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NON-SPECTRAL PROBLEM ON INFINITE BERNOULLI CONVOLUTION

Q. LI[1] and Z.-Y. WU[2,*]

[1]School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan, 430079, P. R. China
e-mail: liqian303606@163.com

[2]School of Mathematics, Sun Yat-sen University, Guangzhou, 510275, P. R. China
e-mail: zhiyiwu@126.com

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Abstract. Let $\{d_k\}_{k=1}^{\infty}$ be an upper-bounded sequence of positive integers and let δ_E be the uniformly discrete probability measure on the finite set E. For $0<\rho<1$, the infinite convolution $\mu_{\rho,\{0,d_k\}} := \delta_{\rho\{0,d_1\}} * \delta_{\rho^2\{0,d_2\}} * \cdots$ is called an infinite Bernoulli convolution. The non-spectral problem on $\mu_{\rho,\{0,d_k\}}$ is to investigate the cardinality of orthogonal exponentials in $L^2(\mu_{\rho,\{0,d_k\}})$. In this paper, we give a characterization of this problem by classifying the values of ρ.

1. Introduction

Given a bounded sequence of positive integers $\{d_k\}_{k=1}^{\infty}$ and $\rho \in (0,1)$, it is well known that the convolution

$$(1.1)\qquad \mu_{\rho,\{0,d_k\}} := \delta_{\rho\{0,d_1\}} * \delta_{\rho^2\{0,d_2\}} * \cdots ,$$

is a Borel probability measure with compact support

$$T_{\rho,\{0,d_k\}} := \Big\{ \sum_{k=1}^{\infty} c_k \rho^k : c_k \in \{0,d_k\} \text{ for } k \geq 1 \Big\},$$

where $\delta_E := \frac{1}{\#E}\sum_{e\in E}\delta_e$, $\#E$ is the cardinality of a set E, δ_e is the Dirac measure at e and the convergence is in the weak sense. In this case, $\mu_{\rho,\{0,d_k\}}$

* Corresponding author.

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is called an *infinite Bernoulli convolution* and *Bernoulli convolution* if all $d_k = 1$.

In the case of the Bernoulli convolution, Jorgensen and Pedersen [13] proved that if $\rho^{-1} \in 2\mathbb{N}$, then there exists a set $\Lambda \subseteq \mathbb{R}$ such that $\{e^{-2\pi i\lambda x} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu_{\rho,\{0,1\}})$. At this time, $\mu_{\rho,\{0,1\}}$ is called a *spectral measure* and the corresponding set Λ is called a *spectrum* for $\mu_{\rho,\{0,1\}}$. In the same paper [13], they also proved that if $\rho^{-1} \in 2\mathbb{N}+1$, then $\mu_{\rho,\{0,1\}}$ is not a spectral measure and there are at most two orthogonal exponentials. The Bernoulli convolution $\mu_{\rho,\{0,1\}}$ is the first example of a singular spectral measure or a non-spectral one for different ratio ρ respectively. Since then, the spectrality or non-spectrality of a singular measure μ has received much attention, please see, e.g., [2,3,5–7,10,11,14,16–20]. Recently, the non-spectral problem on the singular measure μ has become a hot topic. And it may be the start of investigating the completeness of a family of exponential functions in $L^2(\mu)$, e.g., studying the Riesz bases and frames of exponential functions for $L^2(\mu)$ [4,9,15]. The non-spectral problem is to study the cardinality of orthogonal exponentials in $L^2(\mu)$. In general, a non-spectral singular measure μ belongs to one of the following three classes:

- There exists an infinite set of orthogonal exponentials but none of such sets forms a basis for $L^2(\mu)$;
- There are at most a finite number of orthogonal exponentials in $L^2(\mu)$;
- All the cardinality of orthogonal exponentials in $L^2(\mu)$ are bounded, but there is no uniform bound.

Most results of the known cases are concentrated on self-affine measures, see [17,18,20] and the references therein. There are few studies involving one dimensional situation except [7,19]. In this paper, we focus on the infinite Bernoulli convolution $\mu_{\rho,\{0,d_k\}}$ of the above three cases. For simplicity, we call Λ *an orthogonal set* (*resp. a maximal orthogonal set*) for $\mu_{\rho,\{0,d_k\}}$ if $\{e^{-2\pi i\lambda x} : \lambda \in \Lambda\}$ is an orthonormal family (resp. maximal orthonormal family) for $L^2(\mu_{\rho,\{0,d_k\}})$.

Throughout the paper, we make the convention that all fractions have the simplest form, that is, for a fraction $\frac{p}{q}$ we have $\gcd(p,q) = 1$. And r is the smallest integer such that $(\frac{p}{q})^r \in \mathbb{Q}$ (for example, $\rho = (\frac{4}{9})^{1/4} = (\frac{2}{3})^{1/2}$, we will take $r = 2$).

For the infinite Bernoulli convolution $\mu_{\rho,\{0,d_k\}}$, An et al. [1] proved the following:

THEOREM 1.1. *$\mu_{\rho,\{0,d_k\}}$ has an infinitely orthogonal set if and only if $\rho = (\frac{p}{q})^{\frac{1}{r}}$ for some $p, q, r \in \mathbb{N}$, where p is odd and q is even.*

Theorem 1.1 was proved by Hu and Lau [12] for the Bernoulli convolution $\mu_{\rho,\{0,1\}}$. Moreover, An et al. [1] proved that if $\mu_{\rho,\{0,d_k\}}$ is a spectral measure, then $\rho = \frac{1}{N}$ for some even N. This together with Theorem 1.1 suggest that when $\rho \in (0,1)$ is of the forms $\rho = (\frac{p}{q})^{\frac{1}{r}}$ with $p \in 2\mathbb{Z}+1$, $q \in 2\mathbb{Z}$ and $r > 1$ or $\rho = \frac{p}{q}$ with $p \geq 2$, $p \in 2\mathbb{Z}+1$ and $q \in 2\mathbb{Z}$, $\mu_{\rho,\{0,d_k\}}$ has an infinitely orthogonal set, but it is not a spectral measure. We believe that it is a very weak sufficient condition for the first class for $\mu_{\rho,\{0,d_k\}}$. It is worth mentioning that $\mu_{\rho,\{0,1\}}$ is a spectral measure if and only if $\rho = \frac{1}{N}$ with $N \in 2\mathbb{Z}$. Hence $\mu_{\rho,\{0,1\}}$ has an infinitely orthogonal set but is not a spectral measure if and only if ρ is of the form $(\frac{p}{q})^{\frac{1}{r}}$ for some $p,q,r \in \mathbb{N}$ but not of the form $\frac{1}{N}$ with $N \in 2\mathbb{Z}$. This solves the above first class for $\mu_{\rho,\{0,1\}}$. In the following of the paper, we mainly study the remaining two classes above for a non-spectral measure $\mu_{\rho,\{0,d_k\}}$. Specifically, for $\rho \in (0,1)$, we will consider the following three cases:

(i) $\rho = (\frac{p}{q})^{\frac{1}{r}}$ where $p,q \in 2\mathbb{Z}+1$ and $r \in \mathbb{N}$;

(ii) $\rho = (\frac{p}{q})^{\frac{1}{r}}$ where $p \in 2\mathbb{Z}$, $q \in 2\mathbb{Z}+1$ and $r \in \mathbb{N}$;

(iii) ρ is not in the form of $(\frac{p}{q})^{\frac{1}{r}}$ with $p,q,r \in \mathbb{N}$.

If $\rho = (\frac{p}{q})^{\frac{1}{r}}$ with $p,q,r \in \mathbb{N}$, we can conclude from Theorem 1.1 that if q is odd, then any orthogonal set for $\mu_{\rho,\{0,d_k\}}$ is finite. Therefore, we only need to consider the case that $\rho = (\frac{p}{q})^{\frac{1}{r}}$ with $q \in 2\mathbb{Z}+1$. We deal with it by considering the following two cases: $p \in 2\mathbb{Z}+1$ or $p \in 2\mathbb{Z}$. In Theorem 1.2, we prove that if $p \in 2\mathbb{Z}+1$ and all d_k are odd, then the cardinality of any maximal orthogonal set for $\mu_{\rho,\{0,d_k\}}$ is 2. In Theorem 1.3, we show that if $p \in 2\mathbb{Z}$, there are any number of orthogonal exponentials in $L^2(\mu_{\rho,\{0,d_k\}})$ (d_k need not be all odd).

THEOREM 1.2. *Let $\mu_{\rho,\{0,d_k\}}$ be the infinite Bernoulli convolution with $\rho = (\frac{p}{q})^{\frac{1}{r}}$ for some $p,q,r \in \mathbb{N}$ and Λ_ρ be a maximal orthogonal set for $\mu_{\rho,\{0,d_k\}}$. Suppose all d_k are odd. If both p and q are odd, then $\#\Lambda_\rho = 2$.*

If d_k are not all odd, we have some examples (see Examples 2.2 and 2.3 in Section 2) to show that the conclusion of Theorem 1.2 does not hold in general.

THEOREM 1.3. *Let $\mu_{\rho,\{0,d_k\}}$ be the infinite Bernoulli convolution with $\rho = (\frac{p}{q})^{\frac{1}{r}}$ for some $p,q,r \in \mathbb{N}$. If p is even and q is odd, then there are any number of orthogonal exponentials in $L^2(\mu_{\rho,\{0,d_k\}})$.*

If ρ is not in the form of $(\frac{p}{q})^{\frac{1}{r}}$ with $p,q,r \in \mathbb{N}$, Theorem 1.1 also implies that any orthogonal set for $\mu_{\rho,\{0,d_k\}}$ is finite. For this case, motivated by the ideas in [7], we can distinguish two cases: ρ is a trinomial number (see Definition 3.1 in Section 3) or not. Let Λ_ρ be a maximal orthogonal set for

$\mu_{\rho,\{0,d_k\}}$. In Theorem 1.4, supposing all d_k are odd, we prove that if ρ is not in the form of $(\frac{p}{q})^{\frac{1}{r}}$ with $p,q,r \in \mathbb{N}$ but a trinomial number with degree M, then $\#\Lambda_\rho \le 2^{M+1}$. In Theorem 1.5, supposing all d_k are odd, we obtain that if ρ is neither in the form of $(\frac{p}{q})^{\frac{1}{r}}$ with $p,q,r \in \mathbb{N}$ nor a trinomial number, then $\#\Lambda_\rho = 2$.

THEOREM 1.4. *Let $\mu_{\rho,\{0,d_k\}}$ be the infinite Bernoulli convolution and Λ_ρ be a maximal orthogonal set for $\mu_{\rho,\{0,d_k\}}$. Suppose all d_k are odd. If ρ is not in the form of $(\frac{p}{q})^{\frac{1}{r}}$ with $p,q,r \in \mathbb{N}$ but a trinomial number with degree M, then $\#\Lambda_\rho \le 2^{M+1}$.*

We will give an example in Section 3 (see Example 3.3) to illustrate Theorem 1.4.

THEOREM 1.5. *Let $\mu_{\rho,\{0,d_k\}}$ be the infinite Bernoulli convolution and Λ_ρ be a maximal orthogonal set for $\mu_{\rho,\{0,d_k\}}$. Suppose all d_k are odd. If ρ is neither in the form of $(\frac{p}{q})^{\frac{1}{r}}$ with $p,q,r \in \mathbb{N}$ nor a trinomial number, then $\#\Lambda_\rho = 2$.*

Notice that a trinomial number can also be in the form of $(\frac{p}{q})^{\frac{1}{r}}$ with $p,q,r \in \mathbb{N}$. But this does not affect the results in Theorems 1.4 and 1.5.

We organize this paper as follows. In Section 2, we give the proofs of Theorems 1.2 and 1.3. In Section 3, we prove Theorems 1.4 1.5.

2. Proofs of Theorems 1.2 and 1.3

These proofs depend closely on the zero set of the Fourier transform $\hat{\mu}_{\rho,\{0,d_k\}}$.

For a Borel measure μ, the Fourier transform of μ is defined as usual,

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x}\, \mathrm{d}\mu(x)$$

for any $\xi \in \mathbb{R}$. It is easy to show that Λ is an orthogonal set for μ if and only if

$$(\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}(\hat{\mu}), \tag{2.1}$$

where $\mathcal{Z}(\hat{\mu}) = \{\xi \in \mathbb{R} : \hat{\mu}(\xi) = 0\}$ is the zero set of $\hat{\mu}$. By the definition of Fourier transform of $\hat{\mu}_{\rho,\{0,d_k\}}$ and (1.1), for any $\xi \in \mathbb{R}$, we have

$$\hat{\mu}_{\rho,\{0,d_k\}}(\xi) = \prod_{k=1}^{\infty} M_{d_k}(\rho^k \xi),$$

where $M_{d_k}(\xi) = \frac{1}{2} + \frac{1}{2}e^{-2\pi i d_k \xi}$ is the mask polynomial of $\{0, d_k\}$. Then we have

$$\mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}}) = \bigcup_{k=1}^{\infty} \frac{\rho^{-k}(2\mathbb{Z}+1)}{2d_k}.$$

Let Λ be an orthogonal set for $\mu_{\rho,\{0,d_k\}}$. Then by (2.1), we have

$$\Lambda \setminus \{0\} \subseteq (\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}}) = \bigcup_{k=1}^{\infty} \frac{\rho^{-k}(2\mathbb{Z}+1)}{2d_k}. \tag{2.2}$$

As the orthogonality of the set Λ is invariant under translations, in this paper we normalize it by assuming that $0 \in \Lambda$.

We first prove Theorem 1.2. To prove Theorem 1.2, we need the following lemma, which was proved in [8]. For completeness, we give its proof here.

LEMMA 2.1. *Assume that $b \in \mathbb{R}$ admits a minimal integer polynomial $qx^r - p$ $(r > 1)$ and satisfies $a_1 b^l + a_2 b^m = a_3 b^n$, where $a_1, a_2, a_3 \in \mathbb{Z} \setminus \{0\}$ and l, m, n are nonnegative integers. Then we have $l \equiv m \equiv n \pmod r$.*

PROOF. As $r > 1$, we can denote

$$l = k_1 r + t_1, \quad m = k_2 r + t_2, \quad n = k_3 r + t_3,$$

where $k_1, k_2, k_3 \in \mathbb{Z}$ and $0 \le t_1, t_2, t_3 \le r-1$. Let $t_0 = \min\{t_1, t_2, t_3\}$, then $0 \le t_i - t_0 \le r - 1$ for $i = 1, 2, 3$. Since $b^r = \frac{p}{q}$ and $a_1 b^l + a_2 b^m = a_3 b^n$, we have

$$a_1 \Big(\frac{p}{q}\Big)^{k_1} b^{t_1} + a_2 \Big(\frac{p}{q}\Big)^{k_2} b^{t_2} = a_3 \Big(\frac{p}{q}\Big)^{k_3} b^{t_3},$$

that is

$$a_1 p^{k_1} q^{k_2+k_3} b^{t_1-t_0} + a_2 p^{k_2} q^{k_1+k_3} b^{t_2-t_0} = a_3 p^{k_3} q^{k_1+k_2} b^{t_3-t_0}. \tag{2.3}$$

If there exists $i \in \{1, 2, 3\}$ such that $t_i - t_0 \neq 0$, then (2.3) contradicts the fact that $qx^r - p$ is a minimal integer polynomial of b. Hence $t_1 = t_2 = t_3$, which implies $l \equiv m \equiv n \pmod r$. $\square$

Let $\rho = (\frac{p}{q})^{\frac{1}{r}}$ for some $p, q, r \in \mathbb{N}$. Then for any $\xi \in \mathbb{R}$,

$$\hat{\mu}_{\rho,\{0,d_k\}}(\xi) = \prod_{i=1}^{r} \prod_{j=0}^{\infty} M_{d_{jr+i}}\Big(\Big(\frac{p}{q}\Big)^{j} \rho^{i} \xi\Big).$$

Let $\hat{\nu}_i(\xi) = \prod_{j=0}^{\infty} M_{d_{jr+i}}\big((\frac{p}{q})^j \rho^i \xi\big)$ for $1 \le i \le r$. Thus $\mu_{\rho,\{0,d_k\}} = \nu_1 * \nu_2 * \cdots * \nu_r$ and we have

$$\mathcal{Z}(\hat{\nu}_i) = \rho^{-i} \bigcup_{j=0}^{\infty} \Big(\frac{q}{p}\Big)^j \frac{2\mathbb{Z}+1}{2d_{jr+i}}, \tag{2.4}$$

$$\mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}}) = \bigcup_{i=1}^{r} \mathcal{Z}(\hat{\nu}_i) = \bigcup_{i=1}^{r} \rho^{-i} \bigcup_{j=0}^{\infty} \Big(\frac{q}{p}\Big)^j \frac{2\mathbb{Z}+1}{2d_{jr+i}}.$$

Now we are ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Suppose that $\#\Lambda_\rho \ge 3$. Let $\Lambda = \{0, \lambda_1, \lambda_2\}$ be an orthogonal set for $\mu_{\rho,\{0,d_k\}}$. From (2.2) we have $\lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}})$. Let d be the least common multiple of all d_k, then d is odd. Since q is odd, (2.4) implies

$$\mathcal{Z}(\hat{\nu}_i) \subseteq \rho^{-i} \bigcup_{j=0}^{\infty} \frac{2\mathbb{Z}+1}{p^j 2d} \quad \text{and} \quad \mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}}) \subseteq \bigcup_{i=1}^{r} \rho^{-i} \bigcup_{j=0}^{\infty} \frac{2\mathbb{Z}+1}{p^j 2d}. \tag{2.5}$$

Next we need a claim to complete the proof.

CLAIM 1. $(\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}(\hat{\nu}_i)$ *for some* $1 \le i \le r$.

PROOF. By (2.2) and (2.5), we can write $\lambda_1 = \frac{\rho^{-l_1} a}{2p^n d}$ and $\lambda_2 = \frac{\rho^{-l_2} b}{2p^m d}$, where $1 \le l_1, l_2 \le r$, $a, b \in 2\mathbb{Z}+1$ and $n, m \ge 0$. Then there exists $\theta = \frac{\rho^{-l_3} c}{2p^s d} \in \mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}})$ with $1 \le l_3 \le r$, $c \in 2\mathbb{Z}+1$ and $s \ge 0$ such that $\lambda_1 - \lambda_2 = \theta$. Let $N = \max\{l_1, l_2, l_3\}$, then we have

$$\rho^{N-l_1} a p^{m+s} - \rho^{N-l_2} b p^{n+s} = \rho^{N-l_3} c p^{n+m}.$$

If $r > 1$, by Lemma 2.1 we have $N - l_1 \equiv N - l_2 \equiv N - l_3 \pmod r$ and thus $l_1 \equiv l_2 \equiv l_3 \pmod r$. Combining it with $1 \le l_1, l_2, l_3 \le r$, we have $l_1 = l_2 = l_3$. If $r = 1$, it is trivial. Therefore, λ_1, λ_2, $\lambda_1 - \lambda_2$ belong to the same $\mathcal{Z}(\hat{\nu}_i)$ for some $1 \le i \le r$. □

Noting that $0 \in \Lambda$ and by the above claim, we have $\Lambda \setminus \{0\} \subseteq (\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}(\hat{\nu}_i)$ for some $1 \le i \le r$. Then we can write $\lambda_1 = \frac{\rho^{-i} a}{p^j 2d}$, $\lambda_2 = \frac{\rho^{-i} b}{p^{j'} 2d}$ and $\lambda_1 - \lambda_2 = \frac{\rho^{-i} c}{p^{j''} 2d}$, where $a, b, c \in 2\mathbb{Z}+1$ and $j, j', j'' \ge 0$. Hence

$$\frac{\rho^{-i} a}{p^j 2d} - \frac{\rho^{-i} b}{p^{j'} 2d} = \frac{\rho^{-i} c}{p^{j''} 2d}. \tag{2.6}$$

If $j < j'$, then (2.6) implies $\frac{ap^{j'-j} - b}{p^{j'}} = \frac{c}{p^{j''}}$, that is,

$$p^{j'} c = p^{j''}(ap^{j'-j} - b).$$

Since $p \in 2\mathbb{Z}+1$, it follows that the left side of the above equality is odd but the right is even, a contradiction. Similarly, we can get contradictions for the cases $j = j'$ and $j > j'$. Hence $\#\Lambda_\rho \le 2$.

On the other hand, let $\Lambda' = \{0, \lambda\}$ with $\lambda \in \mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}})$. It is obvious that Λ' is an orthogonal set for $\mu_{\rho,\{0,d_k\}}$. Combining with the above conclusion, we have $\#\Lambda_\rho = 2$. □

The following examples show that if d_k are not all odd, the conclusion of Theorem 1.2 does not hold.

Example 2.2. Let $d_{2k-1} = 1$, $d_{2k} = 2$ for $k \ge 1$, and let $\rho = \frac{1}{3}$. Then there exists an orthogonal set for $\mu_{\rho,\{0,d_k\}}$ such that the cardinality of it is strictly greater than 2.

Proof. Denote

$$\nu_1 = \delta_{\frac{1}{3}\{0,1\}} * \delta_{\frac{1}{3^3}\{0,1\}} * \delta_{\frac{1}{3^5}\{0,1\}} * \cdots, \quad \nu_2 = \delta_{\frac{1}{3^2}\{0,2\}} * \delta_{\frac{1}{3^4}\{0,2\}} * \delta_{\frac{1}{3^6}\{0,2\}} * \cdots.$$

Then $\mu_{\rho,\{0,d_k\}} = \nu_1 * \nu_2$. It follows that $\mathcal{Z}(\hat{\nu}_1) = \bigcup_{k=1}^{\infty} \frac{3^{2k-1}(2\mathbb{Z}+1)}{2}$, $\mathcal{Z}(\hat{\nu}_2) = \bigcup_{k=1}^{\infty} \frac{3^{2k}(2\mathbb{Z}+1)}{4}$ and $\mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}}) = \mathcal{Z}(\hat{\nu}_1) \cup \mathcal{Z}(\hat{\nu}_2)$. Let $\lambda_1 = \frac{3^{l_1}a}{2}$ and $\lambda_2 = \frac{3^{l_2}b}{4}$, where $l_1, a, b \in 2\mathbb{Z}+1$, $l_2 \in 2\mathbb{Z}$ and $l_1 > l_2$. Then we have

$$\lambda_1 - \lambda_2 = \frac{3^{l_2}}{4}(2 \cdot 3^{l_1-l_2}a - b) \in \frac{3^{l_2}(2\mathbb{Z}+1)}{4} \subseteq \mathcal{Z}(\hat{\nu}_2).$$

Consequently, $\Lambda = \{0, \lambda_1, \lambda_2\}$ is an orthogonal set for $\mu_{\rho,\{0,d_k\}}$. □

Example 2.3. Let $d_1 = 4$, $d_k = 2$ for $k \ge 2$, and let $\rho = \frac{1}{5}$. Then there exists an orthogonal set for $\mu_{\rho,\{0,d_k\}}$ with cardinality strictly greater than 2.

Proof. Notice that

$$\mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}}) = \frac{5(2\mathbb{Z}+1)}{8} \cup \bigcup_{k=2}^{\infty} \frac{5^k(2\mathbb{Z}+1)}{4}.$$

Let $\lambda_1 = \frac{5a}{8}$ and $\lambda_2 = \frac{5^k b}{4}$, where $a, b \in 2\mathbb{Z}+1$ and $k \ge 2$. Then we have

$$\lambda_1 - \lambda_2 = \frac{5}{8}(a - 2 \cdot 5^{k-1}b) \in \frac{5(2\mathbb{Z}+1)}{8} \subseteq \mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}}).$$

Hence $\Lambda = \{0, \lambda_1, \lambda_2\}$ is an orthogonal set of $\mu_{\rho,\{0,d_k\}}$. Then the conclusion holds. □

At the end of this section, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. Fix $1 \le i \le r$. Since $\{d_{jr+i}\}_{j=0}^{\infty}$ is bounded, there exists a strictly increasing subsequence $\{k_n\}_{n=1}^{\infty} \subseteq \{j\}_{j=0}^{\infty}$ and $d_{k_n r+i}$

are all the same denoted by $\tilde{d}$. For a given N, we define $\Lambda_N := \{\lambda_n\}_{n=0}^N$, where

$$\lambda_n = \rho^{-i}\frac{q^{k_n}a_n}{2p^{k_n}\tilde{d}} \quad \text{with } a_n = \begin{cases} 0, & n=0; \\ q^{k_N}, & 1\le n\le N. \end{cases}$$

Next, we prove that Λ_N is an orthogonal set for $\mu_{\rho,\{0,d_k\}}$. For any $\lambda_n \neq \lambda_m \in \Lambda_N$ with $n<m$, if $\lambda_n \neq 0$, then

$$\lambda_n - \lambda_m = \rho^{-i}\frac{q^{k_n}a_n}{2p^{k_n}\tilde{d}} - \rho^{-i}\frac{q^{k_m}a_m}{2p^{k_m}\tilde{d}} = \rho^{-i}\frac{q^{k_m}(p^{k_m-k_n}q^{k_N+k_n-k_m}-a_m)}{2p^{k_m}\tilde{d}}. \tag{2.7}$$

Since p is even and q is odd, combining (2.4) with (2.7) we have

$$\lambda_n - \lambda_m \in \mathcal{Z}(\hat{\nu}_i) \subseteq \mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}}). \tag{2.8}$$

If $\lambda_n = 0$, it is obvious that (2.8) holds. Accordingly, Λ_N is an orthogonal set for $\mu_{\rho,\{0,d_k\}}$. By the arbitrariness of N, the proof is completed. □

3. Proofs of Theorems 1.4 and 1.5

We first introduce the concept of trinomial number, which plays a crucial role in this section.

DEFINITION 3.1. We say that $\rho \in (0,1)$ is a *trinomial number* if there exist $\alpha,\beta,\gamma \in 2\mathbb{Z}+1$ and $M,m\in\mathbb{N}$ with $M>m$ such that

$$\alpha\rho^{-M} + \beta\rho^{-m} + \gamma = 0. \tag{3.1}$$

Here, we call the smallest M satisfying (3.1) *the degree of the trinomial number* ρ.

To prove Theorem 1.4, we need a lemma involving algebra proved in [7]. For the reader's convenience we give a proof.

LEMMA 3.2. *Let $P(x)$ be an integer polynomial such that all coefficients are even except for one which is odd, and let $Q(x)$ be an integer polynomial whose leading coefficient and constant term are odd. Then $P(x)$ and $Q(x)$ are co-prime.*

PROOF. If $Q(x)$ is irreducible, suppose on the contrary that $(P(x),Q(x)) \neq 1$. Then there exists $H(x)\in\mathbb{Z}[x]$ such that $P(x)=Q(x)H(x)$. Write

$$P(x) = a_0 + a_1x + \cdots + a_nx^n, \quad Q(x) = b_0 + b_1x + \cdots + b_mx^m,$$
$$H(x) = c_0 + c_1x + \cdots + c_lx^l.$$

Note that for any $f(x) \in \mathbb{Z}[x]$, there exists a unique $f'(x) \in \mathbb{Z}_2[x]$ such that the odd coefficients in $f(x)$ become 1 and the even coefficients in $f(x)$ become 0. Hence there exist $P'(x), Q'(x), H'(x) \in \mathbb{Z}_2[x]$ such that

$$P'(x) = Q'(x)H'(x). \tag{3.2}$$

Let a_k be the odd coefficient of $P(x)$ for some $0 \le k \le n$. We denote

$$\{b_0, b_{s_1}, b_{s_2}, \dots, b_{s_i}, b_m\} \text{ and } \{c_{t_1}, c_{t_2}, \dots, c_{t_j}\}$$

as the odd coefficients of $Q(x)$ and $H(x)$, where

$$0 < s_1 < s_2 < \cdots < s_i < m \text{ and } t_1 < t_2 < \cdots < t_j.$$

Then (3.2) implies

$$x^k = (1 + x^{s_1} + x^{s_2} + \cdots + x^{s_i} + x^m)(x^{t_1} + x^{t_2} + \cdots + x^{t_j}). \tag{3.3}$$

Since $t_1 < m + t_j$, the right-hand side of (3.3) have at least two terms but the left-hand side has only one. This is a contradiction, hence $(P(x), Q(x)) = 1$.

If $Q(x)$ is reducible, we have $Q(x) = Q_1(x)Q_2(x)\cdots Q_N(x)$ for some $N \in \mathbb{N}$, where $Q_1(x)$, $Q_2(x)$, ..., $Q_N(x)$ are all irreducible. By the above conclusion, we have

$$(P(x), Q_1(x)) = 1, \quad (P(x), Q_2(x)) = 1, \quad \dots, \quad (P(x), Q_N(x)) = 1.$$

Hence $(P(x), Q(x)) = 1$. The proof is completed. □

Proof of Theorem 1.4. On the contrary, we suppose that the statement was false, then $\#\Lambda_\rho > 2^{M+1}$. Let $\Lambda = \{0, \lambda_1, \dots, \lambda_{2^{M+1}}\}$ be an orthogonal set for $\mu_{\rho,\{0,d_k\}}$. By (2.2), we can write $\lambda_i = \frac{\rho^{-k_i} a_i}{2d_{k_i}}$, where $k_i \in \mathbb{N}$ and $a_i \in 2\mathbb{Z}+1$ for $i = 1, 2, \dots, 2^{M+1}$. Then for any $i \ne j \in \{1, 2, \dots, 2^{M+1}\}$, there exist $k_{i,j} \in \mathbb{N}$ and $a_{i,j} \in 2\mathbb{Z}+1$ such that

$$\lambda_i - \lambda_j = \frac{\rho^{-k_{i,j}} a_{i,j}}{d_{k_{i,j}}}. \tag{3.4}$$

Since ρ is a trinomial number, we can write $\alpha\rho^{-M} = -\beta\rho^{-m} - \gamma$ for some $\alpha, \beta, \gamma \in 2\mathbb{Z}+1$ and $M, m \in \mathbb{N}$ with $M > m$. Then for each $k \in \mathbb{N}$ there exist $c_{k,0}, c_{k,1}, \dots, c_{k,M-1} \in \mathbb{Z}$ satisfying

$$\alpha^k \rho^{-k} = \sum_{s=0}^{M-1} c_{k,s}\rho^{-s}, \tag{3.5}$$

where $c_{k,s}$ is odd or zero. In fact, if $k \le M$, then (3.5) is obviously true. If $k > M$, we can write $k = s_1 M + t_1$ with $s_1, t_1 \in \mathbb{N}$ and $0 \le t_1 < M$, then

$$\alpha^k \rho^{-k} = \alpha^{k-1} \rho^{-(k-M)} (-\beta\rho^{-m} - \gamma) = \alpha^{k-s_1} \rho^{-t_1} (-\beta\rho^{-m} - \gamma)^{s_1}.$$

If $s_1 m + t_1 \le M - 1$, then (3.5) follows. For the case $s_1 m + t_1 \ge M$, denote $s_1 m + t_1 = s_2 M + t_2$ for some $t_1, t_2 \in \mathbb{N}$ and $0 \le t_2 < M$. Then we have

$$\alpha^{k-s_1} \rho^{-(s_1 m + t_1)} = \alpha^{k-s_1} \rho^{-(s_2 M + t_2)} = \alpha^{k-s_1-s_2} \rho^{-t_2} (-\beta\rho^{-m} - \gamma)^{s_2}.$$

Proceeding inductively, then there exists $n \in \mathbb{N}$ such that $s_n m + t_n \le M - 1$. Notice that $k - \sum_{i=1}^{n} s_i \ge 0$, then (3.5) follows. By the pigeonhole principle, there exist $i \ne j \in \{1, 2, \dots, 2^{M+1}\}$ such that

$$c_{k_i,s} - c_{k_j,s} \in 2\mathbb{Z} \quad \text{for } s = 0, 1, \dots, M-1. \tag{3.6}$$

Denote $N = \max\{k_i, k_j\}$. Then

$$\alpha^N (\lambda_i - \lambda_j) = \frac{a_i \alpha^{N-k_i} \alpha^{k_i} \rho^{-k_i}}{2 d_{k_i}} - \frac{a_j \alpha^{N-k_j} \alpha^{k_j} \rho^{-k_j}}{2 d_{k_j}}$$

$$= \frac{a_i \alpha^{N-k_i} d_{k_j} \sum_{s=0}^{M-1} c_{k_i,s} \rho^{-s} - a_j \alpha^{N-k_j} d_{k_i} \sum_{s=0}^{M-1} c_{k_j,s} \rho^{-s}}{2 d_{k_i} d_{k_j}}$$

$$= \frac{\sum_{s=0}^{M-1} (a_i \alpha^{N-k_i} d_{k_j} c_{k_i,s} - a_j \alpha^{N-k_j} d_{k_i} c_{k_j,s}) \rho^{-s}}{2 d_{k_i} d_{k_j}}.$$

Combining it with (3.4), we have

$$\alpha^N \frac{\rho^{-k_{i,j}} a_{i,j}}{2 d_{k_{i,j}}} = \frac{\sum_{s=0}^{M-1} (a_i \alpha^{N-k_i} d_{k_j} c_{k_i,s} - a_j \alpha^{N-k_j} d_{k_i} c_{k_j,s}) \rho^{-s}}{2 d_{k_i} d_{k_j}},$$

that is,

$$\alpha^N \rho^{-k_{i,j}} a_{i,j} d_{k_i} d_{k_j} = \sum_{s=0}^{M-1} d_{k_{i,j}} (a_i \alpha^{N-k_i} d_{k_j} c_{k_i,s} - a_j \alpha^{N-k_j} d_{k_i} c_{k_j,s}) \rho^{-s}.$$

Then we define

$$P(x) = \sum_{s=0}^{M-1} d_{k_{i,j}} (a_i \alpha^{N-k_i} d_{k_j} c_{k_i,s} - a_j \alpha^{N-k_j} d_{k_i} c_{k_j,s}) x^s - \alpha^N a_{i,j} d_{k_i} d_{k_j} x^{k_{i,j}}.$$

Moreover, (3.6) implies that

$$d_{k_{i,j}}(a_i\alpha^{N-k_i}d_{k_j}c_{k_i,s}-a_j\alpha^{N-k_j}d_{k_i}c_{k_j,s})\in 2\mathbb{Z}$$

for all $s=0,1,\dots,M-1$. Denote $Q(x)=\alpha x^M+\beta x^m+\gamma$. Then by Lemma 3.2, we have $(P(x),Q(x))=1$. But from the definition of $P(x)$ and $Q(x)$, we have $P(\rho^{-1})=0$ and $Q(\rho^{-1})=0$, a contradiction. Hence $\#\Lambda_\rho\le 2^{M+1}$. □

Next, we will illustrate Theorem 1.4 by an example.

Example 3.3. Let $\{d_k\}_{k=1}^{\infty}$ be a sequence of bounded odd integers, and let $\rho=\frac{\sqrt{5}-1}{2}$. If Λ_ρ is a maximal orthogonal set for $\mu_{\rho,\{0,d_k\}}$, then $\#\Lambda_\rho\le 8$.

Proof. As $\rho=\frac{\sqrt{5}-1}{2}$, we have $\rho^{-1}=\frac{\sqrt{5}+1}{2}$ (the golden ratio). It follows that $\rho^{-2}-\rho^{-1}-1=0$. Then by Theorem 1.4, we have $\#\Lambda_\rho\le 8$. □

More general examples involve the complex relationship between pisot numbers and trinomial numbers, which makes it difficult to list directly.

Proof of Theorem 1.5. If $\#\Lambda_\rho\ge 3$, let $\Lambda=\{0,\lambda_1,\lambda_2\}$ be an orthogonal set for $\mu_{\rho,\{0,d_k\}}$. Then by (2.2), we can write $\lambda_1=\frac{\rho^{-k_1}a}{2d_{k_1}}$, $\lambda_2=\frac{\rho^{-k_2}b}{2d_{k_2}}$ for some $k_1,k_2\in\mathbb{N}$ and $a,b\in 2\mathbb{Z}+1$. (2.2) also implies that there exist $k_{1,2}\in\mathbb{N}$ and $c\in 2\mathbb{Z}+1$ such that

$$\lambda_1-\lambda_2=\frac{\rho^{-k_1}a}{2d_{k_1}}-\frac{\rho^{-k_2}b}{2d_{k_2}}=\frac{\rho^{-k_{1,2}}c}{2d_{k_{1,2}}}.$$

That is,

$$\rho^{-k_1}ad_{k_2}d_{k_{1,2}}-\rho^{-k_2}bd_{k_1}d_{k_{1,2}}=\rho^{-k_{1,2}}cd_{k_1}d_{k_2}. \tag{3.7}$$

Without loss of generality, we assume $k_1\ge k_2\ge k_{1,2}$. We distinguish the following four cases.

Case 1: $k_1=k_2=k_{1,2}$. (3.7) implies that $a-b=c$. Notice that $a-b$ is even but c is odd, a contradiction.

Case 2: $k_1>k_2=k_{1,2}$. (3.7) implies that

$$ad_{k_2}d_{k_{1,2}}=\rho^{k_1-k_2}(bd_{k_1}d_{k_{1,2}}+cd_{k_1}d_{k_2}).$$

Then $\rho=\left(\frac{ad_{k_2}d_{k_{1,2}}}{bd_{k_1}d_{k_{1,2}}+cd_{k_1}d_{k_2}}\right)^{\frac{1}{k_1-k_2}}$. This contradicts to the condition that ρ is not in the form of $(\frac{p}{q})^{\frac{1}{r}}$ with $p,q,r\in\mathbb{N}$.

Case 3: $k_1=k_2>k_{1,2}$. The proof is similar to that of Case 2, so we omit it here.

Case 4: $k_1 > k_2 > k_{1,2}$. (3.7) implies that

$$\rho^{-(k_1-k_{1,2})}ad_{k_2}d_{k_{1,2}} - \rho^{-(k_2-k_{1,2})}bd_{k_1}d_{k_{1,2}} - cd_{k_1}d_{k_2} = 0. \tag{3.8}$$

Since a, b, c and all d_k are odd, (3.8) implies that ρ is a trinomial number, a contradiction. Hence $\#\Lambda_\rho \le 2$.

Moreover, let $\Lambda' = \{0, \lambda\}$ with $\lambda \in \mathcal{Z}(\hat{\mu}_{\rho,\{0,d_k\}})$. It is obvious that Λ' is an orthogonal set for $\mu_{\rho,\{0,d_k\}}$. Combining with the above conclusion, we have $\#\Lambda_\rho = 2$. $\square$

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