$. But Proposition 3.14 shows that E_0 is not of countable type, hence $E_0 \notin \mathcal{U}_{CNA}$.

However, as the next results show, under (CH) spaces of universal disposition for $\mathcal{U}_{CNA}$ and $\mathcal{U}_{\aleph_1 NA}$ coincide with each other.

Proposition 3.16. (CH) *Let E be a non-archimedean Banach space which is of universal disposition for the class $\mathcal{U}_{CNA}$. Then, E contains an isometric copy of $E_{u\aleph_1}$.*

Proof. First, by proceeding in the same way as in the proof of the part (B) $\Rightarrow$ (A) of Theorem 3.4, we show that for every $g \in G$ the maximal orthogonal set X_g of $\{x \in E : \pi_G(\|x\|) = g\}$ is countable. Assume the contrary and suppose that there is $g \in G$ for which the maximal orthogonal set X_g of $\{x \in E : \pi_G(\|x\|) = g\}$ is finite. Let $F := \overline{[X_g]}$, $H := F \oplus \mathbb{K}_{s_g}$, where $\mathbb{K}_{s_g}$is explained in Section 2.2, and $j \colon F \to H$, $j(x) = (x, 0)$ be the inclusion map. Since, by assumption E is of universal disposition for the class $\mathcal{U}_{CNA}$ and $F, H \in \mathcal{U}_{CNA}$, there exists an isometric embedding $f \colon H \to E$ such that $f(j(x)) = x$ for all $x \in F$. Hence, the set $X_g \cup \{z_0\}$, where $z_0 := f((0, 1))$, is orthogonal and $\pi_G(\|z_0\|) = g$. But, this conclusion contradicts with the maximality of X_g. Denote $X := \bigcup_{g\in G} X_g$. Then $\overline{[X]}$ is isometrically isomorphic with $E_{u\aleph_1}$, see Section 2.2.2, and we are done. □

Theorem 3.17. (CH) *Let E be a non-archimedean Banach space. Then, E is a space of universal disposition for the class $\mathcal{U}_{CNA}$ if and only if E is a space of universal disposition for the class $\mathcal{U}_{\aleph_1 NA}$.*

Proof. "$\Leftarrow$" Since $\mathcal{U}_{CNA} \subset \mathcal{U}_{\aleph_1 NA}$, the conclusion follows from Corollary 3.6.

"$\Rightarrow$" Observe that, if E is a space of universal disposition for the class $\mathcal{U}_{\aleph_1 NA}$ then by Theorem 3.4 E is spherically complete.

Now, assume the contrary and suppose that E is a space of universal disposition for $\mathcal{U}_{CNA}$ but not for $\mathcal{U}_{\aleph_1 NA}$. Hence, by Lemma 3.1 and the above observation there is a closed linear subspace of countable type $D \subset E$ such that there is no spherical completion of D contained in E.

Let $H \subset E$ be a closed linear subspace which is a maximal immediate extension of D in E. Let $\widehat{E}$ be a spherical completion of E and let $i \colon E \to \widehat{E}$

be the natural isometric embedding. Then, by [16, Corollary 4.45], $\widehat{E}$ contains a spherical completion of $i(H)$, which is by [13, Proposition 2.1] a spherical completion of $i(D)$. Denote this spherical completion by D_0. By assumption, $D_0 \nsubseteq i(E)$. Take $u_0 \in D_0 \backslash i(E)$. Then, clearly $u_0 \notin i(H)$ and, as D_0 is an immediate extension of $i(H)$, we have

$$r := \operatorname{dist}(u_0, i(H)) < \|u_0 - i(d)\| \tag{3.3}$$

for every $d \in H$. Hence, we can choose a sequence $(x_n)_n \subset H$ such that $\|i(x_n) - u_0\| > \|i(x_{n+1}) - u_0\|$ for every $n \in \mathbb{N}$ and $\lim_n \|i(x_n) - u_0\| = r > 0$. Set $r_n := \|i(x_n) - u_0\|$, $n \in \mathbb{N}$.

Then the sets $V_n := \{x \in E : \|x_n - x\| \leq r_n\}$, $n \in \mathbb{N}$ form a nested sequence of closed balls in E. First, observe that

$$\Big(\bigcap_n V_n\Big) \cap H = \varnothing. \tag{3.4}$$

Indeed, assume the contrary and suppose that there is $z \in (\bigcap_n V_n) \cap H$. Then, $\|i(z - x_n)\| = \|z - x_n\| \leq r_n$ for each $n \in \mathbb{N}$, so, using Remark 2.1 we get

$$\|i(z) - u_0\| = \|i(z) - i(x_n) + i(x_n) - u_0\|$$
$$\leq \max\big\{\|i(z - x_n)\|, \|i(x_n) - u_0\|\big\} = \max\big\{\|z - x_n\|, \|i(x_n) - u_0\|\big\} \leq r_n$$

for each $n \in \mathbb{N}$. So, $\|i(z) - u_0\| \leq r$, a contradiction with (3.3).

Assume now that there is $z_0 \in \big(\bigcap_n V_n\big) \backslash H$. But, then $H + [z_0]$ is an immediate extension of H (otherwise, there is $h_0 \in H$ with $\|z_0 - h_0\| = \operatorname{dist}(z_0, H) \leq r$ since $\|z_0 - h_0\| \leq \|z - x_n\| \leq r_n$ for each $n \in \mathbb{N}$; thus, for every $n \in \mathbb{N}$ we get $\|h_0 - x_n\| = \|h_0 - z_0 + z_0 - x_n\| \leq r_n$, a contradiction with (3.4). Then, by [13, Proposition 2.1], $H + [z_0]$ is an immediate extension of D, a contradiction with maximality of H. Thus,

$$\bigcap_n V_n = \varnothing. \tag{3.5}$$

Let $F := \overline{i(D) + [i(x_1), i(x_2), \ldots]}$. Then F is a Banach space of countable type. Since, by assumption, E is a space of universal disposition for $\mathcal{U}_{CNA}$ and $F \in \mathcal{U}_{CNA}$, the map $i^{-1} \colon F \to D + [x_1, x_2, \ldots] \subset E$ has an isometric extension $j : \overline{F + [u_0]} \to E$ as $\overline{F + [u_0]} \in \mathcal{U}_{CNA}$. But then, $\|x_n - j(u_0)\| = \|i(x_n) - u_0\| \leq r_n$ for all $n \in \mathbb{N}$; thus, $j(u_0) \in \bigcap_n V_n$, a contradiction with (3.5) . □

LEMMA 3.18. *Let $\mathbb{K}$ be non-spherically complete and $x = (x_i)_{i \in I} \in \ell^\infty(I : s)$. If there exists $p \in I$ such that $\|x\| = s(p) \cdot (x_p)$, then $[x]$ is orthocomplemented in $\ell^\infty(I : s)$. Consequently, for every $z \in \ell^\infty(I : s)$ the two-dimensional linear subspace $[x, z]$ has an orthogonal base.*

PROOF. It is easy to verify that the linear subspace $H := \{(w_i)_{i\in I} \in \ell^\infty(I:s) : w_p = 0\}$ is an orthocomplement of $[x]$ in $\ell^\infty(I:s)$. Then, for $z \in \ell^\infty(I:s)$ we can choose $\lambda_z \in \mathbb{K}$ such that $z = \lambda_z x + (z-\lambda_z x)$. Then $x \perp (z-\lambda_z x)$ and $\{x, z-\lambda_z x\}$ is an orthogonal base of $[x,z]$. □

Recall that if $\mathbb{K}$ is non-spherically complete, the space ℓ^∞ is not of universal disposition for the class $\mathcal{U}_{FNA}$ (see [11, Remark 4.9]). We get the following, more general result.

PROPOSITION 3.19. *Let $\mathbb{K}$ be densely valued, k be a given infinite cardinal number, I be a set with* $\mathrm{card}(I) \geq k$ *and $s\colon I \to (0,\infty)$ be a map. Then, the space $\ell^\infty(I:s)$ is a space of universal disposition for the class $\mathcal{U}_{kNA}$ if and only if $\mathbb{K}$ is spherically complete.*

PROOF. "⇒" Assume that $\mathbb{K}$ is non-spherically complete and suppose that $\ell^\infty(I:s)$ is a space of universal disposition for the class $\mathcal{U}_{kNA}$. Recall (see [16, page 68] and [15, Example 2.3.26]) that if $(B_{r_n}(c_n))_n$ is a nested sequence of closed balls in $\mathbb{K}$ which has an empty intersection, then the formula

$$\|(\lambda_1,\lambda_2)\|_v := \lim_{n\to\infty} |\lambda_1 - \lambda_2 c_n|, \quad (\lambda_1,\lambda_2) \in \mathbb{K}^2,$$

defines a non-archimedean norm on the linear space $\mathbb{K}^2$. The normed space $\mathbb{K}^2_v := (\mathbb{K}^2, \|.\|_v)$ is an immediate extension for each its one-dimensional linear subspace, therefore it has no two orthogonal elements.

Fix $i \in I$, set $X := [e_i] \subset \ell^\infty(I:s)$, $Y := (\mathbb{K}^2, s(i)\cdot\|.\|_v)$ and define the isometric embedding $i\colon X \to Y$ such that $i(e_i) = (1,0)$. Using Lemma 3.18, we imply that every two-dimensional linear subspace of $\ell^\infty(I:s)$ containing e_i has two non-zero orthogonal elements. Thus, as Y has no two non-zero orthogonal elements, there is no isometric embedding $f\colon Y \to \ell^\infty(I:s)$ such that $f(i(x)) = x$ for all $x \in X$, a contradiction.

"⇐" Suppose that $\mathbb{K}$ is spherically complete. Then $\ell^\infty(I:s)$ is spherically complete (see [16, 4.A]). Applying [16, Corollary 4.45], we imply that $\ell^\infty(I:s)$ contains a spherical completion of its every linear subspace which is a member of $\mathcal{U}_{kNA}$. Thus, the condition (ii) of Theorem 3.4 is satisfied.

We prove that for every $g \in G$, $\ell^\infty(I:s)$ contains an orthogonal set X_g such that $\mathrm{card}(X_g) = k$ and $\pi_G(\|x\|_s) = g$ for all $x \in X_g$. Then, using Theorem 3.4, (A) ⇒ (B), we finish the proof. To do it, select an infinite family $\{M_j : j \in J\}$ of infinite, countable and disjoint subsets of I. Then, $\mathrm{card}(J) = \mathrm{card}(I)$. For each $j \in J$ write $M_j = \{i^j_1, i^j_2, \ldots\}$. Next, for every $g \in G$ and $j \in J$ choose a sequence $(\lambda^g_{i^j_n})_n$ in $\mathbb{K}$ such that $s(i^j_n)\cdot|\lambda^g_{i^j_n}| \leq s_g$ and $s(i^j_n)\cdot|\lambda^g_{i^j_n}| \to s_g$ if $n \to \infty$. Define $x^j_g = (x_i)_{i\in I}$, setting $x_i := \lambda^g_{i^j_n}$ if $i = i^j_n$ and $x_i := 0$ otherwise. Then, $W = \{x^j_g : g \in G,\ j \in J\}$ is an orthogonal subset of $\ell^\infty(I:s)$ such that for every $g \in G$ the set $\{w \in W : \pi_G(\|w\|_s) = g\}$ has cardinality greater or equal to k. Hence, we are done. □

PROPOSITION 3.20. *Let $\mathbb{K}$ be discretely valued and k be a given infinite cardinal number. Then,*

- *the space $\ell^\infty(I_{uk} : s_{uk})$ is a space of universal disposition for the class $\mathcal{U}_{kNA}$;*
- *the space $\ell^\infty(I)$ is not a space of universal disposition for the class $\mathcal{U}_{kNA}$.*

PROOF. By [16, 4.A], the space $\ell^\infty(I_{uk} : s_{uk})$ is spherically complete. We show that it satisfies the conditions (i)-(ii) of Theorem 3.4; the condition (i) is clear, by the construction of I_{uk}. By [16, Corollary 4.45], $\ell^\infty(I_{uk} : s_{uk})$ contains a spherical completion of its every linear subspace which is a member of $\mathcal{U}_{kNA}$, thus the condition (ii) is satisfied.

Note that $\|\ell^\infty(I)\| = |\mathbb{K}|$ is a countable set. Thus, $\ell^\infty(I)$ cannot contain a copy of the space E_{uk} since$\|E_{uk}\| = [0,\infty)$; hence, by Corollary 3.6, $\ell^\infty(I)$ is not a space of universal disposition for the class $\mathcal{U}_{kNA}$. □

COROLLARY 3.21. *For any densely valued $\mathbb{K}$ the space $\ell^\infty(I_u, \widehat{\mathbb{K}})$ is a space of universal disposition for the class $\mathcal{U}_{\aleph_0 NA}$.*

PROOF. Since by Theorem 3.10 the space $c_0(I_u, \widehat{\mathbb{K}})$ is a space of universal disposition for the class $\mathcal{U}_{\aleph_0 NA}$, thus for every $g \in G$ it contains an orthogonal set X_g such that $\operatorname{card}(X_g) = k$ and $\pi_G(\|x\|_s) = g$ for all $x \in X_g$. But $c_0(I_u, \widehat{\mathbb{K}}) \subset \ell^\infty(I_u, \widehat{\mathbb{K}})$, hence for every $g \in G$ the space $\ell^\infty(I_u, \widehat{\mathbb{K}})$ contains an orthogonal set X_g with the above property and condition (i) of Theorem 3.4 is satisfied. As $\widehat{\mathbb{K}}$ is spherically complete, $\ell^\infty(I_u, \widehat{\mathbb{K}})$ is spherically complete (see [16, 4.A]) and by [16, Corollary 4.45], $\ell^\infty(I_u, \widehat{\mathbb{K}})$ contains a spherical completion of every its linear subspace H, thus the condition (ii) of Theorem 3.4 is satisfied. Hence, using Theorem 3.4, (A) ⇒ (B) completes the proof. □

4. Injectivity and universality of spaces of universal disposition

A Banach space F is *injective* if for every Banach space E and each linear subspace D of E, every bounded operator $T\colon D \to F$ can be extended to a preserving norm operator $T'\colon E \to F$. Let $\mathcal{U}$ be a given class of Banach spaces. A Banach space F is *universally $\mathcal{U}$-injective* if for every Banach space E and each linear subspace D of E such that $D \in \mathcal{U}$, every bounded operator $T\colon D \to F$ extends to a preserving norm operator $T'\colon E \to F$.

We say that a Banach space E is *isometric $\mathcal{U}$-universal* if for every Banach space $D \in \mathcal{U}$ there is an isometric embedding $D \to E$.

The almost immediate consequence of Ingleton's theorem is the following result.

THEOREM 4.1. *If a non-archimedean Banach space F is of universal disposition for the class $\mathcal{U}_{kNA}$ for some infinite cardinal number k then F is universally $\mathcal{U}_{kNA}$-injective. If $k > \aleph_0$, then F is injective.*

PROOF. Let E be a non-archimedean Banach space, D be a closed linear subspace of E such that $D \in \mathcal{U}_{kNA}$ and $T\colon D \to F$ be a bounded operator. Then, $T(D) \in \mathcal{U}_{kNA}$. Since F is of universal disposition for the class $\mathcal{U}_{kNA}$, by Theorem 3.4, there exists a spherically complete linear subspace $D_0 \subset F$ such that $T(D) \subset D_0$. Now, by Ingleton's theorem (see [16, Theorem 4.10]), T extends to a preserving norm operator $T'\colon E \to D_0 \subset F$. If $k > \aleph_0$, then, by Theorem 3.4, F is spherically complete and the conclusion follows directly from Ingleton's theorem (see [16, Theorem 4.10]). □

Let k be a given infinite cardinal number. By $\mathcal{U}^+_{kNA}$ we will denote the class of non-archimedean Banach spaces over $\mathbb{K}$ satisfying: $E \in \mathcal{U}^+_{kNA}$ if and only if $\operatorname{ort}(E) \le k$.

PROPOSITION 4.2. *Let E be a space of universal disposition for the class $\mathcal{U}_{kNA}$. If $k > \aleph_0$ then E is isometric $\mathcal{U}^+_{kNA}$-universal.*

PROOF. If $k > \aleph_0$, then, by Theorem 3.4 E is spherically complete.

Take a non-archimedean Banach space $D \in \mathcal{U}^+_{kNA}$. Let W be a maximal orthogonal set in D. Then, $\operatorname{card}(W) \le k$. By Theorem 3.4, for every $g \in G$, E contains an orthogonal set X_g such that $\operatorname{card}(X_g) = k$ and $\pi_G(\|x\|_s) = g$ for all $x \in X_g$. Hence, we can establish an isometric map $i\colon [W] \to E$. Then, D is an immediate extension of $[W]$ (see [16, page 167]). By assumption, E is spherically complete. So, we can apply Lemma 3.3 and extend i to the required isometry $D \to E$. □

COROLLARY 4.3. *Every non-archimedean Banach space E can be isometrically embedded into the space $\widehat{E_{uk}}$ for some cardinal number k.*

PROOF. Choose a cardinal number k, $k > \aleph_0$ such that $E \in \mathcal{U}^+_{kNA}$. By Corollary 3.6, $\widehat{E_{uk}}$ is a space of universal disposition for the class $\mathcal{U}_{kNA}$. Now, the conclusion follows directly from Proposition 4.2. □

LEMMA 4.4. *Let $\mathbb{K}$ be densely valued and spherically complete. Then, the space $\widehat{c_0}$, a spherical completion of c_0 has no orthogonal base and* $\operatorname{ort}(\widehat{c_0}) = \aleph_0$.

PROOF. First, extend the set of unit vectors $\{e_i\}_{i\in\mathbb{N}} \subset c_0$, which is an orthogonal base of c_0 (see [15, Theorem 2.3.25]) to a maximal orthogonal set $\{y_i\}_{i\in I_m}$ in $\widehat{c_0}$. Since $\widehat{c_0}$ is an immediate extension of c_0 (see [16, p. 167]), we imply that I_m is countable. Now, assume the contrary and suppose that $\widehat{c_0}$ has an orthogonal base $\{x_i\}_{i\in I}$. Then, by [16, Theorem 5.9] I is countable. But by Proposition 3.14 the space $\widehat{c_0}$ is not of countable type, a contradiction. □

REMARK 4.5. Note that the assumption $k > \aleph_0$ cannot be removed in Proposition 4.2. Let $\mathbb{K}$ be spherically complete and densely valued. Then, by Theorem 3.10, $c_0(I_u, \mathbb{K})$ is a space of universal disposition for the class $\mathcal{U}_{\aleph_0 NA}$. Consider $\widehat{c_0}$, a spherical completion of c_0. Then, by Lemma 4.4 $\widehat{c_0} \in \mathcal{U}^+_{\aleph_0 NA}$. Now, assume that there exists an isometric embedding $T\colon \widehat{c_0} \to c_0(I_u, \mathbb{K})$. Then, by [16, Theorem 5.9], $T(\widehat{c_0})$ as a linear subspace of $c_0(I_u, \mathbb{K})$ has an orthogonal base; hence, we conclude that $\widehat{c_0}$ has an orthogonal base, a contradiction with Lemma 4.4.

5. Transitivity and universal disposition

Let E be a non-archimedean Banach space and let $\mathcal{U}$ be a given class of Banach spaces. Recall (see [3, Definition 3.40]) that E is *$\mathcal{U}$-transitive* if the following property satisfies:

(TR) for any $X, Y \in \mathcal{U}$, linear subspaces of E, and a surjective isometry $i\colon X \to Y$ there exists a surjective isometry $i'\colon E \to E$ which extends i.

We get the following result.

THEOREM 5.1. *Let k be an infinite cardinal number. If E is a space of universal disposition for the class $\mathcal{U}_{kNA}$ then E is $\mathcal{U}_{kNA}$-transitive.*

LEMMA 5.2. *Let E, F be Banach spaces, $\widehat{E}$, $\widehat{F}$ be their spherical completions and $i\colon E \to F$ be a surjective isometry. Then, there exists a surjective isometry $i'\colon \widehat{E} \to \widehat{F}$ such that $i'|_E = i$.*

PROOF. Applying Lemma 3.3, we can extend i to the isometric embedding $i_0\colon \widehat{E} \to \widehat{F}$. Since $F = i(E)$, $i_0(\widehat{E})$, as an isometric range of $\widehat{E}$, is a spherical completion of F. By [16, Theorem 4.43], there exists an isometric isomorphism $j\colon i_0(\widehat{E}) \to \widehat{F}$. Now, the operator $i' := j \circ i_0$ is a required surjective isometry. $\square$

PROOF OF THEOREM 5.1. Let $X, Y \in \mathcal{U}_{kNA}$ and $i\colon X \to Y$ be a surjective isometry. Let $D = X + Y$. Then, $D \in \mathcal{U}_{kNA}$. Since, by assumption, E is a space of universal disposition for $\mathcal{U}_{kNA}$, by Theorem 3.4, there exist linear subspaces $\widehat{X}$, $\widehat{Y}$, $\widehat{D}$ of E such that $\widehat{X}$ is a spherical completion of X, $\widehat{Y}$ is a spherical completion of Y and $\widehat{D}$ is a spherical completion of D, respectively. Obviously, $\widehat{X}, \widehat{Y} \subset \widehat{D}$. Applying Lemma 5.2, we can extend i to the bijective isometry $i'\colon \widehat{X} \to \widehat{Y}$.

Let W_X be a maximal orthogonal set in X and W_Y be a maximal orthogonal set in Y, respectively. Select orthogonal sets $Z_X, Z_Y \subset \widehat{D}$ such that $W_X \cup Z_X$ and $W_Y \cup Z_Y$ are maximal orthogonal sets in $\widehat{D}$. Clearly, $Z_X \cap \widehat{X} = \varnothing$ and $Z_Y \cap \widehat{Y} = \varnothing$. By [16, Theorems 5.2 and 5.4], Z_X and Z_Y have the same cardinality, so we can define the isometric isomorphism

$j\colon [Z_X] \to [Z_Y]$. Then, since $[Z_X] \perp \widehat{X}$ and $[Z_Y] \perp \widehat{Y}$, we can define a surjective isometry $T\colon [Z_X] + \widehat{X} \to [Z_Y] + \widehat{Y}$ by setting

$$T(z_x + x) := j(z_x) + i'(x) \quad (z_x \in [Z_X],\ x \in \widehat{X}).$$

Since $\widehat{D}$ is a spherical completion of $[Z_X] + \widehat{X}$ and $[Z_Y] + \widehat{Y}$, using Lemma 5.2 again, we can establish the surjective isometry $T'\colon \widehat{D} \to \widehat{D}$ such that $T'|_{[Z_X]+\widehat{X}} = T$. Thus, $T'(X) = Y$. But $\widehat{D}$, as a spherically complete linear subspace of E, is orthocomplemented in E (see [16, Exercise 4.H]), so we can find an orthogonal decomposition $E = \widehat{D} \oplus E_0$ and extend T' to the surjective isometry $T''\colon E \to E$ setting $T''(d + x) := T'(d) + x$, where $d \in \widehat{D}$ and $x \in E_0$. Then, $T''(X) = Y$ and the proof is finished. □

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