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ON THE ESTIMATES OF DUNKL KERNELS

B. AMRI* and A. GASMI

Department of Mathematics, College of Sciences, Taibah University,
P.O. Box 30002 Al Madinah Al Munawarah, Saudi Arabia
e-mails: bechiramri69@gmail.com, gasmi1975@yahoo.fr

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Abstract. In this paper, we are interested in estimates of the Dunkl kernels on some special sets. We improving results of M.F.E. de Jeu and M. Rösler [6].

1. Introduction

In [9] Dunkl constructed a commutative set of first-order differential-difference operators on $\mathbb{R}^n$ associated with a root system and positive parameters attached to the roots. These operators, which came to be known as Dunkl operators, have applications in various areas of mathematics and in mathematical physics, such as in the study of integrable quantum many-body systems of Calogero–Moser–Sutherland type (see [16]). Dunkl operators give rise to the so-called Dunkl kernel which extends the usual exponential function and plays a central part in developing the harmonic analysis in the Dunkl setting. It is in particular the integral kernel of the Dunkl transform. Although extensive studies have been devoted to the Dunkl kernel [5,6,8,9,12], its explicit formula is still unknown in general and finding it is a very difficult problem. Recently, there have been an interest in developing formulas in specific cases [1–4]. This paper is concerned with further information about Dunkl kernels in a general setting; we are particularly interested in their asymptotic behavior. Following the topic raised in [6], we obtain an estimate which is uniform in each convex cone contained in an open Weyl chamber (Theorems 2.5 and 2.7).

To begin, we first recall some background from the Dunkl theory. General references are [6,7,9,12,13,15]. Consider the Euclidean space $\mathbb{R}^n$ equipped with the canonical basis $(e_1, e_2, \dots, e_n)$ and the scalar product $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$ with associated norm $|x| = \langle x, x\rangle^{1/2}$. Let $G \subset \mathrm{O}(\mathbb{R}^n)$

* Corresponding author.

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be a finite reflection group associated to a reduced root system R and let $k\colon R \to [0, +\infty)$ be a G-invariant function (called multiplicity function). Let R^+ be a positive root subsystem.

The Dunkl operators $D_\xi(k)$ on $\mathbb{R}^n$ are the following k-deformations of the directional derivatives ∂_ξ by difference operators:

$$D_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\upsilon\in R^+} k(\upsilon)\,\langle \upsilon, \xi\rangle\,\frac{f(x) - f(\sigma_\upsilon.\,x)}{\langle \upsilon,\, x\rangle}\,, \tag{1.1}$$

where σ_υ is the reflection with respect to H_υ, the hyperplane orthogonal to υ. The operators ∂_ξ and $D_\xi(k)$ are intertwined by a Laplace-type operator

$$V_k f(x) = \int_{\mathbb{R}^n} f(y)\, d\nu_x(y)$$

associated to a family of compactly supported probability measures $\{\nu_x \mid x \in \mathbb{R}^n\}$. Specifically, ν_x is supported in the convex hull $\mathrm{co}(G.x)$.

For each $y \in \mathbb{C}^n$, the simultaneous eigenfunction problem

$$D_\xi(k)f = \langle y, \xi\rangle\, f, \quad \text{for all } \xi \in \mathbb{R}^n,$$

has a unique analytic solution $f(x) = E_k(x, y)$ such that $E_k(0, y) = 1$, called the Dunkl kernel, which is given by

$$E_k(x, y) = V_k(e^{\langle \cdot, y\rangle})(x) = \int_{\mathbb{R}^n} e^{\langle z, y\rangle}\, d\nu_x(z) \quad \text{for all } x \in \mathbb{R}^n. \tag{1.2}$$

The Dunkl kernel $E_k(x, y)$ which reduces to the exponential $e^{\langle x,y\rangle}$ in the case $k = 0$ has a holomorphic extension to $\mathbb{C}^n \times \mathbb{C}^n$ and the following holds: for $x, y \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ and $g \in G$ we have

$$E_k(x, y) = E_k(y, x), \quad E_k(\lambda x, y) = E_k(x, \lambda y), \quad E_k(g.x, g.y) = E_k(x, y).$$

Furthermore, for $x, y \in \mathbb{R}^n$ and $z \in \mathbb{C}$

$$|E_k(zx, y)| \le \max_{g\in G} e^{\mathrm{Re}\, z\langle x, g.y\rangle}. \tag{1.3}$$

In particular, if we denote by x^+ the intersection point of any orbit $G.x$ with the closure of the fundamental Weyl chamber $\overline{C}$, then

$$|E_k(x, y)| \le e^{\langle x^+,\, y^+\rangle}.$$

We note here that the chamber C is given by

$$C = \big\{x \in \mathbb{R}^n : \langle x, \upsilon\rangle > 0 \text{ for all } \upsilon \in R^+\big\}.$$

In dimension $n = 1$, the Dunkl kernel E_k can be expressed in terms of the confluent hypergeometric function:

$$E_k(x, y) = E_k(xy) = e^{xy}\,{}_1F_1(k, 2k + 1, -2xy).$$

Using the asymptotic behavior of ${}_1F_1$ (see, e.g., [11]), we state the following estimates: for some constant $c > 0$,

$$|E_k(xy)| \le c\,\frac{e^{|xy|}}{|xy|^k} \quad \text{and} \quad |E_k(ixy)| \le c\,|xy|^{-k}; \quad x, y \in \mathbb{R} \setminus \{0\}. \tag{1.4}$$

Our aim is a generalisation of the estimates (1.4) in higher dimensions and for any reflection group. We claim that the following hold for some constant $c > 0$:

$$|E_k(x, y)| \le c\,\frac{e^{\langle x^+, y^+\rangle}}{\sqrt{w_k(x)w_k(y)}} \quad \text{and} \quad |E_k(ix, y)| \le \frac{c}{\sqrt{w_k(x)w_k(y)}}; \tag{1.5}$$

$x, y \in \mathbb{R}^n \setminus \bigcup_{\upsilon\in R^+} H_\upsilon$, where

$$w_k(x) = \prod_{\upsilon\in R^+} |\langle \upsilon, x\rangle|^{2\,k(\upsilon)} \tag{1.6}$$

is G-invariant and homogeneous of degree $2\gamma_k$, and

$$\gamma_k = \sum_{\upsilon\in R^+} k(\upsilon).$$

For $\delta > 0$ we let

$$C_\delta = \big\{x \in C : \langle x, \upsilon\rangle \ge \delta|x| \text{ for all } \upsilon \in R^+\big\}.$$

In this paper we shall prove that (1.5) holds for every $x, y \in C_\delta$ and for a constant $c = c(\delta)$. We note here that a motivation for extending (1.4) also arises from the work of De Jeu and Rösler in [6] where they proved the following result.

THEOREM 1.1. *There exists a constant non-zero vector* $v = (v_g)_{g\in G} \in \mathbb{C}^G$ *such that for all* $y \in C$ *and* $g \in G$

$$\lim_{|x|\to\infty,\, x\in C_\delta} \sqrt{w_k(x)w_k(y)}\, e^{\langle -ix, gy\rangle} E_k(ix, gy) = v_g.$$

2. The main estimates for the Dunkl kernel

We may assume that $\gamma_k > 0$ and the root system R generates the space $\mathbb{R}^n$. Let Δ be the set of the simple roots of R associated with R^+; it is a basis for $\mathbb{R}^n$ (see [10]). Let $(\lambda_i)_{1\le i\le n}$ be the dual basis of Δ. Then the open fundamental Weyl chamber is given by

$$C = \Big\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{n} x_i\lambda_i,\ x_i > 0,\ \text{for all } i = 1, \dots, n \Big\}.$$

For a family of linearly independent vectors $(v_i)_{1\le i\le n}$ we define the convex polytope

$$\Lambda_{v_1,\dots,v_n} = \Big\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{n} x_i v_i,\ x_i > 0 \Big\}.$$

Lemma 2.1. *For each $\delta > 0$ there exists a family of linearly independent vectors $(v_i)_{1\le i\le n}$ in C such that $C_\delta \subset \Lambda_{v_1,\dots,v_n}$.*

Proof. Let Π_δ be the set

$$\Pi_\delta = \big\{ x \in C_\delta : |x| = 1 \big\}$$

and put $\lambda = \sum_{i=1}^{n} \lambda_i$. For each integer $p \ge 2$ define the vectors

$$v_{p,i} = \lambda_i + \frac{\lambda}{p}, \quad i = 1, \dots, n.$$

It is easy to see that the vectors $v_{p,i}$ are linearly independent and

$$v_{p,i} = v_{p+1,i} + \frac{1}{p(p+n+1)} \sum_{j=1}^{n} v_{p+1,j} \tag{2.1}$$

for all $i = 1, \dots, n$ and $p \ge 2$. Denote $\Lambda^p = \Lambda_{v_{p,1},\dots,v_{p,n}}$. It follows from (2.1) that

$$\Lambda^p \subset \Lambda^{p+1}.$$

Next we claim that

$$\bigcup_p \Lambda^p = C.$$

Clearly $\bigcup_p \Lambda^p \subset C$. If $x \in C$, $x = \sum_{i=1}^{n} x_i\lambda_i$ with $x_i > 0$, then we can find $p \ge 2$ such that

$$x_i - \frac{\sum_{i=1}^{n} x_i}{p+n} > 0$$

and so

$$x = \sum_{i=1}^{n} \Big(x_i - \frac{\sum_{i=1}^{n} x_i}{p+n} \Big) v_{p,i} \in \Lambda^p.$$

Now since

$$\Pi_\delta \subset \bigcup_p \Lambda^p$$

and Π_δ is compact then there exists p_0 such that $\Pi_\delta \subset \Lambda^{p_0}$ which implies that $C_\delta \subset \Lambda^{p_0}$. This concludes the proof of Lemma 2.1. $\square$

REMARK 2.2. Reciprocally, for $\Lambda_{v_1,v_2,\dots,v_n} \subset C$ one can find $\delta > 0$ such that $\Lambda_{v_1,v_2,\dots,v_n} \subset C_\delta$. In fact, for $x \in \Lambda_{v_1,v_2,\dots,v_n}$, $x = \sum_{i=1}^{n} x_i v_i$ and $\upsilon \in R^+$ we have

$$\langle x, \upsilon \rangle = \sum_{i=1}^{n} x_i \langle v_i, \upsilon \rangle \geq \min_i \langle v_i, \upsilon \rangle \sum_{i=1}^{n} x_i \geq c \min_i \langle v_i, \upsilon \rangle |x|,$$

for some constant $c > 0$. Then we put $\delta = c \min_i \langle v_i, \upsilon \rangle$.

LEMMA 2.3. *For all $x, y \in C$ and $g \in G$ the function*

$$t \to t^{\gamma_k} e^{-t\langle x,y \rangle} E_k(tx, gy), \quad t \geq 0$$

is bounded.

PROOF. The lemma follows using the Phragmén–Lindelöf theorem (see, e.g., [14, Section 5.61]) by considering the functions of a complex variable

$$u(z) = z^{\gamma_k} e^{-z\langle x,y \rangle} E_k(zx, gy), \quad z \in H = \{ z \in \mathbb{C} : \mathrm{Re}(z) \geq 0 \}.$$

Indeed, from [6, Corollary 1] it follows that, for regular $x, y \in C$ and $g \in G$,

$$t \to t^{\gamma_k} E_k(itx, gy)$$

is bounded on $[0, \infty)$. Now pick arbitrary $x, y \in \mathbb{R}^n \setminus \bigcup_{\upsilon \in R^+} H_\upsilon$. There exist $x_0, y_0 \in C$ and $g_1, g_2 \in G$ such that $x = g_1 x_0$ and $y = g_2 y_0$. Then, for $t \geq 0$,

$$t^{\gamma_k} E_k(itx, y) = t^{\gamma_k} E_k(itg_1 x_0, g_2 y_0) = t^{\gamma_k} E_k(itx_0, g_1^{-1} g_2 y_0)$$

which is bounded on $[0, \infty)$ by the initial result. This implies that the function u is bounded on the positive imaginary axis as well as on the negative imaginary axis, so that it is bounded on the imaginary axis. Now (1.3) together with the argument given on [6, p. 122] shows that

$$|u(z)| \leq |z|^{\gamma_k}$$

for each z such that $\mathrm{Re}\, z \geq 0$. The Phragmen–Lindelöf theorem can be applied, which implies the statement in the lemma. $\square$

Proposition 2.4. *Given a convex polytope $\Lambda_{v_1,\dots,v_n} \subset C$, there exists a constant $c > 0$ such that for all $g \in G$ and $x, y \in \Lambda_{v_1,\dots,v_n}$ with $x = \sum_{i=1}^n x_i v_i$ and $y = \sum_{i=1}^n y_i v_i$,*

$$0 \le E_k(x, g.y) \le \frac{c\, e^{\langle x,y\rangle}}{\prod_{i,j=1}^n (x_i y_i)^{\gamma_k/n^2}}. \tag{2.2}$$

Proof. The generalized Hölder inequality applied to the integral formula (1.2) yields

$$E_k(x, g.y) = E_k(gy, x) = \int_{\mathbb{R}^n} \prod_{i=1}^n e^{\langle x_i v_i, z\rangle}\, d\nu_{gy}(z)$$
$$\le \Big\{ \prod_{i=1}^n \int_{\mathbb{R}^n} e^{\langle n x_i v_i, z\rangle}\, d\nu_{gy}(z) \Big\}^{1/n} = \prod_{i=1}^n E_k(n x_i v_i, g.y)^{1/n}.$$

We can proceed in the same way to obtain

$$E_k(x, g.y) \le \prod_{i=1}^n \prod_{j=1}^n E_k(n x_i v_i, n y_j g.v_j)^{1/n^2}$$
$$\le \prod_{i=1}^n \prod_{j=1}^n \Big((n^2(x_i y_j))^{\gamma_k} e^{-n^2 x_i y_j \langle v_i, v_j\rangle} E_k(n^2 x_i y_j v_i, g.v_j) \Big)^{1/n^2}$$
$$\times \frac{e^{\langle x,y\rangle}}{n^{2\gamma_k} \prod_{i,j=1}^n (x_i y_j)^{\gamma_k/n^2}}.$$

Then we conclude Proposition 2.4 from Lemma 2.3. $\square$

Now the main estimate is stated as follows:

Theorem 2.5. *Given a convex polytope $\Lambda_{v_1,\dots,v_n} \subset C$, there exists a constant $c > 0$, depending only on the choice of the vectors v_i, such that*

$$E_k(x, gy) \le \frac{c\, e^{\langle x,y\rangle}}{\sqrt{w_k(x) w_k(y)}}, \tag{2.3}$$

for all $x, y \in \Lambda_{v_1,\dots,v_n}$ and $g \in G$.

Proof. From Lemma 2.1 and Remark 2.2 we can find $\delta > 0$ and linearly independent vectors $\xi_1, \dots, \xi_n \in C$ such that

$$\Lambda_{v_1,\dots,v_n} \subset C_\delta \subset \Lambda_{\xi_1,\dots,\xi_n}.$$

As $C_\delta \cup \{0\} = \overline{C_\delta}$ is a closed set then we have

$$\overline{\Lambda_{v_1,\dots,v_n}} \setminus \{0\} \subset \Lambda_{\xi_1,\dots,\xi_n}.$$

Now, define the open convex sets

$$\mathcal{H}_p = \Big\{x \in \Lambda_{\xi_1,\dots,\xi_n} : x = \sum_{i=1}^n x_i \xi_i,\ x_1/p < x_i < p x_1\ i \neq 1\Big\}, \quad p \in \mathbb{N}. \tag{2.4}$$

Clearly $\mathcal{H}_p \uparrow \Lambda_{\xi_1,\dots,\xi_n}$ as $p \to +\infty$. Thus,

$$\overline{\Lambda_{v_1,\dots,v_n}} \setminus \{0\} \subset \mathcal{H}_p \tag{2.5}$$

for some p. Indeed, we write $\overline{\Lambda_{v_1,\dots,v_n}} = \mathbb{R}_+ \mathcal{H}$ where

$$\mathcal{H} = \big\{x \in \overline{\Lambda_{v_1,\dots,v_n}},\ |x| = 1\big\}.$$

As $\mathcal{H}$ is compact and $\mathcal{H} \subset \Lambda_{\xi_1,\dots,\xi_n} = \bigcup_p \mathcal{H}_p$ then there exists p such that $\mathcal{H} \subset \mathcal{H}_p$, from which we get (2.5).

Now take $p \in \mathbb{N}$ such that $\overline{\Lambda_{v_1,\dots,v_n}} \setminus \{0\} \subset \mathcal{H}_p$. Let $x \in \Lambda_{v_1,\dots,v_n}$ with $x = \sum_{i=1}^n x_i v_i$. In view of (2.4) we have that

$$x_i \le p^2 x_j, \quad 1 \le i, j \le n,$$

from which

$$\Big(\sum_{i=1}^n x_i\Big)^n = \sum_{1\le i_1,\dots,i_n \le n} x_{i_1} \cdots x_{i_n} \le n^n p^{2n} \prod_{i=1}^n x_i.$$

It follows that for each $\upsilon \in R^+$ we have

$$\langle x, \upsilon\rangle = \sum_{i=1}^n x_i \langle v_i, \upsilon\rangle \le \max_i \langle v_i, \upsilon\rangle \sum_{i=1}^n x_i \le c\Big(\prod_{i=1}^n x_i\Big)^{1/n}.$$

Hence, according to definition (1.6) of w_k we have that

$$\sqrt{w_k(x)} = \prod_{\upsilon \in R^+} \langle x, \upsilon\rangle^{k(\upsilon)} \le c\Big(\prod_{i=1}^n x_i\Big)^{\gamma_k/n}. \tag{2.6}$$

Finally, using (2.6) in the estimate (2.2) we get

$$E_k(x, g.y) \le \frac{c\, e^{\langle x,y\rangle}}{\prod_{i,j=1}^n (x_i y_j)^{\gamma_k/n^2}} = \frac{c\, e^{\langle x,y\rangle}}{\Big(\prod_{i=1}^n x_i\Big)^{\gamma_k/n}\Big(\prod_{i=1}^n y_i\Big)^{\gamma_k/n}} \le \frac{c\, e^{\langle x,y\rangle}}{\sqrt{w_k(x) w_k(y)}}$$

which is the desired estimate. $\square$

We come now to the second part of this work, that is the behavior of the kernel $E_k(ix,y)$. The main result is the following

Theorem 2.6. *There exists a constant non-zero vector* $v=(v_g)_{g\in G}\in\mathbb{C}^G$ *such that for each* $g\in G$ *and* $\delta>0$

$$\lim_{|x|.|y|\to+\infty;\, x,y\in C_\delta}\sqrt{w_k(x)w_k(y)}e^{-i\langle x,gy\rangle}E_k(ix,gy)=v_g. \tag{2.7}$$

Proof. We proceed as in the proof of [6, Theorem 1]. Keeping the notations of [6] we consider the function

$$F_g(x,y)=\sqrt{w_k(x)w_k(y)}\,e^{-i\langle x,gy\rangle}E_k(ix,gy);\quad (x,y)\in\mathbb{R}^n\times\mathbb{R}^n,\ g\in G$$

and $F=(F_g)_{g\in G}$. According to [6, Lemma 1], if $\xi=(\xi_1,\xi_2)\in\mathbb{R}^n\times\mathbb{R}^n$, then

$$\partial_\xi F_g(x,y)=\sum_{\upsilon\in R^+}k(\upsilon)\Big(\frac{\langle\upsilon,\xi_1\rangle}{\langle\upsilon,x\rangle}+\frac{\langle\upsilon,g\xi_2\rangle}{\langle\upsilon,gy\rangle}\Big)e^{-i\langle\upsilon,x\rangle\langle\upsilon,gy\rangle}F_{\sigma_\upsilon g}(x,y).$$

Here we use the G-invariance of the function k.

Let $\delta>0$ and $\kappa=(\kappa_1,\kappa_2)$ be a curve of $\mathbb{R}^n\times\mathbb{R}^n$ such that $\kappa_1,\kappa_2\colon(0,+\infty)\to C_\delta$ are two admissible curves. Recall here that a C^1-curve $\tau\colon(0,+\infty)\to C_\delta$ is admissible if $\lim_{t\to+\infty}|k(t)|=+\infty$ and $\tau'(t)\in C$ for all $t\in(0+\infty)$. Define $F^\kappa(t)=(F^\kappa_g)_{g\in G}$ where $F^\kappa_g(t)=F_g(\kappa_1(t),\kappa_2(t))$. We have

$$(F^\kappa_g)'(t)=\sum_{\upsilon\in R^+}k(\upsilon)\Big(\frac{\langle\upsilon,\kappa_1'(t)\rangle}{\langle\upsilon,\kappa_1(t)\rangle}+\frac{\langle\upsilon,g\kappa_2'(t)\rangle}{\langle\upsilon,g\kappa_2(t)\rangle}\Big)e^{-i\langle\upsilon,\kappa_1(t)\rangle\langle\upsilon,g\kappa_2(t)\rangle}F^\kappa_{\sigma_\upsilon g}(t)$$

and F^κ satisfies the differential equation

$$(F^\kappa)'(t)=A^\kappa(t)F^\kappa(t)$$

where the matrix $A^\kappa(t)$ is given by $A^\kappa(t)=\sum_{\upsilon\in R^+}k(\upsilon)B^\kappa_\upsilon(t)$ and

$$(B^\kappa_\upsilon(t))_{g,h}(t)=\begin{cases}\Big(\dfrac{\langle\upsilon,\kappa_1'(t)\rangle}{\langle\upsilon,\kappa_1(t)\rangle}+\dfrac{\langle\upsilon,g\kappa_2'(t)\rangle}{\langle\upsilon,g\kappa_2(t)\rangle}\Big)e^{-i\langle\upsilon,\kappa_1(t)\rangle\langle\upsilon,g\kappa_2(t)\rangle} & \text{if } h=\sigma_\upsilon g,\\ 0 & \text{otherwise.}\end{cases}$$

We will try to apply [6, Proposition 1]. For arbitrary $t>0$

$$\int_t^{+\infty}(B^\kappa_\upsilon(s))_{g,\sigma_\upsilon g}\,ds$$
$$=\lim_{T\to+\infty}\int_t^T\Big(\frac{\langle\upsilon,\kappa_1'(s)\rangle}{\langle\upsilon,\kappa_1(s)\rangle}+\frac{\langle\upsilon,g\kappa_2'(s)\rangle}{\langle\upsilon,g\kappa_2(s)\rangle}\Big)e^{-i\langle\upsilon,\kappa_1(s)\rangle\langle\upsilon,g\kappa_2(s)\rangle}\,ds$$

$$= \lim_{T\to+\infty} \int_{\langle \upsilon,\kappa_1(t)\rangle\langle g^{-1}\upsilon,\kappa_2(t)\rangle}^{\langle \upsilon,\kappa_1(T)\rangle\langle g^{-1}\upsilon,\kappa_2(T)\rangle} \frac{e^{-i\,u}}{u}\,du.$$

Let us point out that since $\kappa_2(t)\in C$ for all $t>0$ and the root system R is G-invariant then for each $g\in G$ and $t>0$ we have $\langle g^{-1}\upsilon,\kappa_2(t)\rangle>0$ if $g^{-1}\upsilon\in R^+$ and $\langle g^{-1}\upsilon,\kappa_2(t)\rangle<0$ if $g^{-1}\upsilon\in R^-$. From this fact it follows that

$$\int_{\langle \upsilon,\kappa_1(t)\rangle\langle g^{-1}\upsilon,\kappa_2(t)\rangle}^{\langle \upsilon,\kappa_1(T)\rangle\langle g^{-1}\upsilon,\kappa_2(T)\rangle} \frac{e^{-i\,u}}{u}\,du = \int_{|\langle \upsilon,\kappa_1(t)\rangle\langle \upsilon,g\kappa_2(t)\rangle|}^{|\langle \upsilon,\kappa_1(T)\rangle\langle g^{-1}\upsilon,\kappa_2(T)\rangle|} \frac{e^{-i\,\mathrm{sign}(g^{-1}\upsilon)u}}{u}\,du$$

where for $\beta\in R$

$$\mathrm{sign}(\beta)=\begin{cases}1, & \text{if } \beta\in R^+\\ -1, & \text{if } \beta\in R^-.\end{cases}$$

Now as $\big|\langle \upsilon,\kappa_1(T)\rangle\langle g^{-1}\upsilon,\kappa_2(T)\rangle\big|\ge \delta^2|\kappa_1(T)||\kappa_2(T)|\to+\infty$ as $T\to+\infty$ then we get

$$\int_t^{+\infty}(B^\kappa_\upsilon(s))_{g,\sigma_\upsilon g}\,ds = \int_{|\langle \upsilon,\kappa_1(t)\rangle\langle \upsilon,g\kappa_2(t)\rangle|}^{+\infty} \frac{e^{-i\,\mathrm{sign}(g^{-1}\upsilon)u}}{u}\,du.$$

This integral converges. Next we are led to examine the integrability of

$$I_{\upsilon,\beta,g}(t)=\left(\frac{\langle \upsilon,\kappa_1'(t)\rangle}{\langle \upsilon,\kappa_1(t)\rangle}+\frac{\langle \upsilon,g\kappa_2'(t)\rangle}{\langle \upsilon,g\kappa_2(t)\rangle}\right)e^{-i\langle \upsilon,\kappa_1(t)\rangle\langle \upsilon,gk_2(t)\rangle}$$
$$\times\int_{|\langle \beta,\kappa_1(t)\rangle\langle \beta,\sigma_\upsilon g\kappa_2(t)\rangle|}^{+\infty}\frac{e^{-i\,\mathrm{sign}(g^{-1}\upsilon)\,u}}{u}\,du$$

with $g\in G$ and $\upsilon,\beta\in R^+$. Observe that, by integration by parts,

$$\Big|\int_{|\langle \beta,\kappa_1(t)\rangle\langle \beta,\sigma_\upsilon g\kappa_2(t)\rangle|}^{+\infty}\frac{e^{-i\,\mathrm{sign}(g^{-1}\upsilon)\,u}}{u}\,du\Big|$$
$$\le\frac{2}{|\langle \beta,\kappa_1(t)\rangle\langle \beta,\sigma_\upsilon g\kappa_2(t)\rangle|}\le\frac{c}{|\langle \upsilon,\kappa_1(t)\rangle\langle \upsilon,g\kappa_2(t)\rangle|},$$

since

$$\big|\langle \beta,\sigma_\upsilon g\kappa_2(t)\rangle\big|=\big|\langle g^{-1}\sigma_\upsilon\beta,\kappa_2(t)\rangle\big|\ge\delta\big|\kappa_2(t)\big|\ge\delta\,|\langle \upsilon,g\kappa_2(t)\rangle|/\sqrt{2}$$

and similarly $|\langle \beta,\kappa_1(t)\rangle|=\delta\,|\langle \upsilon,\kappa_1(t)\rangle|/\sqrt{2}$. Then

$$|I_{\upsilon,\beta,g}(t)|\le c\,\mathrm{sign}(g^{-1}\upsilon)\,\frac{\langle \upsilon,\kappa_1'(t)\rangle\langle \upsilon,g\kappa_2(t)\rangle+\langle \upsilon,g\kappa_2'(t)\rangle\langle \upsilon,\kappa_1(t)\rangle}{(\langle \upsilon,\kappa_1(t)\rangle\langle \upsilon,g\kappa_2(t)\rangle)^2}$$

and for $t_0 > 0$

$$\int_{t_0}^{+\infty} |I_{\upsilon,\beta,g}(t)|\, dt \leq \frac{c}{\langle \upsilon, \kappa_1(t_0)\rangle |\langle \upsilon, g\kappa_2(t_0)\rangle|}.$$

Applying now [6, Proposition 1], we conclude that

$$\lim_{t\to+\infty} F_g^{\kappa}(t)$$

exists and is different from zero. Next we prove that this limit is independent of the choice of the admissible curves κ_1 and κ_2. Let ℓ_1 and ℓ_2 be other admissible curves in C_δ. One can construct admissible sequences $(x_n)_n$ and $(y_n)_n$ in C_δ such that $(x_{2n+1}, y_{2n+1}) \in (\kappa_1, \kappa_2)$ and $(x_{2n}, y_{2n}) \in (\ell_1, \ell_2)$. Let r_1 and r_2 be interpolating curves of $(x_n)_n$ and $(y_n)_n$ respectively. Hence we obtain that

$$\lim_{t\to\infty} F_g(\kappa_1(t), \kappa_2(t)) = \lim_{t\to\infty} F_g(r_1(t), r_2(t)) = \lim_{t\to\infty} F_g(\ell_1(t), \ell_2(t)).$$

Now let $x, y \in \mathbb{R}^n \setminus \{0\}$. If we take the admissible curves $\ell_1(t) = tx$ and $\ell_2(t) = ty$ then from [6, Theorem 1] there exists a constant non-zero vector $v = (v_g)_{g\in G} \in \mathbb{G}^{|G|}$ such that

$$\lim_{t\to\infty} F_g(\ell_1(t), \ell_2(t)) = \lim_{t\to\infty} F_g(t^2 x, y) = v_g.$$

Therefore we conclude that

$$\lim_{t\to\infty} F_g(\kappa_1(t), \kappa_2(t)) = v_g \tag{2.8}$$

for any admissible curves κ_1 and κ_2. Now to obtain (2.7) we claim that

$$\lim_{|x|.|y|\to+\infty;\, x,y\in C_\delta} \sqrt{w_k(x) w_k(y)}\, e^{-\langle ix, gy\rangle} E_k(ix, gy) = v_g.$$

Indeed, if it is not true, then we can find $\varepsilon > 0$ and sequences $(x_n)_n$ and $(y_n)_n$ of C_δ such that $|x_n|.|y_n| \to +\infty$ and

$$|F_g(x_n, y_n) - v_g| > \varepsilon.$$

We can assume that $|x_n| \to +\infty$ and $|y_n| \to +\infty$, since we have

$$F_g(x_n, y_n) = F_g(\sqrt{|x_n||y_n|}x'_n, (\sqrt{|x_n||y_n|}y'_n)$$

where $x'_n = x_n/|x_n|$ and $y_n = y'_n/|y_n|$. As at the end of the proof of [6, Theorem 1], we may also assume that $(x_n)_n$ and $(y_n)_n$ are admissible. It follows from (2.8) that

$$\lim_{t\to\infty} F_g(x_n, y_n) = v_g\ ,$$

a contradiction. This ends the proof of Theorem 2.6. $\square$

THEOREM 2.7. *Let $\delta > 0$. There exists a constant $c = c(\delta) > 0$, such that*

$$|E_k(ix, gy)| \leq \frac{c}{\sqrt{w_k(x)w_k(y)}} \tag{2.9}$$

for all $x, y \in C_\delta$ and $g \in G$.

PROOF. From Theorem 2.6 we can find $M > 0$ such that for all $x, y \in C_\delta$, $|x|.|y| \geq M$ we have

$$\sqrt{w_k(x)w_k(y)}\,|E_k(ix, gy)| \leq |v_g| + 1$$

for all $g \in G$. When $|x|.|y| \leq M$, we use (1.3) to obtain

$$\sqrt{w_k(x)w_k(y)}\,|E_k(ix, gy)| \leq \Big(\prod_{v\in R^+} |v|\Big)^{2\gamma_k} (|x|.|y|)^{\gamma_k} \leq M^{\gamma_k}\Big(\prod_{v\in R^+} |v|\Big)^{2\gamma_k}.$$

Hence (2.9) follows. □

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