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ON THE DISTRIBUTION OF MEROMORPHIC FUNCTIONS OF POSITIVE HYPER-ORDER

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Abstract. Let $f(z)$ be a transcendental meromorphic function, whose zeros have multiplicity at least 3. Set $\alpha(z) := \beta(z)\exp(\gamma(z))$, where $\beta(z)$ is a nonconstant elliptic function and $\gamma(z)$ is an entire function. If $\sigma(f(z)) > \sigma(\alpha(z))$, then $f'(z) = \alpha(z)$ has infinitely many solutions in the complex plane.

1. Introduction

Hayman [1] proved the following result.

THEOREM A. *Let f be a transcendental meromorphic function and α be a finite nonzero complex number. If $f(z) \neq 0$ for each z, then $f' = \alpha$ has infinitely many solutions in $\mathbb{C}$.*

A meromorphic function $\alpha(z)$ is called a small function with respect to $f(z)$ provided that $T(r, \alpha(z)) = o\{T(r, f(z))\}$ as $r \to \infty$ outside of a possible exceptional set of r of finite linear measure.

Naturally, we ask that whether Theorem A is valid or not if the finite non-zero complex number α is replaced by a small function $\alpha(z)$ related to $f(z)$.

The defect relation for small functions [10, Corollary 2] due to Yamanoi directly implies the following two theorems.

THEOREM B. *Let f be a transcendental meromorphic function, and let α be a small meromorphic functions with respect to f. Assume that all but finitely many zeros of f' have multiplicity at least 3. Then $f'(z) = \alpha(z)$ has infinitely many solutions in $\mathbb{C}$.*

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Theorem C. *Let f be a transcendental meromorphic function, and let α_1 and α_2 be two small meromorphic functions with respect to f. Then either $f'(z) = \alpha_1(z)$ or $f'(z) = \alpha_2(z)$ has infinitely many solutions in $\mathbb{C}$.*

In 2008, Theorem A was generalized by Pang, Nevo and Zalcman.

Theorem D [5]. *Let f be a transcendental meromorphic function, whose zeros are multiple, and let $\alpha\,(\not\equiv 0)$ be a rational function. Then $f' = \alpha$ has infinitely many solutions in $\mathbb{C}$.*

We wonder if Theorem D still holds provided that $\alpha(z)$ is a transcendental meromorphic function. In this direction, we proved the following result.

Theorem E [12]. *Let $f(z)$ be a transcendental meromorphic function, whose poles are multiple and whose zeros have multiplicity at least 3. Set $\alpha(z) := \beta(z)\exp(\gamma(z))$, where $\beta(z)$ is a nonconstant elliptic function and $\gamma(z)$ is an entire function. If $\sigma(f(z)) > \sigma(\alpha(z))$, then $f'(z) = \alpha(z)$ has infinitely many solutions in $\mathbb{C}$.*

In this paper we show the assumption that all poles of f are multiple in Theorem E is unnecessary. We extend Theorem E as follows.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function, all but finitely many of whose zeros have multiplicity at least 3. Set $\alpha(z) := \beta(z)\exp(\gamma(z))$, where $\beta(z)$ is a nonconstant elliptic function and $\gamma(z)$ is an entire function. If $\sigma(f(z)) > \sigma(\alpha(z))$, then $f'(z) = \alpha(z)$ has infinitely many solutions in $\mathbb{C}$ (including the possibility of infinitely many common poles of $f(z)$ and $\alpha(z)$).*

2. Notation and preliminary lemmas

Let $\mathbb{C}$ be the complex plane and D be a domain in $\mathbb{C}$. For $z_0 \in \mathbb{C}$ and $r > 0$, we write $\Delta(z_0, r) := \{z \mid |z - z_0| < r\}$, $\overline{\Delta}(z_0, r) := \{z \mid |z - z_0| \le r\}$, $\Delta'(z_0, r) := \{z \mid 0 < |z - z_0| < r\}$, $\Delta := \Delta(0, 1)$ and $\Delta' := \Delta'(0, 1)$. Let $n(r, f)$ denote the number of poles of $f(z)$ in $\Delta(0, r)$ (counting multiplicity). We write $f_n \overset{\chi}{\Rightarrow} f$ in D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and $f_n \Rightarrow f$ in D if the convergence is in the Euclidean metric. For f meromorphic in D, we write

$$f^{\#}(z) := \frac{|f'(z)|}{1 + |f(z)|^2}, \tag{2.1}$$

$$S(D, f) := \frac{1}{\pi}\iint_D [f^{\#}(z)]^2\,\mathrm{d}x\,\mathrm{d}y \quad\text{and}\quad S(r, f) := S(\Delta(0, r), f).$$

The Ahlfors–Shimizu characteristic is defined by $T(r,f)=\int_0^r \frac{S(t,f)}{t}\,\mathrm{d}t$. The order $\rho(f)$ and the hyper-order $\sigma(f)$ of a meromorphic function f are defined as follows:

$$\rho(f):=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}\quad\text{and}\quad \sigma(f):=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}.$$

Lemma 2.1 [9]. *Let m be a positive integer and R be a rational function. If $R'(z)\neq z^{-m}$ for each z, then R is a constant function.*

Lemma 2.2 [8]. *Let $R(z)=a_nz^n+a_{n-1}z^{n-1}+\dots+a_0+\frac{Q(z)}{P(z)}$, where a_0, a_1, …, a_n are constants with $a_n\neq 0$, $P(z)$ and $Q(z)$ are two coprime polynomials with $\deg Q(z)<\deg P(z)$. If $R'(z)\neq 1$, then $R(z)=z+a+\frac{b}{(z-c)^m}$, where a, $b\,(\neq 0)$, c are constants and m is a positive integer.*

Lemma 2.3 [8]. *Let f be a nonconstant meromorphic function of finite order in $\mathbb{C}$, whose zeros are multiple. If $f'(z)\neq 1$ for each z, then $f(z)=\frac{(z-a)^2}{z-b}$ for some a and $b\,(\neq a)$.*

Lemma 2.4. *Let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros have multiplicity at least k, and suppose that there exists $A\geq 1$ such that $|f^{(k)}(z)|\leq A$ whenever $f(z)=0$. If $\mathcal{F}$ is not normal at z_0, then there exist*

(a) *points z_n, $z_n\to z_0$;*

(b) *functions $f_n\in\mathcal{F}$; and*

(c) *positive numbers $\rho_n\to 0$*

such that $\rho_n^{-k}f_n(z_n+\rho_n\zeta)=g_n(\zeta)\overset{\chi}{\Rightarrow} g(\zeta)$ in $\mathbb{C}$, where g is a nonconstant meromorphic function in $\mathbb{C}$ such that $g^{\#}(\zeta)\leq g^{\#}(0)=kA+1$. In particular, g has order at most 2.

This is the local version of [6, Lemma 2] (cf. [3, Lemma 1]; [13, pp. 216–217]). The proof consists of a simple change of variable in the result cited from [6] (cf. [4, pp. 299–300]).

Lemma 2.5 [2]. *Let k be a positive integer and let $\{f_n\}$ be a family of meromorphic functions in Δ, all of whose zeros have multiplicity at least $k+1$. If $a_n\to 0$ and $f_n^{\#}(a_n)\to\infty$, then there exist*

(a) *points $z_n\to 0$;*

(b) *a subsequence of $\{f_n\}$ (still denoted by $\{f_n\}$); and*

(c) *positive numbers $\rho_n\leq\frac{M}{\sqrt[k+1]{f_n^{\#}(a_n)}}$, where M is a constant which is independent on n, such that $g_n(\zeta)=\rho_n^{-k}f_n(z_n+\rho_n\zeta)\overset{\chi}{\Rightarrow} g(\zeta)$ in $\mathbb{C}$, where g is a nonconstant meromorphic function in $\mathbb{C}$ such that $g^{\#}(\zeta)\leq g^{\#}(0)=k+1$. In particular, g has order at most 2.*

Lemma 2.6 [7]. *Let f be a meromorphic function in Δ, and let a_1, a_2, a_3 be three distinct complex numbers. Assume that the number of zeros of $\prod_{i=1}^{3}(f(z)-a_i)$ in Δ is $\leq n$, where multiple zeros are counted only once. Then*

$$S(r,f) \leq n + \frac{A}{1-r}, \quad 0 \leq r < 1,$$

where $A>0$ is a constant, which depends on a_1, a_2, a_3 only.

Lemma 2.7 [11]. *Let f be a meromorphic function in $\mathbb{C}$ and α be a nonzero constant. Then $(\alpha f)^{\#}(z) \leq \max\{|\alpha|, 1/|\alpha|\} f^{\#}(z)$.*

Lemma 2.8 [12]. *Let $f(z)$ be a meromorphic function of hyper-order $\sigma(f)>0$, and let $\varepsilon \in (0,\sigma(f))$ denote a fixed constant. Then there exist $a_n \to \infty$ and $\delta_n \to 0$ such that*

$$S(\Delta(a_n,\delta_n),f) \geq \exp\left(|a_n|^{\sigma(f)-\varepsilon}\right),$$

$$f^{\#}(a_n) \geq \exp\left(|a_n|^{\sigma(f)-\varepsilon}\right) \quad \text{and} \quad \delta_n \leq \exp\left(-|a_n|^{\sigma(f)-\varepsilon}\right).$$

Lemma 2.9 [12]. *Let $f(z)$, $g(z)$ be meromorphic functions in $\Delta(0,\rho)$ and let r, R be positive numbers satisfying $r<R<\rho$. Then*

$$S(r,fg) \leq S(R,f) + S(R,g)$$
$$+ \frac{1}{2\pi}\Big(\log\frac{R}{r}\Big)^{-1}\int_0^{2\pi}\log\big(|g(re^{i\theta})| + |g(re^{i\theta})|^{-1}\big)\,d\theta.$$

Lemma 2.10. *Assume that the conditions of Theorem 1.1 are satisfied, $f'(z)-\alpha(z)$ has at most finitely many solutions, and $\beta(z)=z^m\widehat{\beta}(z)$, where $\widehat{\beta}(z)\,(\neq 0)$ is holomorphic in Δ and m is an integer. Let $a_n \to \infty$ and $b_n \to 0$ be sequences of complex numbers such that $\beta(z+a_n-b_n)=\beta(z)$. Let $\{t_n\}$ be a sequence of positive numbers such that $t_n \leq \exp(-|a_n|^{\lambda})$, where $\lambda \in (\sigma(\alpha),\sigma(f))$. Set $T_n(\zeta) := \frac{f(a_n-b_n+t_n\zeta)}{t_n^{m+1}\exp(\gamma(a_n-b_n+t_n\zeta))}$. Then $\{T_n(\zeta)\}$ is normal in $\mathbb{C}\setminus\{0\}$.*

Using the same argument as in the proof of [12, Lemma 3.4], we can show Lemma 2.10 holds. In fact, the condition that all poles of $f(z)$ are multiple is not necessary in the proof of [12, Lemma 3.4].

3. Auxiliary lemmas

Lemma 3.1. *Let f be a nonconstant meromorphic function, whose zeros have multiplicity at least 3. Then for any finite nonzero complex number c, $f'-c$ has at least one zero in $\mathbb{C}$.*

Proof. Suppose that there exists a finite non-zero complex number c such that $f' - c$ has no zeros in $\mathbb{C}$. By Theorem D and Lemma 2.3, $f(z) = \frac{c(z-a)^2}{z-b}$ for some a and $b\,(\neq a)$. This contradicts the fact that all zeros of f have multiplicity at least 3. $\square$

Lemma 3.2. *Let n be a positive integer, and $R(z)$ be a rational function, whose zeros have multiplicity at least 3. If $R'(z) \neq z^n$ for each z, then $n = 1$ and $R(z) = \frac{(z-\frac{c}{3})^3}{2(z-c)}$, where c is a nonzero constant.*

Proof. We consider the following two cases.

Case 1: $R(z)$ is a polynomial. Clearly, we have $R(z) = \frac{z^{n+1}}{n+1} + az + b$ in $\mathbb{C}$, where $a \neq 0$ and b are constant. However $R'(z) = z^n + a$ which contradicts that all zeros of $R(z)$ have multiplicity at least 3.

Case 2: $R(z)$ is not a polynomial. Since $R'(z) \neq z^n$, we have $\big(R(z) - \frac{z^{n+1}}{n+1} + z\big)' \neq 1$ for each $z \in \mathbb{C}$. By Lemma 2.2,

$$R(z) = \frac{z^{n+1}}{n+1} + a + \frac{b}{(z-c)^m} = \frac{z^{n+1}(z-c)^m + a(n+1)(z-c)^m + b(n+1)}{(n+1)(z-c)^m},$$

where a, $b\,(\neq 0)$ and c are constant, m is a positive integer. Then we have

$$R'(z) = z^n - \frac{bm}{(z-c)^{m+1}}, \tag{3.1}$$

$$R''(z) = nz^{n-1} + \frac{bm(m+1)}{(z-c)^{m+2}}. \tag{3.2}$$

By (3.1) and (3.2), we see that $R(z)$ has a unique (multiple) zero $z_0 = \frac{n}{m+n+1}c$.

We claim that $c \neq 0$. Otherwise, substituting $c = 0$ into (3.1), we obtain $R'(z) = \frac{z^{m+n+1} - bm}{z^{m+1}}$ which contradicts that all zeros of $R(z)$ have multiplicity at least 3.

Set

$$P(z) := z^{n+1}(z-c)^m + a(n+1)(z-c)^m + b(n+1).$$

A simple calculation shows that

$$R(z) = \frac{z^{n+1}(z-c)^m + a(n+1)(z-c)^m + b(n+1)}{(n+1)(z-c)^m} = \frac{P(z)}{(n+1)(z-c)^m}.$$

Clearly, $P(z)$ and $R(z)$ have the same zeros with the same multiplicities. Then we have

$$P(z) = z^{n+1}(z-c)^m + a(n+1)(z-c)^m + b(n+1) = \Big(z - \frac{cn}{m+n+1}\Big)^{m+n+1}. \tag{3.3}$$

Comparing the coefficients of the term z^{m+n}, we obtain $m = n$. Take the derivative of both sides of the equation (3.3), we obtain

$$(z-c)^{n-1}\left[(n+1)z^n(z-c) + nz^{n+1} + an(n+1)\right] = (2n+1)\Big(z - \frac{cn}{2n+1}\Big)^{2n}. \tag{3.4}$$

Comparing the constant terms of both sides in (3.4), we see that $(z-c)^{n-1}$ must be constant and thus $m = n = 1$. Then $z_0 = \frac{c}{3}$ and $R(z) = \frac{(z-\frac{c}{3})^3}{2(z-c)}$, where c is a constant. □

LEMMA 3.3. *Let $R(z)\,(\not\equiv 0)$ be a rational function, having a zero of order 2 at the point $z = 0$. If $R'(z) \neq z$ for each $z \in \mathbb{C}\setminus\{0\}$, then $R(z) = cz^2$, where $c\,(\neq 1/2)$ is a nonzero constant.*

PROOF. Clearly, $R(z) - \frac{z^2}{2}$ is not a constant. We assume that $z = 0$ is a zero of $R(z) - \frac{z^2}{2}$ of order $\lambda\,(\geq 2)$. Set $\frac{q(z)}{p(z)} := R(z) - \frac{z^2}{2}$, where $p(z)$ and $q(z)$ are two coprime polynomials.

Case 1: $\deg p(z) \neq \deg q(z)$.

$$R'(z) - z = \Big(\frac{q(z)}{p(z)}\Big)' = \frac{q'(z)p(z) - p'(z)q(z)}{p^2(z)} \neq 0 \text{ for each } z \in \mathbb{C}\setminus\{0\}. \tag{3.5}$$

Let $q(z) = a_m z^m + a_{m-1}z^{m-1} + \dots + a_0$ and $p(z) = b_n z^n + b_{n-1}z^{n-1} + \dots + b_0$, where $a_m\,(\neq 0), \dots, a_1, a_0$ and $b_n\,(\neq 0), \dots, b_1, b_0$ are constants. Clearly, $m \geq \lambda \geq 2$,

$$q'(z)p(z) - p'(z)q(z) = (m-n)a_m b_n z^{m+n-1} + \dots + (a_1 b_0 - a_0 b_1)$$

and $z = 0$ is a zero of order $\lambda - 1$ of $q'(z)p(z) - p'(z)q(z)$. We denote nonzero zeros of $q'(z)p(z) - p'(z)q(z)$ by $c_1, c_2, \dots, c_l$, and the related orders denote by $n_1, n_2, \dots, n_l$.

We deduce from (3.5) that the nonzero zeros of $q'(z)p(z) - p'(z)q(z)$ are the zeros of $p^2(z)$. Since $q(z)$ and $p(z)$ are coprime, we can see from (3.5) that c_i is the zero of $p(z)$ with order $n_i + 1$ $(i = 1, 2, \dots, l)$. Then $n_1 + n_2 + \dots + n_l + \lambda - 1 = m + n - 1$ and $2(n_1 + n_2 + \dots + n_l + l) \leq 2n$. It is easily obtained that $(m - \lambda) + l \leq 0$. We have $l = 0$, $m = \lambda$, $q(z) = a_m z^m$ and $q'(z)p(z) - p'(z)q(z) = (m-n)a_m b_n z^{m+n-1}$. We also have $q'(z)p(z) - p'(z)q(z) = a_m z^{m-1}(mp(z) - zp'(z))$. If $\deg p(z) \neq 0$, then $[mp(z) - zp'(z)]\big|_{z=0} = 0$ and thus $p(0) = 0$ which contradicts the fact $z = 0$ is a zero of $\frac{q(z)}{p(z)}$. Now $\deg p(z) = 0$ and $R(z) = cz^m$, where $c\,(\neq 0)$ is a constant. By (3.5), $cmz^{m-2} - 1 \neq 0$ for each $z \in \mathbb{C}\setminus\{0\}$. Then $R(z) = cz^2$, where $c\,(\neq 1/2)$ is a nonzero constant.

Case 2: $\deg p(z) = \deg q(z)$. Write $R(z) - \frac{z^2}{2} = c + \frac{r(z)}{p(z)}$, where $c\,(\neq 0)$ is a constant, $p(z)$ and $r(z)$ are two coprime polynomials and $\deg p(z) > \deg r(z)$. Now, we have

$$\left(c+\frac{r(z)}{p(z)}\right)' = \left(\frac{r(z)}{p(z)}\right)' = \frac{q'(z)p(z) - p'(z)q(z)}{p^2(z)} \neq 0 \text{ for each } z \in \mathbb{C} \setminus \{0\}. \tag{3.6}$$

Let $r(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_0$ and $p(z) = b_n z^n + b_{n-1} z^{n-1} + \dots + b_0$, where $a_m\,(\neq 0), \dots, a_1, a_0$ and $b_n\,(\neq 0), \dots, b_1, b_0$ are constants. Since $z = 0$ is a zero of $R(z) - \frac{z^2}{2}$ of order $\lambda\,(\geq 2)$, we have $r(0) = a_0 \neq 0$ and $p(0) = b_0 \neq 0$. Using the same argument presented in Case 1, we can show that $r'(z)p(z) - p'(z)r(z) = (m-n)a_m b_n z^{m+n-1}$. Then

$$(m-n)a_m b_n z^{m+n-1} + \dots + (a_1 b_0 - a_0 b_1) = (m-n)a_m b_n z^{m+n-1}. \tag{3.7}$$

Comparing the coefficients of the term z^i in (3.7) for $i = 0, 1, 2, \dots, m-1$, we obtain

$$\frac{a_1}{a_0} = \frac{b_1}{b_0}, \quad \frac{a_2}{a_0} = \frac{b_2}{b_0}, \quad \dots, \quad \frac{a_m}{a_0} = \frac{b_m}{b_0}.$$

Comparing the coefficients of the term z^i in (3.7) for $i = m, m+1, \dots, n-1$, we obtain $b_{m+1} = b_{m+2} = \dots = b_n = 0$, a contradiction. □

4. Proof of Theorem 1.1

We assume that $f'(z) = \alpha(z)$ has at most finitely many solutions and derive a contradiction. In the following part, let $\varepsilon \in (0, (\sigma(f) - \sigma(\alpha))/8)$ denote a fixed constant.

By our assumptions,

$$f'(z) \neq \alpha(z) \quad \text{and} \quad \frac{f'(z)}{\alpha(z)} \neq 1 \quad \text{for sufficiently large } |z|. \tag{4.1}$$

Set $F(z) := \frac{f(z)}{\alpha(z)}$. Clearly, $\sigma(F) = \sigma(f)$. Noting that $\alpha(z) = \beta(z)\exp(\gamma(z))$, we have $\sigma(F) = \sigma(f) > \sigma(\alpha) \geq \sigma(\exp(\gamma)) = \rho(\gamma)$. By an elementary calculation we have

$$\frac{f'(z)}{\alpha(z)} = F'(z) + F(z)\left(\frac{\beta'(z)}{\beta(z)} + \gamma'(z)\right). \tag{4.2}$$

By Lemma 2.8, there exist $a_n \to \infty$ and $\delta_n \to 0$ such that

$$S(\Delta(a_n, \delta_n), F) \geq \exp(|a_n|^{\sigma(f)-\varepsilon}), \tag{4.3}$$

$$F^{\#}(a_n) \geq \exp(|a_n|^{\sigma(f)-\varepsilon}) \quad \text{and} \quad \delta_n \leq \exp(-|a_n|^{\sigma(f)-\varepsilon}).$$

Let ω_1, ω_2 be the two fundamental periods of $\beta(z)$ and $\mathfrak{P}$ be a fundamental parallelogram of $\beta(z)$. There exist integers i_n and j_n such that $b_n \in \mathfrak{P}$, where $b_n = a_n - i_n\omega_1 - j_n\omega_2$. Taking a subsequence and renumbering, we may assume that $b_n \to b^*$ as $n \to \infty$.

Without loss of generality, we may assume that $b^* = 0$, $\Delta \subset \mathfrak{P}$, and $\beta(z) = z^m\widehat{\beta}(z)$ for $z \in \Delta$, where $\widehat{\beta}(0) = 1$, $\widehat{\beta}(z) \neq 0, \infty$ in Δ, and m is an integer. For convenience, we set

$$F_n(z) := F(a_n - b_n + z) \text{ for } z \in \Delta, \quad f_n(z) := f(a_n - b_n + z), \tag{4.4}$$

$$\alpha_n(z) := \alpha(a_n - b_n + z) \text{ and } \gamma_n(z) := \gamma(a_n - b_n + z) \text{ for } z \in \Delta. \tag{4.5}$$

Taking a subsequence and renumbering if necessary, we may assume that

(a1) $f_n'(z) \neq \alpha_n(z) = \beta(z)\gamma_n(z)$ in Δ,

(a2) $S(\Delta(b_n, \delta_n), F_n) \geq \exp\left(|a_n|^{\sigma(f)-\varepsilon}\right)$ and $F_n^{\#}(b_n) \geq \exp\left(|a_n|^{\sigma(f)-\varepsilon}\right)$,

(a3) $1 \neq \frac{f_n'(z)}{\alpha_n(z)} = F_n'(z) + F_n(z)\left(\frac{\beta'(z)}{\beta(z)} + \gamma_n'(z)\right)$ in Δ.

In fact, It follows from (4.1) and (4.5) that (a1) holds. Noting that $S(\Delta(b_n, \delta_n), F_n) = S(\Delta(a_n, \delta_n), F)$ and $F_n^{\#}(b_n) = F^{\#}(a_n)$, we see that (a2) holds by (4.3) and (4.4). Substituting $z = a_n - b_n + z$ into (4.2), we get that (a3) holds by (4.1) and (4.5).

We claim that $\beta(0) = 0$ or $\beta(0) = \infty$. On the contrary, suppose that $\beta(0) \neq 0, \infty$. Clearly, all zeros F_n have multiplicity at least 3 for sufficiently large n in Δ. By (a2) and Marty's criterion, $\{F_n\}$ is not normal at 0. Using Lemma 2.5 for $k = 1$, there exist points $z_n \to 0$, a subsequence of $\{F_n\}$ (still denoted by $\{F_n\}$) and positive numbers $\rho_n \leq \frac{M}{\sqrt{F_n^{\#}(b_n)}}$, where M is a constant which is independent on n, such that

$$G_n(\zeta) = \rho_n^{-1}F_n(z_n + \rho_n\zeta) \overset{\chi}{\Rightarrow} G(\zeta) \text{ in } \mathbb{C}, \tag{4.6}$$

where G is a nonconstant meromorphic function in $\mathbb{C}$, whose zeros have multiplicity at least 3. By (a2), we see that

$$\rho_n \leq M\exp\left(-\frac{1}{2}|a_n|^{\sigma(f)-\varepsilon}\right) \leq \exp\left(-|a_n|^{\sigma(f)-2\varepsilon}\right) \tag{4.7}$$

for sufficiently large n. For any given $R > 0$, we have

$$\begin{aligned} |\gamma'(a_n - b_n + z_n + \rho_n\zeta)| &\leq M(|2a_n|, \gamma') \\ &\leq \exp(|2a_n|^{\rho(\gamma)+\varepsilon}) \leq \exp(|a_n|^{\sigma(\alpha)+2\varepsilon}) \end{aligned} \tag{4.8}$$

for sufficiently large n in $\Delta(0, R)$. By (4.7) and (4.8), we see that

$$\rho_n\gamma'(a_n - b_n + z_n + \rho_n\zeta) \Rightarrow 0 \text{ in } \mathbb{C}. \tag{4.9}$$

Then

$$\text{(4.10)} \quad \frac{\beta'(z_n+\rho_n\zeta)}{\beta(z_n+\rho_n\zeta)} \Rightarrow \frac{\beta'(0)}{\beta(0)} \text{ in } \mathbb{C}, \quad \text{and thus} \quad \rho_n \frac{\beta'(z_n+\rho_n\zeta)}{\beta(z_n+\rho_n\zeta)} \Rightarrow 0 \text{ in } \mathbb{C}.$$

Substituting $z = z_n + \rho_n\zeta$ into (a3), we have

$$\text{(4.11)} \quad 1 \neq \frac{f_n'(z_n+\rho_n\zeta)}{\alpha_n(z_n+\rho_n\zeta)}$$

$$= G_n'(\zeta) + \Big[\rho_n \frac{\beta'(z_n+\rho_n\zeta)}{\beta(z_n+\rho_n\zeta)} + \rho_n\gamma'(a_n - b_n + z_n + \rho_n\zeta)\Big] G_n(\zeta).$$

By (4.9)–(4.11),

$$\text{(4.12)} \quad 1 \neq \frac{f_n'(z_n+\rho_n\zeta)}{\alpha_n(z_n+\rho_n\zeta)} \Rightarrow G'(\zeta) \ \text{ in } \mathbb{C} \setminus G^{-1}(\infty).$$

By Hurwitz's theorem, either $G'(\zeta) \equiv 1$ or $G'(\zeta) \neq 1$ in $\mathbb{C}$. This contradicts to Lemma 3.1.

Next, we consider the cases $\beta(0) = 0$ and $\beta(0) = \infty$. We claim that $b_n \leq \exp(-|a_n|^{\sigma(f)-3\varepsilon})$ for sufficiently large n. Otherwise, taking a subsequence and renumbering, we may assume that $b_n > \exp(-|a_n|^{\sigma(f)-3\varepsilon})$. Set

$$\text{(4.13)} \quad \eta_n := \exp(-|a_n|^{\sigma(f)-2\varepsilon}),$$

$$B_n(z) := \frac{F(a_n+\eta_n z)}{\eta_n} = \frac{F_n(b_n+\eta_n z)}{\eta_n} \quad \text{for } z \in \Delta.$$

Noting that $\eta_n \to 0$, $b_n \to 0$ and $\eta_n/b_n \to 0$ as $n \to \infty$, we see that

$$\text{(4.14)} \quad b_n + \eta_n z \in \Delta \ \text{ and } \ b_n + \eta_n z \neq 0 \ \text{ for sufficiently large } n \text{ in } \Delta,$$

and hence all zeros of $B_n(z)$ have multiplicity at least 3 for sufficiently large n in Δ. By (a2), for sufficiently large n we have

$$\text{(4.15)} \quad B_n^{\#}(0) = \eta_n^2 \frac{|F_n'(b_n)|}{\eta_n^2 + |F_n(b_n)|^2}$$

$$\geq \eta_n^2 \frac{|F_n'(b_n)|}{1 + |F_n(b_n)|^2} = \eta_n^2 F_n^{\#}(b_n) > \exp(|a_n|^{\sigma(f)-2\varepsilon}).$$

Clearly, $B_n^{\#}(0) \to \infty$. By Marty's criterion, $\{B_n(z)\}$ is not normal at 0. Using Lemma 2.4 for $k = 1$, there exist points $z_n \to 0$, a subsequence of $\{B_n(z)\}$ (still denoted by $\{B_n(z)\}$) and positive numbers $\rho_n \to 0$ such that

$$\text{(4.16)} \quad G_n(\zeta) = \frac{B_n(z_n+\rho_n\zeta)}{\rho_n} = \frac{F_n(b_n+\eta_n(z_n+\rho_n\zeta))}{\rho_n\eta_n} \overset{\chi}{\Rightarrow} G(\zeta) \ \text{ in } \mathbb{C},$$

where G is a nonconstant meromorphic function in $\mathbb{C}$ whose zeros have multiplicity at least 3. Noting that $b_n/\eta_n \to \infty$ as $n \to \infty$, we obtain

$$\begin{aligned}(4.17)\qquad &\eta_n \frac{\beta'(b_n+\eta_n(z_n+\rho_n\zeta))}{\beta(b_n+\eta_n(z_n+\rho_n\zeta))}\\ &= \frac{m}{b_n/\eta_n+z_n+\rho_n\zeta} + \eta_n \frac{\widehat{\beta}'(b_n+\eta_n(z_n+\rho_n\zeta))}{\widehat{\beta}(b_n+\eta_n(z_n+\rho_n\zeta))} \Rightarrow 0 \ \text{ in } \mathbb{C}.\end{aligned}$$

For any given $R>0$, we see that

$$|\gamma'(a_n+\eta_n(z_n+\rho_n\zeta))| \le M(|2a_n|,\gamma') \le \exp(|2a_n|^{\rho(\gamma)+\varepsilon}) \le \exp(|a_n|^{\sigma(\alpha)+2\varepsilon})$$

for sufficiently large n in $\Delta(0,R)$. Then

$$(4.18)\qquad \eta_n\gamma'(a_n+\eta_n(z_n+\rho_n\zeta)) \Rightarrow 0 \ \text{ in } \mathbb{C}.$$

Substituting $z=b_n+\eta_n(z_n+\rho_n\zeta)$ into (a3), for sufficiently large n we have

$$\begin{aligned}(4.19)\qquad &1 \ne \frac{f_n'(b_n+\eta_n(z_n+\rho_n\zeta))}{\alpha_n(b_n+\eta_n(z_n+\rho_n\zeta))}\\ &= G_n'(\zeta) + \rho_n\eta_n\Big(\frac{\beta'(b_n+\eta_n(z_n+\rho_n\zeta))}{\beta(b_n+\eta_n(z_n+\rho_n\zeta))} + \gamma'(a_n+\eta_n(z_n+\rho_n\zeta))\Big)G_n(\zeta).\end{aligned}$$

By (4.16)–(4.19), we obtain

$$1 \ne \frac{f_n'(b_n+\eta_n(z_n+\rho_n\zeta))}{\alpha_n(b_n+\eta_n(z_n+\rho_n\zeta))} \overset{\chi}{\Rightarrow} G'(\zeta) \ \text{ in } \mathbb{C}\setminus G^{-1}(\infty).$$

By Hurwitz's theorem, either $G'(\zeta)\equiv 1$ or $G'(\zeta)\ne 1$ in $\mathbb{C}$. This contradicts to Lemma 3.1.

Set $\sigma_n := \exp(-|a_n|^{\sigma(f)-5\varepsilon})$, $\lambda_n := \exp(-|a_n|^{\sigma(f)-6\varepsilon})$, $S_n(z) := \frac{F_n(z)}{z}$ and $\widehat{S}_n(z) := S_n(\lambda_n z)$. We claim that

$$(4.20)\qquad S(1,\widehat{S}_n(z)) \ge \exp(|a_n|^{\sigma(f)-3\varepsilon}) \ \text{ for sufficiently large } n.$$

Clearly, $\Delta(b_n,\delta_n)\subset\Delta(0,\sigma_n)$ and $S(\Delta(0,\sigma_n),F_n) > S(\Delta(b_n,\delta_n),F_n)$ for sufficiently large n. Since $\frac{\sigma_n}{\lambda_n}\to 0$ as $n\to\infty$, for sufficiently large n we have

$$(4.21)\quad S\Big(\frac12,F_n(\lambda_n z)\Big) > S\Big(\frac{\sigma_n}{\lambda_n},F_n(\lambda_n z)\Big) = S(\sigma_n,F_n(z)) \ge \exp(|a_n|^{\sigma(f)-\varepsilon}).$$

It follows from Lemma 2.9 that

$$(4.22)\qquad S\Big(\frac12,F_n(\lambda_n z)\Big) = S\Big(\frac12,\frac{F_n(\lambda_n z)}{z}\cdot z\Big)$$

$$\le S\Big(1, \frac{F_n(\lambda_n z)}{z}\Big) + S(1,z) + \frac{\log 5 - \log 2}{\log 2}$$

for sufficiently large n. (4.21) and (4.22) imply

$$S\Big(1, \frac{F_n(\lambda_n z)}{z}\Big) \ge \exp\big(|a_n|^{\sigma(f)-2\varepsilon}\big) \text{ for sufficiently large } n. \tag{4.23}$$

By (2.1) and Lemma 2.7, for sufficiently large n we have

$$S(1, \widehat{S}_n(z)) = S\Big(1, \frac{F_n(\lambda_n z)}{\lambda_n z}\Big) \ge \lambda_n^2\, S\Big(1, \frac{F_n(\lambda_n z)}{z}\Big) \ge \exp\big(|a_n|^{\sigma(f)-3\varepsilon}\big).$$

We consider the following two cases.

Case 1: $\beta(0)=0$. Set $\mathbb{D}_n := \{z \mid |S_n(z)| = 3,\ |z| \le 2\lambda_n\}$. We claim that $\mathbb{D}_n$ is non-empty set for sufficiently large n. Otherwise, taking a subsequence and renumbering, we may assume that $\mathbb{D}_n$ is empty set. Noting that $S_n(0)=\infty$, we see that $|\widehat{S}_n(z)| > 3$ in $\Delta(0,2)$. Thus we have

$$n\Big(2, \frac{1}{(\widehat{S}_n(z)-1)(\widehat{S}_n(z)-2)(\widehat{S}_n(z)-3)}\Big) = 0.$$

By Lemma 2.6, there exists $M>0$ such that $S(1, \widehat{S}_n(z)) \le M$. This contradicts (4.20).

Set

$$T_n(\zeta) := \frac{f_n(t_n\zeta)}{t_n^{m+1}\exp(\gamma_n(t_n\zeta))} = \zeta^{m+1}\widehat{\beta}(t_n\zeta)S_n(t_n\zeta), \tag{4.24}$$

where t_n is one of an element of $\mathbb{D}_n$ of smallest modulus. Now, we have

(b1) $t_n \ne 0$ and $|t_n| \le 2\lambda_n$ for sufficiently large n, and

(b2) $|S_n(t_n\zeta)| \ge 3$ and $T_n(\zeta) \ne 0$ for sufficiently large n in Δ.

Noting that $S_n(0)=\infty$, we see that $t_n \ne 0$. By the definition of $\mathbb{D}_n$, $|t_n| \le 2\lambda_n$ for sufficiently large n. Thus (b1) holds. Since t_n is one of an element of $\mathbb{D}_n$ of smallest modulus and 0 is a pole of $S_n(z)$, we have $|S_n(t_n\zeta)| \ge 3$ in Δ. By (4.24), $T_n(\zeta) \ne 0$ for sufficiently large n in Δ'. By (a1), $f_n(0) \ne 0$ and hence $T_n(0) \ne 0$ for sufficiently large n. Thus (b2) holds.

By Lemma 2.10, $\{T_n\}$ is normal in $\mathbb{C}\setminus\{0\}$. Taking a subsequence and renumbering, we may assume that $T_n(\zeta) \overset{\chi}{\Rightarrow} T(\zeta)$ in $\mathbb{C}\setminus\{0\}$. By (4.24),

$$|T(1)| = \lim_{n\to\infty} |T_n(1)| = \lim_{n\to\infty} |\widehat{\beta}(t_n)S_n(t_n)| = 3.$$

Thus $T^{-1}(\zeta)$ is a meromorphic function in $\mathbb{C}\setminus\{0\}$. By (b2) and the maximum principle, $T_n^{-1}(\zeta) \Rightarrow T^{-1}(\zeta)$ in Δ. Then we have $T_n(\zeta) \overset{\chi}{\Rightarrow} T(\zeta)$ in $\mathbb{C}$, where $T(\zeta)$ is a meromorphic function in $\mathbb{C}$, whose zeros have multiplicity at least 3.

We claim that either $T'(\zeta) - \zeta^m \equiv 0$ or $T'(\zeta) - \zeta^m \neq 0$ in $\mathbb{C}$. For any $R > 0$, we have

$$\begin{aligned} |\gamma_n'(t_n\zeta)| = |\gamma'(a_n - b_n + t_n\zeta)| &\leq M(|2a_n|, \gamma') \\ &\leq \exp(|2a_n|^{\rho(\gamma)+\varepsilon}) \leq \exp(|a_n|^{\sigma(\alpha)+2\varepsilon}) \end{aligned}$$

for sufficiently large n in $\Delta(0, R)$. Thus we have

$$t_n\gamma_n'(t_n\zeta) \Rightarrow 0 \quad \text{in } \mathbb{C}. \tag{4.25}$$

An elementary calculation shows that

$$T_n'(\zeta) = \frac{f_n'(t_n\zeta)}{t_n^m \exp(\gamma_n(t_n\zeta))} - t_n\gamma_n'(t_n\zeta)\frac{f_n(t_n\zeta)}{t_n^{m+1}\exp(\gamma_n(t_n\zeta))},$$

and then, by (4.25),

$$\frac{f_n'(t_n\zeta)}{t_n^m \exp(\gamma_n(t_n\zeta))} = T_n'(\zeta) + t_n\gamma_n'(t_n\zeta)T_n(\zeta) \Rightarrow T'(\zeta) \quad \text{in } \mathbb{C} \setminus T^{-1}(\infty). \tag{4.26}$$

Set

$$U_n(\zeta) := \frac{f_n'(t_n\zeta) - \alpha(t_n\zeta)}{t_n^m \exp(\gamma_n(t_n\zeta))} = \frac{f_n'(t_n\zeta)}{t_n^m \exp(\gamma_n(t_n\zeta))} - \frac{\beta(t_n\zeta)}{t_n^m}. \tag{4.27}$$

By (4.26) and (4.27), we see that

$$U_n(\zeta) \Rightarrow T'(\zeta) - \zeta^m \quad \text{in } \mathbb{C} \setminus T^{-1}(\infty). \tag{4.28}$$

By (a1), it is easy to see that

$$U_n(\zeta) \neq 0 \quad \text{for sufficiently large } n. \tag{4.29}$$

By (4.28), (4.29) and Hurwitz's theorem, either $T'(\zeta) - \zeta^m \equiv 0$ or $T'(\zeta) - \zeta^m \neq 0$ in $\mathbb{C}$.

Assume that $T'(\zeta) - \zeta^m \equiv 0$ in $\mathbb{C}$. Since all zeros of $T(\zeta)$ have multiplicity at least 3, we obtain $T(\zeta) = \frac{\zeta^{m+1}}{m+1}$ which contradicts the fact that $|T(1)| = 3$. Thus $T'(\zeta) - \zeta^m \neq 0$ in $\mathbb{C}$. By Lemma 3.2, $m = 1$ and $T(\zeta) = \frac{(\zeta - \frac{c_1}{3})^3}{2(\zeta - c_1)}$, where c_1 is a nonzero constant. Then

$$T_n(\zeta) = \frac{f_n(t_n\zeta)}{t_n^2 \exp(\gamma_n(t_n\zeta))} \overset{\chi}{\Rightarrow} \frac{\left(\zeta - \frac{c_1}{3}\right)^3}{2(\zeta - c_1)} \quad \text{in } \mathbb{C}. \tag{4.30}$$

By Hurwitz's theorem, there exist sequences $\zeta_{0,n} \to \frac{c_1}{3}$ and $\zeta_{\infty,n} \to c_1$ such that $T(\zeta_{0,n}) = 0$ and $T(\zeta_{\infty,n}) = \infty$. Set $\widehat{T}_n(\zeta) := \frac{\zeta-\zeta_{0,n}}{(\zeta-\zeta_{\infty,n})^3} \cdot T_n(\zeta)$. By the maximum principle,

$$\widehat{T}_n(\zeta) := \frac{\zeta - \zeta_{0,n}}{(\zeta - \zeta_{\infty,n})^3} \cdot T_n(\zeta) \Rightarrow \frac{1}{2} \text{ in } \mathbb{C}.$$

Set $\mathbb{E}_n := \big\{z \mid |S_n(z)| = 3,\ A|t_n| < |z| \le 2\lambda_n\big\}$, where

$$A = \max\big\{|\zeta| : |T_n(\zeta)| = 1,\ |T_n(\zeta)| = 2 \text{ or } |T_n(\zeta)| = 3\big\}.$$

We claim that $\mathbb{E}_n$ is non-empty set for sufficiently large n. Otherwise, taking a subsequence and renumbering, we may assume that $\mathbb{E}_n$ is empty set. By (4.30) and Hurwitz's theorem,

$$n\Big(2, \frac{1}{(\widehat{S}_n(z)-1)(\widehat{S}_n(z)-2)(\widehat{S}_n(z)-3)}\Big) = 9.$$

By Lemma 2.6, there exists $M > 0$ such that $S(1, \widehat{S}_n(z)) \le M$. This contradicts (4.20).

Set

$$R_n(\xi) := \frac{f_n(r_n\xi)}{r_n^2 \exp(\gamma_n(r_n\xi))} = \xi^2 \widehat{\beta}(r_n\xi) S_n(r_n\xi), \tag{4.31}$$

where r_n is one of an element of $\mathbb{E}_n$ of smallest modulus.

We claim that

(c1) $|r_n| \le 2\lambda_n$ for sufficiently large n,

(c2) $\frac{r_n}{t_n} \to \infty$ as $n \to \infty$, and

(c3) $R_n(\xi)$ has a unique (multiple) zero $\frac{t_n}{r_n} \cdot \zeta_{0,n}$ for sufficiently large n in Δ.

By the definition of $\mathbb{E}_n$, (c1) holds. By (4.30) and Hurwitz's theorem, (c2) and (c3) holds.

By Lemma 2.10, $\{R_n(\xi)\}$ is normal in $\mathbb{C} \setminus \{0\}$. Taking a subsequence and renumbering, we may assume that $R_n(\xi) \overset{\chi}{\Rightarrow} R(\xi)$ in $\mathbb{C} \setminus \{0\}$. By (4.31),

$$|R(1)| = \lim_{n\to\infty} |R_n(1)| = \lim_{n\to\infty} |\widehat{\beta}(t_n) R_n(r_n)| = 3.$$

Thus $R(\xi)$ is a nonzero meromorphic function in $\mathbb{C} \setminus \{0\}$. Using the method of dealing with $\{T_n\}$, we can show either $R'(\xi) - \xi \equiv 0$ or $R'(\xi) - \xi \ne 0$ in $\mathbb{C} \setminus \{0\}$.

Set $\widehat{R}_n(\xi) := \frac{\xi - \frac{t_n}{r_n}\cdot\zeta_{0,n}}{(\xi - \frac{t_n}{r_n}\cdot\zeta_{\infty,n})^3}\cdot R_n(\xi)$. Then

$$\widehat{R}_n(\xi) \stackrel{\chi}{\Rightarrow} \frac{R(\xi)}{\xi^2} \quad \text{in } \mathbb{C}\setminus\{0\}. \tag{4.32}$$

Clearly, $\widehat{R}_n(\xi)$ has no zeros for sufficiently large n in Δ. By the maximum principle, $\{\widehat{R}_n(\xi)\}$ converges in the spherical metric uniformly on $\overline{\Delta}(0,1/2)$. Then we can assume that

$$\widehat{R}_n(\xi) \stackrel{\chi}{\Rightarrow} \widehat{R}(\xi) \quad \text{in } \mathbb{C}. \tag{4.33}$$

(4.32) and (4.33) imply that $R(\xi)$ can be extended to meromorphic function $\xi^2\widehat{R}(\xi)$ in $\mathbb{C}$. Noting that

$$\widehat{R}(0) = \lim_{n\to\infty}\widehat{R}_n(0) = \lim_{n\to\infty}\widehat{T}_n(0) = \frac{1}{2}, \tag{4.34}$$

we see that $\xi = 0$ is a zero of order 2 of $R(\xi)$ and $R''(0) = 1$.

Suppose that $R'(\xi) - \xi \equiv 0$ in $\mathbb{C}\setminus\{0\}$. Then $R'(\xi) - \xi \equiv 0$ in $\mathbb{C}$. Noting that $\xi = 0$ is a zero of order 2 of $R(\xi)$, we have $R(\xi) = \frac{\xi^2}{2}$ which contradicts the fact that $|R(1)| = 3$. Thus $R'(\xi) - \xi \not\equiv 0$ in $\mathbb{C}\setminus\{0\}$. By Lemma 3.3, $R(\xi) = c_1\xi^2$, where c_1 ($\neq 1/2$) is a nonzero constant. A simple calculation shows $R''(0) = R''(\xi)|_{\xi=0} = 2c_1 \neq 1$, a contradiction.

Case 2: $\beta(0) = \infty$. Taking a subsequence and renumbering, we may assume that $S_n(0) \to c_0$ as $n \to \infty$, where c_0 is a finite complex number or $c_0 = \infty$.

Subcase 2.1: $c_0 = 0$. Set $\mathbb{P}_n := \{z \mid |S_n(z)| = 3,\ |z| \le 2\lambda_n\}$. We claim that $\mathbb{P}_n$ is non-empty set for sufficiently large n. Otherwise, taking a subsequence and renumbering, we may assume that $\mathbb{P}_n$ is empty set, and hence $|\widehat{S}_n(z)| < 3$ in $\Delta(0,2)$. Thus we have

$$n\Big(2, \frac{1}{(\widehat{S}_n(z)-3)(\widehat{S}_n(z)-4)(\widehat{S}_n(z)-5)}\Big) = 0.$$

By Lemma 2.6, there exists $M > 0$ such that $S(1, \widehat{S}_n(z)) \le M$. This contradicts (4.20).

Set

$$T_n(\zeta) := \frac{f_n(t_n\zeta)}{t_n^{m+1}\exp(\gamma_n(t_n\zeta))} = \zeta^{m+1}\widehat{\beta}(t_n\zeta)S_n(t_n\zeta), \tag{4.35}$$

where t_n is one of an element of $\mathbb{P}_n$ of smallest modulus. Using a similar argument presented in Case 1, we can get that (d1) $t_n \neq 0$ and $|t_n| \leq 2\lambda_n$ for sufficiently large n, and

(d2) $|S_n(t_n\zeta)| \leq 3$ and $|T_n(\zeta)| < 4$ for sufficiently large n in Δ.

By Lemma 2.10, $\{T_n\}$ is normal in $\mathbb{C}\setminus\{0\}$. Taking a subsequence and renumbering, we may assume that $T_n(\zeta) \overset{\chi}{\Rightarrow} T(\zeta)$ in $\mathbb{C}\setminus\{0\}$. By (4.35), $|T(1)| = \lim_{n\to\infty} |T_n(1)| = \lim_{n\to\infty} |\widehat{\beta}(t_n)S_n(t_n)| = 3$. Thus $T(\zeta)$ is a meromorphic function in $\mathbb{C}\setminus\{0\}$. By (d2) and the maximum principle, $T_n(\zeta) \Rightarrow T(\zeta)$ in Δ. Then we have $T_n(\zeta) \overset{\chi}{\Rightarrow} T(\zeta)$ in $\mathbb{C}$. where $T(\zeta)$ is a meromorphic function in $\mathbb{C}$, whose zeros have multiplicity at least 3. By (d2), we see that $T(z) \neq \infty$ in Δ. The same argument presented in Case 1 show that either $T'(\zeta) - \zeta^m \equiv 0$ or $T'(\zeta) - \zeta^m \neq 0$ in $\mathbb{C}$.

We claim that $T'(\zeta) \neq \zeta^m$ in $\mathbb{C}$. Suppose that $T'(\zeta) - \zeta^m \equiv 0$ in $\mathbb{C}$. It is easy to see that $m \neq -1$. (Otherwise, $T(\zeta)$ is a multivalued function.)

Noting that all zeros of $T(\zeta)$ have multiplicity at least 3, we have $T(\zeta) = \frac{1}{(1+m)\zeta^{-m-1}}$ which contradicts the fact that $|T(1)| = 3$. Thus $T'(\zeta) - \zeta^m \neq 0$ in $\mathbb{C}$. Since $T(\zeta) \neq \infty$ in Δ, we have $T'(\zeta) \neq \zeta^m$ in $\mathbb{C}$.

By Theorem D and Lemma 2.1, we may assume that $T(\zeta) \equiv 3e^{i\theta}$, where θ is a constant. Thus we have

$$T_n(\zeta) \overset{\chi}{\Rightarrow} T(\zeta) = 3e^{i\theta}, \quad S_n(t_n\zeta) = \frac{T_n(\zeta)}{\zeta^{m+1}\widehat{\beta}(t_n\zeta)} \overset{\chi}{\Rightarrow} 3e^{i\theta}\zeta^{-m-1} \text{ in } \mathbb{C}. \tag{4.36}$$

Set $\mathbb{Q}_n := \{z \mid |S_n(z)| = 3,\ A|t_n| < |z| \leq 8\lambda_n\}$, where

$$A = \max\{|\zeta| \mid |T_n(\zeta)| = 1,\ |T_n(\zeta)| = 2 \text{ or } |T_n(\zeta)| = 3\}.$$

We claim that $\mathbb{Q}_n$ is non-empty set for sufficiently large n. Otherwise, taking a subsequence and renumbering, we may assume that $\mathbb{Q}_n$ is empty set. By (4.36) and Hurwitz's theorem, we see that

$$n\Big(7, \frac{1}{(\widehat{S}_n(z)-1)(\widehat{S}_n(z)-2)(\widehat{S}_n(z)-3)}\Big) = -3(m+1).$$

By Lemma 2.6, there exists $M > 0$ such that $S(1, \widehat{S}_n(z)) \leq M$. This contradicts (4.20).

Set

$$V_n(\zeta) := \frac{f_n(r_n\zeta)}{r_n^{m+1}\exp(\gamma_n(r_n\zeta))} = \zeta^{m+1}\widehat{\beta}(r_n\zeta)S_n(r_n\zeta), \tag{4.37}$$

where r_n is one of an element of $\mathbb{Q}_n$ of smallest modulus. We claim that

(e1) $|r_n| \leq 8\lambda_n$ for sufficiently large n,

(e2) $\frac{r_n}{t_n} \to \infty$ as $n \to \infty$, and

(e3) $V_n(\zeta) \neq 0$ in Δ for sufficiently large n.

By the definition of $\mathbb{Q}_n$, (e1) holds. By (4.36) and Hurwitz's theorem, (e2) holds. Since $T_n(\zeta) \overset{\chi}{\Rightarrow} 3e^{i\theta}$ in $\mathbb{C}$, we see that $f_n(z) \neq 0$ in $\Delta(0, 4|t_n|)$ for sufficiently large n. By (4.36) and the definition of $\mathbb{Q}_n$, we have $|S_n(z)| \geq 2$ and hence $f_n(z) \neq 0$ in $\Delta(0, |r_n|) \setminus \overline{\Delta}(0, 3|t_n|)$ for sufficiently large n. Now, $f_n(z) \neq 0$ in $\Delta(0, |r_n|)$ for sufficiently large n. Thus (e3) holds by (4.37).

By Lemma 2.10, $\{V_n\}$ is normal in $\mathbb{C} \setminus \{0\}$. Taking a subsequence and renumbering, we may assume that $V_n(\zeta) \overset{\chi}{\Rightarrow} V(\zeta)$ in $\mathbb{C} \setminus \{0\}$. By (4.37),

$$|V(1)| = \lim_{n\to\infty} |V_n(1)| = \lim_{n\to\infty} |\widehat{\beta}(r_n) S_n(r_n)| = 3.$$

Thus $V(\zeta)$ and $V^{-1}(\zeta)$ are meromorphic functions in $\mathbb{C} \setminus \{0\}$. By (d3) and the maximum principle, $V_n^{-1}(\zeta) \Rightarrow V^{-1}(\zeta)$ in Δ. Then $V_n(\zeta) \overset{\chi}{\Rightarrow} V(\zeta)$ in $\mathbb{C}$, where $V(\zeta)$ is a meromorphic function in $\mathbb{C}$, whose zeros have multiplicity at least 3. The same argument presented in Case 1 show that either $V'(\zeta) - \zeta^m \equiv 0$ or $V'(\zeta) - \zeta^m \neq 0$ in $\mathbb{C}$.

We claim that $V'(\zeta) - \zeta^m \neq 0$ in $\mathbb{C}$. Suppose that $V'(\zeta) - \zeta^m \equiv 0$ in $\mathbb{C}$. Since all zeros of $V(\zeta)$ have multiplicity at least 3, we have $V(\zeta) = \frac{1}{(1+m)\zeta^{-m-1}}$ which contradicts the fact that $|V(1)| = 3$.

By an elementary calculation we have

$$\frac{f_n'(r_n\zeta)}{r_n^m \exp(\gamma_n(r_n\zeta))} = V_n'(\zeta) + r_n \gamma_n'(r_n\zeta) V_n(\zeta). \tag{4.38}$$

Using the same argument presented in Case 1, we can show that

$$r_n \gamma_n'(r_n\zeta) \Rightarrow 0 \quad \text{in } \mathbb{C}. \tag{4.39}$$

By (4.38) and (4.39), we have

$$\begin{aligned} \frac{f_n'(r_n\zeta) - \alpha_n(r_n\zeta)}{r_n^m \exp(\gamma_n(r_n\zeta))} &= \frac{f_n'(r_n\zeta))}{r_n^m \exp(\gamma_n(r_n\zeta))} - \zeta^m \widehat{\beta}_n(r_n\zeta) \\ &\Rightarrow V'(\zeta) - \zeta^m \quad \text{in } \mathbb{C} \setminus V^{-1}(\infty). \end{aligned} \tag{4.40}$$

By (a1), we see that

$$\frac{f_n'(r_n\zeta) - \alpha_n(r_n\zeta)}{r_n^m \exp(\gamma_n(r_n\zeta))} \neq 0 \quad \text{for sufficiently large } n. \tag{4.41}$$

By (4.41) and the maximum principle,

$$L_n(\zeta) = \left[\frac{f_n'(r_n\zeta) - \alpha_n(r_n\zeta)}{r_n^m \exp(\gamma_n(r_n\zeta))}\right]^{-1} \Rightarrow L(\zeta) = \left[V'(\zeta) - \zeta^m\right]^{-1} \quad \text{in } \mathbb{C}. \tag{4.42}$$

By (4.35), (4.37) and (e2), we have

$$V(0)=\lim_{n\to\infty}V_n(0)=\lim_{n\to\infty}\frac{f_n(0)}{r_n^{m+1}\exp(\gamma_n(0))}=\lim_{n\to\infty}T_n(0)\Big(\frac{r_n}{t_n}\Big)^{-m-1}=\infty.$$

We assume that 0 is a pole of $V(\zeta)$ of order k. Clearly, 0 is a zero of $L(\zeta)$ order at most $\max\{k+1,-m\}$. By Hurwitz' theorem, $V_n(\zeta)$ has k poles $\zeta_i\to 0$ and hence $f_n(r_n\zeta)$ has k poles $\zeta_i\to 0$, where $i=1,2,\dots,k$. By (a1), we have $f_n(0)\neq\infty$. Thus $\zeta_i\neq 0$ for $i=1,2,\dots,k$. By (4.42), $L_n(\zeta)$ has at least $k+1$ non-zero zeros $\zeta_i\to 0$ and a zero $\zeta=0$ of order $-m$. By Hurwitz' theorem, 0 is a zero of $L(\zeta)$ of order at least $k+1-m$. Thus we must have $k+1-m\le\max\{k+1,-m\}$. This is a contradiction.

Subcase 2.2: $c_0\neq 0$. In this case, we must have $m=-1$. In fact, 0 is a zero of order $-m-1$ of $S_n(z)$ provided that $m\le -2$, and hence $c_0=0$.

Set

$$\mathbb{Y}_n:=\big\{z\ \big|\ |S_n(z)|=c_0^*,\ |z|\le 2\lambda_n\big\},\quad\text{where}\quad c_0^*=\begin{cases}|c_0|/2 & \text{for } c_0\neq\infty,\\ 1 & \text{for } c_0=\infty.\end{cases}$$

We claim that $\mathbb{Y}_n$ is non-empty set for sufficiently large n. Otherwise, taking a subsequence and renumbering, we may assume that $\mathbb{Y}_n$ is empty set. Thus we have

$$n\Big(2,\frac{1}{(\widehat{S}_n(z)-|c_0^*|/2)(\widehat{S}_n(z)-|c_0^*|/3)(\widehat{S}_n(z)-|c_0^*|/4)}\Big)=0.$$

By Lemma 2.6, there exists $M>0$ such that $S(1,\widehat{S}_n(z))\le M$. This contradicts (4.20).

Set

$$T_n(\zeta):=\frac{f_n(t_n\zeta)}{\exp(\gamma_n(t_n\zeta))}=\widehat{\beta}(t_n\zeta)S_n(t_n\zeta),\tag{4.43}$$

where t_n is one of an element of $\mathbb{Y}_n$ of smallest modulus. Using a similar argument presented in Case 1, we can get that

(f1) $t_n\neq 0$ and $|t_n|\le 2\lambda_n$ for sufficiently large n, and

(f2) $|S_n(t_n\zeta)|\ge c_0^*$ and $|T_n(\zeta)|>c_0^*/2$ for sufficiently large n in Δ.

Using the same argument presented in Subcase 2.1, we may assume that $T_n(\zeta)\overset{\chi}{\Rightarrow}T(\zeta)$ in $\mathbb{C}$. Clearly, all zeros of $T(\zeta)$ have multiplicity at least 3. By (4.43), $|T(1)|=\lim_{n\to\infty}|T_n(1)|=\lim_{n\to\infty}|\widehat{\beta}(t_n)S_n(t_n)|=c_0^*$ and $T(0)=\lim_{n\to\infty}T_n(0)=\lim_{n\to\infty}\widehat{\beta}(0)S_n(0)=c_0$. Thus $T(\zeta)$ is a non-constant meromorphic function.

The same argument presented in Case 1 show that either $T'(\zeta)-\zeta^{-1}\equiv 0$ or $T'(\zeta)-\zeta^{-1}\neq 0$ in $\mathbb{C}$. Suppose that $T'(\zeta)-\zeta^{-1}\equiv 0$ in $\mathbb{C}$. Then $T(\zeta)$

is a multivalued function. A contradiction. Thus $T'(\zeta) - \zeta^{-1} \neq 0$ in $\mathbb{C}$. Suppose that $c_0 \neq \infty$. Noting that $T(0) = c_0$, we have $T'(\zeta) \neq \zeta^{-1}$ in $\mathbb{C}$. By Theorem D and Lemma 2.1, $T(\zeta)$ is a constant function. This is a contradiction. Then we have $c_0 = \infty$. It follows from (4.43) that $T_n(\zeta) \to \infty$ and $T(0) = \infty$. Using the method of dealing with $\{V_n\}$ in Subcase 2.1, we can obtain a contradiction.

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