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DECOMPOSITIONS OF DYNAMICAL SYSTEMS INDUCED BY THE KOOPMAN OPERATOR

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Abstract. For a topological dynamical system we characterize the decomposition of the state space induced by the fixed space of the corresponding Koopman operator. For this purpose, we introduce a hierarchy of generalized orbits and obtain the finest decomposition of the state space into absolutely Lyapunov stable sets. Analogously to the measure-preserving case, this yields that the system is topologically ergodic if and only if the fixed space of its Koopman operator is one-dimensional.

1. Introduction

It is a common strategy to decompose a dynamical system into smaller parts and investigate these instead of the whole system. There exists a variety of such decompositions, e.g., the Conley decomposition (see [4] or [13]), the decomposition of the chain-recurrent set into chain components (see, e.g., [16]) or orbit-closure decompositions in [11] to name a few.

In this paper we study a new decomposition of *topological dynamical systems* $(K;\varphi)$, consisting of a compact Hausdorff space K and a continuous map

$$\varphi\colon K \to K.$$

Our approach is based on the corresponding *Koopman operator*

$$T_\varphi f := f \circ \varphi$$

on the C^*-algebra $\mathrm{C}(K)$ of all continuous complex-valued functions on K. Its fixed space

$$\operatorname{fix} T_\varphi := \big\{ f \in \mathrm{C}(K) : T_\varphi f = f \big\}$$

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yields a decomposition of K into disjoint φ-invariant and closed sets (see Section 2). To characterize this decomposition dynamically, we introduce a transfinite hierarchy of *generalized φ-orbits*. Moreover, we show that $\operatorname{fix} T_\varphi$ induces the finest decomposition of K into *absolutely Lyapunov stable* subsets (see Theorem 5.7).

As a consequence, we obtain that this decomposition is trivial, i.e., the fixed space of T_φ has dimension 1, if and only if the system $(K;\varphi)$ is *topologically ergodic* meaning that there exists $x \in K$ with generalized orbit $\mathcal{S}(x) = K$. This is by analogy with a measure-preserving dynamical system $(\Omega, \Sigma, \mu; \varphi)$ being *ergodic* (i.e., indecomposable) if and only if the fixed space

$$\operatorname{fix} T_\varphi := \left\{ f \in \mathrm{L}^1(\Omega, \Sigma, \mu) : T_\varphi f = f \right\}$$

of the corresponding Koopman operator on $\mathrm{L}^1(\Omega, \Sigma, \mu)$ is one-dimensional. A variety of examples demonstrates the complexity of the topological situation. I thank Nikolai Edeko for providing some of them and Roland Derndinger for many helpful discussions.

2. The decomposition of K corresponding to $\operatorname{fix} T_\varphi$

Since the fixed space $\operatorname{fix} T_\varphi$ is a T_φ-invariant C^*-subalgebra of $\mathrm{C}(K)$, the Gelfand–Naimark theorem shows that it is isomorphic to a space $\mathrm{C}(L)$ for some compact Hausdorff space L, called the *fixed factor* or *maximal trivial factor* of K (see [8]). The embedding $\mathrm{C}(L) \hookrightarrow \mathrm{C}(K)$ is a C^*-algebra homomorphism and hence a Koopman operator T_p for a surjection $p \colon K \twoheadrightarrow L$, called *factor map* (see, e.g., [9, Theorem 4.13 and Chapter 2.2]). This induces a disjoint splitting

$$K = \dot{\bigcup_{l \in L}} p^{-1}(\{l\})$$

into closed φ-invariant sets, hence an equivalence relation $\sim$ on K with equivalence classes $p^{-1}(\{l\})$, $l \in L$. Our problem is the following.

Problem 2.1. *Describe this equivalence relation by dynamical and topological properties of $(K;\varphi)$.*

For this purpose we introduce some technical terms.

Definition 2.2. (a) A nonempty set $M \subseteq K$ is called a *level set of* $\operatorname{fix} T_\varphi$ if $f|_M$ is constant for all $f \in \operatorname{fix} T_\varphi$.

(b) A level set M is called *maximal* if for any other level set $M' \subseteq K$ with $M \subseteq M'$ already $M' = M$.

Remark 2.3. (a) Maximal level sets exist and are closed.

(b) A set $M \subseteq K$ is a maximal level set of $\operatorname{fix} T_\varphi$ if and only if $M = p^{-1}(\{l\})$ for some $l \in L$.

Proof. To show (a), consider the family $\{M : M \text{ level set of } \operatorname{fix} T_\varphi\}$ together with the inclusion "$\subseteq$" and use Zorn's lemma. Clearly, maximal level sets are closed.

For the proof of (b), take $\operatorname{fix} T_\varphi \cong \mathrm{C}(L)$ for some isomorphism $\Phi\colon \mathrm{C}(L) \to \operatorname{fix} T_\varphi$ and $\iota\colon \operatorname{fix} T_\varphi \to \mathrm{C}(K)$ the canonical inclusion. Then $\iota \circ \Phi$ is a C^*-algebra homomorphism and hence a Koopman operator $\iota \circ \Phi = T_p$ for the factor map $p\colon K \to L$. For $f \in \operatorname{fix} T_\varphi$, there is a unique $\widehat{f} \in \mathrm{C}(L)$ such that $f = \Phi \widehat{f}$. Hence

$$f = \Phi \widehat{f} = \iota \circ \Phi \widehat{f} = T_p \widehat{f} = \widehat{f} \circ p,$$

and for $l \in L$ and $x, y \in p^{-1}(\{l\})$ we obtain $f(x) = \widehat{f}(l) = f(y)$. This shows that $p^{-1}(\{l\})$ is a level set of $\operatorname{fix} T_\varphi$ for all $l \in L$. Now assume that there is some level set $M \subseteq K$ such that $p^{-1}(\{l_1\}) \cup p^{-1}(\{l_2\}) \subseteq M$ for some $l_1 \neq l_2 \in L$. Then for $x_1 \in p^{-1}(\{l_1\})$ and $x_2 \in p^{-1}(\{l_2\})$ we have $f(x_1) = f(x_2)$ for all $f \in \operatorname{fix} T_\varphi$. This implies

$$\widehat{f}(l_1) = \widehat{f}(p(x_1)) = \widehat{f}(p(x_2)) = \widehat{f}(l_2)$$

for all $\widehat{f} \in \mathrm{C}(L)$ which is a contradiction. Conversely, it is clear that each maximal level set M of $\operatorname{fix} T_\varphi$ is of the form $M = p^{-1}(\{l\})$ for some $l \in L$. □

Proposition 2.4. *Let $(K;\varphi)$ be a topological dynamical system and identify $\operatorname{fix} T_\varphi$ with $\mathrm{C}(L)$. Let $\sim$ be any equivalence relation on K with canonical projection $\pi\colon K \to K/{\sim}$ satisfying*

(i) *$\varphi(x) \sim x$ and*

(ii) *$[x]$ is a level set of $\operatorname{fix} T_\varphi$*

for all $x \in K$. Then the following are equivalent.

(a) *For each $x \in K$ the equivalence class $[x]$ is a maximal level set of $\operatorname{fix} T_\varphi$.*

(b) *With respect to the quotient topology $K/{\sim}$ is Hausdorff.*

(c) *$K/{\sim} \cong L$.*

Proof. (a) $\Rightarrow$ (c): By assumption we have $[x] = p^{-1}(\{l\})$ for some $l \in L$, where $p\colon K \to L$ is the factor map. Hence for $x, y \in K$ we have $\pi(x) = \pi(y)$ if and only if $p(x) = p(y)$. By the universal property of the quotient topology there are unique continuous maps $h\colon K/{\sim} \to L$ and $g\colon L \to K/{\sim}$ such that $h \circ \pi = p$ and $g \circ p = \pi$. Then $g = h^{-1}$ since

$$g \circ h(\pi(x)) = g(p(x)) = \pi(x)$$

and

$$h \circ g(p(x)) = h(\pi(x)) = p(x)$$

for all $x \in K$. Hence h is a homeomorphism between $K/{\sim}$ and L.

(c) $\Rightarrow$ (b): Since $K/{\sim}$ is homeomorphic to the space L it is Hausdorff.

(b) $\Rightarrow$ (a): It suffices to show that $p^{-1}(p(x)) \subseteq [x]$ for all $x \in K$, because $p^{-1}(p(x))$ are the maximal level sets of $\operatorname{fix} T_\varphi$. Assume for a contradiction there are some $y, z \in K$ such that $y \in p^{-1}(p(z)) \setminus [z]$. Since $K/{\sim}$ is Hausdorff, $[y]$ and $[z]$ are closed. By Urysohn's lemma, there is some $\widetilde{f} \in \mathrm{C}(K/{\sim})$ such that $\widetilde{f}([y]) \neq \widetilde{f}([z])$. Then $f \coloneqq \widetilde{f} \circ \pi \in \operatorname{fix} T_\varphi$, since f is continuous and $x \sim \varphi(x)$ implies

$$f(x) = \widetilde{f}([x]) = \widetilde{f}([\varphi(x)]) = T_\varphi f(x)$$

for all $x \in K$. By the universal property of the quotient topology there is some $\widehat{f} \in \mathrm{C}(L)$ such that $f = \widehat{f} \circ p$. Since $y \in p^{-1}(p(z))$, we obtain

$$f(y) = \widehat{f}(p(y)) = \widehat{f}(p(z)) = f(z)$$

which contradicts $\widetilde{f}([y]) \neq \widetilde{f}([z])$. $\square$

Remark 2.5. For a topological dynamical system $(K;\varphi)$ both conditions (i) and (ii) in Proposition 2.4 are needed. This can be seen by the following trivial equivalence relations.

(a) $x \sim y$ for all $x, y \in K$ shows that $\varphi(x) \sim x$ does not imply that $[x]$ is a level set of $\operatorname{fix} T_\varphi$ for $x \in K$.

(b) $x \sim y$ only for $x = y$ shows that also the converse implication does not hold true in general.

3. Equivalence relations induced by generalized orbits

Our goal is to dynamically describe $\operatorname{fix} T_\varphi$ for a topological dynamical system $(K;\varphi)$ (see Problem 2.1). To do so, we use Lemma 2.4 and search for an equivalence relation $\sim$ on K such that $K/{\sim}$ is Hausdorff, $\varphi(x) \sim x$ and $[x]$ are level sets for all $x \in K$.

A first observation is the following. If we take the closed *orbit*

$$\overline{\operatorname{orb}}(x) \coloneqq \overline{\{\varphi^n(x) \colon n \in \mathbb{N}_0\}}$$

for $x \in K$, then $f|_{\overline{\operatorname{orb}}(x)}$ is constant for all $f \in \operatorname{fix} T_\varphi$. Thus, every closed orbit is a level set of $\operatorname{fix} T_\varphi$. If K admits a decomposition into mutually disjoint closed orbits, this clearly induces an equivalence relation $\sim$ with $\varphi(x) \sim x$ for all $x \in K$. But the corresponding quotient space may not be Hausdorff as the following example shows.

Example 3.1. Let $K \coloneqq D$ be the closed unit disk in $\mathbb{C}$ and

$$\varphi(x) \coloneqq r\mathrm{e}^{2\pi\mathrm{i}(\alpha+r)}$$

for $x := re^{2\pi i\alpha} \in K$ with $r \in [0,1]$, $\alpha \in [0,1)$. Denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$. Then the closed orbits

$$\overline{\mathrm{orb}}(x) = \begin{cases} \{re^{2\pi i(\alpha+nr)} \colon n = 1,\dots,q-1\} & \text{for } r = \frac{p}{q} \text{ rational,} \\ & \quad p \text{ and } q \text{ coprime,} \\ r\mathbb{T} & \text{for } r \text{ irrational} \end{cases}$$

form a non-trivial decomposition of K. However, the fixed space of T_φ is

$$\operatorname{fix} T_\varphi = \big\{ f \in \mathrm{C}(K) \colon f|_{c\mathbb{T}} \equiv \text{const. for all } c \in [0,1] \big\},$$

so the maximal level sets are the circles $c\mathbb{T}$ for $c \in [0,1]$. This shows that even mutually disjoint closed orbits may induce a quotient space that is not Hausdorff.

Our approach to obtain the Hausdorff quotient space corresponding to $\operatorname{fix} T_\varphi$ is based on the following characterization.

REMARK 3.2. A topological space X is Hausdorff if and only if each point is the intersection of its closed neighborhoods, i.e. for all $x \in X$ we have

$$\{x\} = \bigcap_{U \in \mathcal{U}(x) \text{ closed}} U$$

where $\mathcal{U}(x)$ denotes the neighborhood filter of x.

Moreover, we need the following definition.

DEFINITION 3.3. Let $(K_x)_{x\in K}$ be a covering of K satisfying $x \in K_x$ for all $x \in K$. Define an equivalence relation $\sim$ on K via $x \sim y$ for $x, y \in K$ if there is some $k \in \mathbb{N}$, $x_1, \dots, x_k \in K$ such that $x_1 = x$ and $x_k = y$ and

$$K_{x_i} \cap K_{x_{i+1}} \neq \emptyset \quad \text{for } i = 1, \dots, k-1.$$

We call $\sim$ the *equivalence relation generated by* $(K_x)_{x\in K}$.

REMARK 3.4. For the equivalence relation $\sim$ generated by $(K_x)_{x\in K}$ we have $[x] = \bigcup_{y\in K, y\sim x} K_y$.

We now outline our strategy. Starting from a quotient space $K/{\sim_0}$ we successively achieve the Hausdorff property by the following steps. We first build the intersection of closed neighborhoods of each equivalence class (cf. Remark 3.2). The preimages under the canonical projection of these intersections yield a covering of K. We obtain a new quotient space $K/{\sim_1}$ taking the equivalence relation generated by this covering. We then repeat the steps above with the new equivalence relation and so forth. We show that by repeating sufficiently often we arrive at a Hausdorff space.

Remark 3.5. For a similar approach to a *Hausdorffization* we refer to [18]. Here, the author starts from an arbitrary topological space X and constructs the smallest equivalence relation $\sim$ on X such that $X/\sim$ is Hausdorff. See also [14] or [12].

3.1. Approximating orbits and superorbits. We apply this strategy to our situation in order to reach the assumptions (i) and (ii) of Proposition 2.4 to characterize the fixed factor of T_φ.

Definition 3.6. (a) We define the *approximating orbit* of x for each $x \in K$ as

$$\mathcal{A}(x) \coloneqq \bigcap_{\substack{U \in \mathcal{U}(x) \text{ closed} \\ \varphi(U) \subseteq U}} U \,.$$

(b) Let $\sim$ be the equivalence relation on K generated by $(\mathcal{A}(x))_{x \in K}$. The *superorbit* of x is

$$\mathcal{S}(x) \coloneqq [x] = \bigcup_{\substack{y \in K \\ y \sim x}} \mathcal{A}(y) \,.$$

For this equivalence relation the assumptions (i) and (ii) of Proposition 2.4 are fulfilled.

Proposition 3.7. *For each $x \in K$ we have that*

(a) $\varphi(x) \sim x$ *and*

(b) *the superorbit $\mathcal{S}(x)$ is a level set of* $\operatorname{fix} T_\varphi$.

Proof. The proof of (a) is clear. For (b), it suffices to show that $\mathcal{A}(x)$ is a level set of $\operatorname{fix} T_\varphi$ for all $x \in K$. For $x \in K$, $f \in \operatorname{fix} T_\varphi$ and $\varepsilon > 0$ define the closed neighborhood $U \coloneqq \{y \in K : |f(y) - f(x)| \leq \varepsilon\}$ of x which is φ-invariant because f is a fixed function of T_φ. This implies $\mathcal{A}(x) \subseteq U$ by the definition of the approximating orbit. If $z \in \mathcal{A}(x)$, then $z \in U$, hence $|f(z) - f(x)| \leq \varepsilon$ for each $\varepsilon > 0$ showing $f(z) = f(x)$. $\square$

We now give examples for approximating orbits, respectively, superorbits and analyze the corresponding quotient space.

Example 3.8. Take $K \coloneqq [0, 1]$ and $\varphi(x) \coloneqq x^2$ for $x \in K$. Then

$$\mathcal{A}(x) = \begin{cases} \overline{\operatorname{orb}}(x) & \text{for } x \in [0, 1), \\ [0, 1] & \text{for } x = 1. \end{cases}$$

By $\mathcal{A}(1) = K$ we obtain the trivial decomposition of K. The corresponding quotient space is a singleton and therefore Hausdorff, hence corresponds to the fixed factor L by Proposition 2.4. This is in accordance with $\dim \operatorname{fix} T_\varphi = 1$.

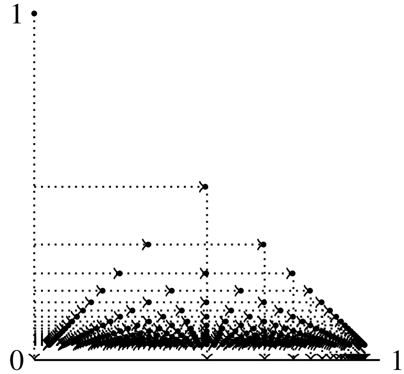

Fig. 1: $(K;\varphi)$

EXAMPLE 3.9. Take the compact space

$$K := \left\{(c,0)\colon c\in[0,1]\right\} \dot\cup \left\{\left(\tfrac{k}{n},\tfrac{1}{n}\right) : n\in\mathbb{N},\ k=0,\dots,n-1\right\} \subseteq \mathbb{R}^2$$

and consider on K the continuous dynamics

$$\varphi(x) := \begin{cases} x & \text{if } x=(c,0) \text{ for some } c\in[0,1],\\ \left(\frac{k+1}{n},\frac{1}{n}\right) & \text{if } x=\left(\frac{k}{n},\frac{1}{n}\right) \text{ for some } n\in\mathbb{N} \text{ and } k\in\{0,\dots,n-2\},\\ \left(\frac{n-1}{n},0\right) & \text{if } x=\left(\frac{n-1}{n},\frac{1}{n}\right) \text{ for some } n\in\mathbb{N} \end{cases}$$

(see Fig. 1).

(a) The approximating orbits are

$$\mathcal{A}(x) = \begin{cases} \{(a,0) : a\in[c,1]\} & \text{if } x=(c,0) \text{ for some } c\in[0,1],\\ \operatorname{orb}(x) & \text{otherwise} \end{cases}$$

where

$$\operatorname{orb}(x) = \left\{\left(\tfrac{k+m}{n},\tfrac{1}{n}\right) : m=0,\dots,n-k-1\right\} \cup \left\{\left(\tfrac{n-1}{n},0\right)\right\}$$

for $x=(\frac{k}{n},\frac{1}{n})$ with $n\in\mathbb{N}$ and $k\in\{0,\dots,n-1\}$.

(b) If $x=(\frac{k}{n},\frac{1}{n})\in K$ for some $n\in\mathbb{N}$ and $k\in\{0,\dots,n-2\}$, we have

$$\mathcal{A}(x)\cap\mathcal{A}\left(\tfrac{n-1}{n},0\right) = \left(\tfrac{n-1}{n},0\right) \neq \emptyset.$$

This implies

$$\mathcal{A}\left(\tfrac{n-1}{n},0\right)\cap\mathcal{A}(x_1,0) = \left\{(c,0)\colon c\in\left[\tfrac{n-1}{n},1\right]\right\} \cap \left\{(c,0)\colon c\in[x_1,1]\right\} \neq \emptyset$$

for all $x_1\in[0,1]$. Hence $x\sim y$ for all $y\in K$ yielding $\mathcal{S}(x)=K$.

Therefore, the quotient space induced by the superorbits is a singleton and hence a Hausdorff space, thus corresponds to the one-dimensional fixed space of T_φ.

While the superorbits in the above examples were sufficient to characterize the fixed space of T_φ, the next example reveals that this is not always the case.

EXAMPLE 3.10. Let $K \coloneqq [0,\infty]$ be the one-point compactification of $[0,\infty)$ and

$$\varphi\colon K \to K, \quad x \mapsto \begin{cases} (x-n)^2+n & \text{for } x \in [n,n+1),\ n \in \mathbb{N}_0, \\ \infty & \text{for } x = \infty \end{cases}.$$

(see Fig. 2). Then the approximating orbits are

$$\mathcal{A}(x) = \begin{cases} \{0\} & \text{for } x = 0, \\ [n-1,n] & \text{for } x = n \in \mathbb{N}, \\ \overline{\operatorname{orb}}(x) = \{(x-n)^{2k}+n : k \in \mathbb{N}_0\} \cup \{n\} & \text{for } x \in (n,n+1),\ n \in \mathbb{N}_0, \\ \{\infty\} & \text{for } x = \infty. \end{cases}$$

This yields the superorbits

$$\mathcal{S}(x) = \begin{cases} [0,\infty) & \text{for } 0 \le x < \infty, \\ \{\infty\} & \text{for } x = \infty. \end{cases}$$

However, since $\dim \operatorname{fix} T_\varphi = 1$, the maximal level set of $\operatorname{fix} T_\varphi$ is $[0,\infty]$. Hence the quotient space induced by the superorbits is not Hausdorff.

3.2. Superorbits of finite degree. To obtain a Hausdorff quotient space, we iterate the process of building intersections of certain neighborhoods (approximating orbits) and then defining an equivalence relation yielding superorbits.

DEFINITION 3.11. Let $n \in \mathbb{N}_0$ and $x \in K$.

Base case: For $n = 0$, define the *approximating orbit of x of degree* 0 as

$$\mathcal{A}_0(x) \coloneqq \mathcal{A}(x)$$

and the *superorbit of x of degree* 0 as

$$\mathcal{S}_0(x) \coloneqq \mathcal{S}(x)$$

as in Definition 3.6.

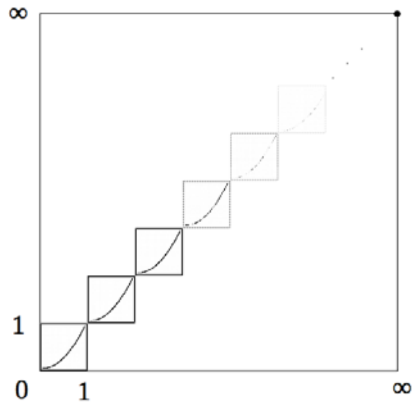

Fig. 2: $(K;\varphi)$

Successor case: (i) Let $n \geq 1$. The *approximating orbit of* x *of degree* n is

$$\mathcal{A}_n(x) := \bigcap_{\substack{U \in \mathcal{U}(x) \text{ open} \\ \mathcal{S}_{n-1}(U) \subseteq U}} \overline{U}^{\mathcal{S}_{n-1}}$$

with $\mathcal{S}_{n-1}(U) := \bigcup_{y \in U} \mathcal{S}_{n-1}(y)$ for $U \subseteq K$ and

$$\overline{U}^{\mathcal{S}_{n-1}} := \bigcap_{\substack{U \subseteq F \text{ closed} \\ \mathcal{S}_{n-1}(F) \subseteq F}} F,$$

called the $\mathcal{S}_{n-1}$-*closure of* U.

(ii) Let $\sim_n$ be the equivalence relation generated by $(\mathcal{A}_n(x))_{x \in K}$. The *superorbit of* x *of degree* n is

$$\mathcal{S}_n(x) := [x]_n = \bigcup_{\substack{y \in K, \\ y \sim_n x}} \mathcal{A}_n(y).$$

We collect some basic properties of approximating orbits, superorbits and the $\mathcal{S}_n$-closure.

PROPOSITION 3.12. *Let* $n \in \mathbb{N}_0$ *and* $U \subseteq K$.

(a) *For all* $x, y \in K$ *with* $y \sim_n x$ *we have*

$$\mathcal{A}_n(y) \subseteq \mathcal{S}_n(x), \quad \mathcal{S}_n(y) \subseteq \mathcal{A}_{n+1}(x)$$

and

$$\mathcal{A}_n(y) \subseteq \mathcal{A}_{n+1}(x), \quad \mathcal{S}_n(y) \subseteq \mathcal{S}_{n+1}(x).$$

In particular, these inclusions hold true for $x = y$.

(b) *For all* $x \in K$ *we have that* $\mathcal{A}_n(x)$ *and* $\mathcal{S}_n(x)$ *are* φ*-invariant.*
(c) *The following assertions are equivalent.*
(i) $\mathcal{S}_n(U) \subseteq U$.
(ii) $\mathcal{S}_n(U) = U$.
(iii) *There is some* $M \subseteq K$ *such that* $U = \bigcup_{y \in M} \mathcal{S}_n(y)$.
(d) *We have*

$$\mathcal{S}_n(\overline{U}^{\mathcal{S}_n}) = \overline{U}^{\mathcal{S}_n} = \overline{\overline{U}^{\mathcal{S}_n}}^{\mathcal{S}_n} = \overline{\overline{U}}^{\mathcal{S}_n} = \overline{U^\circ}^{\mathcal{S}_n}.$$

Proof. (a) is clear by definition. For (b) it suffices to show that $\mathcal{S}_n(x)$ is φ-invariant for all $x \in K$ and $n \in \mathbb{N}$. We give a proof by induction on n. For $n = 0$, see Proposition 3.7. If $\mathcal{S}_n(x)$ is φ-invariant for all $x \in K$ and $n \in \mathbb{N}_0$, then $\mathcal{S}_{n+1}(x)$ is also φ-invariant since $\mathcal{S}_n(y) \subseteq \mathcal{A}_{n+1}(y) \subseteq \mathcal{S}_{n+1}(x)$ for $x, y \in K$ with $y \in \mathcal{S}_{n+1}(x)$.

In (c) the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) are trivial. (iii) $\Rightarrow$ (ii): By assumption we have $U \subseteq \bigcup_{y \in M} \mathcal{S}_n(y)$ and thus for all $z \in U$ there is some $y \in M$ such that $z \in \mathcal{S}_n(y)$. Since $\sim_n$ is an equivalence relation, we have $\mathcal{S}_n(z) = \mathcal{S}_n(y)$, hence $\mathcal{S}_n(U) = \bigcup_{y \in U} \mathcal{S}_n(y) \subseteq \bigcup_{y \in M} \mathcal{S}_n(y)$. The converse inclusion is clear.

We now show (d). A simple calculation and (c) show

$$\mathcal{S}_n(\overline{U}^{\mathcal{S}_n}) = \overline{U}^{\mathcal{S}_n}. \tag{1}$$

From (1) and the closedness of $\overline{U}^{\mathcal{S}_n}$ then follows

$$\overline{\overline{U}^{\mathcal{S}_n}}^{\mathcal{S}_n} = \bigcap_{\substack{\overline{U}^{\mathcal{S}_n} \subseteq F \text{ closed} \\ \mathcal{S}_{n-1}(F) \subseteq F}} F = \overline{U}^{\mathcal{S}_n}. \tag{2}$$

From $U \subseteq \overline{U} \subseteq \overline{U}^{\mathcal{S}_n}$ and (2) we obtain

$$\overline{\overline{U}}^{\mathcal{S}_n} = \overline{U}^{\mathcal{S}_n}. \tag{3}$$

Finally, (3) and $\overline{U^\circ} = \overline{U}$ imply

$$\overline{U^\circ}^{\mathcal{S}_n} = \overline{U}^{\mathcal{S}_n}. \quad \square$$

Remark 3.13. Proposition 3.12(c) holds for equivalence classes of any equivalence relation.

As before, we check assumptions (i) and (ii) in Proposition 2.4.

Proposition 3.14. *For each* $n \in \mathbb{N}_0$ *and* $x \in K$ *we have that*
(a) $\varphi(x) \sim_n x$ *and*
(b) *the superorbit* $\mathcal{S}_n(x)$ *of degree* n *is a level set of* $\operatorname{fix} T_\varphi$.

PROOF. Assertion (a) follows from the φ-invariance of $\mathcal{S}_n(x)$ for each $n \in \mathbb{N}_0$ and $x \in K$ shown in Proposition 3.12(b).

We use induction on n to show (b). For $n = 0$ see Proposition 3.7. For $n \in \mathbb{N}_0$ and $x \in K$ assume that $\mathcal{S}_n(x)$ is a level set of T_φ. We show that the assertion holds true for $n+1$. As in the base case, consider $U := \{y \in K : |f(y) - f(x)| \leq \varepsilon\} = f^{-1}(\overline{B_\varepsilon(f(x))})$ for some $f \in \operatorname{fix} T_\varphi$ and some $\varepsilon > 0$. To prove $\mathcal{A}_{n+1}(x) \subseteq U$, we need to find some open $V \in \mathcal{U}(x)$ with $\mathcal{S}_n(V) \subseteq V$ such that $\overline{V}^{\mathcal{S}_n} \subseteq U$.

Define

$$V := f^{-1}(B_\varepsilon(f(x))).$$

Then $V \in \mathcal{U}(x)$, V is open and $V \subseteq U$. By the induction hypothesis we have for $x' \in K$ with $x \sim_n x'$ that $f(x) = f(x')$. Hence by the universal property of the quotient topology there is some unique continuous function $\widehat{f} \colon K/{\sim_n} \to \mathbb{C}$ such that $f = \widehat{f} \circ \pi_n$ for the canonical projection $\pi_n \colon K \to K/{\sim_n}$. This implies $V = \pi_n^{-1}\big(\widehat{f}^{-1}\big(B_\varepsilon(f(x))\big)\big)$, hence

$$\mathcal{S}_n(V) = \pi_n^{-1}(\pi_n(V)) = V. \tag{4}$$

This yields $\mathcal{A}_{n+1}(x) \subseteq \overline{V}^{\mathcal{S}_n}$.

We now show that $\overline{V}^{\mathcal{S}_n} \subseteq U$. For $f \in \operatorname{fix} T_\varphi$ and $C \subseteq \mathbb{C}$, we have $\mathcal{S}_n(f^{-1}(C)) = f^{-1}(C)$ by the universal property of the quotient topology as above. Moreover, if $B_\varepsilon(f(x)) \subseteq C$ then $V \subseteq f^{-1}(C)$ by definition.

Whence we conclude that

$$U = f^{-1}(\overline{B_\varepsilon(f(x))}) = \bigcap_{\substack{C \subseteq \mathbb{C} \text{ closed} \\ B_\varepsilon(f(x)) \subseteq C}} f^{-1}(C)$$

$$\supseteq \bigcap_{\substack{C \subseteq \mathbb{C} \\ f^{-1}(C) \text{ closed} \\ f^{-1}(B_\varepsilon(f(x))) \subseteq f^{-1}(C)}} f^{-1}(C) \supseteq \bigcap_{\substack{F \subseteq K \text{ closed} \\ \mathcal{S}_n(F) \subseteq F \\ V \subseteq F}} F = \overline{V}^{\mathcal{S}_n}.$$

Hence $\mathcal{A}_{n+1}(x) \subseteq \overline{V}^{\mathcal{S}_n} \subseteq U$. This implies that for $z \in \mathcal{A}_{n+1}(x)$ we have $|f(z) - f(x)| \leq \varepsilon$ by definition of U. Since ε is arbitrary, this implies $f(z) = f(x)$. $\square$

We now give a concrete example for these new orbits and analyze the corresponding quotient space.

EXAMPLE 3.15. (a) Let $K := [0, \infty]$ be the one-point compactification of $[0, \infty)$ and $\varphi_1 \colon K \to K$ with $\varphi_1 := \varphi$ as in Example 3.10. As seen before,

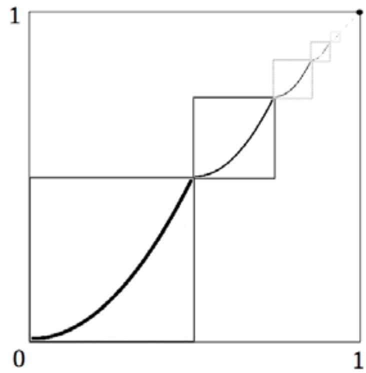

Fig. 3: $(\widetilde{K};\widetilde{\varphi}_1)$

$\dim \operatorname{fix} T_{\varphi_1} = 1$ and

$$\mathcal{S}_0(x) = \begin{cases} [0,\infty) & \text{for } 0 \le x < \infty, \\ \{\infty\} & \text{for } x = \infty. \end{cases}$$

Since $[0,\infty)$ is the only $\mathcal{S}_0$-invariant open subset of K, we have for all $x \in K$

$$\mathcal{A}_1(x) = \mathcal{S}_1(x) = K.$$

Next we define an isomorphic system $(\widetilde{K};\widetilde{\varphi}_1)$ by $\widetilde{K} := [0,1]$ and

$$\widetilde{\varphi}_1 \colon \widetilde{K} \to \widetilde{K}, \quad \widetilde{\varphi}_1 := h \circ \varphi_1 \circ h^{-1}$$

for a homeomorphism $h\colon K \to \widetilde{K}$ with $h(0) = 0$ and $h(\infty) = 1$. For this "compressed" system $(\widetilde{K};\widetilde{\varphi}_1)$ (see Fig. 3) we still have $\dim \operatorname{fix} T_{\widetilde{\varphi}_1} = 1$.

(b) Analogously, we construct a system $(K;\varphi_2)$ on $K = [0,\infty]$ with $\mathcal{S}_2(x) = K$ and $\mathcal{S}_1(x) \subsetneq K$ for all $x \in K$ via

$$\varphi_2 \colon K \to K, \quad x \mapsto \begin{cases} \widetilde{\varphi}_1(x-m)+m & \text{for } x \in [m,m+1),\ m \in \mathbb{N}_0, \\ \infty & \text{for } x = \infty. \end{cases}$$

We iterate this procedure of compressing systems and lining up copies of these on $K = [0,\infty]$ (cf. Fig. 4). By this procedure we obtain systems $(K;\varphi_n)$ with $\mathcal{S}_{n-1}(x) \subsetneq K$ and $\mathcal{S}_n(x) = K$ for some $n \in \mathbb{N}$ and all $x \in K$. Hence the quotient space $K/{\sim_n}$ is a singleton, thus homeomorphic to the fixed factor L by Proposition 2.4.

This construction leads to an example in which even superorbits of arbitrary degree $n \in \mathbb{N}$ are not sufficient to characterize the fixed space.

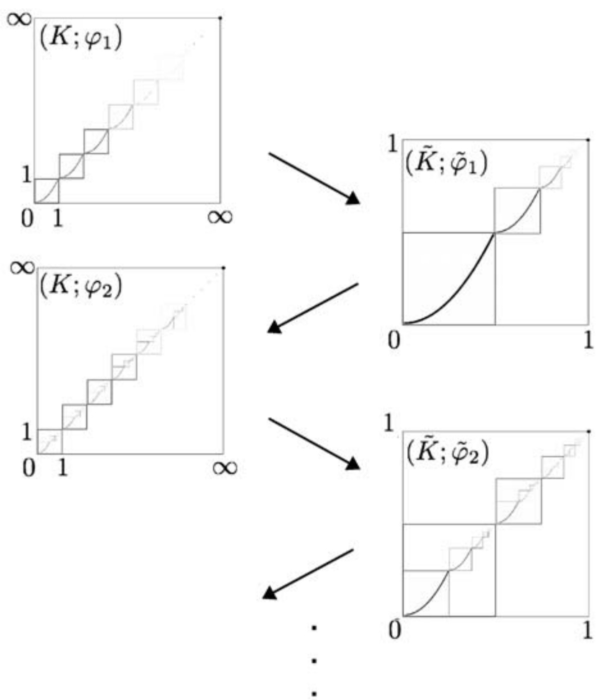

Fig. 4: Construction of systems with $\mathcal{S}_n(x) = K$ for $x \in K$, $n \in \mathbb{N}$

EXAMPLE 3.16. Let $K := [0, \infty]$ and define

$$\varphi(x) := \widetilde{\varphi}_k(x)$$

for $x \in [k-1, k)$, $k \in \mathbb{N}$, with $\widetilde{\varphi}_k$ as in Example 3.15 (see Fig. 5). Then for $n \in \mathbb{N}_0$ the superorbit of degree n is

$$\mathcal{S}_n(x) = \begin{cases} [0, n+1) & \text{for } x \in [0, n+1), \\ \{\infty\} & \text{for } x = \infty \end{cases}$$

and

$$\mathcal{S}_n(x) \subseteq [n+1, \infty) \quad \text{for } x \in [n+1, \infty).$$

Hence $\mathcal{S}_n(x) \neq K$ for all $x \in K$ and $n \in \mathbb{N}_0$ which implies that the corresponding quotient space $K/{\sim_n}$ contains more than one element. Thus, it does not correspond to the fixed factor L, which is a singleton since $\dim \operatorname{fix} T_\varphi = 1$.

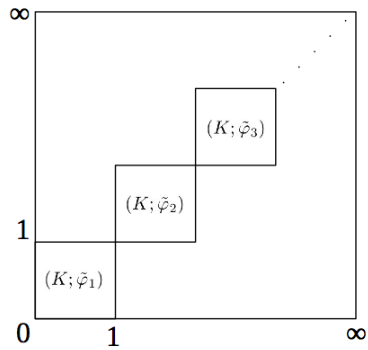

Fig. 5: $(K;\varphi)$

3.3. Superorbits of non-finite degree. Because superorbits of arbitrary finite degree do not, in general, yield a Hausdorff quotient space, we introduce superorbits of non-finite degree using ordinal numbers. We propose the following definition, where ω denotes the first non-finite ordinal number (see, e.g., [7, p. 42, Definition 6.1]).

DEFINITION 3.17. For any $x \in K$ the *approximating orbit of x of degree ω* is

$$\mathcal{A}_\omega(x) \coloneqq \bigcup_{n\in\mathbb{N}} \mathcal{S}_n(x),$$

and the *superorbit of x of degree ω* is

$$\mathcal{S}_\omega(x) \coloneqq [x]_\omega = \bigcup_{\substack{y\in K\\ y\sim_\omega x}} \mathcal{A}_\omega(y),$$

where $\sim_\omega$ is the equivalence relation generated by $(\mathcal{A}_\omega(x))_{x\in K}$.

By Proposition 3.14, the assumptions (i) and (ii) of Proposition 2.4 are satisfied for this equivalence relation.

PROPOSITION 3.18. *For each $x \in K$ we have that*

(a) $\varphi(x) \sim_\omega x$ *and*

(b) *the superorbit $\mathcal{S}_\omega(x)$ of degree ω is a level set of* $\operatorname{fix} T_\varphi$.

Even superorbits of degree ω do not, in general, yield a Hausdorff quotient space as the following example shows.

EXAMPLE 3.19. Let again $K \coloneqq [0,\infty]$ be the one-point compactification of $[0,\infty)$ and $\widetilde{K} \coloneqq [0,1]$. Consider the system $(K;\varphi)$ as in Example 3.16 and the isomorphic system

$$\widetilde{\varphi}\colon \widetilde{K} \to \widetilde{K}, \quad \widetilde{\varphi} \coloneqq h \circ \varphi \circ h^{-1}$$

for a homeomorphism $h\colon K \to \widetilde{K}$ with $h(0) = 0$ and $h(\infty) = 1$. Analogously to Example 3.15, we construct a system $(K;\psi)$ via

$$\psi\colon K \to K, \quad x \mapsto \begin{cases} \widetilde{\varphi}(x-n)+n & \text{for } x \in [n,n+1),\ n \in \mathbb{N}_0, \\ \infty & \text{for } x = \infty, \end{cases}$$

by putting copies of the compressed system in a row. Then the fixed factor L is a singleton, while $\mathcal{S}_\omega(x) \neq K$ for all $x \in K$, thus the corresponding quotient space $K/{\sim_\omega}$ contains more than one point.

4. Characterization of the fixed space via transfinite superorbits

To achieve our goal to characterize the fixed space $\operatorname{fix} T_\varphi$ dynamically, we need superorbits for arbitrary ordinal numbers. We define the base case, successor case and limit case analogously to Definitions 3.11 and 3.17. The class of ordinal numbers is denoted by Ord.

Definition 4.1. Let $x \in K$.
Base case: (i) The *approximating orbit of x of degree* 0 is

$$\mathcal{A}_0(x) \coloneqq \bigcap_{\substack{U \in \mathcal{U}(x) \text{ closed} \\ \varphi(U) \subseteq U}} U.$$

(ii) Let $\sim_0$ be the equivalence relation on K generated by $(\mathcal{A}_0(x))_{x\in K}$. The *superorbit of x of degree* 0 is

$$\mathcal{S}_0(x) \coloneqq [x]_0 = \bigcup_{\substack{y \in K \\ y \sim_0 x}} \mathcal{A}_0(y).$$

Successor case: (i) Let $\gamma \in \mathrm{Ord}$ a successor. Then the *approximating orbit of degree γ* is

$$\mathcal{A}_\gamma(x) \coloneqq \bigcap_{\substack{U \in \mathcal{U}(x) \text{ open} \\ \mathcal{S}_{\gamma-1}(U) \subseteq U}} \overline{U}^{\mathcal{S}_{\gamma-1}}$$

where $\mathcal{S}_{\gamma-1}(U) \coloneqq \bigcup_{y\in U} \mathcal{S}_{\gamma-1}(y)$ and

$$\overline{U}^{\mathcal{S}_{\gamma-1}} \coloneqq \bigcap_{\substack{U \subseteq F \text{ closed} \\ \mathcal{S}_{\gamma-1}(F) \subseteq F}} F \tag{5}$$

denotes the $\mathcal{S}_{\gamma-1}$*-closure of U*.

(ii) As before let $\sim_\gamma$ be the equivalence relation on K generated by $(\mathcal{A}_\gamma(x))_{x\in K}$. Finally, the *superorbit of x of degree γ* is

$$\mathcal{S}_\gamma(x) := [x]_\gamma = \bigcup_{\substack{y\in K\\ y\sim_\gamma x}} \mathcal{A}_\gamma(y).$$

Limit case: Let $0 \neq \gamma \in \mathrm{Ord}$ be a limit ordinal. Then the *approximating orbit of x of degree γ* is

$$\mathcal{A}_\gamma(x) := \bigcup_{\beta<\gamma} \mathcal{S}_\beta(x) = \bigcup_{\substack{\beta<\gamma\\ y\in K,\, y\sim_\beta x}} \mathcal{A}_\beta(x) = \bigcup_{\substack{\beta<\gamma\\ y\in K,\, y\sim_\beta x}} \mathcal{S}_\beta(x).$$

The equivalence relation $\sim_\gamma$ on K and the superorbit $\mathcal{S}_\gamma(x)$ of degree γ are defined as in the successor case,

$$\mathcal{S}_\gamma(x) := [x]_\gamma = \bigcup_{\substack{y\in K\\ y\sim_\gamma x}} \mathcal{A}_\gamma(y).$$

Before proving that superorbits of arbitrary degree are level sets, we list some basic properties by analogy with Proposition 3.12.

Proposition 4.2. *Let $\beta,\gamma \in \mathrm{Ord}$ with $\beta \leq \gamma$ and $U \subseteq K$.*

(a) *For all $x, y \in K$ with $y \sim_\gamma x$ we have*

$$\mathcal{A}_\gamma(y) \subseteq \mathcal{S}_\gamma(x), \quad \mathcal{S}_\gamma(y) \subseteq \mathcal{A}_{\gamma+1}(x)$$

and

$$\mathcal{A}_\beta(y) \subseteq \mathcal{A}_\gamma(x), \quad \mathcal{S}_\beta(y) \subseteq \mathcal{S}_\gamma(x).$$

In particular, these inclusions hold true for $x = y$.

(b) *For all $x \in K$ we have that $\mathcal{A}_\gamma(x)$ and $\mathcal{S}_\gamma(x)$ are φ-invariant.*

(c) *The following assertions are equivalent.*

(i) $\mathcal{S}_\gamma(U) \subseteq U$.

(ii) $\mathcal{S}_\gamma(U) = U$.

(iii) *There is some $M \subseteq K$ such that $U = \bigcup_{y\in M} \mathcal{S}_\gamma(y)$.*

(d) *We have*

$$\mathcal{S}_\gamma(\overline{U}^{\mathcal{S}_\gamma}) = \overline{U}^{\mathcal{S}_\gamma} = \overline{\overline{U}^{\mathcal{S}_\gamma}}^{\mathcal{S}_\gamma} = \overline{\overline{U}}^{\mathcal{S}_\gamma} = \overline{U^\circ}^{\mathcal{S}_\gamma}.$$

Proof. Take $x, y \in K$ with $x \sim_\gamma y$. The inclusions $\mathcal{A}_\gamma(y) \subseteq \mathcal{S}_\gamma(x)$ and $\mathcal{S}_\gamma(y) \subseteq \mathcal{A}_{\gamma+1}(x)$ in (a) are clear by definition. We show $\mathcal{S}_\beta(y) \subseteq \mathcal{S}_\gamma(x)$ for all $\beta \leq \gamma$ using transfinite induction. For $\gamma = 0$ the statement is trivial. For

$\gamma \in \mathrm{Ord}$ assume $\mathcal{S}_\beta(y) \subseteq \mathcal{S}_\gamma(x)$ for all $\beta \leq \gamma$. We show $\mathcal{S}_\beta(y) \subseteq \mathcal{S}_{\gamma+1}(x)$ for all $\beta \leq \gamma+1$. This follows immediately from

$$\mathcal{S}_\gamma(y) \subseteq \mathcal{A}_{\gamma+1}(y) \subseteq \mathcal{S}_{\gamma+1}(x).$$

Now let γ be a limit and $\beta < \gamma$. Then $\mathcal{S}_\beta(y) \subseteq \mathcal{S}_\gamma(x)$ by the definition of $\mathcal{S}_\gamma(x)$.

Similarly, one can show $\mathcal{A}_\beta(y) \subseteq \mathcal{A}_\gamma(x)$ for all $\beta \leq \gamma$.

To show (b) we use again transfinite induction. For $\gamma = 0$ or $\gamma \in \mathrm{Ord}$ a successor see the proof of Proposition 3.12. For $\gamma \in \mathrm{Ord}$ a limit the assertion follows by definition directly from the induction hypothesis. (c) and (d) can be proved by analogy with Proposition 3.12. □

We now show that the properties (i) and (ii) in Proposition 2.4 hold for all superorbits (of transfinite order).

Proposition 4.3. *For each* $\gamma \in \mathrm{Ord}$ *and* $x \in K$ *we have that*
(a) $\varphi(x) \sim_\gamma x$ *and*
(b) *the superorbit* $\mathcal{S}_\gamma(x)$ *of degree* γ *is a level set of* $\operatorname{fix} T_\varphi$.

Proof. We use transfinite induction. For the base case $\gamma = 0$ see Proposition 3.7. If $\gamma \in \mathrm{Ord}$ is a successor, the proof works analogously to Proposition 3.14. Let thus $\gamma \in \mathrm{Ord}$ be a limit. Clearly, $\varphi(x) \sim_\gamma x$ for all $x \in K$. Assume that $\mathcal{S}_\beta(x)$ is a level set of T_φ for all $x \in K$, $\beta < \gamma$. Then $\mathcal{S}_\gamma(x)$ is a level set of T_φ by definition and Proposition 4.2(a). □

The next proposition is crucial for the proof of our Main Theorem 4.6. It shows that an approximating orbit corresponds to the intersection of closed neighborhoods in the quotient space.

Proposition 4.4. *If* $x \in K$, $\gamma \in \mathrm{Ord}$ *a successor and* $\pi_\gamma\colon K \to K/{\sim_\gamma}$ *the canonical projection, then*

$$\pi_\gamma(\mathcal{A}_{\gamma+1}(x)) = \bigcap_{\substack{U_\sim \in \mathcal{U}([x]_\gamma)\\ \text{closed}}} U_\sim \quad \text{and} \quad \pi_\gamma^{-1}\Bigg(\bigcap_{\substack{U_\sim \in \mathcal{U}([x]_\gamma)\\ \text{closed}}} U_\sim\Bigg) = \mathcal{A}_{\gamma+1}(x).$$

Proof. Since π_γ is surjective, it suffices to show the following inclusions:
(a) $\pi_\gamma(\mathcal{A}_{\gamma+1}(x)) \subseteq \bigcap_{\substack{U_\sim \in \mathcal{U}([x]_\gamma)\\ \text{closed}}} U_\sim$,
(b) $\pi_\gamma^{-1}\Bigg(\bigcap_{\substack{U_\sim \in \mathcal{U}([x]_\gamma)\\ \text{closed}}} U_\sim\Bigg) \subseteq \mathcal{A}_{\gamma+1}(x)$.

By the definition of an approximating orbit and the results obtained in Proposition 4.2, we conclude

$$\pi_\gamma\big(\mathcal{A}_{\gamma+1}(x)\big) \overset{4.2(c)}{=} \pi_\gamma\Bigg(\bigcap_{\substack{U\in\mathcal{U}(x)\text{ open}\\ \mathcal{S}_\gamma(U)=U}} \overline{U}^{\mathcal{S}_\gamma}\Bigg) \underset{4.2(d)}{\overset{3.4}{=}} \pi_\gamma\Bigg(\bigcap_{\substack{U\in\mathcal{U}(x)\text{ open}\\ \mathcal{S}_\gamma(U)=U}} \pi_\gamma^{-1}\big(\pi_\gamma(\overline{U}^{\mathcal{S}_\gamma})\big)\Bigg)$$

$$= \bigcap_{\substack{U\in\mathcal{U}(x)\text{ open}\\ \pi_\gamma^{-1}(\pi_\gamma(U))=U}} \pi_\gamma\big(\overline{U}^{\mathcal{S}_\gamma}\big) \subseteq \bigcap_{U_\sim\in\mathcal{U}([x]_\gamma)\text{ closed}} U_\sim$$

which proves (a).

To show (b), let

$$[z]_\gamma \in \bigcap_{U_\sim\in\mathcal{U}([x]_\gamma)\text{ closed}} U_\sim\,.$$

Since $\mathcal{S}_\gamma(z) = [z]_\gamma = \pi_\gamma^{-1}([z]_\gamma)$, we show $\mathcal{S}_\gamma(z) \subseteq \mathcal{A}_{\gamma+1}(x)$. By the definition of $\mathcal{A}_{\gamma+1}(x)$ it suffices to show $\mathcal{S}_\gamma(z) \subseteq \overline{U}^{\mathcal{S}_\gamma}$ for $U \in \mathcal{U}(x)$ open with $\mathcal{S}_\gamma(U) \subseteq U$.

We now move to the quotient space and define

$$V_\sim := \pi_\gamma(\overline{U}^{\mathcal{S}_\gamma}) := \big\{\pi_\gamma(y) : y \in \overline{U}^{\mathcal{S}_\gamma}\big\}.$$

To show $V_\sim \in \mathcal{U}([x]_\gamma)$, we check the following.

(i) $[x]_\gamma \in V_\sim$ and

(ii) there is some subset $W_\sim \subseteq V_\sim$ which is open in $K/\!\sim_\gamma$ and $[x]_\gamma \in W_\sim$.

We have $\pi_\gamma^{-1}([x]_\gamma) = \mathcal{S}_\gamma(x) \subseteq \overline{U}^{\mathcal{S}_\gamma}$ since $x \in \overline{U}^{\mathcal{S}_\gamma}$ and $\mathcal{S}_\gamma(\overline{U}^{\mathcal{S}_\gamma}) \subseteq \overline{U}^{\mathcal{S}_\gamma}$ (see Proposition 4.2(d)). Therefore,

$$\{[x]_\gamma\} = \pi_\gamma(\pi_\gamma^{-1}([x]_\gamma)) \subseteq \pi_\gamma(\overline{U}^{\mathcal{S}_\gamma}) = V_\sim,$$

hence $[x]_\gamma \in V_\sim$. This shows (i).

Define $W_\sim := \pi_\gamma(U)$. Then $U \subseteq \overline{U}^{\mathcal{S}_\gamma}$ implies $W_\sim = \pi_\gamma(U) \subseteq \pi_\gamma(\overline{U}^{\mathcal{S}_\gamma}) = V_\sim$. Furthermore, $W_\sim$ is open with respect to the quotient topology since $\pi_\gamma^{-1}(W_\sim) = \pi_\gamma^{-1}(\pi_\gamma(U)) = \mathcal{S}_\gamma(U) \overset{4.2\,(c)}{=} U$ is open. Clearly, $[x]_\gamma \in W_\sim$. This shows (ii).

Analogously, we see $\pi_\gamma^{-1}(V_\sim) = \overline{U}^{\mathcal{S}_\gamma}$. Hence $V_\sim$ is closed with respect to the quotient topology.

Summarizing, we obtain $V_\sim \in \mathcal{U}([x]_\gamma)$ and $V_\sim$ closed, hence $[z]_\gamma \in V_\sim$ by assumption. This implies $\mathcal{S}_\gamma(z) = \pi_\gamma^{-1}([z]_\gamma) \subseteq \pi_\gamma^{-1}(V_\sim) = \overline{U}^{\mathcal{S}_\gamma}$ which proves assertion (b). □

To obtain a Hausdorff quotient space corresponding to the fixed factor L, the process of building superorbits must become stationary.

THEOREM 4.5. *There is some ordinal number* $\gamma \in \mathrm{Ord}$ *such that* $\mathcal{S}_\gamma(x) = \mathcal{S}_{\gamma+1}(x)$ *for all* $x \in K$.

PROOF. For all $\beta \in \mathrm{Ord}$ we have $\left|\{\mathcal{S}_\alpha(x) : x \in K, \alpha \leq \beta\}\right| \leq |\mathfrak{P}(K)|$ for the power set $\mathfrak{P}(K)$ of K. Moreover, by Proposition 4.2(a), if $\mathcal{S}_\alpha(x) = \mathcal{S}_{\alpha+1}(x)$ for some $\alpha \in \mathrm{Ord}$ and some $x \in K$, then $\mathcal{S}_\alpha(x) = \mathcal{S}_{\alpha'}(x)$ for all $\alpha' \in \mathrm{Ord}$ with $\alpha \leq \alpha'$. This implies for a $\gamma \in \mathrm{Ord}$ with $|\gamma| > |\mathfrak{P}(K)|$ that $\mathcal{S}_\gamma(x) = \mathcal{S}_{\gamma+1}(x)$ for all $x \in K$. □

We can now describe the fixed space of T_φ in terms of $(K;\varphi)$.

MAIN THEOREM 4.6. *Let* $\operatorname{fix} T_\varphi \cong \mathrm{C}(L)$ *for a compact Hausdorff space* L. *Then* L *is homeomorphic to* $K/{\sim_\alpha}$ *for some* $\alpha \in \mathrm{Ord}$.

PROOF. Choose $\alpha \in \mathrm{Ord}$ such that $\mathcal{S}_\alpha(x) = \mathcal{S}_{\alpha+1}(x)$ for all $x \in K$ (see Theorem 4.5) and assume, without loss of generality, that α is a successor. By Proposition 2.2 and Theorem 2.4 it remains to show that $K/{\sim_\alpha}$ is Hausdorff, i.e.,

$$\{[x]_\alpha\} = \bigcap_{U \in \mathcal{U}([x]_\alpha) \text{ closed}} U$$

for all $x \in K$ (see Lemma 3.2). To do so, let

$$[z]_\alpha \in \bigcap_{U \in \mathcal{U}([x]_\alpha) \text{ closed}} U.$$

As seen in the proof of Proposition 4.4 we have $\mathcal{S}_\alpha(z) \subseteq \mathcal{A}_{\alpha+1}(x)$. Consequently,

$$\mathcal{S}_\alpha(z) \subseteq \mathcal{A}_{\alpha+1}(x) \overset{4.2(a)}{\subseteq} \mathcal{S}_{\alpha+1}(x) = \mathcal{S}_\alpha(x).$$

This implies $\mathcal{S}_\alpha(z) \subseteq \mathcal{S}_\alpha(x)$ and hence $\mathcal{S}_\alpha(z) = \mathcal{S}_\alpha(x)$ since $\sim_\alpha$ is an equivalence relation. Therefore, also $[z]_\alpha = \pi(\mathcal{S}_\alpha(z)) = \pi(\mathcal{S}_\alpha(x)) = [x]_\alpha$ which shows that $K/{\sim_\alpha}$ is Hausdorff. □

From this, we obtain a characterization of a one-dimensional fixed space of T_φ in terms of its underlying dynamical system $(K;\varphi)$.

DEFINITION 4.7. We call a topological dynamical system $(K;\varphi)$ *topologically ergodic* if there is some $x \in K$ and $\gamma \in \mathrm{Ord}$ such that

$$K = \mathcal{S}_\gamma(x).$$

MAIN THEOREM 4.8. *The fixed space of* T_φ *is one-dimensional if and only if* $(K;\varphi)$ *is topologically ergodic.*

Remark 4.9. Topological ergodicity is a global property depending on the dynamical behavior of φ on the entire space K. Recall that a measure-preserving dynamical system $(\Omega, \Sigma\mu; \varphi)$ is *ergodic* if and only if the fixed space

$$\operatorname{fix} T_\varphi := \big\{ f \in \mathrm{L}^1(\Omega, \Sigma, \mu) : T_\varphi f = f \big\}$$

of the corresponding Koopman operator on $\mathrm{L}^1(\Omega, \Sigma, \mu)$ is one-dimensional. This motivates our choice of terminology even if there exist other meanings of "topological ergodicity", compare, e.g., [10, p. 2144], [15, p. 151] or [5, p. 31].

Remark 4.10. (a) In continuous-time dynamical systems there is a transfinite construction yielding so-called *prolongations* (cf., e.g., [2], [3, Chapters II.4 and VII] or [17]). These are – if adapted to the discrete-time setting – different from approximating orbits and superorbits as can be seen from Example 3.9. Here we have for the first prolongation

$$\mathcal{D}_1(x) := \mathcal{A}_0(x) = \begin{cases} \{(c,0) : c \in [a,1]\} & \text{if } x = (a,0) \text{ for some } a \in [0,1], \\ \overline{\operatorname{orb}}(x) & \text{elsewhere} \end{cases}$$

and for the second prolongation

$$\mathcal{D}_2(x) := \bigcap_{U \in \mathcal{U}(x)} \overline{\bigcup_{n\in\mathbb{N}} \mathcal{D}_1^n(U)} = \bigcap_{U \in \mathcal{U}(x)} \mathcal{D}_1(U) = \mathcal{D}_1(x)$$

for all $x \in K$ because $\mathcal{D}_1(\mathcal{D}_1(U)) = \mathcal{D}_1(U)$ and $\mathcal{D}_1(U)$ is closed for all $U \in \mathcal{U}(x)$. This implies that all prolongations of higher degree are equal to $\mathcal{D}_1(x)$ for all $x \in K$, while $\dim \operatorname{fix} T_\varphi = 1$. Hence the decomposition induced by $\operatorname{fix} T_\varphi$ is not obtained by the prolongations.

Also *chain prolongations* (see, e.g., [6]) are in general different from our superorbits.

(b) By a completely different approach, Akin and Wiseman in [1, Theorem 7.11] also obtain the equivalence relation $\sim_\alpha$ but do not relate it to the fixed space of the Koopman operator.

5. Lyapunov stability of higher order

It is an interesting problem to decompose a topological dynamical system $(K; \varphi)$ into disjoint, φ-invariant and "stable" sets. We now suggest a hierarchy of stability notions which are closely linked to the fixed space $\operatorname{fix} T_\varphi$ of a Koopman operator T_φ.

We first recall the following standard definition.

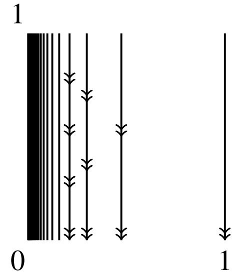

Fig. 6: $(K;\varphi)$

DEFINITION 5.1. A set $M \subseteq K$ is called *Lyapunov stable* if it is the intersection of its φ-invariant neighborhoods, i.e.,

$$M = \bigcap_{\substack{U\in\mathcal{U}(M)\\ \varphi(U)\subseteq U}} U.$$

The maximal level sets of $\operatorname{fix} T_\varphi$ are Lyapunov stable and yield a decomposition of K. However, it may happen that there exist finer decompositions into Lyapunov stable sets as the following example shows.

EXAMPLE 5.2.

$$K := \left\{(\tfrac{1}{n},c) : n\in\mathbb{N},\ c\in[0,1]\right\} \cup \left\{(0,c) : c\in[0,1]\right\}$$

with the subspace topology of $\mathbb{R}^2$ and the dynamics φ given for $x\in K$ by

$$\varphi(x) := \begin{cases} \left(\frac{1}{n}, n(c-\frac{m}{n})^2+\frac{m}{n}\right) & \text{for } x=(\frac{1}{n},c) \text{ with } c\in[\frac{m}{n},\frac{m+1}{n}] \\ & \quad \text{for some } n\in\mathbb{N} \text{ and } m\in\mathbb{N}_0, \\ (0,c) & \text{for } x=(0,c) \text{ with } c\in[0,1] \end{cases}$$

(see Fig. 6). Here, the decomposition of K induced by the fixed space $\operatorname{fix} T_\varphi$ is

$$K = \dot{\bigcup_{n\in\mathbb{N}}} \left\{(\tfrac{1}{n},c) : c\in[0,1]\right\} \dot{\cup} \left\{(0,c) : c\in[0,1]\right\}$$

while a finer decomposition into Lyapunov stable sets is given by

$$K = \dot{\bigcup_{n\in\mathbb{N}}} \left\{(\tfrac{1}{n},c) : c\in[0,1]\right\} \dot{\cup} \dot{\bigcup_{c\in[0,1]}} \left\{(0,c)\right\}.$$

To explain the difference between these decompositions, we use our concept of superorbits from Chapters 3 and 4 to generalize Lyapunov stability to a hierarchy of stability notions. This can produce decompositions of K which are coarser than a decomposition into Lyapunov stable sets but finer than the decomposition induced by $\operatorname{fix} T_\varphi$.

Definition 5.3. (a) A set $M \subseteq K$ is called *Lyapunov stable of degree α* for some $\alpha \in \mathrm{Ord}$ if

$$M = \bigcap_{\substack{U \in \mathcal{U}(M) \text{ open} \\ \mathcal{S}_\alpha(U) \subseteq U}} \overline{U}^{\mathcal{S}_\alpha}.$$

(b) A set $M \subseteq K$ is called *absolutely Lyapunov stable* if M is Lyapunov stable of degree α for all $\alpha \in \mathrm{Ord}$.

Remark 5.4. If a set M is Lyapunov stable of degree α, then it is Lyapunov stable of degree β for all $\beta \leq \alpha$.

Lemma 5.5. *Let $M \subseteq K$ and $\alpha \in \mathrm{Ord}$.*

(a) *For M Lyapunov stable and $x \in M$, also $\mathcal{A}_0(x) \subseteq M$ and $\mathcal{S}_0(x) \subseteq M$.*

(b) *For M Lyapunov stable of degree α and $x \in M$, also $\mathcal{A}_{\alpha+1}(x) \subseteq M$ and $\mathcal{S}_{\alpha+1}(x) \subseteq M$.*

Proof. It suffices to show the assertions for the approximating orbits. We have

$$\mathcal{A}_0(x) = \bigcap_{\substack{U \in \mathcal{U}(x) \text{ closed} \\ \varphi(U) \subseteq U}} U \subseteq \bigcap_{\substack{U \in \mathcal{U}(M) \text{ closed} \\ \varphi(U) \subseteq U}} U = \bigcap_{\substack{U \in \mathcal{U}(M) \\ \varphi(U) \subseteq U}} U = M.$$

since K is a Hausdorff space yielding (a). Assertion (b) follows by definition from

$$\mathcal{A}_{\alpha+1}(x) = \bigcap_{\substack{U \in \mathcal{U}(x) \text{ open} \\ \mathcal{S}_\alpha(U) \subseteq U}} \overline{U}^{\mathcal{S}_\alpha} \subseteq \bigcap_{\substack{U \in \mathcal{U}(M) \text{ open} \\ \mathcal{S}_\alpha(U) \subseteq U}} \overline{U}^{\mathcal{S}_\alpha} = M. \qquad \square$$

Lemma 5.6. *Let $x \in K$, $\alpha \in \mathrm{Ord}$ such that $L \cong K/{\sim_\alpha}$ for the fixed factor L and $\pi\colon K \to K/{\sim_\alpha}$ the canonical projection. Then for each closed $V \subseteq K/{\sim_\alpha}$ we have*

$$\overline{\pi^{-1}(V)^\circ}^{\mathcal{S}_\alpha} = \overline{\pi^{-1}(V^\circ)}^{\mathcal{S}_\alpha}.$$

Proof. The inclusion "$\supseteq$" is clear. To prove "$\subseteq$" note that for $V \subseteq K/{\sim_\alpha}$ there exists some $W \subseteq K$ such that $V = \pi(W)$. This implies that $\pi^{-1}(V)$ is $\mathcal{S}_\alpha$-invariant by

$$\begin{aligned}\mathcal{S}_\alpha(\pi^{-1}(V)) = \pi^{-1}(\pi(\pi^{-1}(V))) &= \pi^{-1}\big(\pi\big(\pi^{-1}(\pi(W))\big)\big) \\ &= \pi^{-1}(\pi(W)) = \pi^{-1}(V).\end{aligned}$$

Moreover, $\pi^{-1}(V)$ is closed, $\pi^{-1}(V)^\circ \subseteq \pi^{-1}(V)$ and $\pi^{-1}(V^\circ) \subseteq \pi^{-1}(V)$. From this we obtain

$$\overline{\pi^{-1}(V)^\circ}^{\mathcal{S}_\alpha} \subseteq \pi^{-1}(V) \tag{6}$$

and

$$\overline{\pi^{-1}(V^\circ)}^{\mathcal{S}_\alpha} \subseteq \pi^{-1}(V) \tag{7}$$

by definition of the $\mathcal{S}_\alpha$-closure.

If "$\supsetneq$" is strict then there is some closed $F \subseteq K$ such that $\mathcal{S}_\alpha(F) = F$ and $\pi^{-1}(V^\circ) \subseteq F$ but $\pi^{-1}(V)^\circ \nsubseteq F$. By (6) and (7) we can take $F \subsetneq \pi^{-1}(V)$ without loss of generality. By $\mathcal{S}_\alpha(F) = \pi^{-1}(\pi(F)) = F$ we obtain

$$\pi^{-1}(V^\circ) \subseteq \pi^{-1}(\pi(F)) \subsetneq \pi^{-1}(V),$$

which implies $V^\circ \subseteq \pi(F) \subsetneq V$. Since V and $\pi(F)$ are closed this yields

$$\overline{V^\circ} \subseteq \pi(F) \subsetneq V$$

which contradicts $\overline{V^\circ} = V$. $\square$

Remark 5.4 and the Lemmas 5.5 and 5.6 yield the following result.

THEOREM 5.7. *The finest decomposition into absolutely Lyapunov stable sets is induced by* $\operatorname{fix} T_\varphi$.

PROOF. We first show that the maximal level sets of $\operatorname{fix} T_\varphi$ are absolutely Lyapunov stable. By Remark 5.4 it suffices to show that a maximal level set M is Lyapunov stable of degree α where $L \cong K/{\sim_\alpha}$ for the fixed factor L. Let $x \in K$ such that $M = \pi^{-1}([x])$ where $\pi\colon K \to K/{\sim_\alpha}$ denotes the canonical projection. We first show that for all closed $V \in \mathcal{U}([x])$ there is some open $U \subseteq K$ with $\mathcal{S}_\alpha(U) \subseteq U$ such that $\pi^{-1}(V) = \overline{U}^{\mathcal{S}_\alpha}$.

Take $U := \pi^{-1}(V)^\circ$. Clearly, U is open and $\mathcal{S}_\alpha(U) = U$ because of $\mathcal{S}_\alpha(U) = \pi^{-1}(\pi(U))$. We show that $\pi^{-1}(V) = \overline{U}^{\mathcal{S}_\alpha}$. By continuity of π, we have

$$\pi^{-1}(V) \subseteq \overline{\pi^{-1}(V)}^{\mathcal{S}_\alpha} \overset{4.2(d)}{=} \overline{\pi^{-1}(V)^\circ}^{\mathcal{S}_\alpha} \overset{5.6}{=} \overline{\pi^{-1}(V^\circ)}^{\mathcal{S}_\alpha} = \overline{U}^{\mathcal{S}_\alpha}.$$

Conversely,

$$\overline{U}^{\mathcal{S}_\alpha} \overset{4.2(d)}{=} \overline{\pi^{-1}(V)}^{\mathcal{S}_\alpha} = \bigcap_{\substack{F \text{ closed} \\ \pi^{-1}(V)\subseteq F \\ \mathcal{S}_\alpha(F)\subseteq F}} F \overset{F:=\pi^{-1}(V)}{\subseteq} \pi^{-1}(V).$$

From this we obtain that

$$\bigcap_{\substack{U\in\mathcal{U}(M)\text{ open}\\ \mathcal{S}_\alpha(U)\subseteq U}} \overline{U}^{\mathcal{S}_\alpha} \subseteq \bigcap_{V\in\mathcal{U}([x])\text{ closed}} \pi^{-1}(V)$$

$$= \pi^{-1}\Bigg(\bigcap_{V\in\mathcal{U}([x])\text{ closed}} V\Bigg) \overset{K/\sim_\alpha \text{ Hausdorff}}{=} \pi^{-1}([x]) = M.$$

Together with the converse inclusion

$$M \subseteq \bigcap_{\substack{U\in\mathcal{U}(M)\text{ open}\\ \mathcal{S}_\alpha(U)\subseteq U}} \overline{U}^{\mathcal{S}_\alpha}$$

we obtain that M is Lyapunov stable of degree α.

That there is no finer decomposition into absolutely Lyapunov stable sets follows from Lemma 5.5 because a finer decomposition contradicts $\mathcal{S}_{\alpha+1}(x) \subseteq M'$ for $x \in M'$ with $M' \subseteq K$ Lyapunov stable of degree α. □

As a final result, we link absolute Lyapunov stability and topological ergodicity.

THEOREM 5.8. *A topological dynamical system $(K;\varphi)$ is topologically ergodic if and only if there is no nontrivial decomposition of K into absolutely Lyapunov stable sets.*

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