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# THE PERIODICITY ON A TRANSCENDENTAL ENTIRE FUNCTION WITH ITS DIFFERENTIAL POLYNOMIALS

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**Abstract.** We discuss the relationship on the periodicity of a transcendental entire function with its differential polynomials. For example, we obtain that if $f$ is a transcendental entire function, $k$ is a non-negative integer and if $(a_n f^n + \dots + a_1 f)^{(k)}$ is a periodic function, then $f$ is also a periodic function, where $a_1, \dots, a_n$ $(\neq 0)$ are constants. Our results are related to Yang's Conjecture on the periodicity of transcendental entire functions.

## 1. Introduction and main results

Let us start by recalling a basic fact on the periodicity between $f(z)$ and $f^{(k)}(z)$. Obviously, if $f(z)$ is a periodic function with period $c$, then $f^{(k)}(z)$ is a periodic function with the same period $c$ as well. However, the converse is not true, see $f(z) = e^{e^z} + z$ or $f(z) = \sin z + z$. In fact, if $f^{(k)}(z)$ is a periodic function with the same period $c$, then $f(z) = \varphi(z) + p(z)$, where $\varphi(z)$ is a periodic function with period $c$ and $p(z)$ is a polynomial with degree at most $k$. Considering the square function of $f(z)$, Wang and Hu [15, Theorem 1] obtained the following result.

Theorem A. *Let $f$ be a transcendental entire function and $k$ be a positive integer. If $(f^2)^{(k)}$ is a periodic function, then $f$ is also a periodic function.*

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In fact, Theorem A shows that Yang's Conjecture below, given in [9] and [15], is true for the case $k = 1$.

Yang's Conjecture. *Let $f$ be a transcendental entire function and $k$ be a positive integer. If $ff^{(k)}$ is a periodic function, then $f$ is also a periodic function.*

Recently, Liu, Korhonen and Liu [13, Theorem 1] also considered Yang's Conjecture for transcendental meromorphic functions in the case $k = 1$. Related to Yang's Conjecture and its variants, several authors have considered the periodicity on the transcendental entire functions with their differential or difference or differential-difference polynomials, see, e.g., [10–12,15]. We assume that the reader is familiar with basic results of Nevanlinna theory [3,16].

Recall that $f^2$ in Theorem A can be replaced by $f^n$, see Liu and Yu [10, Corollary 1.3] and $f^2$ can be replaced by $a_2f^2 + a_1f$, see [10, Theorem 1.5], where $a_1$ and $a_2$ are constants and $a_2 \neq 0$. Using the difference analogue of the logarithmic derivative lemma [2, Theorem 5.1], Liu and Yu [10, Theorem 1.7] also considered the problem that $f^2$ is replaced by any polynomial $a_nf^n + \cdots + a_1f$, where $a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. This result can be stated as follows.

Theorem B [10, Theorem 1.7]. *Let $f$ be a transcendental entire function of hyper-order $\rho_2(f) < 1$ and $N(r, \frac{1}{f}) = S(r, f)$, let $k$ be a positive integer and $n \geq 2$. If $(a_nf^n + \cdots + a_1f)^{(k)}$ is a periodic function, then $f$ is also a periodic function, where $a_n \neq 0$.*

Rényi and Rényi [14, Theorem 2] obtained the following result on the periodicity of $P(f)$, where $P(z)$ is a non-constant polynomial.

Theorem C. *If $P(z)$ is a non-constant polynomial and $f(z)$ is an entire function which is not periodic, then $P(f(z))$ cannot be periodic either.*

Theorem C implies that $f(z)$ must be a periodic function if $f(z)$ satisfies the functional equation

$$a_nf(z)^n + \cdots + a_1f(z) = a_nf(z + c)^n + \cdots + a_1f(z + c).$$

Thus, the conditions $\rho_2(f) < 1$ and $N(r, \frac{1}{f}) = S(r, f)$ can be removed in Theorem B in the case $k = 0$. We proceed to give our first result to consider the case $k \geq 1$.

Theorem 1.1. *Let $f$ be a transcendental entire function and $k$, $n$ be positive integers and let $a_1, \ldots, a_n$ $(\neq 0)$ be constants. If $(a_nf^n + \cdots + a_1f)^{(k)}$ is a periodic function and $n \geq 2$, then $f$ is also a periodic function.*

Li and Yang [7] introduced the following definition:

Definition 1.2. A non-constant polynomial $P(z)$ is called a uniqueness polynomial for entire functions (UPE), if $f = g$ whenever $P(f) = P(g)$ for any two non-constant entire functions $f$ and $g$.

Similarly as the above definition, we also introduce the definition as follows:

Definition 1.3. A non-constant polynomial $P(z)$ is called a uniqueness polynomial for periodicity of entire functions (UPPE) if $f$ is a periodic function with period $c$ or $mc$ whenever $[P(f(z))]^{(k)} = [P(f(z+c))]^{(k)}$, where $k$ is any non-negative integer and $m$ is a non-zero integer.

Remark 1.4. (1) Obviously, UPE must be UPPE. However, UPPE may not UPE, for example, $z^2 \circ (f(z)) = z^2 \circ (f(z+c))$ implies that $f(z)=f(z+c)$ or $f(z) = -f(z+c)$ $(f(z) = f(z+2c))$, hence $z^2$ is UPPE but not UPE. Theorem 1.1 and Theorem C imply that any non-linear polynomial is UPPE.

(2) Theorem 1.1 is also true for transcendental meromorphic functions if $n \geq 4$ by the proof of Theorem 1.1 below.

(3) Theorem 1.1 is not valid if the polynomial $P(z) := a_n z^n + \cdots + a_1 z$ is replaced by any transcendental entire function. For example, take $P(z) = e^z - z$ and $k \geq 1$, then

$$[e^{f(z)} - f(z)]^{(k)} = [e^{f(z+c)} - f(z+c)]^{(k)},$$

admits an entire solution $f(z) = e^z + z$ and $c = 2ki\pi$, here $f(z)$ is not a periodic function.

The following corollary is obtained immediately by considering the polynomial $\frac{1}{n+m+1} f^{n+m+1} - \frac{1}{n+1} f^{n+1}$ in Theorem 1.1.

Corollary 1.5. *Assume that $f$ is a transcendental entire function. If $f^n(f^m - 1)f'$ is a periodic function, then $f$ is also a periodic function.*

Based on Yang's Conjecture and Corollary 1.5, we also can consider the periodicity of $f^n(f^m - 1)f^{(k)}$.

Theorem 1.6. *Let $f$ be a transcendental entire function and $n$, $m$ and $k$ be positive integers. If $f^n(f^m - 1)f^{(k)}$ is a periodic function with period $c$, and one of following conditions is satisfied:*

(i) *$f(z) = e^{h(z)}$, where $h(z)$ is an entire function;*

(ii) *$f(z)$ has a non-zero Picard exceptional value and $f(z)$ is of finite order;*

(iii) *$f^n(f^m - 1)f^{(k+1)}$ is a periodic function with period $c$;*

*then $f(z)$ is also a periodic function.*

We proceed to present the following problem inspired by Yang's Conjecture and Theorem 1.6. Very recently, Liu, Korhonen and Liu [13] also considered the periodicity of $\frac{f^{(k)}}{f^n}$.

Question 1. *Let $f$ be a transcendental entire function and $k$ be a positive integer. If $\frac{f^{(k)}}{f^n(f^m-1)}$ $(m,n\geq 1)$ is a periodic function, is it true that $f$ is also a periodic function?*

Theorem 1.7. *Let $f$ be a transcendental entire function. If $\frac{f'}{f(f-1)}$ is a periodic function, then $f$ is also a periodic function.*

Remark 1.8. Assume we replace $f(f-1)$ by $f^n(f^m-1)$ in Theorem 1.7, where $m,n\geq 1$ and at least one of them is more than 1. We may attempt to factor $\frac{f'}{f^n(f^m-1)}$ as in the basic method of the proof of Theorem 1.7 below, but it is difficult to find the relationship of $f(z)$ and $f(z+c)$ for any positive integers $m,n$.

Theorem 1.9. *Let $f$ be a transcendental entire function and $n$, $m$, $k$ be positive integers. If $\frac{f^{(k)}}{f^n(f^m-1)}$ and $\frac{f^{(k+1)}}{f^n(f^m-1)}$ are periodic functions, then $f$ is also a periodic function.*

## 2. Lemmas

We recall a definition [1, Definition 3.4] that is to call an entire function $F(z)$ periodic mod $g$ with period $c$, if and only if $F(z+c)-F(z)=g$.

Lemma 2.1 [1, Theorem 3.4]. *If $P(z)$ is any non-linear polynomial and $g(z)$ is an arbitrary transcendental entire function, then $P(g)$ is not periodic mod a non-constant polynomial.*

To prove Theorem 1.1, we also need the following result.

Lemma 2.2 [8, Theorem 1]. *Let $P(z)$ and $Q(z)$ be two polynomials of degree $p$, $q$, respectively. Then there does not exist non-constant meromorphic functions $f$ and $g$ satisfying*

$$P(f)=Q(g)$$

*if $P(z)$ and $Q(z)$ satisfy one of the following conditions:*

(i) *$p\leq q$ and there exist a zero $r_1$ of $P'(z)$ and a multiple zero $r_2$ of $P'(z)$ such that both $Q(z)-P(r_j)=0$ $(j=1,2)$ have no multiple roots.*

(ii) *$p\leq q$ and there exist three zeros $r_j (j=1,2,3)$ of $P'(z)$ such that all $Q(z)-P(r_j)=0$ $(j=1,2,3)$ have no multiple roots.*

(iii) *$p\leq q$ and there exist a zero $r_0$ of $P'(z)$ with multiplicity $k\geq 3$ such that $Q(z)-P(r_0)=0$ has no multiple roots.*

Lemma 2.3 [16, Theorem 1.62]. *Suppose that $f_j$ $j=1,2,\ldots,n$ $(n\geq 3)$ are meromorphic functions which are not constants except for $f_n$. If*

$$\sum_{j=1}^{n} f_j = 1$$

*and*

$$\sum_{j=1}^{n} N\Big(r,\frac{1}{f_j}\Big)+(n-1)\sum_{j=1}^{n}\overline{N}(r,f_j)<(\lambda+o(1))T(r,f_k)\quad(r\in I),$$

*where* $\lambda<1$ *and* $I$ *is a set whose linear measure is infinite,* $k\in\{1,2,\dots,n-1\}$, *then* $f_n\equiv 1$.

Lemma 2.4 [16, Theorem 1.56]. *Suppose that* $f_j (j=1,2,3)$ *are meromorphic functions and* $f_1$ *is not a constant. If*

$$\sum_{j=1}^{3} f_j=1$$

*and*

$$\sum_{j=1}^{3} N\Big(r,\frac{1}{f_j}\Big)+2\sum_{j=1}^{n}\overline{N}(r,f_j)<(\lambda+o(1))T(r)\quad(r\in I),$$

*where* $\lambda<1$, $I$ *is a set whose linear measure is infinite and*

$$T(r)=\max_{1\le j\le 3}\{T(r,f_j)\},$$

*then* $f_2\equiv 1$ *or* $f_3\equiv 1$.

## 3. Proofs of Theorems

Proof of Theorem 1.1. We first consider the case $n\ge 4$. Suppose that $(a_nf^n+\cdots+a_1f)^{(k)}$ is a periodic function with period $c$ and denote $P(z)=a_nz^n+\cdots+a_1z$ with $\deg(P(z))=n\ge 4$. By Lemma 2.1, we have

$$P(f(z))=P(f(z+c))-d \tag{3.1}$$

for some $d\in\mathbb{C}$. If $d=0$, using Theorem C, then $f(z)$ must be a periodic function. Suppose now that $d\neq 0$. Then (3.1) yields

$$P(f(z))=P(f(z+kc))-kd \tag{3.2}$$

for all $k\in\mathbb{N}$. Let $r_l$ ($l\in\{1,2,3\}$) be a zero of $P'(z)$, and assume that there are infinitely many $k\in\mathbb{N}$ such that the polynomial $P(z)-kd-P(r_l)$ has multiple roots. For each such $k$, let $\beta_k$ be any one of the multiple roots of $P(z)-kd-P(r_l)$ (there may in principle be more than one). We affirm that the roots $\beta_k$ are distinct. Towards this end suppose on the contrary that $\beta_i=\beta_j$ for $i\neq j$, $i,j\in\mathbb{N}$. Then

$$P(\beta_i)-id-P(r_l)=P(\beta_j)-jd-P(r_l)=0,$$

hence $(j-i)d=0$, which is a contradiction. Thus $\beta_i \neq \beta_j$ for all $i \neq j$. But now $P'(\beta_k)=0$ for infinitely many distinct $\beta_k$ which implies that $P(z)$ is a constant, a contradiction. Thus the polynomial $P(z)-kd-P(r_l)$ can have multiple roots for only finitely many $k \in \mathbb{N}$ for all choices of $l \in \{1,2,3\}$. Hence, there exists $k_0 \in \mathbb{N}$ such that $P(z)-k_0 d-P(r_l)$ has only simple roots for all $l \in \{1,2,3\}$. By letting $Q(z)=P(z)-k_0 d$, it follows by (3.2) and Lemma 2.2 that $f$ is a constant, which is a contradiction.

We proceed to proving the case $n \leq 3$. Suppose that $(a_3 f^3+a_2 f^2+a_1 f)^{(k)}$ is a periodic function with period $c$. Then

$$a_3 f(z)^3+a_2 f(z)^2+a_1 f(z)=a_3 f(z+c)^3+a_2 f(z+c)^2+a_1 f(z+c)+p(z),$$

where $p(z)$ is a polynomial with degree less than $k$. Applying Lemma 2.1, we have $p(z)$ is a constant. Then, we obtain

$$\begin{aligned}(3.3)\qquad &\big(f(z)-f(z+c)\big)\big(a_3\big(f(z)^2+f(z)f(z+c)+f(z+c)^2\big)\\ &\qquad +a_2\big(f(z)+f(z+c)\big)+a_1\big)=p.\end{aligned}$$

*Case* 1: $p=0$ in (3.3). In fact, Theorem C implies that $f(z)$ is a periodic function in this case. Another basic method will be given to show the possible periods when $p=0$. From (3.3), we have

$$(3.4)\qquad f(z)-f(z+c)=0$$

or

$$(3.5)\qquad a_3\big(f(z)^2+f(z)f(z+c)+f(z+c)^2\big)+a_2\big(f(z)+f(z+c)\big)+a_1=0.$$

The equation (3.4) implies that $f(z)$ is a periodic function with period $c$. Shifting forward the equation (3.5), we have

$$\begin{aligned}(3.6)\qquad &a_3\big(f(z+c)^2+f(z+c)f(z+2c)+f(z+2c)^2\big)\\ &\qquad +a_2\big(f(z+c)+f(z+2c)\big)+a_1=0.\end{aligned}$$

Combining (3.5) with (3.6), we obtain

$$(3.7)\qquad \big(f(z)-f(z+2c)\big)\big(a_3\big(f(z)+f(z+c)+f(z+2c)\big)+a_2\big)=0.$$

Thus, we have

$$(3.8)\qquad f(z)-f(z+2c)=0$$

or

$$(3.9)\qquad a_3\big(f(z)+f(z+c)+f(z+2c)\big)+a_2=0.$$

The equation (3.8) implies that $f(z)$ is a periodic function with period $2c$ and the equation (3.9) implies that $f(z)$ is a periodic function with period $3c$.

*Case* 2: $p \neq 0$ in (3.3). Since $f(z)$ is an entire function, then the equation (3.3) implies that both $f(z)-f(z+c)$ and

$$a_3\big(f(z)^2+f(z)f(z+c)+f(z+c)^2\big)+a_2\big(f(z)+f(z+c)\big)+a_1$$

have no zeros. Using Hadamard factorization theorem, we have

$$\begin{cases} f(z)-f(z+c)=c_1e^{P(z)}, \\ a_3\big(f(z)^2+f(z)f(z+c)+f(z+c)^2\big) \\ \quad +a_2\big(f(z)+f(z+c)\big)+a_1=c_2e^{-P(z)}. \end{cases} \tag{3.10}$$

where $c_1$, $c_2$ are non-zero constants, $c_1c_2=p$ and $P(z)$ is an entire function. By shifting forward the equation (3.10), we obtain

$$\begin{cases} f(z+c)-f(z+2c)=c_1e^{P(z+c)}, \\ a_3\big(f(z+c)^2+f(z+c)f(z+2c)+f(z+2c)^2\big) \\ \quad +a_2\big(f(z+c)+f(z+2c)\big)+a_1=c_2e^{-P(z+c)}. \end{cases} \tag{3.11}$$

By adding the first equation of (3.10) to the first equation of (3.11), we obtain

$$f(z)-f(z+2c)=c_1e^{P(z)}+c_1e^{P(z+c)}. \tag{3.12}$$

By subtracting the second equation of (3.11) from the second equation of (3.10), we obtain

$$\begin{aligned} \big(f(z)-f(z+2c)\big)\big(a_3\big(f(z)+f(z+c)+f(z+2c)\big)+a_2\big) \\ =c_2e^{-P(z)}-c_2e^{-P(z+c)}. \end{aligned} \tag{3.13}$$

A basic computation from (3.10), (3.12) and (3.13) gives

$$\begin{cases} f(z)=\dfrac{c_2}{3a_3c_1}e^{-P(z)-P(z+c)}G(z)+\dfrac{2}{3}c_1e^{P(z)}+\dfrac{1}{3}c_1e^{P(z+c)}-\dfrac{a_2}{3a_3}, \\ f(z+c)=\dfrac{c_2}{3a_3c_1}e^{-P(z)-P(z+c)}G(z)-\dfrac{1}{3}c_1e^{P(z)}+\dfrac{1}{3}c_1e^{P(z+c)}-\dfrac{a_2}{3a_3}, \end{cases} \tag{3.14}$$

where $G(z)=\frac{e^{P(z+c)-P(z)}-1}{e^{P(z+c)-P(z)}+1}$.

Since $f(z)$ is an entire function, the expression of $f(z)$ in (3.14) implies that $G(z)$ must be a transcendental entire function or a constant.

Obviously, $G(z)$ is not a non-constant entire function for the reason that $e^{P(z+c)-P(z)}-1$ and $e^{P(z+c)-P(z)}+1$ must have infinitely many zeros unless $P(z+c)-P(z)$ is a constant, in which case $G(z)$ reduces to a constant. Let $G(z)=D$ and $P(z+c)-P(z)=d$ where $D,d$ are constants. If $D=0$, then $e^{P(z+c)-P(z)}-1=e^d-1=0$, which means that $d=2s\pi i$ $(s=0,\pm1,\pm2,\dots)$. Then

$$\begin{aligned} f(z+c) &= \frac{2}{3}c_1e^{P(z+c)}+\frac{1}{3}c_1e^{P(z+2c)}-\frac{a_2}{3a_3} \\ &= \frac{2}{3}c_1e^{P(z)+2s\pi i}+\frac{1}{3}c_1e^{P(z+c)+2s\pi i}-\frac{a_2}{3a_3} \\ &= \frac{2}{3}c_1e^{P(z)}+\frac{1}{3}c_1e^{P(z+c)}-\frac{a_2}{3a_3}=-\frac{1}{3}c_1e^{P(z)}+\frac{1}{3}c_1e^{P(z+c)}-\frac{a_2}{3a_3}, \end{aligned}$$

which means that $c_1e^{P(z)}=0$, which is impossible. If $D\neq 0$, then $d\neq 2k\pi i$, and substituting $P(z+c)=P(z)+d$ into the formulas of $f(z)$ and $f(z+c)$, we have

$$f(z)=\frac{Dc_2}{3a_3c_1}e^{-2P(z)-d}+\frac{2}{3}c_1e^{P(z)}+\frac{1}{3}c_1e^{P(z)+d}-\frac{a_2}{3a_3}, \tag{3.15}$$

$$f(z+c)=\frac{Dc_2}{3a_3c_1}e^{-2P(z+c)+d}-\frac{1}{3}c_1e^{P(z+c)-d}+\frac{1}{3}c_1e^{P(z+c)}-\frac{a_2}{3a_3}. \tag{3.16}$$

Shifting forward (3.15) and combining with (3.16) we have

$$Ce^{-3P(z+c)}=B,$$

where $C=\frac{Dc_2}{a_3c_1^2}(e^d-e^{-d})$ and $B=1+e^d+e^{-d}$. Obviously, $e^{-3P(z+c)}$ is not a constant, otherwise $f(z)$ is a constant. If $e^{-3P(z+c)}$ is not a constant, then $B=C=0$, by a basic computation from the expressions of $B$ and $C$, we can get a contradiction. Thus, we have the proof of Theorem 1.1. □

Proof of Theorem 1.6. (i) Assume that $f(z)=e^{h(z)}$, where $h(z)$ is an entire function. Obviously, $T(r,h(z))=S(r,e^{h(z)})$. If $f^n(f^m-1)f^{(k)}$ is a periodic function with period $c$, then

$$f(z)^n(f(z)^m-1)f^{(k)}(z)=f(z+c)^n(f(z+c)^m-1)f^{(k)}(z+c). \tag{3.17}$$

Substituting $f(z)=e^{h(z)}$ into (3.17), we obtain

$$\begin{aligned} &H(z,h(z))e^{(n+m+1)h(z)}-H(z,h(z))e^{(n+1)h(z)} \\ &= H(z,h(z+c))e^{(n+m+1)h(z+c)}-H(z,h(z+c))e^{(n+1)h(z+c)}, \end{aligned}$$

where $H(z,h(z))$ is a differential polynomial of $h(z)$. The above equation also implies that

$$T(r,e^{h(z+c)}) = (1+o(1))T(r,e^{h(z)}). \tag{3.18}$$

Thus

$$-\frac{H(z,h(z))}{H(z,h(z+c))}e^{(n+m+1)h(z)-(n+1)h(z+c)} + \frac{H(z,h(z))}{H(z,h(z+c))}e^{(n+1)(h(z)-h(z+c))} + e^{mh(z+c)} = 1.$$

Clearly, $e^{mh(z+c)}$ is not a constant. From (3.18) and Lemma 2.4, we have two cases.

*Case* 1. The first possibility is

$$\begin{cases} -\dfrac{H(z,h(z))}{H(z,h(z+c))}e^{(n+m+1)h(z)-(n+1)h(z+c)} = 1, \\ \dfrac{H(z,h(z))}{H(z,h(z+c))}e^{(n+1)(h(z)-h(z+c))} + e^{mh(z+c)} = 0. \end{cases}$$

The above system implies that $e^{m(h(z)+h(z+c))} = 1$. Therefore, $h(z+c) = A - h(z)$, where $A$ is a constant. The first equation of the above system means that

$$\frac{H(z,h(z))}{H(z,h(z+c))}e^{(2n+m+2)h(z)-(n+1)A} = -1,$$

so

$$T(r,e^{h(z)}) = S(r,e^{h(z)}),$$

which is impossible.

*Case* 2. The second possibility is

$$\begin{cases} \dfrac{H(z,h(z))}{H(z,h(z+c))}e^{(n+1)(h(z)-h(z+c))} = 1, \\ -\dfrac{H(z,h(z))}{H(z,h(z+c))}e^{(n+m+1)h(z)-(n+1)h(z+c)} + e^{mh(z+c)} = 0. \end{cases}$$

This gives $e^{mh(z)} = e^{mh(z+c)}$. Thus, $f(z)^m = f(z+c)^m$, therefore, $f(z) = tf(z+c)$ and $t^m = 1$. It means that $f(z)$ is a periodic function with period $mc$.

(ii) Assume that $d$ is a non-zero Picard exceptional value of $f(z)$, then $f(z) = e^{h(z)} + d$ follows by the Hadamard factorization theorem, where $h(z)$

is a non-constant polynomial. We also assume that $f^n(f^m-1)f^{(k)}$ is a periodic function with period $c$. A basic computation from (3.17) implies that

$$H_1(z,h(z))(e^{h(z)}+d)^n((e^{h(z)}+d)^m-1)e^{h(z)} = H_1(z,h(z+c))(e^{h(z+c)}+d)^n((e^{h(z+c)}+d)^m-1)e^{h(z+c)}, \tag{3.19}$$

where $H_1(z,h(z))$ is a differential polynomial of $h(z)$. Then

$$e^{(n+m+1)h(z)}+b_{n+m}e^{(n+m)h(z)}+\cdots+b_2e^{2h(z)}+b_1e^{h(z)} = P_1(z)\big[e^{(n+m+1)h(z+c)}+b_{n+m}e^{(n+m)h(z+c)}+\cdots+b_2e^{2h(z+c)}+b_1e^{h(z+c)}\big],$$

where $P_1(z)=\frac{H_1(z,h(z+c))}{H_1(z,h(z))}$, $b_i$ $(i=1,2,\dots,n+m)$ are constants and

$$T(r,P_1(z))=S(r,f).$$

We also have

$$\frac{P_1(z)}{b_1}e^{(n+m+1)h(z+c)-h(z)}+\cdots+P_1(z)e^{h(z+c)-h(z)} - \frac{e^{(n+m)h(z)}}{b_1}-\cdots-\frac{b_2}{b_1}e^{h(z)}=1. \tag{3.20}$$

Since $h(z)$ is a non-constant polynomial, then $sh(z+c)-h(z)$ $(s=1,2,\dots,(n+m+1))$ can not be constants except for $h(z+c)-h(z)$. From Lemma 2.3 and (3.20), we obtain

$$\frac{H_1(z,h(z+c))e^{h(z+c)-h(z)}}{H_1(z,h(z))}\equiv 1.$$

Hence, we have that $h(z)$ is a linear polynomial. Moreover, we obtain that $H_1(z,h(z))$ and $H_1(z,h(z+c))$ are the same constants. So we obtain $e^{h(z+c)}=e^{h(z)}$, hence, $f(z)=f(z+c)$ and $f(z)$ is a periodic function with period $c$.

(iii) Since $f^n(f^m-1)f^{(k)}$ and $f^n(f^m-1)f^{(k+1)}$ are periodic functions with the same period $c$, then

$$\begin{cases} f(z)^n(f(z)^m-1)f^{(k)}(z)=f(z+c)^n(f(z+c)^m-1)f^{(k)}(z+c),\\ f(z)^n(f(z)^m-1)f^{(k+1)}(z)=f(z+c)^n(f(z+c)^m-1)f^{(k+1)}(z+c). \end{cases} \tag{3.21}$$

Let $F(z)=f^{(k)}(z)$. So $F'(z)=f^{(k+1)}(z)$, $F(z+c)=f^{(k)}(z+c)$, $F'(z+c)=f^{(k+1)}(z+c)$. From (3.21), we have

$$\frac{F'(z)}{F(z)}=\frac{F'(z+c)}{F(z+c)}.$$

Integrating the above equation, we have

$$F(z+c) = BF(z), \tag{3.22}$$

which means

$$f^{(k)}(z+c) = Bf^{(k)}(z).$$

Integrating again, we obtain

$$f(z+c) = B(f(z)+p(z)), \tag{3.23}$$

where $p(z)$ is a polynomial with $\deg(p(z)) \le k-1$. By the first equation of (3.21), we obtain

$$f(z)^{n+m} - f(z)^n = B^{n+m+1}(f(z)+p(z))^{n+m} - B^{n+1}(f(z)+p(z))^n. \tag{3.24}$$

Assume that $p(z) \not\equiv 0$. If $B^{n+m+1} \neq 1$, we can get $T(r,f) = S(r,f)$ from (3.24) and Valiron-Mohon'ko [5, Theorem 2.2.5], which is impossible. If $B^{n+m+1} = 1$ and $m \ge 2$, we also get $T(r,f) = S(r,f)$, which is impossible. If $m = 1$, from (3.24), then

$$\begin{cases} (n+1)p(z) - B^{n+1} = -1 \\ p(z)^{n+1} - B^{n+1}p(z)^n = 0. \end{cases} \tag{3.25}$$

We get $B^{n+1} = \frac{1}{-n}$. Since $B^{n+2} = 1$, then it follows that $n = 1$ and $B = -1$, which is also impossible. So $p(z) \equiv 0$. From (3.24), we have $B^{n+1} = 1$ and $B^{n+m+1} = 1$. Thus, we obtain that $f(z)$ is a periodic function with period $(n+1)c$ or $mc$. □

Proof of Theorem 1.7. If $\frac{f'}{f(f-1)}$ is a periodic function with period $c$, then

$$\frac{f'(z)}{f(z)(f(z)-1)} = \frac{f'(z+c)}{f(z+c)(f(z+c)-1)}.$$

Integrating the above equation, we have

$$\frac{f(z)-1}{f(z)} = A\frac{f(z+c)-1}{f(z+c)},$$

where $A$ is a non-zero constant. Then we have

$$(A-1)f(z)f(z+c) = Af(z) - f(z+c). \tag{3.26}$$

*Case* 1. Assuming $A = 1$, we have $f(z) = f(z+c)$, so $f(z)$ is a periodic function with period $c$.

*Case* 2. Assuming $A \neq 1$, from (3.26), we have

$$f(z+c) = \frac{Af(z)}{(A-1)f(z)+1}. \tag{3.27}$$

Since $f(z)$ is a transcendental entire function, so $f(z) = \frac{1}{A-1}(e^{p(z)}-1)$, where $p(z)$ is an entire function. Substituting $f(z)$ into (3.27) gives

$$e^{P(z+c)} + Ae^{-p(z)} = A+1.$$

Clearly, $e^{P(z+c)}$ and $Ae^{-p(z)}$ are transcendental entire functions. If $A \neq -1$, from the second main theorem of Nevanlinna theory, we obtain

$$T(r, e^{P(z+c)}) \le N\Big(r, \frac{1}{e^{P(z+c)}}\Big) + N\Big(r, \frac{1}{e^{P(z+c)} - A - 1}\Big) + S(r, e^{P(z+c)})$$
$$= N\Big(r, \frac{1}{e^{P(z+c)}}\Big) + N\Big(r, \frac{1}{-Ae^{-p(z)}}\Big) + S(r, e^{P(z+c)}) = S(r, e^{P(z+c)}),$$

which is impossible. Thus, we have $A = -1$ and $e^{P(z+c)} = e^{-p(z)}$. Hence, $f(z)$ is a periodic function with period $2c$. □

Proof of Theorem 1.9. Since $\frac{f^{(k)}}{f^n(f^m-1)}$ and $\frac{f^{(k+1)}}{f^n(f^m-1)}$ are periodic functions with the same period $c$, then

$$\begin{cases} \dfrac{f^{(k)}(z)}{f(z)^n(f(z)^m-1)} = \dfrac{f^{(k)}(z+c)}{f(z+c)^n(f(z+c)^m-1)}, \\ \dfrac{f^{(k+1)}(z)}{f(z)^n(f(z)^m-1)} = \dfrac{f^{(k+1)}(z+c)}{f(z+c)^n(f(z+c)^m-1)}. \end{cases} \tag{3.28}$$

Let $F(z) = f^{(k)}(z)$. Using a similar method as in the proof of Theorem 1.6(iii), we also get

$$f(z+c) = B(f(z)+p(z)),$$

where $p(z)$ is a polynomial with $\deg(p(z)) \le k-1$ and $B$ is a constant. Furthermore, from the first equation of (3.28), we have

$$f(z)^{n+m} - f(z)^n = B^{n+m-1}(f(z)+p(z))^{n+m} - B^{n-1}(f(z)+p(z))^n.$$

We also get $p(z) \equiv 0$ and $B^{n-1} = 1$, so $f(z)$ is a periodic function with period $(n-1)c$. □

## 4. Discussion

We mainly present results on the transcendental entire functions in this paper. In fact, for the transcendental meromorphic functions, Theorem

1.6(iii) and Theorem 1.9 remain true with the same proofs. Theorem 1.7 remains open for transcendental meromorphic functions. We give the following additional observations. By a transformation $z \to cz$ and letting $g(z) = f(cz)$, the equation (3.27) changes into

$$g(z+1) = \frac{Ag(z)}{(A-1)g(z)+1}.$$

This equation is also called Pielou logistic equation (a special case of a Riccati difference equation), which has the following non-constant solutions with finite order

$$g(z) = \frac{e^{z\log A}}{-1+c+e^{z\log A}},$$

where $c \neq 1$, is a complex parameter, see [6, p. 563]. We see that $g(z)$ is a periodic function with period $\eta$, where $e^{\eta\log A} = 1$. We also remark that Ishizaki [4, Theorem 5.1] considered the difference Riccati equation

$$g(z+1) = \frac{A(z)+g(z)}{1-g(z)},$$

where $A(z) \neq -1$ to show that either $g(z)$ has infinitely many simple poles or $g(z)$ is a periodic function of period 3 or 4. The above example also shows that the case where $g(z)$ has infinitely many simple poles occurs. It is open for us that does it follow that all meromorphic solutions with infinitely many simple poles are also periodic functions? In addition, extending Theorem 1.1 for $n = 2, 3$ and Theorem 1.6(i)(ii) to meromorphic cases will bring some difficulties, which are worthwhile to consider further.

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