Analysis Math., **46 (3)** (2020), 517–533
DOI: 10.1007/s10476-020-0036-8
First published online June 17, 2020

DIFFERENCE OF WEIGHTED COMPOSITION OPERATORS ON WEIGHTED-TYPE SPACES IN THE UNIT BALL

B. HU[1] and S. LI[2,*]

[1]Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388, USA
e-mail: bhu32@wisc.edu

[2]Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, 610054, Chengdu, Sichuan, P.R. China
e-mail: jyulsx@163.com

(Received October 23, 2019; accepted December 16, 2019)

Abstract. In this paper, a new characterization is provided for the boundedness, compactness and essential norm of the difference of two weighted composition operators on weighted-type spaces in the unit ball of $\mathbb{C}^n$.

1. Introduction

Let $\mathbb{B}$ be the open unit ball of $\mathbb{C}^n$ and $\partial\mathbb{B}$ the boundary of $\mathbb{B}$. For $a \in \mathbb{B}\backslash\{0\}$, the automorphism of $\mathbb{B}$ is defined by

$$\Phi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}, \quad z \in \mathbb{B},$$

where

$$s_a = \sqrt{1 - |a|^2}\,, \quad P_a z = \frac{\langle z, a\rangle}{\langle a, a\rangle} a, \quad \text{and} \quad Q_a z = z - P_a z, \quad z \in \mathbb{B},$$

and $\Phi_0(z) = -z$. For $z, w \in \mathbb{B}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z, w) = |\Phi_w(z)|.$$

* Corresponding author.

This project was funded by the Science and Technology Development Fund, Macau SAR (file no. 186/2017/A3) and NNSF of China (No. 11720101003).

Key words and phrases: weighted composition operator, difference, weighted-type space.
Mathematics Subject Classification: 32A37, 47B38.

It is clear that $\rho(z,w) \le 1$, moreover, it is invariant under automorphism, that is,

$$\rho(\phi(z),\phi(w)) = \rho(z,w),$$

for all $z,w \in \mathbb{B}$ and $\phi \in \mathrm{Aut}(\mathbb{B})$.

Let $H(\mathbb{B})$ be the space of all holomorphic functions on $\mathbb{B}$. Let $\alpha > 0$. An $f \in H(\mathbb{B})$ is said to belong to the weighted-type space, denoted by H_α^∞, if

$$\|f\|_\alpha := \sup_{z\in\mathbb{B}} (1-|z|^2)^\alpha |f(z)| < \infty.$$

It is well known that H_α^∞ is a Banach space under the norm $\|\cdot\|_\alpha$.

Let φ be a holomorphic self-map of $\mathbb{B}$ and $u \in H(\mathbb{B})$. The weighted composition operator $uC_\varphi \colon H(\mathbb{B}) \mapsto H(\mathbb{B})$ is defined by

$$uC_\varphi(f)(z) = u(z) f(\varphi(z)), \quad f \in H(\mathbb{B}),\ z \in \mathbb{B}.$$

Observe that $uC_\varphi(f) = M_u \circ C_\varphi(f)$, where $M_u(f) = uf$ is the multiplication operator with symbol u and $C_\varphi(f) = f\circ\varphi$ is the composition operator with symbol φ.

The boundedness and compactness of the operator uC_φ are always important in the study of such operators (see, e.g., [3]). Recently, it is known that such properties can be merely captured by polynomials. More precisely, for a operator uC_φ from $\mathbb{X}$ into $\mathbb{Y}$, where $\mathbb{X}$ and $\mathbb{Y}$ are some "nice" analytic function spaces that are defined on the unit disc $\mathbb{D}$ (or $\mathbb{B}$, respectively),

1. $uC_\varphi \colon \mathbb{X} \mapsto \mathbb{Y}$ is bounded if and only if

$$\sup_{j\in\mathbb{N}} \frac{\|u\varphi^j\|_{\mathbb{Y}}}{\|z^j\|_{\mathbb{X}}} < \infty \quad \Big(\text{or } \sup_{j\in\mathbb{N}}\sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\|_{\mathbb{Y}}}{\|\langle z,\xi\rangle^j\|_{\mathbb{X}}} < \infty,\ \text{respectively}\Big);$$

2. $uC_\varphi \colon \mathbb{X} \mapsto \mathbb{Y}$ is compact if and only if $uC_\varphi \colon \mathbb{X} \mapsto \mathbb{Y}$ is bounded and

$$\limsup_{j\to\infty} \frac{\|u\varphi^j\|_{\mathbb{Y}}}{\|z^j\|_{\mathbb{X}}} = 0 \quad \Big(\text{or } \limsup_{j\to\infty}\sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\|_{\mathbb{Y}}}{\|\langle z,\xi\rangle^j\|_{\mathbb{X}}} = 0,\ \text{respectively}\Big).$$

Such a phenomenon was first found by Wulan, Zheng and Zhu in [18] in the setting of the unit disk. They showed that C_φ is compact on $\mathcal{B}(\mathbb{D})$ if and only if $\lim_{j\to\infty}\|\varphi^j\|_{\mathcal{B}(\mathbb{D})} = 0$. Here $\mathcal{B}(\mathbb{D})$ is the Bloch space, which consists of all analytic functions f on $\mathbb{D}$ satisfying $\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$. In [4], Dai extended the main result in [18] to the unit ball. He showed that C_φ is compact on $\mathcal{B}(\mathbb{B})$ if and only if $\lim_{j\to\infty}\sup_{\xi\in\partial\mathbb{B}}\|\langle\varphi,\xi\rangle^j\|_{\mathcal{B}(\mathbb{B})} = 0$. See [2,4,6,7,10,16–19] for more results on such characterization of composition operators and weighted composition operators on some analytic function spaces.

Recently, the difference of composition operators (as well as the weighted composition operators) draws great attention of lots of researchers, as it can be used to study the topological structure of the set of composition operators (as well as the weighted composition operators). For example, given uC_φ and vC_ψ two bounded operators acting from $\mathbb{X}$ to $\mathbb{Y}$ as above, one may ask whether uC_φ and vC_ψ are in the same path component in $\mathcal{W}(\mathbb{X}, \mathbb{Y})$, the set of all bounded weighted composition operators between $\mathbb{X}$ and $\mathbb{Y}$, equipped with the topology induced by operator norm. More precisely, we are interested in that whether there exists a continuous mapping $\gamma\colon [0,1] \mapsto \mathcal{W}(\mathbb{X}, \mathbb{Y})$, such that $\gamma(0) = uC_\varphi$ and $\gamma(1) = vC_\psi$. In general, this is a hard question and it turns out that one should first understand the behavior of the difference of two weighted composition operators.

The line of this research was first started by Berkson [1]. In [16], Shi and Li obtained several estimates for the essential norm of the difference of composition operators on $\mathcal{B}(\mathbb{D})$. Among others, they showed that

$$\|C_\varphi - C_\psi\|_{e,\mathcal{B}(\mathbb{D})\to\mathcal{B}(\mathbb{D})} \simeq \lim_{j\to\infty} \|\varphi^j - \psi^j\|_{\mathcal{B}(\mathbb{D})}.$$

For further results of the difference under various settings, we refer the readers to [1,3,5,6,8,9,11–17] and the references therein.

In [12], Nieminen obtained a characterization of the compactness of differences of weighted composition operators on weighted-type spaces. Motivated by the results in [18] and [12], Hu, Li and Shi [6] gave a new characterization for the boundedness, compactness and essential norm of the operator $uC_\varphi - vC_\psi\colon H^\infty_\alpha \to H^\infty_\beta$ in the unit disk. More precisely, they showed that $uC_\varphi - vC_\psi\colon H^\infty_\alpha(\mathbb{D}) \to H^\infty_\beta(\mathbb{D})$ is bounded (respectively, compact) if and only if the sequence $\big(\frac{\|u\varphi^j - v\psi^j\|_\beta}{\|z^j\|_\alpha}\big)_{j=0}^\infty$ is bounded (respectively, convergent to 0 as $j \to \infty$).

In this paper, we study the difference of two weighted composition operators between different weighted-type spaces in the unit ball, namely, the operator $uC_\varphi - vC_\psi\colon H^\infty_\alpha \to H^\infty_\beta$, where $u, v \in H(\mathbb{B})$ and φ, ψ are two holomorphic self-maps of $\mathbb{B}$. We characterize the boundedness, compactness and essential norm of the operator $uC_\varphi - vC_\psi$ by using

$$\sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}$$

and

$$\sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha}.$$

For a non-polynomial description, we refer the readers to the paper [5] for details.

Throughout this paper, for $a, b \in \mathbb{R}$, $a \lesssim b$ ($a \gtrsim b$, respectively) means there exists a positive number C, which is independent of a and b, such that

$a \le Cb$ ($a \ge Cb$, respectively). Moreover, if both $a \lesssim b$ and $a \gtrsim b$ hold, then we say $a \simeq b$.

2. Boundedness of $uC_\varphi - vC_\psi \colon H_\alpha^\infty \mapsto H_\beta^\infty$

In this section, we characterize the boundedness of the difference of weighted composition operators from H_α^∞ to H_β^∞. For all $z, w \in \mathbb{B}$, define

$$\flat_\alpha(z,w) = \sup_{\|f\|_{H_\alpha^\infty} \le 1} \left|(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)\right|.$$

Let φ and ψ be holomorphic self-maps of $\mathbb{B}$, $u, v \in H(\mathbb{B})$. We denote

$$\mathcal{D}_{u,\varphi}(z) = \frac{(1-|z|^2)^\beta u(z)}{(1-|\varphi(z)|^2)^\alpha}, \quad \mathcal{D}_{v,\psi}(z) = \frac{(1-|z|^2)^\beta v(z)}{(1-|\psi(z)|^2)^\alpha}.$$

Moreover, for each $a \in \mathbb{B}$, we define the following families of test functions on $\mathbb{B}$:

$$f_a(z) = \frac{(1-|a|^2)^\alpha}{(1-\langle z, a\rangle)^{2\alpha}},$$

$$g_{\varphi,\psi,a}(z) = \begin{cases} f_{\varphi(a)}(z) \cdot \frac{\langle \Phi_{\varphi(a)}(z), \Phi_{\varphi(a)}(\psi(a))\rangle}{|\Phi_{\varphi(a)}(\psi(a))|}, & \Phi_{\varphi(a)}(\psi(a)) \neq 0; \\ 0, & \Phi_{\varphi(a)}(\psi(a)) = 0, \end{cases}$$

and

$$g_{\psi,\varphi,a}(z) = \begin{cases} f_{\psi(a)}(z) \cdot \frac{\langle \Phi_{\psi(a)}(z), \Phi_{\psi(a)}(\varphi(a))\rangle}{|\Phi_{\psi(a)}(\varphi(a))|}, & \Phi_{\psi(a)}(\varphi(a)) \neq 0; \\ 0, & \Phi_{\psi(a)}(\varphi(a)) = 0. \end{cases}$$

It is clear that f_a, $g_{\varphi,\psi,a}$ and $g_{\psi,\varphi,a}$ are holomorphic with $g_{\varphi,\psi,a}(\varphi(a)) = 0$ and $g_{\psi,\varphi,a}(\psi(a)) = 0$. Moreover, we have

$$\sup_{a\in\mathbb{B}} \|f_a\|_\alpha \le 1, \quad \sup_{a\in\mathbb{B}} \|g_{\varphi,\psi,a}\|_\alpha \le 1, \quad \sup_{a\in\mathbb{B}} \|g_{\psi,\varphi,a}\|_\alpha \le 1.$$

To state and prove our main results in this paper, we need some lemmas. The following well-known estimate can be found in [5, Lemma 3.2].

Lemma 2.1. *For $f \in H_\alpha^\infty$ and $z, w \in \mathbb{B}$,*

$$\left|(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)\right| \lesssim \|f\|_\alpha \rho(z,w).$$

Lemma 2.2. *Let $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{B})$. Further, let φ and ψ be holomorphic self-maps of $\mathbb{B}$. Then the following inequalities hold*:

(i) *For each* $a \in \mathbb{B}$,

$$|\mathcal{D}_{u,\varphi}(a)|\rho(\varphi(a),\psi(a)) \le \|(uC_\varphi - vC_\psi)f_{\varphi(a)}\|_\beta + \|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\|_\beta;$$

(ii) *For each* $a \in \mathbb{B}$,

$$|\mathcal{D}_{v,\psi}(a)|\rho(\varphi(a),\psi(a)) \le \|(uC_\varphi - vC_\psi)f_{\psi(a)}\|_\beta + \|(uC_\varphi - vC_\psi)g_{\psi,\varphi,a}\|_\beta.$$

(iii) *For each* $a \in \mathbb{B}$,

$$|\mathcal{D}_{u,\varphi}(a) - \mathcal{D}_{v,\psi}(a)| \lesssim \|(uC_\varphi - vC_\psi)f_{\varphi(a)}\|_\beta + \|(uC_\varphi - vC_\psi)f_{\psi(a)}\|_\beta + \min\big\{\|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\|_\beta, \|(uC_\varphi - vC_\psi)g_{\psi,\varphi,a}\|_\beta\big\}.$$

PROOF. The idea of the proof follows from [6, Lemma 2.1].

(i) We only consider the case that $\Phi_{\varphi(a)}(\psi(a)) \ne 0$. Otherwise,

$$\big|\mathcal{D}_{u,\varphi}(a)|\rho(\varphi(a),\psi(a))\big| = 0$$

and hence there is nothing to prove. Then, by a simple calculation, we have

$$\big\|(uC_\varphi - vC_\psi)f_{\varphi(a)}\big\|_\beta \ge |\mathcal{D}_{u,\varphi}(a)| - \frac{(1-|\varphi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\langle\psi(a),\varphi(a)\rangle|^{2\alpha}}|\mathcal{D}_{v,\psi}(a)|$$

and

$$\big\|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\big\|_\beta \ge \frac{(1-|\varphi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\langle\psi(a),\varphi(a)\rangle|^{2\alpha}}|\mathcal{D}_{v,\psi}(a)|\rho(\varphi(a),\psi(a)).$$

Hence

$$\begin{aligned}
&|\mathcal{D}_{u,\varphi}(a)|\rho(\varphi(a),\psi(a))\\
&\le \big\|(uC_\varphi - vC_\psi)f_{\varphi(a)}\big\|_\beta \cdot \rho(\varphi(a),\psi(a)) + \big\|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\big\|_\beta\\
&\le \big\|(uC_\varphi - vC_\psi)f_{\varphi(a)}\big\|_\beta + \big\|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\big\|_\beta.
\end{aligned}$$

(ii) The proof of (ii) is similar to (i), by interchanging the role of φ and ψ, and hence we omit it.

(iii) By Lemma 2.1, we have

$$\begin{aligned}
\|(uC_\varphi - vC_\psi)f_{\varphi(a)}\|_\beta &\ge \Big|\mathcal{D}_{u,\varphi}(a) - \frac{(1-|\varphi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\langle\psi(a),\varphi(a)\rangle|^{2\alpha}}\mathcal{D}_{v,\psi}(a)\Big|\\
&\ge |\mathcal{D}_{u,\varphi}(a) - \mathcal{D}_{v,\psi}(a)| - \Big|1 - \frac{(1-|\varphi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\langle\psi(a),\varphi(a)\rangle|^{2\alpha}}\Big||\mathcal{D}_{v,\psi}(a)|
\end{aligned}$$

$$= |\mathcal{D}_{u,\varphi}(a) - \mathcal{D}_{v,\psi}(a)|$$
$$- \big|(1-|\varphi(a)|^2)^\alpha f_{\varphi(a)}(\varphi(a)) - (1-|\psi(a)|^2)^\alpha f_{\varphi(a)}(\psi(a))\big||\mathcal{D}_{v,\psi}(a)|$$
$$\geq |\mathcal{D}_{u,\varphi}(a) - \mathcal{D}_{v,\psi}(a)| - \flat_\alpha(\varphi(a),\psi(a))|\mathcal{D}_{v,\psi}(a)|$$
$$\gtrsim |\mathcal{D}_{u,\varphi}(a) - \mathcal{D}_{v,\psi}(a)| - |\mathcal{D}_{v,\psi}(a)|\rho(\varphi(a),\psi(a)).$$

Thus, by (ii), we have

$$|\mathcal{D}_{u,\varphi}(a) - \mathcal{D}_{v,\psi}(a)| \lesssim \|(uC_\varphi - vC_\psi)f_{\varphi(a)}\|_\beta + |\mathcal{D}_{v,\psi}(a)|\rho(\varphi(a),\psi(a))$$
$$\leq \|(uC_\varphi - vC_\psi)f_{\varphi(a)}\|_\beta + \|(uC_\varphi - vC_\psi)f_{\psi(a)}\|_\beta + \|(uC_\varphi - vC_\psi)g_{\psi,\varphi,a}\|_\beta.$$

Interchanging the role of φ and ψ, it is easy to see that

$$|\mathcal{D}_{u,\varphi}(a) - \mathcal{D}_{v,\psi}(a)| \leq \|(uC_\varphi - vC_\psi)f_{\varphi(a)}\|_\beta$$
$$+ \|(uC_\varphi - vC_\psi)f_{\psi(a)}\|_\beta + \|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\|_\beta.$$

Thus, we have

$$|\mathcal{D}_{u,\varphi}(a) - \mathcal{D}_{v,\psi}(a)| \leq \|(uC_\varphi - vC_\psi)f_{\varphi(a)}\|_\beta + \|(uC_\varphi - vC_\psi)f_{\psi(a)}\|_\beta$$
$$+ \min\big\{\|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\|_\beta, \|(uC_\varphi - vC_\psi)g_{\psi,\varphi,a}\|_\beta\big\}.$$

The proof is complete. □

Next, we introduce the following condition with respect to φ and ψ: there exists a $C > 0$, such that

$$\inf_{a\in\mathbb{B}} \frac{1-|\varphi(a)|^2}{|1-\langle\varphi(a),\xi_a\rangle|} > C, \tag{2.1}$$

where $\xi_a = \frac{\Phi_{\varphi(a)}(\psi(a))}{|\Phi_{\varphi(a)}(\psi(a))|}$. We also need its dual version, namely, there exists a $C > 0$, such that

$$\inf_{a\in\mathbb{B}} \frac{1-|\psi(a)|^2}{|1-\langle\psi(a),\zeta_a\rangle|} > C, \tag{2.2}$$

where $\zeta_a = \frac{\Phi_{\psi(a)}(\varphi(a))}{|\Phi_{\psi(a)}(\varphi(a))|}$. We refer the reader Remark 2.4 below for some motivations on introducing these conditions.

Lemma 2.3. *Let $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{B})$. Let, further, φ and ψ be holomorphic self-maps of $\mathbb{B}$. Then the following inequalities hold*:

(i)

$$\sup_{a\in\mathbb{B}} \|(uC_\varphi - vC_\psi)f_a\|_\beta \lesssim \sup_{j\in\mathbb{N}} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha};$$

(ii) *Suppose* (2.1) *holds. Then*

$$\sup_{a\in\mathbb{B}}\|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\|_\beta \lesssim \sup_{j\in\mathbb{N}}\sup_{\xi\in\partial\mathbb{B}}\frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}$$
$$+\sup_{j\in\mathbb{N}}\sup_{\xi,\xi'\in\partial\mathbb{B}}\frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha};$$

(iii) *Suppose* (2.2) *holds. Then*

$$\sup_{a\in\mathbb{B}}\|(uC_\varphi - vC_\psi)g_{\psi,\varphi,a}\|_\beta \lesssim \sup_{j\in\mathbb{N}}\sup_{\xi\in\partial\mathbb{B}}\frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}$$
$$+\sup_{j\in\mathbb{N}}\sup_{\xi,\xi'\in\partial\mathbb{B}}\frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha}.$$

PROOF. (i) If $a=0$, then $f_a(z)=1$, and hence

$$\|(uC_\varphi - vC_\psi)f_a\|_\beta = \|u-v\|_\beta \le \sup_{j\in\mathbb{N}}\sup_{\xi\in\partial\mathbb{B}}\frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha},$$

where in the above inequality, we simply consider the case $j=0$ and use the fact that $\|1\|_\alpha = 1$.

For any $a\in\mathbb{B}$ with $a\neq 0$, we have

$$f_a(z) = \frac{(1-|a|^2)^\alpha}{(1-\langle z,a\rangle)^{2\alpha}} = (1-|a|^2)^\alpha\sum_{k=0}^\infty \frac{\Gamma(k+2\alpha)}{k!\,\Gamma(2\alpha)}\langle z,a\rangle^k$$
$$= (1-|a|^2)^\alpha\sum_{k=0}^\infty \frac{\Gamma(k+2\alpha)}{k!\,\Gamma(2\alpha)}|a|^k\Big\langle z,\frac{a}{|a|}\Big\rangle^k.$$

Moreover, for each $\xi\in\partial\mathbb{B}$, it is easy to see

$$\|\langle z,\xi\rangle^k\|_\alpha = \sup_{z\in\mathbb{B}}|\langle z,\xi\rangle|^k(1-|z|^2)^\alpha \simeq k^{-\alpha}, \tag{2.3}$$

uniformly in ξ. Thus,

$$\|(uC_\varphi - vC_\psi)f_a\|_\beta$$
$$\le (1-|a|^2)^\alpha\sum_{k=0}^\infty \frac{\Gamma(k+2\alpha)}{k!\,\Gamma(2\alpha)}|a|^k\Big\|u\Big\langle\varphi,\frac{a}{|a|}\Big\rangle^k - v\Big\langle\psi,\frac{a}{|a|}\Big\rangle^k\Big\|_\beta$$
$$= (1-|a|^2)^\alpha\sum_{k=0}^\infty \frac{\Gamma(k+2\alpha)}{k!\,\Gamma(2\alpha)}k^{-\alpha}|a|^k k^\alpha\Big\|u\Big\langle\varphi,\frac{a}{|a|}\Big\rangle^k - v\Big\langle\psi,\frac{a}{|a|}\Big\rangle^k\Big\|_\beta$$

$$\le (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\,\Gamma(2\alpha)} k^{-\alpha}|a|^k \cdot \sup_{j\in\mathbb{N}} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle\|_\alpha}$$

$$\lesssim \sup_{j\in\mathbb{N}} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle\|_\alpha},$$

where in the last inequality, we use the fact that

$$\sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\,\Gamma(2\alpha)} k^{-\alpha}|a|^k \simeq (1-|a|^2)^{-\alpha}.$$

The desired result follows from by talking supremum of a over $\mathbb{B}$.

(ii) First, we observe that for any $\xi,\xi'\in\partial\mathbb{B}$ and $k\in\mathbb{N}$,

$$\|\langle z,\xi\rangle^k\langle z,\xi'\rangle\|_\alpha \lesssim k^{-\alpha},$$

which implies

$$1 \lesssim \frac{k^{-\alpha}}{\|\langle z,\xi\rangle^k\langle z,\xi'\rangle\|_\alpha}. \tag{2.4}$$

Indeed, the above claim follows from (2.3) and the fact that $\|\langle z,\xi\rangle^k\langle z,\xi'\rangle\|_\alpha \le \|\langle z,\xi\rangle^k\|_\alpha$.

Take and fix some $a\in\mathbb{B}$. Again, without the loss of generality, we may assume that $\Phi_{\varphi(a)}(\psi(a))\neq 0$, otherwise the statement is trivial.

We may also assume that $\varphi(a)\neq 0$. Indeed, if $\varphi(a)=0$, we have $g_{\varphi,\psi,a}(z) = -\langle z,\xi_a\rangle$, where $\xi_a = \frac{\Phi_{\varphi(a)}(\psi(a))}{|\Phi_{\varphi(a)}(\psi(a))|}\in\partial\mathbb{B}$. Then it is clear that

$$\|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\|_\beta = \|u\langle\varphi,\xi_a\rangle - v\langle\psi,\xi_a\rangle\|_\beta$$

$$\lesssim \sup_{j\in\mathbb{N}} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}.$$

By [20, Lemma 1.3], we have

$$\frac{\langle\Phi_{\varphi(a)}(z),\Phi_{\varphi(a)}(\psi(a))\rangle}{|\Phi_{\varphi(a)}(\psi(a))|} = \langle\Phi_{\varphi(a)}(z),\xi_a\rangle$$

$$= 1 + \big(\big\langle\Phi_{\varphi(a)}(z),\xi_a\big\rangle - 1\big) = 1 - \Big(1 - \Big\langle\Phi_{\varphi(a)}(z),\Phi_{\varphi(a)}(\Phi_{\varphi(a)}(\xi_a))\Big\rangle\Big)$$

$$= 1 - \frac{(1-|\varphi(a)|^2)(1-\langle z,\xi_a'\rangle)}{(1-\langle z,\varphi(a)\rangle)(1-\langle\varphi(a),\xi_a'\rangle)},$$

where $\xi_a' := \Phi_{\varphi(a)}(\xi_a) \in \partial\mathbb{B}$. Note that another application of [20, Lemma 1.3] gives

$$1 - \langle \varphi(a), \xi_a' \rangle = \frac{1 - |\varphi(a)|^2}{1 - \langle \varphi(a), \xi_a \rangle}.$$

Thus, for each $z \in \mathbb{B}$, we have

$$\begin{aligned} g_{\varphi,\psi,a}(z) &= f_{\varphi(a)}(z) \cdot \frac{\langle \Phi_{\varphi(a)}(z), \Phi_{\varphi(a)}(\psi(a)) \rangle}{|\Phi_\varphi(a)(\psi(a))|} \\ &= f_{\varphi(a)}(z) - f_{\varphi(a)}(z) \cdot \frac{(1 - |\varphi(a)|^2)(1 - \langle z, \xi_a' \rangle)}{(1 - \langle z, \varphi(a) \rangle)(1 - \langle \varphi(a), \xi_a' \rangle)}, \end{aligned}$$

which implies

$$\begin{aligned} &\|(uC_\varphi - vC_\psi) g_{\varphi,\psi,a}\|_\beta \le \|(uC_\varphi - vC_\psi)(f_{\varphi(a)})\|_\beta \\ &+ \Big\|(uC_\varphi - vC_\psi)\Big(f_{\varphi(a)} \cdot \frac{(1 - |\varphi(a)|^2)(1 - \langle z, \xi_a' \rangle)}{(1 - \langle z, \varphi(a) \rangle)(1 - \langle \varphi(a), \xi_a' \rangle)}\Big)\Big\|_\beta. \end{aligned}$$

By part (i), the first term is bounded by

$$\sup_{j \in \mathbb{N}} \sup_{\xi \in \partial\mathbb{B}} \frac{\|u\langle \varphi, \xi \rangle^j - v\langle \psi, \xi \rangle^j\|_\beta}{\|\langle z, \xi \rangle\|_\alpha}.$$

To bound the second term, we first note that

$$\begin{aligned} &f_{\varphi(a)}(z) \cdot \frac{(1 - |\varphi(a)|^2)(1 - \langle z, \xi_a' \rangle)}{(1 - \langle z, \varphi(a) \rangle)(1 - \langle \varphi(a), \xi_a' \rangle)} \\ &= \Big[(1 - |\varphi(a)|^2)^\alpha \sum_{k=0}^\infty \frac{\Gamma(k + 2\alpha)}{k!\,\Gamma(2\alpha)} \langle z, \varphi(a) \rangle^k\Big] \\ &\times \Big[\frac{1 - |\varphi(a)|^2}{1 - \langle \varphi(a), \xi_a' \rangle} \cdot (1 - \langle z, \xi_a' \rangle) \cdot \sum_{k=0}^\infty \langle z, \varphi(a) \rangle^k\Big] = J_1(z) - J_2(z), \end{aligned}$$

where both $J_1(z)$ and $J_2(z)$ are holomorphic on $\mathbb{B}$, defined by

$$\begin{aligned} J_1(z) &:= \frac{(1 - |\varphi(a)|^2)^{1+\alpha}}{1 - \langle \varphi(a), \xi_a' \rangle} \Big[\sum_{k=0}^\infty \frac{\Gamma(k + 2\alpha)}{k!\,\Gamma(2\alpha)} \langle z, \varphi(a) \rangle^k\Big] \cdot \Big[\sum_{k=0}^\infty \langle z, \varphi(a) \rangle^k\Big] \\ &= \frac{(1 - |\varphi(a)|^2)^{1+\alpha}}{1 - \langle \varphi(a), \xi_a' \rangle} \cdot \sum_{k=0}^\infty \Big(\sum_{j=0}^k \frac{\Gamma(j + 2\alpha)}{j!\Gamma(2\alpha)}\Big) \langle z, \varphi(a) \rangle^k \end{aligned}$$

and

$$J_2(z) := \frac{(1-|\varphi(a)|^2)^{1+\alpha}}{1-\langle \varphi(a), \xi_a' \rangle} \cdot \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} \frac{\Gamma(j+2\alpha)}{j!\Gamma(2\alpha)} \Big) \langle z, \varphi(a) \rangle^k \langle z, \xi_a' \rangle.$$

Hence,

$$\Big\| (uC_\varphi - vC_\psi) \Big(f_{\varphi(a)} \cdot \frac{(1-|\varphi(a)|^2)(1-\langle z, \xi_a' \rangle)}{(1-\langle z, \varphi(a) \rangle)(1-\langle \varphi(a), \xi_a' \rangle)} \Big) \Big\|_\beta \\ \leq \| (uC_\varphi - vC_\psi) J_1 \|_\beta + \| (uC_\varphi - vC_\psi) J_2 \|_\beta .$$

By Stirling's formula, we have

$$\sum_{j=0}^{k} \frac{\Gamma(j+2\alpha)}{j!\Gamma(2\alpha)} \simeq \sum_{j=0}^{k} j^{2\alpha-1} \simeq k^{2\alpha}$$

for k large enough. Thus, by (2.1) and (2.3), we have

$$\begin{aligned} & \| (uC_\varphi - vC_\psi) J_1 \|_\beta \\ & \lesssim (1-|\varphi(a)|^2)^{1+\alpha} \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} \frac{\Gamma(j+2\alpha)}{j!\Gamma(2\alpha)} \Big) \big\| (uC_\varphi - vC_\psi) \langle z, \varphi(a) \rangle^k \big\|_\beta \\ & \lesssim (1-|\varphi(a)|^2)^{1+\alpha} \sum_{k=1}^{\infty} k^{2\alpha} |\varphi(a)|^k \Big\| (uC_\varphi - vC_\psi) \Big(\Big\langle z, \frac{\varphi(a)}{|\varphi(a)|} \Big\rangle^k \Big) \Big\|_\beta \\ & \lesssim (1-|\varphi(a)|^2)^{1+\alpha} \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \cdot k^{2\alpha} |\varphi(a)|^k \cdot \frac{\Big\| (uC_\varphi - vC_\psi) \big(\big\langle z, \frac{\varphi(a)}{|\varphi(a)|} \big\rangle^k \big) \Big\|_\beta}{\Big\| \big\langle z, \frac{\varphi(a)}{|\varphi(a)|} \big\rangle^k \Big\|_\alpha} \\ & \lesssim \sup_{j \in \mathbb{N}} \sup_{\xi \in \partial \mathbb{B}} \frac{\| u \langle \varphi, \xi \rangle^j - v \langle \psi, \xi \rangle^j \|_\beta}{\| \langle z, \xi \rangle \|_\alpha} . \end{aligned}$$

The estimation of $\|(uC_\varphi - vC_\psi)J_2\|_\beta$ is similar as the previous part, by replacing the role of (2.3) by (2.4), and $\langle z, \varphi(a) \rangle^k$ by $\langle z, \varphi(a) \rangle^k \langle z, \xi_a' \rangle$. We omit the details. The desired estimation follows from combining the above two estimations.

(iii) The proof of (iii) is similar as (ii), by replacing (2.1) by it dual version (2.2). □

REMARK 2.4. In the above lemma, the assumptions (2.1) and (2.2) can be dropped when $n = 1$. Indeed, by the definition of $\Phi_{\varphi(a)}(z)$, we have

$$\begin{aligned}\Phi_{\varphi(a)}(z) &= \big(\varphi(a) - P_{\varphi(a)}(z) - s_{\varphi(a)}Q_{\varphi(a)}(z)\big)\sum_{k=0}^{\infty}\langle z,\varphi(a)\rangle^k \\ &= \big(\varphi(a) - P_{\varphi(a)}(z)\big)\sum_{k=0}^{\infty}\langle z,\varphi(a)\rangle^k,\end{aligned}$$

where in the last equality we use the fact that the orthogonal projection Q_a vanishes when $n = 1$. Hence

$$\begin{aligned}\frac{\langle\Phi_{\varphi(a)}(z),\Phi_{\varphi(a)}(\psi(a))\rangle}{|\Phi_{\varphi(a)}(\psi(a))|} &= \langle\Phi_{\varphi(a)}(z),\xi_a\rangle \\ &= \frac{|\varphi(a)|^2-1}{|\varphi(a)|^2}\cdot\langle\varphi(a),\xi_a\rangle\cdot\sum_{k=0}^{\infty}\langle z,\varphi(a)\rangle^k.\end{aligned}$$

The desired claim then follows from a similar argument as the one in Lemma 2.3(ii) and an application of Cauchy's inequality. We omit the details.

THEOREM 2.5. *Let $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{B})$. Let further, φ and ψ be holomorphic self-maps of $\mathbb{B}$ satisfying (2.1) or (2.2). Then $uC_\varphi - vC_\psi \colon H^\infty_\alpha \mapsto H^\infty_\beta$ is bounded if and only if*

$$\sup_{j\in\mathbb{N}}\sup_{\xi\in\partial\mathbb{B}}\frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha} < \infty \tag{2.5}$$

and

$$\sup_{j\in\mathbb{N}}\sup_{\xi,\xi'\in\partial\mathbb{B}}\frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha} < \infty. \tag{2.6}$$

PROOF. *Necessity*. Suppose $uC_\varphi - vC_\psi$ is bounded. For any $j \in \mathbb{N}$ and $\xi, \xi' \in \partial\mathbb{B}$, consider the probe funcitons

$$f_{j,\xi}(z) = \frac{\langle z,\xi\rangle^j}{\|\langle z,\xi\rangle^j\|_\alpha} \quad\text{and}\quad f_{j,\xi,\xi'}(z) = \frac{\langle z,\xi\rangle^j\langle z,\xi'\rangle}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha}.$$

Then $\|f_{j,\xi}\|_\alpha = \|f_{j,\xi,\xi'}\|_\alpha = 1$. Thus, by the boundedness of $uC_\varphi - vC_\psi$, we have

$$\frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha} = \|(uC_\varphi - vC_\psi)f_{j,\xi}\|_\beta \le \|uC_\varphi - vC_\psi\| < \infty,$$

and

$$\frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha}$$
$$= \|(uC_\varphi - vC_\psi)f_{j,\xi,\xi'}\|_\beta \le \|uC_\varphi - vC_\psi\| < \infty,$$

The desired result then follows by take the supremum of j, ξ and ξ' on both sides of the above inequalities.

Sufficiency. Suppose (2.5) and (2.6) hold. Moreover, without loss of generality, we assume (2.1) holds. Then for any $f \in H_\alpha^\infty$ with $\|f\|_{H_\alpha^\infty} \le 1$ and using Lemma 2.1, we have

$$\|(uC_\varphi - vC_\psi)f\|_\beta = \sup_{z\in\mathbb{B}} \big|u(z)f(\varphi(z)) - v(z)f(\psi(z))\big|(1-|z|^2)^\beta$$
$$\le \sup_{z\in\mathbb{B}} \big|f(\varphi(z))(1-|\varphi(z)|^2)^\alpha - f(\psi(z))(1-|\psi(z)|^2)^\alpha\big||\mathcal{D}_{u,\varphi}(z)|$$
$$+ \sup_{z\in\mathbb{B}} |f(\psi(z))|(1-|\psi(z)|^2)^\alpha|\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)|$$
$$\le \sup_{z\in\mathbb{B}} \flat_\alpha(\varphi(z),\psi(z))|\mathcal{D}_{u,\varphi}(z)| + \sup_{z\in\mathbb{B}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)|$$
$$\lesssim \sup_{z\in\mathbb{B}} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z),\psi(z)) + \sup_{z\in\mathbb{B}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)|.$$

Hence, by Lemmas 2.2 and 2.3, we have

$$\|(uC_\varphi - vC_\psi)f\|_\beta \lesssim \sup_{a\in\mathbb{B}} \|(uC_\varphi - vC_\psi)f_{\varphi(a)}\|_\beta$$
$$+ \sup_{a\in\mathbb{B}}\sup_{a\in\mathbb{B}} \|(uC_\varphi - vC_\psi)f_{\psi(a)}\|_\beta + \sup_{a\in\mathbb{B}} \|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\|_\beta$$
$$\lesssim \sup_{j\in\mathbb{N}}\sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}$$
$$+ \sup_{j\in\mathbb{N}}\sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha} < \infty.$$

Therefore, $uC_\varphi - vC_\psi \colon H_\alpha^\infty \mapsto H_\beta^\infty$ is bounded. The proof is complete. □

3. Essential norm estimates

In this section, we give an estimate for the essential norm of $uC_\varphi - vC_\psi$ from H_α^∞ to H_β^∞.

Recall that the essential norm $\|T\|_{e,\mathbb{X}\to\mathbb{Y}}$ of a bounded linear operator $T\colon \mathbb{X}\to\mathbb{Y}$ is defined as the distance from T to the set of compact operators K

mapping $\mathbb{X}$ into $\mathbb{Y}$, that is, $\|T\|_{e,\mathbb{X}\to\mathbb{Y}} = \inf\{\|T-K\|_{\mathbb{X}\to\mathbb{Y}} : K$ is compact$\}$, where $\|\cdot\|_{\mathbb{X}\to\mathbb{Y}}$ is the operator norm.

LEMMA 3.1. *Let $0<\alpha,\beta<\infty$, $u,v\in H(\mathbb{B})$. Let further, φ and ψ be holomorphic self-maps of $\mathbb{B}$. Then the following inequalities hold*:

(i)

$$\limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi)f_a\|_\beta \lesssim \limsup_{j\to\infty} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha};$$

(ii) *Suppose* (2.1) *holds. Then*

$$\limsup_{|\varphi(a)|\to 1} \|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\|_\beta \lesssim \limsup_{j\to\infty} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}$$
$$+ \limsup_{j\to\infty} \sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha};$$

(iii) *Suppose* (2.2) *holds. Then*

$$\limsup_{|\psi(a)|\to 1} \|(uC_\varphi - vC_\psi)g_{\psi,\varphi,a}\|_\beta \lesssim \limsup_{j\to\infty} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}$$
$$+ \limsup_{j\to\infty} \sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha}.$$

PROOF. (i) For each $N\in\mathbb{N}$ and $a\in\mathbb{B}$ with $a\neq 0$, the proof of (i) in Lemma 2.3 gives

$$\big\|(uC_\varphi - vC_\psi)f_a\big\|_\beta$$
$$\lesssim (1-|a|^2)^\alpha \sum_{k=0}^N \frac{\Gamma(k+2\alpha)}{k!\,\Gamma(2\alpha)}|a|^k \Big\|u\Big\langle\varphi,\frac{a}{|a|}\Big\rangle^k - v\Big\langle\psi,\frac{a}{|a|}\Big\rangle^k\Big\|_\beta$$
$$+ (1-|a|^2)^\alpha \sum_{N+1}^\infty \frac{\Gamma(k+2\alpha)}{k!\,\Gamma(2\alpha)} k^{-\alpha}|a|^k \sup_{j\ge N+1} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}.$$

Now for this fixed N, by letting $|a|\to 1$, we have

$$\limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi)f_a\|_\beta \lesssim \sup_{j\ge N+1} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha},$$

which implies

$$\limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi)f_a\|_\beta \lesssim \limsup_{j\to\infty} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}.$$

(ii) Again, from the proof of Lemma 2.3, we have for each $N \in \mathbb{N}$,

$$\begin{aligned} &\big\|(uC_\varphi - vC_\psi)(g_{\varphi,\psi,a} - f_{\varphi(a)})\big\|_\beta \\ &\lesssim (1-|\varphi(a)|^2)^{1+\alpha} \sum_{k=1}^{N} k^{2\alpha}|\varphi(a)|^k \Big\|(uC_\varphi - vC_\psi)\Big(\langle z, \frac{\varphi(a)}{|\varphi(a)|}\rangle^k\Big)\Big\|_\beta \\ &+ (1-|\varphi(a)|^2)^{1+\alpha} \sum_{k=1}^{N} k^{2\alpha}|\varphi(a)|^k \\ &\times \Big\|(uC_\varphi - vC_\psi)\Big(\Big\langle z, \frac{\varphi(a)}{|\varphi(a)|}\Big\rangle^k \langle z,\varphi(a)\rangle^k \langle z,\xi_a'\rangle\Big)\Big\|_\beta \\ &+ (1-|\varphi(a)|^2)^{1+\alpha} \sum_{k=N+1}^{\infty} k^{\alpha}|\varphi(a)|^k \sup_{k\ge N}\sup_{\xi\in\partial\mathbb{B}} \frac{\|(uC_\varphi - vC_\psi)(\langle z,\xi\rangle^k)\|_\beta}{\|\langle z,\xi\rangle\|_\alpha} \\ &+ (1-|\varphi(a)|^2)^{1+\alpha} \sum_{k=N+1}^{\infty} k^{\alpha}|\varphi(a)|^k \sup_{k\ge N}\sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|(uC_\varphi - vC_\psi)(\langle z,\xi\rangle^k\langle z,\xi'\rangle)\|_\beta}{\|\langle z,\xi\rangle^k\langle z,\xi'\rangle\|_\alpha}. \end{aligned}$$

Let $|\varphi(a)| \to 1$, then we have

$$\begin{aligned} &\limsup_{|\varphi(a)|\to 1} \big\|(uC_\varphi - vC_\psi)(g_{\varphi,\psi,a} - f_{\varphi(a)})\big\|_\beta \\ &\lesssim \sup_{k\ge N}\sup_{\xi\in\partial\mathbb{B}} \frac{\|(uC_\varphi - vC_\psi)(\langle z,\xi\rangle^k)\|_\beta}{\|\langle z,\xi\rangle\|_\alpha} \\ &+ \sup_{k\ge N}\sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|(uC_\varphi - vC_\psi)(\langle z,\xi\rangle^k\langle z,\xi'\rangle)\|_\beta}{\|\langle z,\xi\rangle^k\langle z,\xi'\rangle\|_\alpha}, \end{aligned}$$

which clearly implies the desired estimate.

(iii) The proof for (iii) again is similar as (ii), and hence we omit it. □

THEOREM 3.2. *Let $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{B})$. Let further, φ and ψ be holomorphic self-maps of $\mathbb{B}$ satisfying (2.1) and (2.2). Suppose that $uC_\varphi \colon H_\alpha^\infty \mapsto H_\beta^\infty$ and $uC_\varphi \colon H_\alpha^\infty \mapsto H_\beta^\infty$ are bounded, then*

$$\begin{aligned} \|uC_\varphi - vC_\psi\|_{e,H_\alpha^\infty\mapsto H_\beta^\infty} &\simeq \limsup_{j\to\infty}\sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha} \\ &+ \limsup_{j\to\infty}\sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha}. \end{aligned}$$

Proof. Following a standard argument (see, e.g., [6, Theorem 3.1]), one can easily establish that

$$\|uC_\varphi - vC_\psi\|_{e,H^\infty_\alpha\mapsto H^\infty_\beta} \lesssim \lim_{r\to 1}\sup_{|\varphi(z)|>r} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z),\psi(z))$$

$$+\lim_{r\to 1}\sup_{|\psi(z)|>r} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)) + \lim_{r\to 1}\sup_{|\varphi(z)|,|\psi(z)|>r} |\mathcal{D}_{u,\varphi}(z)-\mathcal{D}_{v,\psi}(z)|.$$

Thus, by Lemma 3.1, we have

$$\|uC_\varphi - vC_\psi\|_{e,H^\infty_\alpha\mapsto H^\infty_\beta}$$

$$\lesssim \limsup_{|\varphi(a)|\to 1}\|(uC_\varphi - vC_\psi)f_{\varphi(a)}\|_\beta + \limsup_{|\psi(a)|\to 1}\|(uC_\varphi - vC_\psi)f_{\psi(a)}\|_\beta$$

$$+\limsup_{|\varphi(a)|\to 1}\|(uC_\varphi - vC_\psi)g_{\varphi,\psi,a}\|_\beta + \limsup_{|\psi(a)|\to 1}\|(uC_\varphi - vC_\psi)g_{\psi,\varphi,a}\|_\beta$$

$$\lesssim \limsup_{j\to\infty}\sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}$$

$$+\limsup_{j\to\infty}\sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha}.$$

Next, we shall prove that

$$\|uC_\varphi - vC_\psi\|_{e,H^\infty_\alpha\mapsto H^\infty_\beta} \gtrsim \limsup_{j\to\infty}\sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}$$

$$+\limsup_{j\to\infty}\sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha}.$$

To see this, recall the test functions $f_{j,\xi}$ and $f_{j,\xi,\xi'}$ defined in the proof of Theorem 2.5, namely, for $j\in\mathbb{N}$ and $\xi,\xi'\in\partial\mathbb{B}$, we write

$$f_{j,\xi}(z) = \frac{\langle z,\xi\rangle^j}{\|\langle z,\xi\rangle^j\|_\alpha} \quad\text{and}\quad f_{j,\xi,\xi'}(z) = \frac{\langle z,\xi\rangle^j\langle z,\xi'\rangle}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha}.$$

We first show that

$$\|uC_\varphi - vC_\psi\|_{e,H^\infty_\alpha\mapsto H^\infty_\beta} \ge \limsup_{j\to\infty}\sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha}. \tag{3.1}$$

Since $\|f_{j,\xi}\|_\alpha = 1$ for all $j\in\mathbb{N}$ and $\xi\in\partial\mathbb{B}$, by the boundedness assumption on both uC_φ and vC_ψ, we have

$$\sup_{j\in\mathbb{N}}\sup_{\xi\in\partial\mathbb{B}}\|(uC_\varphi - vC_\psi)(f_{j,\xi})\|_\beta < \infty.$$

Take and fix any $\varepsilon > 0$, then for each $k \in \mathbb{N}$, we can take a $\xi_k \in \partial\mathbb{B}$, such that

$$\|(uC_\varphi - vC_\psi)(f_{k,\xi_k})\|_\beta \geq \sup_{\xi\in\partial\mathbb{B}} \|(uC_\varphi - vC_\psi)(f_{k,\xi})\|_\beta - \varepsilon.$$

Now we consider the set $\{f_{k,\xi_k}\}_{k\in\mathbb{N}}$. It is clear that $f_{k,\xi_k} \to 0$ uniformly on compact subsets of $\mathbb{B}$. Hence, if K is any compact operator from H_α^∞ to H_β^∞, then

$$\lim_{k\to\infty} \|K f_{k,\xi_k}\|_\beta = 0\,.$$

Hence

$$\|uC_\varphi - vC_\psi - K\|_{H_\alpha^\infty \mapsto H_\beta^\infty} \geq \limsup_{k\to\infty} \|(uC_\varphi - vC_\psi - K)(f_{k,\xi_k})\|_\beta$$

$$= \limsup_{k\to\infty} \|(uC_\varphi - vC_\psi)(f_{k,\xi_k})\|_\beta \geq \limsup_{k\to\infty} \sup_{\xi\in\partial\mathbb{B}} \|(uC_\varphi - vC_\psi)(f_{k,\xi})\|_\beta - \varepsilon,$$

which implies the desired result by first letting $\varepsilon \to 0$ and then taking the infimum of K over all the compact operators.

Similarly, we can show that

$$\|uC_\varphi - vC_\psi\|_{e,H_\alpha^\infty \mapsto H_\beta^\infty} \geq \limsup_{j\to\infty} \sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha}.$$

Finally, combining the above estimate with (3.1), we get the desired result. □

As a corollary, we have the following result on the compactness of $uC_\varphi - vC_\psi$, which is simply due to the essential norm of a compact operator is 0.

Corollary 3.3. *Let $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{B})$. Let further, φ and ψ be holomorphic self-maps of $\mathbb{B}$ satisfying (2.1) and (2.2). Suppose that $uC_\varphi\colon H_\alpha^\infty \mapsto H_\beta^\infty$ and $uC_\varphi\colon H_\alpha^\infty \mapsto H_\beta^\infty$ are bounded, then $uC_\varphi - vC_\psi$ is compact if and only if*

$$\limsup_{j\to\infty} \sup_{\xi\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j - v\langle\psi,\xi\rangle^j\|_\beta}{\|\langle z,\xi\rangle^j\|_\alpha} = 0$$

and

$$\limsup_{j\to\infty} \sup_{\xi,\xi'\in\partial\mathbb{B}} \frac{\|u\langle\varphi,\xi\rangle^j\langle\varphi,\xi'\rangle - v\langle\psi,\xi\rangle^j\langle\psi,\xi'\rangle\|_\beta}{\|\langle z,\xi\rangle^j\langle z,\xi'\rangle\|_\alpha} = 0.$$

References

[1] E. Berkson, Composition operators isolated in the uniform operator topology, *Proc. Amer. Math. Soc.*, **81** (1981), 230–232.

[2] F. Colonna, New criteria for boundedness and compactness of weighted composition operators mapping into the Bloch space, *Cent. Eur. J. Math.*, **11** (2013), 55–73.

[3] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press (Boca Raton, FL, 1995).

[4] J. Dai, Compact composition operators on the Bloch space of the unit ball, *J. Math. Anal. Appl.*, **386** (2012), 294–299.

[5] J. Dai and C. Ouyang, Differences of weighted composition operators on $H_\alpha^\infty(B_N)$, *J. Inequal. Appl.*, **2009** (2009), Article ID 127431, 19 pp.

[6] Q. Hu, S. Li and Y. Shi, A new characterization of differences of weighted composition operators on weighted-type spaces, *Comput. Methods Funct. Theory*, **17** (2017), 303–318.

[7] Q. Hu, S. Li and H. Wulan, New essential norm estimates of weighted composition operators from H^∞ into the Bloch space, *Complex Var. Elliptic Equ.*, **62** (2017), 600–615.

[8] S. Li, Differences of generalized composition operators on the Bloch space, *J. Math. Anal. Appl.*, **394** (2012), 706–711.

[9] M. Lindström and E. Wolf, Essential norm of the difference of weighted composition operators, *Monatsh. Math.*, **153** (2008), 133–143.

[10] X. Liu and S. Li, Norm and essential norm of a weighted composition operator on the Bloch space, *Integr. Equ. Oper. Theory*, **87** (2017), 309–325.

[11] J. Moorhouse, Compact differences of composition operators, *J. Funct. Anal.*, **219** (2005), 70–92.

[12] P. Nieminen, Compact differences of composition operators on Bloch and Lipschitz spaces, *Comput. Method Funct. Theory*, **7** (2007), 325–344.

[13] E. Saukko, Difference of composition operators between standard weighted Bergman spaces, *J. Math. Anal. Appl.*, **381** (2011), 789–798.

[14] E. Saukko, An application of atomic decomposition in Bergman spaces to the study of differences of composition operators, *J. Funct. Anal.*, **262** (2012), 3872–3890.

[15] J. Shapiro and C. Sundberg, Isolation amongst the composition operators, *Pacific J. Math.*, **145** (1990), 117–152.

[16] Y. Shi and S. Li, Essential norm of the differences of composition operators on the Bloch space, *Math. Ineqal. Appl.*, **20** (2017), 543–555.

[17] Y. Shi and S. Li, Differences of composition operators on Bloch type spaces, *Complex Anal. Oper. Theory*, **11** (2017), 227–242.

[18] H. Wulan, D. Zheng and K. Zhu, Compact composition operators on BMOA and the Bloch space, *Proc. Amer. Math. Soc.*, **137** (2009), 3861–3868.

[19] R. Zhao, Essential norms of composition operators between Bloch type spaces, *Proc. Amer. Math. Soc.*, **138** (2010), 2537–2546.

[20] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag (2004).