ON A THEOREM OF WIEGERINCK

R. SZŐKE

Department of Analysis, Institute of Mathematics, ELTE Eötvös Loránd University,
Pázmány Péter sétány 1/c, Budapest 1117, Hungary
e-mail: rszoke@caesar.elte.hu

Dedicated to László Lempert on the occasion of his 70th birthday

Abstract. A theorem of Wiegerinck says that the Bergman space over any domain in $\mathbb{C}$ is either trivial or infinite dimensional. We generalize this theorem in the following form. Let E be a Hermitian, holomorphic vector bundle over $\mathbb{P}^1$, the later equipped with a volume form and D an arbitrary domain in $\mathbb{P}^1$. Then the space of holomorphic L^2 sections of E over D is either equal to $H^0(M, E)$ or it has infinite dimension.

1. Introduction

Let D be a domain in $\mathbb{C}^n$ and $\mathcal{O}L^2(D)$ the Bergman space, that is the space of those holomorphic functions on D that also belong to $L^2(D)$, with respect to the Lebesgue measure. Wiegerinck proved in [16], that for $n = 1$ and for any domain D in $\mathbb{C}$, either $\mathcal{O}L^2(D)$ is trivial ($= \{0\}$) or has infinite dimension. In the same paper for any $k > 0$ he constructed a Reinhardt domain in $\mathbb{C}^2$ with k-dimensional Bergman space. These examples were not logarithmically convex, hence they were not Stein domains. These results give rise to the following natural conjecture.

CONJECTURE 1.1 (Wiegerinck). *Let D be a Stein domain in $\mathbb{C}^n$. Then its Bergman space is either trivial or infinite dimensional.*

Despite of a lot of work and partial results (cf. [1,4,7,11–13], as far as we can tell, the conjecture is still open.

The purpose of this paper is to show that there is another very natural direction in which Wiegerinck's result could be generalized. More precisely we mean to consider the following situation. Let M be a compact,

connected Riemann surface, $E \to M$ a holomorphic vector bundle equipped with a smooth Hermitian metric h, and dV a smooth volume form on M. For a domain $D \subset M$, denote by $\mathcal{O}L^2(E|_D)$ the set of those holomorphic sections of E over D for which

$$\|s\|^2 := \int_D h(s,s)\, dV < \infty.$$

As is well known $\mathcal{O}L^2(E|_D)$ is a Hilbert space and the vector space $H^0(M, E)$ of global holomorphic sections of E is finite dimensional ([9]). Clearly $H^0(M, E) \subset \mathcal{O}L^2(E|_D)$.

CONJECTURE 1.2. $\mathcal{O}L^2(E|_D)$ *is either equal to* $H^0(M, E)$ *or it has infinite dimension.*

Supporting this conjecture we have the following observations. First of all Wiegerinck's result in [16] implies that Conjecture 1.2 is true when $M = \mathbb{P}^1$ and E is the canonical bundle of $\mathbb{P}^1$. It is also true when M and E are arbitrary and K is either small, i.e., locally polar (cf. Proposition 2.1) or K is big, i.e., has nonempty interior (cf. Proposition 2.2).

Our main result is that for $M = \mathbb{P}^1$, Conjecture 1.2 holds.

THEOREM 1. *Let* $M = \mathbb{P}^1$ *and* D *and* E *be arbitrary. Then Conjecture 1.2 is true.*

2. Proofs

Suppose M is an arbitrary Riemann surface. Recall that a subset $K \subset M$ is called locally polar, if for each point $p \in K$, there exists a connected neighborhood U of p and a subharmonic function $\varphi \not\equiv -\infty$ on U such that $K \cap U \subset \{y \in U | \varphi(y) = -\infty\}$. Now back to our original setting, let us assume that M is a compact Riemann surface, $(E, h) \to M$ a holomorphic Hermitian vector bundle over M, dV a smooth volume form on M, $D \subset M$ a domain and $K = M \setminus D$. When K is small, every element of $\mathcal{O}L^2(E|_D)$ extends holomorphically to a global section of E yielding:

PROPOSITION 2.1. *Suppose* K *is locally polar. Then* $\mathcal{O}L^2(E|_D) = H^0(M, E)$.

The special case when M is $\mathbb{P}^1$ and E is its canonical bundle, this is a reformulation of a result of Carleson in [2]. Indeed Carleson proves that in this case the Bergman space over $D = \mathbb{P}^1 \setminus K$ is trivial. On the other hand the canonical bundle of $\mathbb{P}^1$ is negative so its only holomorphic section is the zero section, i.e., $H^0(M, E)$ is trivial as well. The general case is also essentially known, but for the readers' sake we include a proof here.

PROOF. We need to show that each element of $\mathcal{O}L^2(E|_D)$ extends holomorphically to yield a global holomorphic section of E. It suffices to prove local extension. But locally the bundle E is trivial and the volume form of M is equivalent to the Euclidean volume form (in a coordinate neighborhood). So the problem reduces to an extension problem on holomorphic functions.

Let $W \subset \mathbb{C}$ be open, $X \subset W$ a closed, locally polar set in W and f a holomorphic L^2 function on $W \setminus X$. We need to show that for each point $p \in X$, f extends holomorphically to a neighborhood of p. According to [3, Ch. 21, Proposition 5.5], X is a polar subset of $\mathbb{C}$. In light of [3, Ch. 21,Theorem 9.5(c)], it suffices to find an open set V, that contains p, so that $K = V \cap X$ is compact. To find such a set, one can argue as follows. Since X is a polar subset of $\mathbb{C}$, by [14, Corollary 3.8.5], X is a totally disconnected space. Now X is closed in W, hence it is also locally compact. Let $F \subset X$ be a compact neighborhood of p and $U \subset X$ open with $p \in U \subset F$. A general result on totally disconnected, locally compact spaces ([5, Theorem 6.2.9]) tells us that there exists a set $K \subset U$ that is both open and closed in X and $p \in K$. Let $V \subset \mathbb{C}$ be open with $V \cap X = K$ (can assume $V \subset W$). Since K is a closed subset of a compact space (namely F), it is also compact.   □

The other extreme case is when K is big in the sense, that its interior is nonempty.

PROPOSITION 2.2. *Suppose the interior of K is nonempty. Then $\mathcal{O}L^2(E|_D)$ is infinite dimensional.*

PROOF. Let p be an interior point of K and l be the rank of E. Let $G = M \setminus \{p\}$. Since G is an open Riemann surface, $E|_G$ is holomorphically trivial ([6, Theorem 30.4]), hence $H^0(G, E|_G) = \mathrm{Hol}(G, \mathbb{C}^l)$, the space of holomorphic maps from G to $\mathbb{C}^l$. G being Stein, the latter space is infinite dimensional. But $\overline{D}$ is a compact subset in G, hence all these holomorphic maps (sections), via restriction to D, belong to $L^2(E|_D)$.   □

Denote by $H^\infty(D)$ the set of bounded holomorphic functions on D. We shall say that $H^\infty(D)$ is nontrivial, if it contains an element not identically constant.

PROPOSITION 2.3. *Suppose $H^\infty(D)$ is nontrivial. Then Conjecture* 1.2 *holds.*

PROOF. Suppose there exists an $s \in \mathcal{O}L^2(E|_D)$, with $s \notin H^0(M, E)$. Let $g \in H^\infty(D)$ be a nonconstant element and $w \in D$ be arbitrary. Then $f := g - g(w) \in H^\infty(D)$ vanishes at w with finite, nonzero multiplicity. This yields

$$f^m s \in \mathcal{O}L^2(L|_D), \quad m \in \mathbb{N},$$

implying the infinite dimensionality of $\mathcal{O}L^2(L|_D)$.   □

For an arbitrary holomorphic line bundle $L \to M$, denote by $\mathcal{M}(L)$ the set of meromorphic sections. Whereas for an open subset $D \subset M$ the set of meromorphic functions on D is denoted by $\mathcal{M}(D)$.

PROOF OF THEOREM 1. Let $E \to \mathbb{P}^1$ be a holomorphic vector bundle of rank r. According to Grothendieck's theorem ([10]) E holomorphically splits as a direct sum of holomorphic line bundles:

$$E = L_1 \oplus \cdots \oplus L_r.$$

Therefore it is enough to prove Conjecture 1.2 when $E = L$ is a holomorphic line bundle. The proof follows similarly as in [16].

As is well known ([15, Prop. 4.6.2]), there exists an integer k, so that $L = \mathcal{O}(k)$, where $\mathcal{O}(k)$ denotes the line bundle associated to the divisor kp, p being an (arbitrary) point in $\mathbb{P}^1$. Suppose there exists an $s \in \mathcal{O}L^2(L|_D)$, with $s \notin H^0(\mathbb{P}^1, L)$.

Case 1: $s \in \mathcal{M}(L)$. Since isolated L^2 singularities are removable and $s \in \mathcal{M}(L) \setminus H^0(\mathbb{P}^1, L)$, we get

$$(2.1) \qquad\qquad \int_{\mathbb{P}^1} h(s,s) \, dV = \infty.$$

Now $s|_D$ is in L^2, hence

$$(2.2) \qquad\qquad \lambda(\mathbb{P}^1 \setminus D) > 0,$$

where λ denotes Lebesgue measure on $\mathbb{P}^1$. (The fact that a subset of a smooth manifold has Lebesgue measure zero or positive does not depend on any particular choice of a Riemannian metric or a volume form.) From (2.2) and a well known result about the Cauchy transform (cf. [8, p. 2]) we get that $H^\infty(D)$ is nontrivial, so in light of Proposition 2.3 we are done.

Case 2: $s \notin \mathcal{M}(L)$ and $k \geq 0$. Pick an arbitrary point $\infty \neq z_0 \in D$. Since $L = \mathcal{O}(k)$, L admits a holomorphic section $s_L : \mathbb{P}^1 \to L$ with the only possible zero at z_0, the multiplicity at z_0 being k. Then

$$s = f s_L,$$

with some $f \in \mathcal{M}(D)$. Here f must be holomorphic in D except perhaps at z_0, where f may have a pole at most of order k. Since s is not in $\mathcal{M}(L)$, f cannot be in $\mathcal{M}(\mathbb{P}^1)$. Let $z_1 \in D \setminus \{z_0, \infty\}$ be another arbitrary point. Define the function g_1 by

$$(2.3) \qquad\qquad g_1(z) = (f(z) - f(z_1)) \frac{(z - z_0)}{(z - z_1)}.$$

CLAIM 2.4. $s_1 := g_1 s_L \in \mathcal{O}L^2(L|_D)$

Proof. g_1 is holomorphic in $D \setminus \{z_0\}$ and in z_0 it has either a removable singularity or a pole of order at worst $k-1$. Hence $g_1 s_L$ is a holomorphic section of L over D. Let $U \subset D$ be an open subset with $z_0, z_1 \in U$ and $\bar{U} \subset D$ compact. Holomorphicity of $g_1 s_L$ implies that $g_1 s_L$ is an L^2 section of L over U.

Since $(f - f(z_1)) s_L$ is an L^2 section over D and $\frac{z - z_0}{z - z_1}$ is bounded on $\mathbb{P}^1 \setminus U$, we get that $g_1 s_L$ is in L^2 over $D \setminus U$ as well. Since $g_1 s_L$ is holomorphic, it is also in L^2 over the compact set $\bar{U}$ which proves Claim 2.4. □

Back to the proof of our theorem, let now N be an arbitrary positive integer and $z_1, z_2, \ldots, z_N \in D \setminus \{z_0, \infty\}$ be arbitrary different points. Denote the corresponding functions defined by (2.3) by g_j and let $s_j := g_j s_L$, where $j = 1, \ldots, N$. Due to Claim 2.4 they all belong to $\mathcal{O}L^2(L|_D)$.

Claim 2.5. $s_1, \ldots, s_N$ are linearly independent.

Proof. Indeed if they were linearly dependent, that would yield that f is a meromorphic function on $\mathbb{P}^1$, a contradiction. □

Claim 2.5 shows that $\mathcal{O}L^2(L|_D)$ has infinite dimension when $k \geq 0$, finishing the proof of Case 2.

Case 3: $s \notin \mathcal{M}(L)$ and $k < 0$. Similarly to Case 2, pick a point $\infty \neq z_0 \in D$. $L = \mathcal{O}(k)$ implies that L admits a meromorphic section m_L with z_0 being the only singularity, a pole of order k and that m_L does not vanish on $\mathbb{P}^1 \setminus \{z_0\}$.

Because $s \in \mathcal{O}L^2(L|_D)$, we can write

$$s = f m_L,$$

where f now is holomorphic in D but is not in $\mathcal{M}(\mathbb{P}^1)$. Since s is holomorphic, f must vanish at z_0. Denote by l its multiplicity (necessarily $l \geq k$). Then s vanishes at z_0 with multiplicity $l - k$.

Let $N > l$ be an arbitrary positive integer and $z_1, z_2, \ldots, z_N \in D \setminus \{z_0, \infty\}$ arbitrary but different points. Define the corresponding functions g_j by formula 2.3. Then all g_j are holomorphic in D, but $g_j m_L$ has a pole at z_0 of order $k - 1$ if $f(z_j) \neq 0$. Let $\lambda_j \in \mathbb{C}$, $j = 1, \ldots, N$, $\lambda = (\lambda_1, \ldots, \lambda_N)$ and define g_λ by

$$(2.4) \qquad g_\lambda(z) := \sum_{j=1}^{N} \lambda_j g_j(z)$$

$$= f(z) \left(\sum_{j=1}^{N} \frac{\lambda_j}{z - z_j} \right)(z - z_0) - \left(\sum_{j=1}^{N} \frac{\lambda_j f(z_j)}{z - z_j} \right)(z - z_0).$$

Since $f \notin \mathcal{M}(\mathbb{P}^1)$, $g_\lambda \notin \mathcal{M}(\mathbb{P}^1)$ except if all the λ_j's are zero. In (2.4) the first term is holomorphic near z_0 and vanishes at z_0 with multiplicity at least $l + 1$. The function

$$(2.5) \qquad\qquad f_\lambda := \sum_{j=1}^{N} \frac{\lambda_j f(z_j)}{z - z_j}$$

is also holomorphic near z_0 and the coefficients of its Taylor series around z_0 depend linearly on the $\lambda_j s$. Therefore to impose the condition on f_λ to vanish at z_0 with multiplicity l means solving a homogeneous system of linear equations in N unkowns and l equations. Since $N > l$, this system will have a nontrivial solution. Let λ be such a solution. Then the corresponding g_λ will vanish at z_0 with multiplicity at least $l + 1$. Hence the holomorphic section $s_\lambda := g_\lambda m_L$ vanishes at z_0 with higher multiplicity than s does. A similar argument as in the proof of Claim 2.4 shows that $s_\lambda \in \mathcal{O}L^2(L|_D)$. Moreover $g_\lambda \notin \mathcal{M}(\mathbb{P}^1)$ implies that $s_\lambda \notin \mathcal{M}(L)$. Hence we can now repeat the whole process to s_λ. That yields indeed that $\mathcal{O}L^2(L|_D)$ has infinite dimension. $\square$

Note added in proof.   During the refereeing process, a closely related paper appeared in the arxiv: A-K. Gallagher, P. Gupta and L. Vivas, On the dimension of Begman spaces on $\mathbb{P}^1$, arXiv:2110.02324v2. Here the authors give potential theoretic characterizations of the dimension of the Bergman space. Furthermore, by generalizing Wiegerinck's example, they show that in higher dimensions the dichotomy we considered in our paper, does not hold in general for certain line bundles over $\mathbb{P}^2$.

References

[1] Z. Błocki and W. Zwonek, Generalizations of the higher dimensional Suita conjecture and its relation with a problem of Wiegerinck, *J. Geom. Anal.*, **30** (2020), 1259–1270.

[2] L. Carleson, *Selected Problems on Exceptional Sets*, Van Nostrand (Princeton, 1967).

[3] J. B. Conway, *Functions of Complex Variable. II*, Springer-Verlag (1995).

[4] Z. Dinew, The Ohsawa–Takegoshi extension theorem on some unbounded sets, *Nagoya Math. J.*, **188** (2007), 19–30.

[5] R. Engelking, *General Topology*, Heldermann Verlag (1989).

[6] O. Forster, *Lectures on Riemann Surfaces*, Springer-Verlag (1981).

[7] A. K. Gallagher, T. Harz and G. Herbort, On the dimension of the Bergman space for some unbounded domains, *J. Geom. Anal.*, **27** (2017), 1435–1444.

[8] J. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Math., vol. 297, Springer (1972).

[9] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley (1978).

[10] A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann, *Amer. J. Math.*, **79** (1957), 121–138.

[11] P. Jucha, A remark on the dimension of the Bergman space of some Hartogs domains, *J. Geom. Anal.*, **22** (2012), 23–37.

[12] P. Pflug and W. Zwonek, L_h^2-domains of holomorphy in the class of unbounded Hartogs domains, *Illinois J. Math.*, **51** (2007), 617–624.

[13] P. Pflug and W. Zwonek, L_h^2-functions in unbounded balanced domains, *J. Geom. Anal.*, **3** (2017), 2118–2130.

[14] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press (1995).

[15] D. Varolin, *Riemann surfaces by way of complex analytic geometry*, Graduate Studies in Math., vol. 25, Amer. Math. Soc. (2011).

[16] J. J. O. O. Wiegerinck, Domains with finite dimensional Bergman space, *Math. Z.*, **187** (1984), 559–562.