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A NEW INTEGRAL EQUATION AND INTEGRALS ASSOCIATED WITH NUMBER THEORY

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Abstract. We utilize a combination of integral transforms, including the Laplace transform, with some classical results in analytic number theory concerning the Riemann ξ-function, to obtain a new integral equation. This integral equation is generalized to self-dual principal automorphic L-functions. We also provide a new proof of known functional-type identities from analytic number theory, and recast some criteria associated with the RH. An application of our integral equation to the Dirichlet problem in the half plane is stated, giving a new application of the Riemann ξ-function integral.

1. Introduction

In Titchmarsh [16, p. 35], we find a famous relation connecting Fourier cosine transforms with Mellin transforms

$$\int_0^\infty f(t)\lambda(t)\cos(xt)\,dt \tag{1.1}$$

$$= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\Big(s-\frac{1}{2}\Big)\phi\Big(\frac{1}{2}-s\Big)(s-1)\Gamma\Big(1+\frac{s}{2}\Big)\pi^{-\frac{s}{2}}\zeta(s)y^s\,ds,$$

where $f(t)=|\phi(it)|^2$ ($\phi(s)$ is analytic). $\zeta(s)$ is the Riemann zeta function [8,16], $\Gamma(s)$ is the gamma function, $\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, $\lambda(t) := \xi(\frac{1}{2}+it)$ (Riemann's Ξ function), and $y=e^x$. Recall [8,16] the famous Riemann ξ function satisfies the functional equation

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s). \tag{1.2}$$

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We denote by $\Re(s)$ and $\Im(s)$ the real and complex parts of $s \in \mathbb{C}$, respectively. Many authors [5,6,10,11,14–16] have utilized equation (1.1) and its variations to obtain many interesting relations among other special functions, as well as results on $\lambda(t)$. See also [12] for some interesting ideas on integrals related to Riemann's ξ-function. The most widely cited example in the literature appears to be [16, (2.16.1)] (for example, see [2,8]), and its variant [16, (2.16.2)]:

$$\Lambda(x) := \int_0^\infty \frac{\lambda(t)}{t^2 + \frac{1}{4}} \cos(xt)\, dt = \frac{\pi}{2}\left(e^{x/2} - 2e^{-x/2}\psi(e^{-2x})\right). \tag{1.3}$$

Here we have a slightly modified Jacobi theta function $\psi(x) = \sum_{n\geq 1} e^{-\pi n^2 x}$. The purpose of this paper is to offer some further consequences of the integral formula (1.3) that appear to be overlooked. The main one being a new integral equation that has some resemblance to the Fredholm integral equation of the second kind, which may be helpful in studying the zero's of the Riemann ξ-function. The integral equation implies several known important facts, including the integral representation for the Riemann ξ-function, and a convergent series involving the incomplete gamma function. We believe it is most likely that, developing properties of its solutions will lead to further information about the Riemann ξ-function.

Throughout this paper, we will define the Dirichlet series

$$L(s,a) := \sum_{n\geq 1} \frac{a_n}{n^s},$$

to have the following properties:

(i) $L(s,a)$ has real Dirichlet coefficients a_n of moderate growth,

(ii) $L(s,a)$ has a meromorphic continuation to the complex plane with at most a simple pole at $s=1$,

(iii) $L(s,a)$ satisfies a standard functional equation of the form $F(s,a) = F(1-s,a)$, where $F(s,a) := C^{s/2}\bar{\gamma}(s)L(s,a)$ is the complete L-function; C is a positive constant, and $\bar{\gamma}(s)$ a product of finitely many shifts of $\pi^{-s/2}\Gamma(s/2)$.

In particular, all self-dual principal automorphic L-functions satisfy these properties. Examples include the L-functions of real Dirichlet characters, and the L-functions of cusp forms (holomorphic and Maass) with trivial nebentypus (see [8, p. 356] for a definition) on the upper half-plane. For the functional relationship for L-functions with Dirichlet characters see [8, p. 84, Theorem 4.15] (more generally [8, p. 94, (5.5)]), and for some background on L-functions for cusp forms see [8, p. 131].

We define the functions

$$\mathfrak{g}(s,x) := \int_0^x t^{s-1}\left(\frac{1}{2\pi i}\int_{(c)} \bar{\gamma}(s')t^{-s'}\, ds'\right) dt,$$

$$\mathfrak{G}(s,x) := \int_x^\infty t^{s-1}\Big(\frac{1}{2\pi i}\int_{(c)} \bar{\gamma}(s')t^{-s'}\,ds'\Big)\,dt,$$

where c is restricted to the region where $\bar{\gamma}(s')$ is analytic, so that $\bar{\gamma}(s) = \mathfrak{g}(s,x) + \mathfrak{G}(s,x)$. When $\bar{\gamma}(s) = \pi^{-s/2}\Gamma(s/2)$, this leads to the classic incomplete gamma function relation $\Gamma(s) = \gamma(s,x) + \Gamma(s,x)$, where

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt, \quad \text{and} \quad \Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt.$$

THEOREM 1.1. *Let C_1 denote the residue of the completed L-function $F(s,a)$ at the simple pole $s=1$. When $\Re(s) > 1$, we have*

$$\text{(1.4)}\quad \Big(s-\frac{1}{2}\Big)\int_0^\infty \frac{F(\frac{1}{2}+it,a)}{(t^2+(s-\frac{1}{2})^2)}\,dt = \frac{\pi}{2}\Big(\frac{C_1}{s-1} - \frac{F(s,a)}{C^{s/2}} + \sum_{n\ge 1}\frac{a_n}{n^s}\mathfrak{G}(s,n)\Big).$$

Or equivalently, the L-function satisfies the integral equation

$$\text{(1.5)}\qquad \int_0^\infty F\Big(\frac{1}{2}+it,a\Big)K(s,t)\,dt = \frac{\pi}{2}\Big(-\frac{F(s,a)}{C^{s/2}} + r(s)\Big),$$

with the kernel $K(s,t) = (t^2 + (s-\frac{1}{2})^2)^{-1}(s-\frac{1}{2})$, for some analytic function $r(s)$ which depends on $L(s,a)$.

PROOF. As usual, we define the Laplace transform by

$$\text{(1.6)}\qquad \mathcal{L}(f)(s) := \int_0^\infty f(t)e^{-st}\,dt.$$

Let $\Lambda(x,a)$ denote the integral

$$\text{(1.7)}\quad \Lambda(x,a) := \int_0^\infty \cos(xt)F\Big(\frac{1}{2}+it,a\Big)\,dt = \frac{\pi}{2}\big(C_1e^{x/2} - e^{-x/2}\psi(e^{-x},a)\big).$$

Here we have a generalized theta function for $d > 1$,

$$\psi(x,a) = \sum_{n\ge 1} a_n\Big(\frac{1}{2\pi i}\int_{(d)}\bar{\gamma}(s)(nx)^{-s}\,ds\Big).$$

A proof of (1.7) is straightforward and analogous to the one for (1.3), with the only notable difference that we apply the residue C_1 at the simple pole $s=1$ due to our $L(s,a)$, which is known to be 1 in the case of real Dirichlet characters [8, Theorem 4.15]. If we assume $\Re(s) > 1$, we may take the Laplace transform of (1.7) and write

$$\text{(1.8)}\qquad \mathcal{L}(\Lambda(x,a))\Big(s-\frac{1}{2}\Big)$$

$$=\int_0^\infty e^{-\left(s-\frac{1}{2}\right)x}\int_0^\infty F(\frac{1}{2}+it,a)\cos(xt)\,dt\,dx \tag{1.9}$$

$$=\left(s-\frac{1}{2}\right)\int_0^\infty \frac{F(\frac{1}{2}+it,a)}{(t^2+(s-\frac{1}{2})^2)}\,dt \tag{1.10}$$

$$=\frac{\pi}{2}\int_0^\infty e^{-sx}\big(C_1e^x-\psi(e^{-x},a)\big)\,dx \tag{1.11}$$

$$=\frac{\pi}{2}\left(\frac{C_1}{s-1}-\int_0^\infty e^{-sx}\psi(e^{-x},a)\right)dx \tag{1.12}$$

$$=\frac{\pi}{2}\left(\frac{C_1}{s-1}+\lim_{r\to 0^+}\int_1^r t^{s-1}\psi(t,a)\,dt\right) \tag{1.13}$$

$$=\frac{\pi}{2}\left(\frac{C_1}{s-1}-\sum_{n\geq 1}\frac{a_n}{n^s}\mathfrak{g}(s,n)\right) \tag{1.14}$$

$$=\frac{\pi}{2}\left(\frac{C_1}{s-1}-\sum_{n\geq 1}\frac{a_n}{n^s}\big(\bar{\gamma}(s)-\mathfrak{G}(s,n)\big)\right) \tag{1.15}$$

$$=\frac{\pi}{2}\left(\frac{C_1}{s-1}-\frac{F(s,a)}{C^{s/2}}+\sum_{n\geq 1}\frac{a_n}{n^s}\mathfrak{G}(s,n)\right). \tag{1.16}$$

In the line (1.11) we used (1.9) together with (1.7). In the line (1.13) we made a change of variables $x=-\log t$, and in subsequent lines employed the definition of our functions $\mathfrak{g}(s,x)$, $\mathfrak{G}(s,x)$. The assertion that $r(s)$ depends on $L(s,a)$ follows directly from properties of Mellin transforms. □

In the case of $a_n=1$ for $n\geq 1$, the completed L-function reduces to the Riemann ξ-function, which in turn reduces to the following integral equation with kernel $K(s,t)=(t^2+\frac{1}{4})^{-1}(t^2+(s-\frac{1}{2})^2)^{-1}(s-\frac{1}{2})$.

Corollary 1.2. *When $\Re(s)>1$, the Riemann ξ-function satisfies the integral equation*

$$\Upsilon(s):=\left(s-\frac{1}{2}\right)\int_0^\infty \frac{\lambda(t)}{(t^2+\frac{1}{4})(t^2+(s-\frac{1}{2})^2)}\,dt \tag{1.17}$$
$$=\frac{\pi}{2}\left(\frac{1}{s-1}-\frac{\xi(s)}{s(s-1)}+\pi^{-s/2}\sum_{n\geq 1}n^{-s}\Gamma\Big(\frac{s}{2},\pi n^2\Big)\right).$$

Note that by an instance $F(y)=e^{-ty^2}$ of the Müntz formula [16, (2.11.1)], for $0<\sigma<1$,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)\big(\sqrt{t}\,x\big)^{-s}\,ds=\sum_{n\geq 1}e^{-tn^2x^2}-\frac{1}{2x}\sqrt{\frac{\pi}{t}}. \tag{1.18}$$

To see this, we first note that $F'(x)$ is bounded in any finite interval, continuous, and is $O(x^{-M})$, for $M>1$ as $x\to\infty$. Hence

$$\begin{aligned}\zeta(s)\int_0^\infty y^{s-1}F(y)\,dy&=\frac{1}{2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)(\sqrt{t})^{-s}\\ &=\int_0^\infty x^{s-1}\Big(\sum_{n\geq 1}F(nx)-\frac{1}{x}\int_0^\infty F(w)\,dw\Big)dx\\ &=\int_0^\infty x^{s-1}\Big(\sum_{n\geq 1}e^{-tn^2x^2}-\frac{1}{2x}\sqrt{\frac{\pi}{t}}\Big)dx,\end{aligned}$$

for $0<\Re(s)<1$. Equation (1.18) now follows from Mellin inversion. Multiplying through by x^{v-1}, with $\Re(v)>1$, and integrating over the interval $[0,z]$, $z>0$, we obtain

$$\begin{aligned}&\frac{z^v}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)\frac{(\sqrt{t}z)^{-s}}{(v-s)}\,ds\\ &=\frac{t^{-v/2}}{2}\sum_{n\geq 1}n^{-v}\gamma\Big(\frac{v}{2},tz^2n^2\Big)-\frac{z^{v-1}}{2(v-1)}\sqrt{\frac{\pi}{t}},\end{aligned} \tag{1.19}$$

again for $0<\sigma<1$. (Note this integration is justified on the left-hand side, since if $\Re(v-s)>0$, then $0^{v-s}=0$.) On the other hand the integral on the left-hand side of (1.19) is precisely $-\frac{1}{\pi}\Upsilon(v)$ defined in Corollary 1.2, when $t=\pi$, $z=1$, $\sigma=\frac{1}{2}$. Hence we have again arrived at Corollary 1.2.

From Titchmarsh [16, p. 257] we have $\lambda(t)\ll t^Ae^{-\frac{\pi}{4}t}$, which gives us

$$\Upsilon(s)\ll\int_0^\infty\frac{t^Ae^{-\frac{\pi}{4}t}}{(t^2+\frac{1}{4})(t^2+(s-\frac{1}{2})^2)}\,dt<+\infty.$$

Clearly we have that $\Upsilon(s)=-\Upsilon(1-s)$. Upon noticing this fact, we may use (1.19) to obtain a well-known result concerning $\zeta(s)$. Computing the residue $R_{s=0}=\zeta(0)/v=-1/(2v)$ out from the integral, we may then extract the series $\frac{1}{2}\pi^{-(1-v)/2}\sum_{n\geq 1}n^{-(1-v)}\Gamma(\frac{1-v}{2},\pi n^2)$. Collecting these observations, we see that adding (1.17) evaluated at s and at $1-s$, we obtain the following expansion [1, p. 256, (30)] as a direct corollary to Corollary 1.2:

$$\pi^{-s/2}\zeta(s)\Gamma\Big(\frac{s}{2}\Big) \tag{1.20}$$
$$=\frac{1}{s(s-1)}+\pi^{-s/2}\sum_{n\geq1}n^{-s}\Gamma\Big(\frac{s}{2},\pi n^2\Big)+\pi^{-(1-s)/2}\sum_{n\geq1}n^{-(1-s)}\Gamma\Big(\frac{1-s}{2},\pi n^2\Big).$$

2. Imaginary quadratic forms and the associated integral equation

In this section we give another example of our main integral equation in Theorem 1.1, which we believe is worthy of note. Following [8, p. 511] put

$$L_K(s,\chi)=\sum_{\mathfrak{a}}\chi(\mathfrak{a})(N\mathfrak{a})^{-s}, \tag{2.1}$$

where $\Re(s)>1$, and χ maps the class group $\mathfrak{H}$ to the complex plane $\mathbb{C}$. In [10,11], we find N. S. Koshlyakov investigating integrals related to L-functions associated with number fields as well as [14,16]. We consider his work coupled with that of the ideas in the introduction. Let D denote a discriminant with respect to a primitive ideal $\mathfrak{a}$. Then we have the functional equation [8, (22.51)]

$$\Omega_K(s,\chi)=\Omega_K(1-s,\chi), \tag{2.2}$$

where $\Omega_K(s,\chi)=(2\pi)^{-s}\Gamma(s)|D|^{\frac{s}{2}}L_K(s,\chi)$.

An important integral representation relevant to our study, which continues $L_K(s,\chi)$ to the entire complex plane, is given by Hecke [8, (22.52)]:

$$\Omega_K(s,\chi)=\frac{|\mathfrak{H}|\delta(\chi)}{\bar{D}s(s-1)}+\int_1^\infty(t^{s-1}+t^{-s})\sum_{\mathfrak{a}}\chi(\mathfrak{a})e^{-2\pi tN\mathfrak{a}/\sqrt{|D|}}\,dt, \tag{2.3}$$

where $|\mathfrak{H}|$ is the class number, and $\bar{D}$ is 6 if $D=-3$, 4 if $D=-4$, and 2 if $D<-4$. We shall prove an equivalent form of this integral representation toward the end using Theorem 2.1. Put $\Omega_K(\frac{1}{2}+it,\chi)=\mathfrak{O}_K(t)$, and note that

$$\int_0^\infty f(t)\mathfrak{O}_K(t)\cos(xt)\,dt \tag{2.4}$$
$$=\frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\Big(s-\frac{1}{2}\Big)\phi\Big(\frac{1}{2}-s\Big)(2\pi)^{-s}|D|^{s/2}\Gamma(s)L_K(s,\chi)y^s\,ds.$$

N. S. Koshlyakov appears to be the first to consider instances of this type of general integral with L-functions associated with number fields (see, especially, [11, pp. 217–220]). In this section we give an equivalent integral equation for imaginary quadratic forms, using all these ideas.

THEOREM 2.1. *For $\Re(s) > 1$, we have that*

$$\mathfrak{u}(s) = \int_0^\infty \mathfrak{O}_K(t)\bar{K}(s,t)\,dt \tag{2.5}$$

$$= \frac{\pi}{2}\left(\frac{|\mathfrak{H}|\delta(\chi)}{\bar{D}(s-1)} - \Omega_K(s,\chi) + \sum_{\mathfrak{a}} \chi(\mathfrak{a})\left(\frac{\sqrt{|D|}}{2\pi N\mathfrak{a}}\right)^s \Gamma\left(s, \frac{2\pi N\mathfrak{a}}{\sqrt{|D|}}\right)\right),$$

with the kernel $\bar{K}(s,t) = (s-\frac{1}{2})(t^2 + (s-\frac{1}{2})^2)^{-1}$.

PROOF. We may easily evaluate the case $\phi(s) = 1$, and so $f(t) = 1$, for all t. Consider

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} (2\pi)^{-s}|D|^{s/2}\Gamma(s)L_K(s,\chi)y^s\,ds, \tag{2.6}$$

for $d > 1$, and move the line of integration to $d = \frac{1}{2}$. Since $L_K(s,\chi)$ has a simple pole at $s = 1$ when χ is trivial, we compute [8, p. 512, (22.50)]

$$R_{s=1} = \frac{|\mathfrak{H}|\delta(\chi)}{\bar{D}}y,$$

(where $\delta(\chi) = 0$ when χ is non-trivial, 1 otherwise) and obtain

$$\int_0^\infty \mathfrak{O}_K(t)\cos(xt)\,dt = \pi\left(\frac{|\mathfrak{H}|\delta(\chi)e^{x/2}}{\bar{D}} - e^{-x/2}\Psi(e^{-x})\right), \tag{2.7}$$

where $\Psi(x) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})e^{-2\pi(N\mathfrak{a})x/\sqrt{|D|}}$. (Note that (2.6) is precisely $\Psi(y^{-1})$.) Applying the same concepts as in our introduction, we may obtain the theorem. □

Due to the kernel, we again have $\mathfrak{u}(s) = -\mathfrak{u}(1-s)$, and as a direct corollary we have [8, p. 512, (22.54)]

$$\Omega_K(s,\chi) = \frac{|\mathfrak{H}|\delta(\chi)}{\bar{D}s(s-1)} + \sum_{\mathfrak{a}} \chi(\mathfrak{a})\left(\frac{\sqrt{|D|}}{2\pi N\mathfrak{a}}\right)^s \Gamma\left(s, \frac{2\pi N\mathfrak{a}}{\sqrt{|D|}}\right) \tag{2.8}$$

$$+ \sum_{\mathfrak{a}} \chi(\mathfrak{a})\left(\frac{\sqrt{|D|}}{2\pi N\mathfrak{a}}\right)^{1-s} \Gamma\left(1-s, \frac{2\pi N\mathfrak{a}}{\sqrt{|D|}}\right).$$

3. Dirichlet's problem in the half plane and other applications

Here we discuss some results related to (1.3) and (1.5) that we hope will encourage further interest. First we discuss the relationship to the solution of Dirichlet's problem in the half plane,

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}=0, \tag{3.1}$$

where $y\in\mathbb{R}$, $x\geq 0$, initial condition $u(y,0)=h(y)$, and growth condition $u(y,x)\to 0$, as $|y|\to\infty$. The known solution is given as the Poisson integral (see [4, p. 36])

$$u(y,x)=\frac{x}{\pi}\int_{\mathbb{R}}\frac{h(t)}{(t-y)^2+x^2}\,dt.$$

Making the change of variables $t=t+y$, $x=s-\frac{1}{2}$, we may state this as

$$u\Big(y,s-\frac{1}{2}\Big)=\frac{s-\frac{1}{2}}{\pi}\int_{\mathbb{R}}\frac{h(t+y)}{t^2+(s-\frac{1}{2})^2}\,dt, \tag{3.2}$$

Hence, to incorporate our integral equation, we could impose the additional condition to the Dirichlet problem that,

$$u\Big(0,s-\frac{1}{2}\Big)=\frac{s-\frac{1}{2}}{\pi}\int_{\mathbb{R}}\frac{h(t)}{t^2+(s-\frac{1}{2})^2}\,dt=-h\Big(-i\Big(s-\frac{1}{2}\Big)\Big)+r(s), \tag{3.3}$$

where $r(s)$ is analytic for $\Re(s)>1$. Our results are applicable if one sets

$$r(s)=\frac{C_1}{s-1}+\sum_{n\geq 1}\frac{a_n}{n^s}\mathfrak{G}(s,n),$$

where C is a constant, and the coefficients a_n depend on $L(s)$, and subsequently the solution to Theorem 1.1 gives

$$h(t)=C^{(\frac{1}{2}+it)/2}\bar{\gamma}\Big(\frac{1}{2}+it\Big)L\Big(\frac{1}{2}+it,a\Big),$$

with the associated gamma factor $\bar{\gamma}(s)$. The particular case of Corollary 1.2, we have $h(t)=\lambda(t)(t^2+\frac{1}{4})^{-1}=\pi^{-(\frac{1}{2}+it)/2}\Gamma(\frac{1}{4}+\frac{it}{2})\zeta(\frac{1}{2}+it)$. We summarize our observation in the following.

Theorem 3.1. *The solution of Dirichlet's problem in the half plane,*

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}=0,$$

where $y \in \mathbb{R}$, $x \geq 0$, *initial condition* $u(y,0) = C^{(\frac{1}{2}+it)/2}\bar{\gamma}(\frac{1}{2}+iy)L(\frac{1}{2}+iy,a)$, $u(y,x) \to 0$, *as* $|y| \to \infty$, *is given by the Poisson integral. Furthermore, the solution also satisfies the condition* $u(0, s-\frac{1}{2}) = -h(-i(s-\frac{1}{2})) + r_1(s)$, *for an analytic function* $r_1(s)$ *that satisfies* $h(-i(s-1/2)) = C_1/(s(s-1)) + r_1(s) + r_1(1-s)$.

Next we mention other tangential results concerning criteria for the Riemann Hypothesis. First we recall for $0 < \Re(s) < 1$, the well-known formula

$$\int_0^\infty t^{s-1}\cos(xt)\,dt = \frac{\Gamma(s)\cos(\frac{\pi}{2}s)}{x^s}. \tag{3.4}$$

Hence for $0 < \Re(s) < 1$,

$$\int_0^\infty t^{s-1}\Big(\int_0^\infty \cos(xt)\Big(\psi(x,a) - \frac{C_1}{x}\Big)\,dx\Big)\,dt = 2F(s,a)\Gamma(s)\cos\Big(\frac{\pi}{2}s\Big). \tag{3.5}$$

This gives us $(0 < c < 1)$

$$\begin{aligned} \psi(x,a) - \frac{C_1}{x} &= \int_0^\infty \cos(xt)\Big(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s,a)\Gamma(s)\cos\Big(\frac{\pi}{2}s\Big)t^{-s}\,ds\Big)\,dt \\ &= \int_0^\infty \cos(xt)\dot{I}(t)\,dt, \end{aligned} \tag{3.6}$$

say. So we may now compute (using (1.7))

$$-F\Big(\frac{1}{2}+it,a\Big) = \int_0^\infty \cos(xt)e^{-x/2}\Big(\int_0^\infty \cos(e^{-x}u)\dot{I}(u)\,du\Big)\,dx. \tag{3.7}$$

It is also possible to prove (3.5) using Parseval's theorem for Mellin transforms and the functional equation (iii).

We now recast the known criteria for the RH concerning (3.6) outlined in [3] in a different form. (See [3, Definition 1.1] for a criterion for an entire function to belong to the Laguerre–Pólya class.)

Theorem 3.2. *The* (*Grand*) *Riemann Hypothesis for* $L(s,a)$ *is equivalent to the statement that the integral*

$$\ddot{I}(t) := \int_0^\infty \cos(xt)\Big(-\partial_x^2 + \frac{1}{4}\Big)e^{-x/2}\Big(\int_0^\infty \cos(e^{-x}u)\dot{I}(u)\,du\Big)\,dx,$$

has only real zeros. Consequently, the Riemann Hypothesis is true if and only if $\ddot{I}(t)$ *is in the Laguerre–Pólya class.*

Proof. The proof uses the integral (3.7), the operator from [6], coupled with comparing the kernel of $\ddot{I}(t)$, given by

$$\ddot{k}(x) := \Big(-\partial_x^2 + \frac{1}{4}\Big)e^{-x/2}\Big(\int_0^\infty \cos(e^{-x}u)\dot{I}(u)du\Big)dx,$$

to what is considered an 'admissible' kernel according to the definition given in [3, Definition 1.2]. For example, it is easily verified that $\ddot{k}(x)$ is even by the functional equation (iii) for the modified Jacobi theta function $\psi(x,a)$ (and (3.6)), $\ddot{k}(x) > 0$ for all $x \in \mathbb{R}$. The fact that the kernel satisfies these properties tells us that $\ddot{I}(t)$ is a real entire function, and it is possible to have only real zeros. The rest of the proof utilizes the well-known criterion that the (Grand) Riemann hypothesis is equivalent to the statement that all the zeros of $F(\frac{1}{2}+it,a)$ are real (or equivalently, all zeros of $L(s,a)$ in the critical strip lie on $\Re(s) = \frac{1}{2}$ [8, p. 113]), together with [3, Definition 1.1]. □

We can also include the Hankel transform in our re-stating of the RH criteria found in [3] using the above ideas. Recall that the Hankel transform of a suitable function $f(r)$ is given by $H(k) = \int_0^\infty f(r)J_v(kr)r\,dr$, and $J_v(x)$ is the Bessel function.

Theorem 3.3. *Let $\dot{H}(t)$ be defined in (3.8). The (Grand) Riemann Hypothesis for $L(s,a)$ is equivalent to the statement that the function*

$$\bar{G}(t) := \int_0^\infty \cos(xt)\Big(-\partial_x^2 + \frac{1}{4}\Big)e^{-3x/2}\Big(\int_0^\infty J_0(e^{-x}u)\dot{H}(u)\,du\Big)dx,$$

has only real zeros. Consequently, the Riemann Hypothesis is true if and only if $\bar{G}(t)$ is in the Laguerre–Pólya class.

Proof. We mimic the proof of the last theorem. First, for $0 < \Re(s) < 1$ we have that

$$\int_0^\infty t^{s-1}\Big(\int_0^\infty J_0(kt)\Big(\psi(k,a) - \frac{C_1}{k}\Big)\,dk\Big)\,dt = \frac{2^{s-1}F(s,a)\Gamma(\frac{s}{2})}{\Gamma(1-\frac{s}{2})}.$$

Hence, using the inverse Mellin and inverse Hankel transform (with $v = 0$), we find

$$\frac{1}{k}(\psi(k,a) - \frac{C_1}{k})) = \int_0^\infty J_0(kt)t\dot{H}(t)\,dt,$$

where for $c \in (0,1)$,

$$\dot{H}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{t^{-s}2^{s-1}F(s,a)\Gamma(\frac{s}{2})}{\Gamma(1-\frac{s}{2})}\,ds. \tag{3.8}$$

Putting $k = e^{-x}$, multiplying by $e^{-3x/2}$, taking the Fourier cosine transform and the applying the same arguments we used previously gives the result. □

At this point we are left with some questions regarding the solutions of Theorem 1.1. First, in light of equation (1.5) resembling a "modified" Fredholm integral of the second kind, does the integral equation in Theorem 1.1 admit application of a kind of Fredholm theory? Can anything be said regarding the uniqueness of solution of (1.5) for each $r(s)$? It is clear through Mellin transforms that uniqueness of $F(s,a)$ follows for the choices we have made for $r(s)$. However, it is still open if this is true for any analytic $r(s)$, as the general Fredholm theory doesn't clearly offer an answer.

More interesting relations may be obtained by expanding $L_K(s,\chi)$ in (2.4) into a sum of Epstein zeta functions, with formulas related to the work in [9,10,17]. This would allow us to obtain integrals related to a theta functions of the form $\sum_{n,m\in\mathbb{Z}} \chi_{n,m} q^{an^2+bnm+cm^2}$, $4ac-b^2>0$, where $q=e^{-\pi x}$, $x>0$. It may also be of interest to look at the integral $\ddot{I}(t)$ by explicitly computing $\dot{I}(u)$ using the Residue Theorem.

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