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THE PERIODICITY OF DIFFERENTIAL-DIFFERENCE POLYNOMIALS OF ENTIRE FUNCTIONS

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Abstract. We present some results on the periodicity of an entire function $f(z)$ with its differential-difference polynomials. For instance, we obtain that if $f^n + L(f)$ is a periodic function with period c, then $f(z)$ must be a periodic function with period c, where $f(z)$ is a transcendental entire function with hyper-order less than one and $N(r, \frac{1}{f}) = S(r, f)$, $L(f)$ is a differential-difference polynomial in f with constant coefficients and degree less than n.

1. Introduction and main results

Recently, the periodicity of transcendental meromorphic functions, related to complex differential polynomials or complex difference polynomials, is considered in [7,9–12,14,16], where the main intention is to obtain $f(z)$ is a periodic function when the complex differential polynomials or complex difference polynomials of f are periodic functions. These works were inspired by Yang's Conjecture (see [8,16]), which is given by observing the meromorphic solutions on complex differential equation $f(z)f''(z) = p(z)\sin^2 z$, where $p(z)$ is a non-zero polynomial with real coefficients and real zeros, see more details on the meromorphic solutions in [8]. Yang's Conjecture can be stated as follows.

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Yang's Conjecture. Let f be a transcendental entire function and k be a positive integer. If $ff^{(k)}$ is a periodic function, then f is also a periodic function.

In the paper, we proceed to consider the periodicity of transcendental entire functions related to complex differential-difference polynomials. Observe the non-linear differential equation

$$f(z)^3 + \frac{3}{4}f''(z) = -\frac{1}{4}\sin 3z, \tag{1.1}$$

$f(z) = \sin z$ is a transcendental entire solution of (1.1). Heittokangas, Korhonen and Laine [5, p. 338] showed that (1.1) has at least two meromorphic solutions, namely, $f_1(z) = \sin z$ and $f_2(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$. Afterwards, Yang and Li [19, Theorem 4] obtained

Theorem A. *There exist exactly three entire solutions of* (1.1), *namely* $f_1(z) = \sin z,\ f_2(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z,\ f_3(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z.$

Our proceeding observation, the existence of entire solutions of the difference counterpart of (1.1), is obtained directly by Yang and Laine [18, Theorem 2.5].

Theorem B. *The non-linear difference equation*

$$f(z)^3 + \frac{3}{4}f(z+1) = -\frac{1}{4}\sin 3\pi z \tag{1.2}$$

exists exactly the following three entire solutions: $f_1(z) = \sin \pi z,\ f_2(z) = -\frac{1}{2}\sin \pi z + \frac{\sqrt{3}}{2}\cos \pi z$ *and* $f_3(z) = -\frac{1}{2}\sin \pi z - \frac{\sqrt{3}}{2}\cos \pi z.$

The recent improvements on equations (1.1) and (1.2) can be found in [1,6,13]. It is interesting to find that all entire solutions of (1.1) and (1.2) are periodic functions, and the functions in the right hand sides of (1.1) and (1.2) are periodic as well. In addition, (1.1) may be extended to

$$f(z)^3 + \frac{3}{4}f''(z) + f^{(k-2)}(z+2\pi) + f^{(k)}(z+2\pi) = -\frac{1}{4}\sin 3z \quad (k \geq 3),$$

and (1.2) also may be extended to

$$f(z)^3 + \frac{3}{4}f(z+1) + f(z+2k) + f(z+2k+1) = -\frac{1}{4}\sin 3\pi z \quad (k \geq 1).$$

Let $L(f)$ be a nonconstant differential-difference polynomial in f with constant coefficients, that is,

$$L(f) = \sum_{ij} a_{ij} f^{(j)}(z+c_i)^{m_{ij}},$$

where a_{ij} are constants and m_{ij} are positive integers. Hence, it is natural to ask the following question by the above observations.

QUESTION 1.1. Let $L(f)$ be defined as the above and $\max\{m_{ij}\} < n$. Does it follow that the entire function $f(z)$ is a periodic function if

$$f^n + L(f) \tag{1.3}$$

is a periodic function?

We first to discuss two cases which are not included in Question 1.1.

(i) If $\max\{m_{ij}\} = n = 1$, then Question 1.1 is not true. For example, we assume that

$$f(z) = c_1 e^z + c_2 e^{(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)z} + c_3 e^{(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)z} + e^{-z},$$

where c_1, c_2, c_3 are constants, then

$$f - f''' = 2e^{-z}.$$

Obviously, $2e^{-z}$ is a periodic function, however, these $f(z)$ are not periodic functions when c_1, c_2, c_3 are non-zero constants. Another example, the function $f(z) = ze^z$ satisfies $f(z) + f(z+\pi) = -\pi e^z$ and $f(z) - f'(z) = -e^z$ here, $-\pi e^z$ and $-e^z$ are periodic functions, but ze^z is a non-periodic function.

(ii) Recall a result given by Rényi and Rényi [15, Theorem 2], namely, if $P(f(z))$ is a periodic function, where $P(z)$ is a non-zero polynomial, then the transcendental entire function $f(z)$ is also a periodic function. Hence, if $L(f) \equiv 0$, then Question 1.1 is true for all transcendental entire solutions by Rényi and Rényi's result.

In what follows, we assume that the reader is familiar with the basic notations and fundamentals of Nevannlinna theory [3,20]. Denote $\rho_2(f)$ the hyper-order of growth of a meromorphic function $f(z)$. The preliminary results on Question 1.1 are also considered by Liu, Wei and Yu [11], Lü and Zhang [14], Latreuch and Zemirni [7]. We recall a result given by Lü and Zhang [14, Theorem 1.2, Remark 1.2] as follows.

THEOREM C. *Let $f(z)$ be a transcendental entire function of $\rho_2(f) < 1$, $n \geq 2$ and $k \geq 1$. If $f(z)^n + a_k f^{(k)}(z) + \cdots + a_1 f'(z)$ is a periodic function, then $f(z)$ is periodic as well. Furthermore, if $n = 2$ or $n \geq 4$, the hypothesis $\rho_2(f) < 1$ is not necessary, where $a_1, \ldots, a_k$ are constants, at least one of them is non-zero.*

In addition, Question 1.1 can also be considered from the uniqueness problem of meromorphic solutions of complex differential-difference equations. Such results can be found in Yang [17], Heittokangas, Korhonen and Laine [5], Yang and Laine [18]. In most cases, the uniqueness results can be obtained for large n. For example, see Yang and Laine [18, Theorem 2.6] as follows.

Theorem D. *Let $n \geq 4$ be an integer and $L(z,f)$ be a linear differential-difference polynomial of f, not vanishing identically, and let $h(z)$ be a meromorphic function of finite order. Then the differential-difference equation $f^n + L(z,f) = h$ possesses at most one admissible transcendental entire solution of finite order such that all coefficients of $L(z,f)$ are small functions of f. If such a solution f exists, then f is of the same order as h.*

From Theorem D, we obtain that if $f(z)^n + L(f)$ is a periodic function with period c, that is

$$f(z)^n + L(f) = f(z+c)^n + L(f(z+c)) =: h(z).$$

Question 1.1 is true with the conditions that $f(z)$ is of finite order, $n \geq 4$ and $L(f)$ is a linear differential-difference polynomial. Now, we state our first result as follows.

Theorem 1.2. *Let $f(z)$ be a transcendental entire function of $\rho_2(f) < 1$ with $N(r, \frac{1}{f}) = S(r,f)$, $L(f)$ be defined as the above and $n \geq 2$. If $f^n + L(f)$ is a periodic function with period c, then $f(z)$ is a periodic function with period c.*

Theorem 1.2 improves Theorem C from the differential polynomials to differential-difference polynomials. And we provide a different proof with the proof of Theorem C in [14]. We also remark that Lü and Zhang 's method to deal with the case $n = 2, 4$ and our method in the paper are not valid for transcendental entire functions $\rho_2(f) \geq 1$ when $L(f)$ includes the difference operator or differential-difference operator. Here, we proceed to consider the periodicity of $f(z)^2 + f(z+c)$. Assume that $f(z)^2 + f(z+c)$ is a periodic function with period η, namely,

$$f(z)^2 + f(z+c) = f(z+\eta)^2 + f(z+c+\eta),$$

then

$$(f(z) - f(z+\eta))(f(z) + f(z+\eta)) = f(z+c+\eta) - f(z+c).$$

We conjecture that all the solutions of the above equation are periodic functions with period η. But we have not succeed in doing that. With an additional condition, we obtain

Theorem 1.3. *If $f(z)^2 + f(z+c)$ is a nonconstant periodic entire function with period c and $f(z) = p(z) + e^{h(z)}$, where $p(z)$ is a polynomial and $h(z)$ is an entire function. Then $p(z)$ must be a constant and $f(z)$ is a periodic function with period c or $2c$.*

Finally, the results on Fermat type functional equations [4] can be applied to consider the periodicity of non-linear difference polynomials $f(z)^n + f(z+c)^n + f(z+2c)^n + \cdots + f(z+kc)^n$. We obtain

THEOREM 1.4. *Let $f(z)$ be a transcendental meromorphic function and n be positive integers. If $n \geq (2k+1)^2$ and $f(z)^n + f(z+c)^n + f(z+2c)^n + \cdots + f(z+kc)^n$ is periodic function with period η, then $f(z)$ is a periodic function with period $2nc$ or $n\eta$.*

REMARK 1.5. It is an open question for us on the periodicity of $f(z)^n + f(z+c_1)^n + f(z+c_2)^n + \cdots + f(z+c_k)^n$ with $f(z)$, where $c_1, c_2, \ldots, c_k$ are distinct constants.

2. Lemmas

The following two lemmas are important in difference analogues of Nevanlinna theory, also play an important part in the present paper.

LEMMA 2.1 [2, Lemma 8.3]. *Let f be a nonconstant meromorphic function with hyper-order $\rho_2(f) < 1$, $c \in \mathbb{C}$. Then*

$$T(r, f(z+c) = T(r, f) + S(r, f),$$

outside of a possible exceptional set with finite logarithmic measure.

LEMMA 2.2 [2, Theorem 5.1]. *Let f be a nonconstant meromorphic function with hyper-order $\rho_2(f) < 1$, $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f),$$

outside of a possible exceptional set with finite logarithmic measure.

We can obtain the following result by the above two lemmas.

LEMMA 2.3. *Let $f(z)$ be a meromorphic function with hyper-order $\rho_2(f) < 1$ and let $L(f)$ be a nonconstant linear differential-difference polynomial of $f(z)$ and $\Delta_c f = f(z+c) - f(z)$. Then*

$$m\left(r, \frac{L(\Delta_c f)}{\Delta_c f}\right) = S(r, f),$$

outside of a possible exceptional set with finite logarithmic measure.

To prove Theorem 1.2, we need the following lemma, which can be obtained from the proof of the classical result $T(r, P(f)) = nT(r, f) + S(r, f)$, see [20, Theorem 1.12].

LEMMA 2.4. *Let $f(z)$ be a meromorphic function and $P(f) = a_n(z)f(z)^n + \cdots + a_1(z)f(z) + a_0(z)$, where $a_n(z)$ ($\not\equiv 0$), $a_{n-1}(z)$, ..., $a_0(z)$ are small functions with respect to $f(z)$ in the sense of $T(r, a_j) = S(r, f)$, $j = 1, 2, \dots, n$. Then*

$$m(r, P(f)) = nm(r, f) + S(r, f).$$

LEMMA 2.5 [20, Theorem 1.62]. *Let $f_j(z)$ be meromorphic functions, $f_k(z)$ ($k = 1, 2, \dots, n-1$) be nonconstant functions satisfying $\sum_{j=1}^{n} f_j = 1$ and $n \geq 3$. If $f_n(z) \not\equiv 0$, and*

$$\sum_{j=1}^{n} N\Big(r, \frac{1}{f_j}\Big) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$

where $\lambda < 1$, $k = 1, 2, \dots, n-1$, then $f_n(z) \equiv 1$.

LEMMA 2.6 [20, Theorem 1.60]. *Let $f_1(z)$, $f_2(z)$, $f_3(z)$ and $f_4(z)$ be meromorphic functions and $f_1(z)$, $f_2(z)$ be nonconstant, satisfying*

$$\sum_{j=1}^{4} f_j(z) \equiv 1.$$

If

$$\sum_{j=1}^{4} N\Big(\frac{1}{f_j}\Big) + 3\sum_{j=1}^{4} N(r, f_j) < (\lambda + o(1))T(r, f_k), \quad k = 1, 2,$$

where $\lambda < 1$, then $f_3 \equiv 1$ or $f_4 \equiv 1$ or $f_3 + f_4 \equiv 1$.

3. Proofs of theorems

PROOF OF THEOREM 1.2. Assume that $f^n + L(f)$ is a periodic function with period c. Then

$$f(z)^n + L(f(z)) = f(z+c)^n + L(f(z+c)),$$

that is

$$f(z)^n - f(z+c)^n = L(f(z+c)) - L(f(z)). \tag{3.1}$$

We discuss two cases for $L(f)$ in the following.

Case 1. If $L(f)$ is a linear differential-difference polynomial, that is all $m_{ij} = 1$, then (3.1) can be rewritten as follows,

$$\big(f(z) - f(z+c)\big)\big(f(z)^{n-1} + f(z)^{n-2}f(z+c) + \cdots + f(z+c)^{n-1}\big) = L(\Delta_c f). \tag{3.2}$$

If $f(z)-f(z+c)\equiv 0$, then $f(z)$ is a periodic function with period c. If $f(z)-f(z+c)\not\equiv 0$, then (3.2) implies that

$$f(z)^{n-1}+f(z)^{n-2}f(z+c)+\cdots+f(z+c)^{n-1}$$
$$=f(z)^{n-1}\Big(1+\frac{f(z+c)}{f(z)}+\cdots+\frac{f(z+c)^{n-1}}{f(z)^{n-1}}\Big)=-\frac{L(\Delta_c f)}{\Delta_c f}.$$

Using Lemma 2.2 and $N\big(r,\frac{1}{f}\big)=S(r,f)$, we obtain

$$T\Big(r,1+\frac{f(z+c)}{f(z)}+\cdots+\frac{f(z+c)^{n-1}}{f(z)^{n-1}}\Big)=S(r,f).$$

By Lemma 2.3 and Lemma 2.4, we obtain

$$(n-1)m(r,f)=S(r,f),$$

which is impossible for $n\geq 2$, since $f(z)$ is a transcendental entire function.

Case 2. If $L(f)$ is a non-linear differential-difference polynomial with $1<\max\{m_{ij}\}<n$, then

$$f(z)^n-f(z+c)^n \tag{3.3}$$
$$=\big(f(z)-f(z+c)\big)\big(f(z)^{n-1}+f(z)^{n-2}f(z+c)+\cdots+f(z+c)^{n-1}\big)$$
$$=\sum_{ij}a_{ij}f^{(j)}(z+c_i+c)^{m_{ij}}-\sum_{ij}a_{ij}f^{(j)}(z+c_i)^{m_{ij}}$$
$$=\sum_{ij}a_{ij}\big(f^{(j)}(z+c_i+c)^{m_{ij}}-f^{(j)}(z+c_i)^{m_{ij}}\big).$$

If $f(z)-f(z+c)\equiv 0$, then $f(z)$ is a periodic function with period c. If $f(z)-f(z+c)\not\equiv 0$, then we see that

$$\frac{f^{(j)}(z+c_i+c)^{m_{ij}}-f^{(j)}(z+c_i)^{m_{ij}}}{f(z+c)-f(z)}=\frac{f^{(j)}(z+c_i+c)-f^{(j)}(z+c_i)}{f(z+c)-f(z)}L_1(f),$$

where

$$L_1(f)=f^{(j)}(z+c_i+c)^{m_{ij}-1}$$
$$+f^{(j)}(z+c_i+c)^{m_{ij}-2}f^{(j)}(z+c_i)+\cdots+f^{(j)}(z+c_i)^{m_{ij}-1}$$
$$=\frac{f^{(j)}(z+c_i+c)^{m_{ij}-1}}{f(z)^{m_{ij}-1}}f(z)^{m_{ij}-1}+\cdots+\frac{f^{(j)}(z+c_i)^{m_{ij}-1}}{f(z)^{m_{ij}-1}}f(z)^{m_{ij}-1}.$$

From the lemma of the logarithmic derivative, Lemma 2.2, Lemma 2.4 and the condition $N\big(r,\frac{1}{f}\big)=S(r,f)$, we see $m(r,L_1(f))=(m_{ij}-1)m(r,f)+S(r,f)$. We also obtain

$$m\big(r,f(z)^{n-1}+f(z)^{n-2}f(z+c)+\cdots+f(z+c)^{n-1}\big)$$
$$=m\Big(r,f(z)^{n-1}\Big(1+\frac{f(z+c)}{f(z)}+\cdots+\frac{f(z+c)^{n-1}}{f(z)^{n-1}}\Big)\Big).$$

Thus, from (3.3), we get

$$(n-1)m(r,f)=(m_{ij}-1)m(r,f)+S(r,f),$$

and so

$$(n-m_{ij})m(r,f)=S(r,f),$$

which is impossible for transcendental entire function f. □

Proof of Theorem 1.3. We only need to consider the case that $f(z)$ is of $\rho_2(f)\geq 1$, the case $\rho_2(f)<1$ is proved in Theorem 1.2. From $f(z)=p(z)+e^{h(z)}$, where $h(z)$ is a transcendental entire function and $f(z)^2+f(z+c)$ is a periodic function with period c, we obtain

$$\begin{aligned}(3.4)\qquad & e^{2h(z)}+2p(z)e^{h(z)}+(1-2p(z+c))e^{h(z+c)}-e^{2h(z+c)}-e^{h(z+2c)}\\ &=p(z+c)^2+p(z+2c)-p(z)^2-p(z+c).\end{aligned}$$

If $p(z)$ is not a constant, then $p(z+c)^2+p(z+2c)-p(z)^2-p(z+c)\not\equiv 0$, since that $p(z)^2+p(z+c)$ remains a nonconstant polynomial, which is not a periodic function. By Lemma 2.5, we get that one of the exponential functions in (3.4) is a rational function, which is impossible, since $f(z)$ is a transcendental entire function.

If $p(z)$ is a constant p, then $p(z+c)^2+p(z+2c)-p(z)^2-p(z+c)\equiv 0$. We rewrite (3.4) as

$$(3.5)\qquad e^{2h(z)}+2pe^{h(z)}+(1-2p)e^{h(z+c)}-e^{2h(z+c)}-e^{h(z+2c)}=0.$$

If $p\neq 0$, we have

(3.6)
$$-\frac{1}{2p}e^{h(z)}-\frac{(1-2p)}{2p}e^{h(z+c)-h(z)}+\frac{1}{2p}e^{2h(z+c)-h(z)}+\frac{1}{2p}e^{h(z+2c)-h(z)}=1.$$

Obviously, $-\frac{1}{2p}e^{h(z)}$ is not a constant, otherwise $f(z)$ is a constant. If $p\neq\frac{1}{2}$ and we assume that $-\frac{(1-2p)}{2p}e^{h(z+c)-h(z)}=A$, where A is a non-zero constant.

Substitute the above into (3.6), we have

$$\Big(\frac{1-2p}{4p^2A}+\frac{A}{2p-1}\Big)e^{h(z+c)}=1-A-\frac{2pA^2}{(1-2p)^2}.$$

For avoiding a contradiction, we should have

$$\frac{1-2p}{4p^2A}+\frac{A}{2p-1}=0,\quad 1-A-\frac{2pA^2}{(1-2p)^2}=0.$$

Therefore, $\frac{1-2p}{2Ap}=-1$ follows by the above system, so we have $e^{h(z)}=e^{h(z+c)}$, thus $f(z)$ is a periodic function with period c.

We proceed to consider $-\frac{(1-2p)}{2p}e^{h(z+c)-h(z)}$ is not a constant. From Lemma 2.6, we will discuss three cases as follows:

Case 1. If $\frac{1}{2p}e^{2h(z+c)-h(z)}=1$, then (3.6) implies that

$$-(1-2p)e^{h(z+c)-2h(z)}+e^{h(z+2c)-2h(z)}=1. \tag{3.7}$$

For avoiding a contradiction to the second main theorem of Nevanlinna, $e^{h(z+c)-2h(z)}$ and $e^{h(z+2c)-2h(z)}$ are non-zero constants, so $e^{h(z+2c)-h(z+c)}$ is also a non-zero constant, so $e^{h(z+c)-h(z)}$ is. Thus, we have $e^{h(z+c)}$ is a constant with the assumption in this case, a contradiction to f is transcendental.

Case 2. If $\frac{1}{2p}e^{h(z+2c)-h(z)}=1$, then

$$-(1-2p)e^{h(z+c)-2h(z)}+e^{2h(z+c)-2h(z)}=1. \tag{3.8}$$

For avoiding a contradiction to the second main theorem of Nevanlinna, we can get $e^{h(z+c)-2h(z)}$ and $e^{2h(z+c)-2h(z)}$ are non-zero constants, then $e^{h(z+c)}$ is a constant, which is a contradiction again to $f(z)$ is transcendental.

Case 3. If $\frac{1}{2p}e^{2h(z+c)-h(z)}+\frac{1}{2p}e^{h(z+2c)-h(z)}=1$, then $e^{2h(z+c)-h(z)}$ and $e^{h(z+2c)-h(z)}$ are non-zero constants. From (3.6), we obtain

$$-\frac{1}{2p}e^{h(z)}-\frac{(1-2p)}{2p}e^{h(z+c)-h(z)}=0.$$

We also have

$$(2p-1)e^{h(z+c)-2h(z)}=(2p-1)e^{h(z+2c)-2h(z+c)}=1,$$

hence, we have $e^{h(z+2c)-4h(z)}=\frac{1}{(2p-1)^3}$. Combining $e^{h(z+2c)-h(z)}$ is a constant, we get $e^{h(z)}$ is a constant, a contradiction again.

If $p=\frac{1}{2}$, from (3.6), we have

$$-e^{h(z)}+e^{2h(z+c)-h(z)}+e^{h(z+2c)-h(z)}=1. \tag{3.9}$$

Obviously, $-e^{h(z)} \not\equiv 1$. By Lemma 2.5,

$$e^{2h(z+c)-h(z)} = 1, \quad -e^{h(z)} + e^{h(z+2c)-h(z)} = 0, \tag{3.10}$$

or

$$e^{h(z+2c)-h(z)} = 1, \quad -e^{h(z)} + e^{2h(z+c)-h(z)} = 0. \tag{3.11}$$

The system (3.10) give a contradiction. In fact, rewrite the system (3.10) as follows:

$$e^{2h(z+c)} = e^{h(z)}, \quad e^{h(z+2c)} = e^{2h(z)}, \tag{3.12}$$

Then, we have

$$e^{8h(z)} = e^{4h(z+2c)} = e^{2h(z+c)} = e^{h(z)},$$

which is a contradiction with $f(z)$ being transcendental. The second equation of (3.11) implies that $e^{2h(z+c)} = e^{2h(z)}$, then $f(z)$ is a periodic function with period $2c$ by the system (3.11).

The final case $p = 0$. From (3.5), we have

$$e^{2h(z)} + e^{h(z+c)} - e^{2h(z+c)} - e^{h(z+2c)} = 0, \tag{3.13}$$

that is

$$e^{2h(z)-2h(z+c)} + e^{-h(z+c)} - e^{h(z+2c)-2h(z+c)} = 1. \tag{3.14}$$

Obviously, $e^{-h(z+c)} \not\equiv 1$. By Lemma 2.5,

$$e^{2h(z)-2h(z+c)} = 1, \quad e^{-h(z+c)} - e^{h(z+2c)-2h(z+c)} = 0, \tag{3.15}$$

or

$$-e^{h(z+2c)-2h(z+c)} = 1, \quad e^{2h(z)-2h(z+c)} + e^{-h(z+c)} = 0. \tag{3.16}$$

The equations of (3.15) give that $f(z)$ is a periodic function with period c. The system (3.16) implies that $e^{2h(z)} = -e^{h(z+c)}$, that is $f(z)^2 + f(z+c) = 0$, which is a contradiction to the condition of Theorem 1.3. □

Proof of Theorem 1.4. If $f(z)^n + f(z+c)^n + \cdots + f(z+kc)^n$ is a periodic function with period η, then

$$f(z)^n + f(z+c)^n + \cdots + f(z+kc)^n$$
$$= f(z+\eta)^n + f(z+c+\eta)^n + \cdots + f(z+kc+\eta)^n.$$

Since $f(z)$ is a transcendental meromorphic function, we have

$$\tag{3.17} \Big(\frac{f(z+\eta)}{f(z)}\Big)^n + \Big(\frac{f(z+c+\eta)}{f(z)}\Big)^n + \dots + \Big(\frac{f(z+kc+\eta)}{f(z)}\Big)^n + \Big(\sqrt[n]{-1}\,\frac{f(z+c)}{f(z)}\Big)^n + \dots + \Big(\sqrt[n]{-1}\,\frac{f(z+kc)}{f(z)}\Big)^n = 1.$$

Using the classical result for Fermat functional equations [4], we know that $n \geq (2k+1)^2$, we obtain $\frac{f(z+\eta)}{f(z)}$, $\frac{f(z+c+\eta)}{f(z)}$, …, $\frac{f(z+kc+\eta)}{f(z)}$ and $\frac{f(z+c)}{f(z)}$, …, $\frac{f(z+kc)}{f(z)}$ are all non-zero constants. Set $\frac{f(z+c)}{f(z)} = A$, $\frac{f(z+\eta)}{f(z)} = B$, then

$$\frac{f(z+c+\eta)}{f(z)} = \frac{f(z+c)}{f(z)}\frac{f(z+c+\eta)}{f(z+c)} = AB,$$

and

$$\frac{f(z+kc+\eta)}{f(z)} = \frac{f(z+c)}{f(z)}\frac{f(z+2c)}{f(z+c)} \cdots \frac{f(z+kc+\eta)}{f(z+(k-1)c)} = A^k B.$$

Hence, (3.17) can be written as

$$B^n(1 + A^n + A^{2n} + \dots + A^{kn}) = 1 + A^n + A^{2n} + \dots + A^{kn}.$$

If $B^n = 1$, which means that $f(z)^n = f(z+\eta)^n$. Hence, $f(z)$ is a periodic function with period $n\eta$.

If $B^n \neq 1$, then $1 + A^n + A^{2n} + \dots + A^{kn} = 0$, which implies that $A^{(k+1)n} = 1$, thus we have $f(z)$ is a periodic function with period $(k+1)n$. □

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References

[1] X. J. Dong and L. W. Liao, Meromorphic solutions of certain type of differential-difference equations, *J. Math. Anal. Appl.*, **470** (2019), 327–339.
[2] R. G. Halburd, R. J. Korhonen and K. Tohge, Holomorphic curves with shift-invariant hyperplane preimages, *Trans. Amer. Math. Soc.*, **366** (2014), 4267–4298.
[3] W. K. Hayman, *Meromorphic Functions*, Clarendon Press (Oxford, 1964).
[4] W. K. Hayman, Waring's problem für analytische Funktionen, *Bayer. Akad. Wiss. Math. Natur. Kl. Sitzungsber.*, **1984** (1985), 1–13.
[5] J. Heittokangas, R. J. Korhonen and I. Laine, On meromorphic solutions of certain nonlinear differential equations, *Bull. Aust. Math. Soc.*, **66** (2002), 331–343.
[6] J. Heittokangas, Z. Latreuch, J. Wang and M. A. Zemirni, On meromorphic solutions of non-linear differential equations of Tumura–Clunie type, *Math. Nachr.*, **294** (2021), 748–773.

[7] Z. Latreuch and M. A. Zemirni, On a theorem of A. and C. Rényi and a conjecture of C. C. Yang concerning periodicity of entire function, *Anal. Math.*, **48** (2022), xx–yy.

[8] P. Li, W. R. Lü and C. C. Yang, Entire solutions of certain types of nonlinear differential equations, *Houston J. Math.*, **45** (2019), 431–437.

[9] X. L. Liu and R. Korhonen, The periodicity of transcendental entire functions, *Bull. Aust. Math. Soc.*, **101** (2020), 453–465.

[10] X. L. Liu, R. Korhonen and K. Liu, Variations on a Conjecture of C. C. Yang concerning periodicity, *Comput. Methods Funct. Theory*, doi.org/10.1007/s40315-020-00359-0.

[11] K. Liu, Y. M. Wei and P. Y. Yu, Generalized Yang's Conjecture on the periodicity of entire functions, *Bull. Korean Math. Soc.*, **57** (2020), 1259–1267.

[12] K. Liu and P. Y. Yu, A note on the periodicity of entire functions, *Bull. Aust. Math. Soc.*, **100** (2019), 290–296.

[13] W. R. Lü, L. L. Wu, D. D. Wang and C. C. Yang, The existence of solutions to certain type of non-linear difference-differential equations, *Open Math.*, **16** (2018), 806–815.

[14] W. R. Lü and X. X. Zhang, On the periodicity of entire functions, *Results Math.*, **75** (2020), Paper No. 176, 13 pp.

[15] A. Rényi and C. Rényi, Some remarks on periodic entire functions, *J. Analyse Math.*, **14** (1965), 303–310.

[16] Q. Wang and P. C. Hu, On zeros and periodicity of entire functions, *Acta. Math. Sci. Ser. A (Chin. Ed.)*, **38** (2018), 209–214.

[17] C. C. Yang, On entire solutions of a certain type of nonlinear differential equations, *Bull. Aust. Math. Soc.*, **64** (2001), 377–380.

[18] C. C. Yang and I. Laine, On analogies between nonlinear difference and differential equations, *Proc. Japan Acad. Ser. A*, **86** (2010), 10–14.

[19] C. C. Yang and P. Li, On the transcendental solutions of a certain type of nonlinear differential equations, *Arch. Math.*, **82** (2004), 442–448.

[20] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers (2003).