Analysis Math., **46 (2)** (2020), 293–322
DOI: 10.1007/s10476-020-0025-y
First published online April 5, 2020

GENERATION OF RELATIVELY UNIFORMLY CONTINUOUS SEMIGROUPS ON VECTOR LATTICES

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(Received January 2, 2019; revised November 25, 2019; accepted November 27, 2019)

Abstract. We prove a Hille–Yosida type theorem for relatively uniformly continuous positive semigroups on vector lattices. We introduce the notions of relatively uniformly continuous, differentiable, and integrable functions on $\mathbb{R}_+$. These notions allow us to study the generators of relatively uniformly continuous semigroups. Our main result provides sufficient and necessary conditions for an operator to be the generator of an exponentially order bounded, relatively uniformly continuous, positive semigroup.

1. Introduction

The Hille–Yosida Theorem is a milestone in the theory of one-parameter semigroups of operators and was originally proved in 1948 by E. Hille in [4] and K. Yosida [14], independently. It enables the identification of a strongly continuous operator semigroup on a Banach space through the resolvents of its generator. Our main goal here is to prove a counterpart of the Hille–Yosida Theorem for relatively uniformly continuous positive semigroups.

The presented paper can be viewed as a companion paper to [5] where the notion of a relatively uniformly continuous semigroup on vector lattices is introduced. This notion is motivated by various examples such as the heat

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The authors acknowledge financial support from the Slovenian Research Agency, Grant No. P1-0222.

Key words and phrases: vector lattice, positive operator semigroup, relative uniform convergence, generator, Hille–Yosida theorem.

Mathematics Subject Classification: primary 47D06, 46A40, 47B65.

semigroup, the (left) translation semigroup, and Koopman semigroups on the space of continuous functions on the real line $\mathrm{C}(\mathbb{R})$ and on its sublattices such as the space of uniformly continuous functions $\mathrm{UC}(\mathbb{R})$, Lipschitz continuous functions $\mathrm{Lip}(\mathbb{R})$, and continuous functions with compact support $\mathrm{C}_c(\mathbb{R})$. It is shown that many basic results from the strongly continuous operator semigroup theory on Banach spaces can be translated in an analogous way to this setting and foundations are laid for further studies. We build upon these results and focus on generation properties of such semigroups.

The paper is structured as follows. In Section 2 we recall some basic notions and facts about relative uniform convergence. In Section 3 we introduce the notions of relatively uniformly continuous, differentiable, and integrable functions. We develop the appropriate calculus fitting to this context and show a version of the Fundamental Theorem of Calculus. In Section 4 we use these concepts to study the generators of relatively uniformly continuous positive semigroups. There we introduce the notions of an ru-closed and ru-densely defined operator on a vector lattice and show that every generator of a relatively uniformly continuous positive semigroup is such. The proofs presented here have similarities to the C_0-semigroup case, see e.g. [3], however, due to convergence with respect to a regulator, dealing with ru-continuous semigroups is more difficult. While the orbit maps of strongly continuous semigroups on Banach spaces grow at most exponentially in norm, relatively uniformly continuous semigroups a priori do not experience such a behavior. Hence, we introduce the notion of exponentially order bounded semigroups. In Section 5 we prove that the resolvent of the generator of such a semigroup is its Laplace transform and that it satisfies a certain property related to the exponential growth of the semigroup. The rest of this section is devoted to the proof of our main result, Theorem 5.4, using the so called Yosida approximations. We conclude by showing that every exponentially order bounded, relatively uniformly continuous, positive semigroup is uniquely determined by its generator, see Proposition 5.8.

2. Preliminaries

A net $(x_\alpha)_\alpha$ in a vector lattice X *converges relatively uniformly to* $x \in X$ if one can find a (non-unique) *regulator* $u \in X$ such that for each $\varepsilon > 0$ there exists α_0 such that

$$|x_\alpha - x| \le \varepsilon \cdot u \quad \text{holds for all } \alpha \ge \alpha_0.$$

In this case we write $x_\alpha \xrightarrow{ru} x$ (with respect to u) and $\text{ru-}\lim_\alpha x_\alpha := x$. We call x the *relative uniform limit* (or *ru-limit*, for short) of $(x_\alpha)_\alpha$.

A vector lattice is said to be *Archimedean* if for each $x, y \in X$ from $0 \le nx \le y$ for all $n \in \mathbb{N}$ it follows that $x = 0$. Throughout this paper we denote by X an Archimedean vector lattice.

The following properties for relatively convergent nets in X are easy to verify; see e.g. [8, Theorem 16.2].

LEMMA 2.1. *Let X be an Archimedean vector lattice.*

(i) *If $(x_\alpha)_\alpha$ converges relatively uniformly to x as well as to y, then $x = y$.*

(ii) *If $x_\alpha \xrightarrow{ru} x$ with respect to u_x, $y_\alpha \xrightarrow{ru} y$ with respect to u_y and $a, b \in \mathbb{R}$, then*

- *$ax_\alpha + by_\alpha \xrightarrow{ru} ax + by$ with respect to $|a|u_x + |b|u_y$,*
- *$x_\alpha \vee y_\alpha \xrightarrow{ru} x \vee y$ with respect to $u_x + u_y$,*
- *$x_\alpha \wedge y_\alpha \xrightarrow{ru} x \wedge y$ with respect to $u_x + u_y$,*
- *$x_\alpha^+ \xrightarrow{ru} x^+$ with respect to u_x,*
- *$|x_\alpha| \xrightarrow{ru} |x|$ with respect to u_x, and*
- *if x_α is positive for all α, then x is positive.*

It is evident that relative uniform convergence implies order convergence and, by [10, Ch.1, Proposition 5.9], the converse is true for sequences if the vector lattice is σ-order complete and has the diagonal property.

For vector lattices X and Y a map $T\colon X \to Y$ *preserves ru-convergence* if for every $x_\alpha \xrightarrow{ru} x$ in X one has $Tx_\alpha \xrightarrow{ru} Tx$ in Y. By [12, Theorem 5.1], a linear operator between Archimedean vector lattices preserves ru-convergence if and only if it is order bounded. In particular, if $T\colon X \mapsto X$ is a positive operator and $x_\alpha \xrightarrow{ru} x$ with respect to a regulator u, then $Tx_\alpha \xrightarrow{ru} Tx$ with respect to regulator Tu.

A subset S of X is called *relatively uniformly closed* whenever $(x_n)_{n\in\mathbb{N}} \subset S$ and $x_n \xrightarrow{ru} x$ imply $x \in S$. By [6, Section 3], the relatively uniformly closed sets are exactly the closed sets of a certain topology in X, the *relative uniform topology* which we denote by τ_{ru}. The relative uniform topology has been first studied by W. A. J. Luxemburg and L. C. Moore in [6]; see also [9]. It is evident that relative uniform convergence implies convergence in the relative uniform topology.

EXAMPLE 2.2. On a vector lattice X with an order unit $u \in X$ the relative uniform topology τ_{ru} is generated by the norm

$$\|x\|_u := \inf\left\{\lambda > 0 : |x| \le \lambda \cdot u\right\},$$

since $x_\alpha \xrightarrow{ru} x$ if and only if $x_\alpha \xrightarrow{\|\cdot\|_u} x$. It is well-known that in a completely metrizable, locally solid vector lattice (X, τ) every convergent sequence has a subsequence which converges relatively uniformly to the same limit, see [1, Lemma 2.30]. This immediately yields that a subset of X is relatively uniformly closed if and only if it is τ-closed, so that topologies τ_{ru} and τ agree. In particular, if X is a Banach lattice, then τ_{ru} agrees with norm topology.

For $0<p<1$ the vector lattice $L^p(\mathbb{R})$ equipped with the topology τ induced by the metric

$$d_p(f_1,f_2):=\int_{\mathbb{R}}|f_1(x)-f_2(x)|^p\,dx$$

is a completely metrizable, locally solid vector lattice which is not locally convex. Hence, the relative uniform topology need not be locally convex in general.

As mentioned above, a linear operator is order bounded if and only if it preserves ru-convergence. For convergence in the relative uniform topology we have the following result.

LEMMA 2.3. *Let X and Y be Archimedean vector lattices. If a linear operator $T\colon X\to Y$ preserves ru-convergence, then $T\colon(X,\tau_{ru})\to(Y,\tau_{ru})$ is continuous.*

PROOF. It suffices to show that for a fixed relatively uniformly closed set $V\subset Y$ the set $T^{-1}(V)$ is relatively uniformly closed in X. Pick $(x_n)_{n\in\mathbb{N}}\subset T^{-1}(V)$ and $x\in X$ such that $x_n\xrightarrow{ru}x$. By assumption, $Tx_n\xrightarrow{ru}Tx$ and, since V is relatively uniformly closed in Y, we have $Tx\in V$, i.e., $x\in T^{-1}(V)$. □

Note, that we even obtain an equivalence in the lemma above if the space Y has an order unit. In the rest of this paper we focus on relative uniform convergence.

We say that a sequence $(x_n)_{n\in\mathbb{N}}\subset X$ is a *relatively uniform Cauchy sequence* (or *ru-Cauchy sequence*, for short) if one can find a regulator $u\in X$ such that for each $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|x_n-x_m|\le\varepsilon\cdot u$ holds for all $n,m\ge N$. We call X *relatively uniformly complete* (or *ru-complete*, for short) if each relatively uniform Cauchy sequence in X converges relatively uniformly in X.

It is known that a vector lattice X is ru-complete if and only if its every principal ideal is ru-complete and hence, also every ideal of X is ru-complete; see e.g. [8, Exercise 59.5].

EXAMPLE 2.4. By [8, Theorem 42.5], every Dedekind σ-complete vector lattice is ru-complete and hence, for each $0<p<\infty$ the vector lattice $L^p(\mathbb{R})$ is ru-complete. By [8, Theorem 43.1], the vector lattice $\mathrm{C}(\mathbb{R})$ is ru-complete and hence, its ideals $\mathrm{C}_c(\mathbb{R})$ and the space of continuous functions vanishing at infinity $\mathrm{C}_0(\mathbb{R})$ are also ru-complete. Furthermore, it is easy to see that the space of uniformly continuous bounded functions $\mathrm{UCB}(\mathbb{R})$ and the space of continuous bounded functions $\mathrm{C}_b(\mathbb{R})$ are ru-complete.

For the unexplained terminology and basic results on vector lattices and relative uniform convergence we refer to [8, Ch. 9], [10, Sec. 1.5 and 4.1] and [13].

3. Relative uniform calculus

In this section we introduce the concepts of continuity, differentiability, and integrability of a function from $\mathbb{R}_+$ to X in terms of relative uniform convergence and discuss their relationships.

A function $f\colon \mathbb{R}_+ \to X$ is called *relatively uniformly continuous* (or *ru-continuous*, or *ruc*, for short) if one can find a *continuity regulator* $u\colon \mathbb{R}_+ \to X$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(h+t) - f(t)| \le \varepsilon \cdot u(t)$$

holds for all $t \ge 0$ and $h \in [-\min\{\delta, t\}, \delta]$. In this case we write

$$f(h+t) \xrightarrow{ru} f(t) \quad \text{as } h \to 0 \quad \text{or} \quad \text{ru-}\lim_{h\to 0} f(h+t) = f(t).$$

A function $f\colon \mathbb{R}_+ \to X$ is called *relatively uniformly differentiable* (or *ru-differentiable*, for short) if one can find a function $f'\colon \mathbb{R}_+ \to X$ and a *differentiation regulator* $u\colon \mathbb{R}_+ \to X$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Big|\frac{f(h+t) - f(t)}{h} - f'(t)\Big| \le \varepsilon \cdot u(t)$$

holds for all $t \ge 0$ and $h \in [-\min\{\delta, t\}, \delta]$. We call f' the *ru-derivative of* f.

REMARK 3.1. (i) By Lemma 2.1, if $f\colon \mathbb{R}_+ \to X$ and $g\colon \mathbb{R}_+ \to X$ are two ru-differentiable functions with ru-derivatives f', g' and differentiation regulators u_f, u_g, respectively, and $a, b \in \mathbb{R}$, then the function $af + bg$ is ru-differentiable with ru-derivative $af' + bg'$ and differentiation regulator $|a|u_f + |b|u_g$.

(ii) If X is a Banach lattice, then ru-differentiability implies differentiability with respect to the norm.

PROPOSITION 3.2. *Every ru-differentiable function is also ru-continuous.*

PROOF. Let $f\colon \mathbb{R}_+ \to X$ be an ru-differentiable function with differentiation regulator $u\colon \mathbb{R}_+ \to X$. Then for each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that

$$|f(h+t) - f(t)| \le |h| \cdot \Big|\frac{f(h+t) - f(t)}{h} - f'(t)\Big| + |h| \cdot |f'(t)| \le \varepsilon \cdot (u(t) + |f'(t)|)$$

holds for all $t \ge 0$ and $h \in [-\min\{\delta, t\}, \delta]$. $\square$

Let $s \ge 0$. A function $f\colon \mathbb{R}_+ \to X$ is called *relatively uniformly integrable on the interval* $[0, s]$ if one can find $I_s \in X$ and a regulator $u_s \in X$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Big|\sum_{i=1}^{n} (s_i - s_{i-1}) f(t_i) - I_s\Big| \le \varepsilon \cdot u_s$$

holds for every partition $\{s_0, s_1, \dots, s_n\}$ of the interval $[0, s]$ with

$$\max_{1\le i\le n} |s_i - s_{i-1}| \le \delta \quad \text{and} \quad t_i \in [s_{i-1}, s_i], \quad 1 \le i \le n.$$

Since I_s is defined as an ru-limit, by Lemma 2.1.(i), it is unique and we write $\int_0^s f(t)\,\mathrm{d}t := I_s$. We say that $f\colon \mathbb{R}_+ \to X$ is *relatively uniformly integrable* (or *ru-integrable*, for short) if it is relatively uniformly integrable on the interval $[0, s]$ for all $s \ge 0$ and call the map $s \mapsto \int_0^s f(t)\,\mathrm{d}t$ the *ru-integral* of f.

The following proposition states some important properties of ru-integrals which we shall use later on.

Proposition 3.3. *Let $f\colon \mathbb{R}_+ \to X$ and $g\colon \mathbb{R}_+ \to X$ be ru-integrable functions, $a, b \in \mathbb{R}$, $x, s \in \mathbb{R}_+$, and T a positive linear operator on X. Then the following assertions hold.*

(i) *The function $af + bg$ is ru-integrable and we have*

$$\int_0^s (af(t) + bg(t))\,\mathrm{d}t = a\int_0^s f(t)\,\mathrm{d}t + b\int_0^s g(t)\,\mathrm{d}t.$$

(ii) *We have*

$$\int_0^s f(x+t)\,\mathrm{d}t = \int_0^{x+s} f(t)\,\mathrm{d}t - \int_0^x f(t)\,\mathrm{d}t.$$

(iii) *If $|f(t)| \le g(t)$ for each $0 \le t \le s$, then*

$$\left|\int_0^s f(t)\,\mathrm{d}t\right| \le \int_0^s g(t)\,\mathrm{d}t.$$

(iv) *We have*

$$T\int_0^s f(t)\,\mathrm{d}t = \int_0^s Tf(t)\,\mathrm{d}t.$$

If, in addition,

$$\text{ru-}\lim_{s\to\infty} \int_0^s f(t)\,\mathrm{d}t =: \int_0^\infty f(t)\,\mathrm{d}t \quad \textit{and} \quad \text{ru-}\lim_{s\to\infty} \int_0^s g(t)\,\mathrm{d}t =: \int_0^\infty g(t)\,\mathrm{d}t$$

exist, then the above results also hold for $s = \infty$.

Proof. Assertion (i) follows directly from Lemma 2.1.(ii).

In order to show (ii), take any partitions $\{s_0, s_1, \dots, s_n\}$, $\{x_0, x_1, \dots, x_m\}$ of the intervals $[0, s]$, $[0, x]$, respectively, $t_i \in [s_{i-1}, s_i]$ for $1 \le i \le n$, and $y_j \in [x_{j-1}, x_j]$ for $1 \le j \le m$. Then, choosing $r_i := x_i$ for $0 \le i \le m$ and

$r_i := x + s_i$ for $m+1 \le i \le m+n$ we obtain a partition $\{r_0, r_1, \dots, r_{m+n}\}$ of the interval $[0, x+s]$ and

$$\sum_{i=1}^{n}(s_i - s_{i-1})f(t_i + x) = \sum_{i=1}^{m+n}(r_i - r_{i-1})f(\tau_i) - \sum_{i=1}^{m}(x_i - x_{i-1})f(y_i)$$

where $\tau_i := y_i$ for $1 \le i \le m$ and $\tau_i := x + t_i$ for $m+1 \le i \le m+n$. This proves (ii).

We now verify assertion (iii). By assumption, there exist regulator functions $u_f, u_g \colon \mathbb{R}_+ \to X$ such that for each $\varepsilon > 0$ and each appropriate partition $\{s_0, s_1, \dots, s_n\}$ of the interval $[0, s]$ and $t_i \in [s_{i-1}, s_i]$, $1 \le i \le n$, we have

$$\Big|\sum_{i=1}^{n}(s_i - s_{i-1})f(t_i) - \int_0^s f(t)\,\mathrm{d}t\Big| \le \varepsilon \cdot u_f(s)$$

and

$$\Big|\sum_{i=1}^{n}(s_i - s_{i-1})g(t_i) - \int_0^s g(t)\,\mathrm{d}t\Big| \le \varepsilon \cdot u_g(s).$$

Hence,

$$\Big|\int_0^s f(t)\,\mathrm{d}t\Big| \le \Big|\int_0^s f(t)\,\mathrm{d}t - \sum_{i=1}^{n}(s_i - s_{i-1})f(t_i)\Big| + \sum_{i=1}^{n}(s_i - s_{i-1})|f(t_i)|$$

$$\le \varepsilon \cdot u_f(s) + \sum_{i=1}^{n}(s_i - s_{i-1})g(t_i) \le \varepsilon \cdot (u_f(s) + u_g(s)) + \int_0^s g(t)\,\mathrm{d}t.$$

Since X is Archimedean, we obtain (iii).

Assertion (iv) follows from the fact that positive operators preserve relative uniform limits. □

We now show a version of the Fundamental Theorem of Calculus for ru-integrals and ru-derivatives.

Proposition 3.4. *Let $f \colon \mathbb{R}_+ \to X$ be an ru-continuous and ru-integrable function. Then the ru-integral of f is ru-differentiable and its ru-derivative equals f.*

Proof. By assumption, there exists a map $u \colon \mathbb{R}_+ \to X$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(t+s) - f(s)| \le \varepsilon \cdot u(s)$ holds for all

$s \geq 0$ and $t \in [-\min\{\delta, s\}, \delta]$. Hence, by Proposition 3.3.(ii)-(iii), we obtain

$$\left| \frac{\int_0^{s+h} f(t)\,\mathrm{d}t - \int_0^s f(t)\,\mathrm{d}t}{h} - f(s) \right|$$

$$\leq \frac{1}{h} \left| \int_0^h (f(t+s) - f(s))\,\mathrm{d}t \right| \leq \frac{1}{h} \int_0^h \varepsilon \cdot u(s)\,\mathrm{d}t = \varepsilon \cdot u(s)$$

for all $s \geq 0$ and $h \in [-\min\{\delta, s\}, \delta]$. $\square$

The following result will be used in the proof of Lemma 4.8. It is a version of the Newton–Leibniz theorem in the relatively uniform context.

Proposition 3.5. *Let $f\colon \mathbb{R}_+ \to X$ be ru-differentiable with differentiation regulator $u\colon \mathbb{R}_+ \to X$ such that its ru-derivative f' is ru-continuous with continuity regulator $\tilde{u}\colon \mathbb{R}_+ \to X$. If u and $\tilde{u}$ are ru-integrable, then f' is ru-integrable and for each $s > 0$ we have*

$$\int_0^s f'(t)\,\mathrm{d}t = f(s) - f(0).$$

Proof. By assumption, there exists $w\colon \mathbb{R}_+ \to X$ such that for each $s \geq 0$ and $\varepsilon > 0$ one can find $\delta_s > 0$ such that for all partitions $\{s_0, s_1, \dots, s_n\}$ of the interval $[0, s]$ with $\max_{1 \leq i \leq n} |s_i - s_{i-1}| \leq \delta_s$ and $t_i \in [s_{i-1}, s_i]$, $1 \leq i \leq n$, we have

$$\left| \sum_{i=1}^n (s_i - s_{i-1}) u(t_i) - \int_0^s u(t)\,\mathrm{d}t \right| \leq \varepsilon \cdot w(s)$$

and

$$\left| \sum_{i=1}^n (s_i - s_{i-1}) \tilde{u}(t_i) - \int_0^s \tilde{u}(t)\,\mathrm{d}t \right| \leq \varepsilon \cdot w(s).$$

Fix $s > 0$ and $\varepsilon > 0$. By assumption, there exists $0 < \delta < \delta_s$ such that

$$\left| \frac{f(h+t) - f(t)}{h} - f'(t) \right| \leq \varepsilon \cdot u(t) \quad \text{and} \quad \left| f'(h+t) - f'(t) \right| \leq \varepsilon \cdot \tilde{u}(t)$$

hold for all $t \geq 0$ and $h \in [-\min\{\delta, t\}, \delta]$. Now we estimate

$$\left| \sum_{i=1}^n (s_i - s_{i-1}) f'(t_i) - (f(s) - f(0)) \right|$$

$$\leq \sum_{i=1}^n (s_i - s_{i-1}) \left| f'(t_i) - \frac{f(s_i) - f(s_{i-1})}{s_i - s_{i-1}} \right|$$

$$\le \sum_{i=1}^{n}(s_i - s_{i-1})\big|f'(t_i) - f'(s_{i-1})\big| + \sum_{i=1}^{n}(s_i - s_{i-1})\left|f'(s_{i-1}) - \frac{f(s_i) - f(s_{i-1})}{s_i - s_{i-1}}\right|$$

$$\le \varepsilon \cdot \left(\sum_{i=1}^{n}(s_i - s_{i-1})\tilde{u}(s_{i-1}) + \sum_{i=1}^{n}(s_i - s_{i-1})u(s_{i-1})\right)$$

$$\le \varepsilon \cdot \left(2\varepsilon \cdot w(s) + \int_0^s \tilde{u}(t)\,\mathrm{d}t + \int_0^s u(t)\,\mathrm{d}t\right). \quad \square$$

4. Relatively uniformly continuous semigroups and generators

As defined in [5], a family $(T(t))_{t\ge 0}$ of linear operators on X is called a *relatively uniformly continuous semigroup* (or *ruc-semigroup*, for short) if it satisfies the following two conditions.

(i) For each $t, s \ge 0$ we have $T(s+t) = T(t)T(s)$ and $T(0) = I_X$.

(ii) For each $x \in X$ and $t \ge 0$ the orbit map $t \mapsto T(t)x$ is ru-continuous, i.e.,

$$T(h+t)x \xrightarrow{ru} T(t)x \quad \text{as } h \to 0.$$

If, in addition, $T(t)$ is a positive operator on X for each $t \ge 0$, the semigroup $(T(t))_{t\ge 0}$ is called *positive*.

Remark 4.1. Since relatively uniform convergence implies convergence in the relatively uniform topology, the notion of relative uniform continuity allows us to study continuous semigroups on non-locally convex spaces such as $L^p(\mathbb{R})$ for $0 < p < 1$; see Example 2.2.

It was shown in [5, Proposition 3.5] that for a positive semigroup it suffices to check the ru-continuity of the orbit maps only at $t = 0$ and for positive vectors, i.e.,

$$T(t)x \xrightarrow{ru} x \qquad \text{as } t \searrow 0 \text{ for } x \in X_+.$$

Another crucial property of ruc-semigroups is that orbit maps are order bounded on finite intervals; see [5, Proposition 3.4]. For the general theory of positive operator semigroups and ruc-semigroups we refer to [2], [3] and [5], respectively.

Next, we study the ru-integrability of the orbit maps of a positive ruc-semigroup on an ru-complete vector lattice.

Lemma 4.2. *Let $(T(t))_{t\ge 0}$ be a relatively uniformly continuous positive semigroup on a relatively uniformly complete vector lattice X. Then the following assertions hold for each $x \in X$ and $s \ge 0$.*

(i) *The orbit map $t \mapsto T(t)x$ is ru-integrable.*

(ii) *The operator* $y \mapsto \int_0^s T(\tau)y \,\mathrm{d}\tau$ *on* X *is well-defined and positive.*

(iii) *We have* $y_h := \frac{1}{h}\big(\int_0^h T(\tau)x \,\mathrm{d}\tau\big) \xrightarrow{ru} x$ *as* $h \searrow 0$.

Proof. To prove (i) fix $\varepsilon > 0$. By assumption, there exist a positive element $u \in X$, independent of ε, and $\delta > 0$ such that $|T(h)x - x| \le \varepsilon \cdot u$ holds for all $h \in [0, \delta]$. Furthermore, by [5, Proposition 3.4], there exists $v \in X$ such that $T(t)(u \vee x) \le v$ holds for all $t \in [0, s]$. In particular, for each $t \in [0, s]$ we have $T(t)x \in I_{\{v\}}$, where $I_{\{v\}}$ is the order ideal generated by v. Pick $0 \le t' \le t \le s$ with $|t - t'| \le \delta$. Then

$$|T(t)x - T(t')x| \le T(t')|T(t - t')x - x| \le \varepsilon \cdot T(t')u \le \varepsilon \cdot v.$$

Hence, the mapping

$$\varphi\colon [0, s] \to I_{\{v\}}, \quad t \mapsto T(t)x,$$

is continuous with respect to the AM-norm on $I_{\{v\}}$ defined by

$$\|y\|_v := \inf\big\{\lambda > 0 : |y| \le \lambda \cdot v\big\}.$$

Since X is Archimedean and ru-complete, the order ideal $I_{\{v\}}$ is complete with respect to the norm $\|\cdot\|_v$ and so there exists the unique Riemann integral in X which is the ru-limit of the Riemann sums of the orbit map $t \mapsto T(t)x$ on $[0, s]$.

To prove (ii) fix $y \in X_+$. We show that $\int_0^s T(\tau)y \,\mathrm{d}\tau \in X_+$. Indeed, for each $t \ge 0$ the operator $T(t)$ is positive and thus, for any partition $\{s_0, s_1, \dots, s_n\}$ of the interval $[0, s]$ and $t_i \in [s_{i-1}, s_i]$, $1 \le i \le n$, the Riemann sum

$$\sum_{i=1}^{n} (s_i - s_{i-1})T(t_i)y$$

is positive in X. The element $\int_0^s T(\tau)y \,\mathrm{d}\tau$ is the ru-limit of a net of positive elements in X and hence, $\int_0^s T(\tau)y \,\mathrm{d}\tau \in X_+$.

To show (iii) fix $\varepsilon > 0$. By assumption, there exist $u \in X$, independent of ε, and $\delta > 0$ such that $|T(h)x - x| \le \varepsilon \cdot u$ holds for all $h \in [0, \delta]$ and hence, by Proposition 3.3.(iii), we have

$$|y_h - x| = \frac{1}{h}\left| \int_0^h (T(\tau)x - x) \,\mathrm{d}\tau \right| \le \varepsilon \cdot u$$

for all $h \in [0, \delta]$. □

Example 4.3. For a function $f\colon \mathbb{R} \to \mathbb{R}$ and $t \ge 0$, we consider the translation operator

$$(T_l(t)f)(x) = f(t + x), \quad x \in \mathbb{R}.$$

It is evident that by fixing a translation invariant space Y of functions on $\mathbb{R}$ one obtains a semigroup $(T_l(t))_{t\geq 0}$ on Y which we call the *(left) translation semigroup* on Y. This semigroup is ru-continuous on $\mathrm{UCB}(\mathbb{R})$ which is an ru-complete vector lattice. Indeed, if we fix $f \in \mathrm{UCB}(\mathbb{R})$ and pick $\varepsilon > 0$, then there exists $\delta > 0$ such that $|f(h+t) - f(t)| \leq \varepsilon \cdot 1$ holds for all $h \in [0,\delta]$ and $t \in \mathbb{R}$, and since the constant function is in $\mathrm{UCB}(\mathbb{R})$ we obtain the claim. Hence, by [5, Proposition 3.11] and [5, Proposition 6.2], the (left) shift semigroup is relatively uniformly continuous on $\mathrm{C}_c(\mathbb{R})$ and $\mathrm{C}(\mathbb{R})$ which, by Example 2.4, are ru-complete vector lattices. So, $(T_l(t))_{t\geq 0}$ on $\mathrm{C}_c(\mathbb{R})$, $\mathrm{C}(\mathbb{R})$, and $\mathrm{UCB}(\mathbb{R})$ satisfies the assumptions of Lemma 4.2.

Since we will repeatedly use Lemma 4.2, in this section X will denote an ru-complete vector lattice. Contrary to ru-integrability, the ru-differentiability of the orbit maps does not always hold. On the set of vectors, for which the orbits are ru-differentiable we can define the generator of an ruc-semigroup as follows.

The *generator* $A\colon D(A) \subset X \to X$ *of a relatively uniformly continuous semigroup* $(T(t))_{t\geq 0}$ on X is the operator

$$Ax := \text{ru-}\lim_{h\searrow 0} \frac{1}{h}(T(h)x - x),$$

$$D(A) := \left\{ x \in X : \text{ru-}\lim_{h\searrow 0} \frac{1}{h}(T(h)x - x) \text{ exists in } X \right\}.$$

REMARK 4.4. Obviously, every ruc-semigroup determines its generator uniquely. Proposition 5.8 will show that under additional assumptions the converse is also true.

EXAMPLE 4.5. The generator of the (left) translation semigroup $(T_l(t))_{t\geq 0}$ on $\mathrm{C}_c(\mathbb{R})$ is the first derivative operator $A := \frac{\mathrm{d}}{\mathrm{d}x}$ with the domain

$$D(A) = \left\{ f \in \mathrm{C}_c(\mathbb{R}) \mid f \text{ is continuously differentiable} \right\}.$$

Indeed, if $(B, D(B))$ is the generator of $(T_l(t))_{t\geq 0}$, then, by definition, for fixed $f \in D(B)$ there exists $u \in \mathrm{C}_c(\mathbb{R})$ such that for each $\varepsilon > 0$ there exists $0 < \delta < 1$ such that we have

$$\left| \frac{f(h+x) - f(x)}{h} - (Bf)(x) \right| \leq \varepsilon \cdot u(x)$$

for all $x \in \mathbb{R}$ and $h \in [0,\delta]$ and hence, f is left differentiable with left derivative Bf. Since Bf is a continuous function, f is differentiable and $Bf = Af$. In particular, we have $D(B) \subset D(A)$.

Now, let $f \in D(A)$. Then $Af \in \mathrm{C}_c(\mathbb{R})$ and hence, there exists $n \in \mathbb{N}$ such that $f(x) = 0$ and $Af(x) = 0$ for all $x \in [-n,n]^c$. Furthermore, since

$Af = f'$ is continuous, by [11, Exercise 5.8], for fixed $\varepsilon > 0$ there exists $0 < \delta < 1$ such that

$$\Big|\frac{f(h+x)-f(x)}{h} - f'(x)\Big| \le \varepsilon$$

holds for all $h \in [0,\delta]$ and $x \in [-n,n]$. By Urysohn's lemma, there exists $u \in \mathrm{C}_c(\mathbb{R})$ such that $u(x) = 1$ holds for all $x \in [-n-1, n+1]$ and hence, we obtain

$$\Big|\frac{(T_l(h)f)(x)-f(x)}{h} - (Af)(x)\Big| = \Big|\frac{f(h+x)-f(x)}{h} - f'(x)\Big| \le \varepsilon \cdot u(x)$$

for all $h \in [0,\delta]$ and $x \in \mathbb{R}$, i.e., $f \in D(B)$.

The following lemma captures some of the important properties of generators of positive ruc-semigroups. It is motivated by properties from the classical theory of strongly continuous semigroups; cf. [3, II.1.3].

LEMMA 4.6. *Let A be the generator of a relatively uniformly continuous positive semigroup $(T(t))_{t\ge 0}$ on an ru-complete vector lattice X. The following assertions hold for each $s \ge 0$.*

(i) *The operator $A\colon D(A) \subset X \to X$ is linear.*

(ii) *For $x \in D(A)$ we have $T(s)x \in D(A)$ and $AT(s)x = T(s)Ax$. Futhermore, the orbit map $t \mapsto T(t)x$ is ru-differentiable with ru-derivative $t \mapsto T(t)Ax$.*

(iii) *For each $x \in X$ we have*

$$\int_0^s T(\tau)x \,\mathrm{d}\tau \in D(A).$$

(iv) *We have*

$$T(s)x - x = \begin{cases} A\int_0^s T(\tau)x\,\mathrm{d}\tau & \text{if } x \in X, \\ \int_0^s T(\tau)Ax\,\mathrm{d}\tau & \text{if } x \in D(A). \end{cases}$$

PROOF. Assertion (i) follows directly from the linearity of the operators $T(t)$ and Lemma 2.1.

To prove (ii) fix $x \in D(A)$. By assumption, there exists $u \in X$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|T(h)Ax - Ax| \le \varepsilon \cdot u \quad \text{and} \quad \Big|\frac{1}{h}(T(h)x - x) - Ax\Big| \le \varepsilon \cdot u$$

hold for all $h \in [0, \delta]$. Furthermore, by [5, Proposition 3.4], there exists $v \in X$ such that $T(t)u \le v$ holds for all $t \in [0, s]$. Therefore

$$\begin{aligned}&\Big|\frac{1}{h}(T(h)T(s)x - T(s)x) - T(s)Ax\Big|\\ &\le T(s)\Big|\frac{1}{h}(T(h)x - x) - Ax\Big| \le \varepsilon \cdot T(s)u \le \varepsilon \cdot 2v\end{aligned}$$

holds for all $h \in [0, \delta]$. Hence, we obtain $T(s)x \in D(A)$ and $AT(s)x = T(s)Ax$. Moreover,

$$\begin{aligned}&\Big|\frac{1}{h}\big(T(h)T(s)x - T(s)x\big) - T(s)Ax\Big|\\ &\le T(h+s)\Big(\Big|\frac{1}{-h}\big(T(-h)x - x\big) - Ax\Big| + \big|Ax - T(-h)Ax\big|\Big)\\ &\le \varepsilon \cdot T(h+s)(2u) \le \varepsilon \cdot (2v)\end{aligned}$$

holds for all $h \in [-\min\{\delta, s\}, 0]$. This proves that $t \mapsto T(t)x$ is ru-differentiable with ru-derivative $t \mapsto T(t)Ax$.

To prove (iii) and (iv) fix $x \in X$. Using Proposition 3.3.(iv) and Proposition 3.3.(ii) twice we obtain

$$\begin{aligned}&\frac{1}{h}\Big(T(h)\int_0^t T(\tau)x\,\mathrm{d}\tau - \int_0^t T(\tau)x\,\mathrm{d}\tau\Big)\\ &= \frac{1}{h}\Big(\int_0^{t+h} T(\tau)x\,\mathrm{d}\tau - \int_0^h T(\tau)x\,\mathrm{d}\tau - \int_0^t T(\tau)x\,\mathrm{d}\tau\Big)\\ &= \frac{1}{h}\int_0^h T(\tau)T(t)x\,\mathrm{d}\tau - \frac{1}{h}\int_0^h T(\tau)x\,\mathrm{d}\tau\end{aligned}$$

for each $h > 0$. By Lemma 4.2.(iii), the right-hand side converges relatively uniformly to $T(t)x - x$ as $h \searrow 0$. This proves (iii) and the first identity of (iv). Furthermore, we have

$$\frac{1}{h}\Big(T(h)\int_0^t T(\tau)x\,\mathrm{d}\tau - \int_0^t T(\tau)x\,\mathrm{d}\tau\Big) = \int_0^t T(\tau)\Big(\frac{1}{h}(T(h)x - x)\Big)\mathrm{d}\tau$$

for each $h > 0$. Since, by Lemma 4.2.(ii), the operator $y \mapsto \int_0^s T(\tau)y\,\mathrm{d}\tau$ is positive on X it preserves ru-convergence and, hence, the right-hand side converges relatively uniformly to $\int_0^t T(\tau)Ax\,\mathrm{d}\tau$ as $h \searrow 0$. This proves the second identity of (iv). □

The generator of a strongly continuous semigroup on a Banach space is closed and densely defined; see e.g. [3, II.1.4]. Before we state an analogue to this result in our setting we need to introduce the appropriate notions.

A set $D \subset X$ is called *ru-dense* if for each $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ such that $x_n \xrightarrow{ru} x$. We call an operator P on X *ru-densely defined* if its domain $D(P)$ is ru-dense in X. An operator P on X with domain $D(P)$ is called *ru-closed* if $x_n \xrightarrow{ru} x$ and $Px_n \xrightarrow{ru} y$ imply that $x \in D(P)$ and $Px = y$.

PROPOSITION 4.7. *The generator of a positive relatively uniformly continuous semigroup is an ru-densely defined and ru-closed operator.*

PROOF. Consider a positive ruc-semigroup $(T(t))_{t\geq 0}$ on X with generator A. Take $x \in X$ and define $y_n := n\int_0^{\frac{1}{n}} T(\tau)x \,\mathrm{d}\tau$. By Lemma 4.6.(iii), $y_n \in D(A)$ for for each $n \in \mathbb{N}$ and, by Lemma 4.2.(iii), we have $y_n \xrightarrow{ru} x$ as $n \to \infty$. This proves that A is ru-densely defined.

To show that A is ru-closed pick $x, y \in X$ and $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that $x_n \xrightarrow{ru} x$ and $Ax_n \xrightarrow{ru} y$. By Lemma 4.6.(iv), the identity

$$T(h)x_n - x_n = \int_0^h T(\tau)Ax_n \,\mathrm{d}\tau$$

holds for each $h > 0$ and $n \in \mathbb{N}$. Furthermore, since for each $h > 0$ the operators $T(h)$ and $y \mapsto \int_0^h T(\tau)y \,\mathrm{d}\tau$ preserve relative uniform convergence, we have

$$T(h)x_n - x_n \xrightarrow{ru} T(h)x - x \quad \text{and} \quad \int_0^h T(\tau)Ax_n \,\mathrm{d}\tau \xrightarrow{ru} \int_0^h T(\tau)y \,\mathrm{d}\tau$$

as $n \to \infty$. Hence, the identity

$$\frac{1}{h}(T(h)x - x) = \frac{1}{h}\int_0^h T(\tau)y \,\mathrm{d}\tau$$

holds for each $h > 0$. By Lemma 4.2.(iii), the right-hand side converges relatively uniformly to y as $h \searrow 0$ and, hence $x \in D(A)$. Since at the same time the left-hand side converges relatively uniformly to Ax, we obtain $Ax = y$. This proves that A is ru-closed. □

The following result can be interpreted as the ‘product rule’ for the ru-derivative of commuting semigroups and is vital for the proof of the main result of this paper, Theorem 5.4.

LEMMA 4.8. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be relatively uniformly continuous positive semigroups on X with generators A and B, respectively. If $D(A) \subset D(B)$ and for each $s, t \geq 0$ the operators $T(t)$ and $S(s)$ commute, then for each $x \in D(A)$ and $t \geq 0$ we have*

$$T(t)x - S(t)x = \int_0^t T(t-\tau)S(\tau)(B-A)x \,\mathrm{d}\tau.$$

PROOF. Fix $x \in D(A) \subset D(B)$ and $t \geq 0$. We will prove that the function

$$f\colon \tau \mapsto T(t-\tau)S(\tau)x$$

is ru-differentiable with the ru-derivative

$$f'\colon \tau \mapsto T(t-\tau)S(\tau)(B-A)x,$$

which is ru-continuous, and that there exists $w_t \in X$ such that the constant function $\tau \mapsto w_t$ is a differentiation regulator of f and a continuity regulator of f'. If so, then f satisfies the assumptions of Proposition 3.5 which yields the result.

By assumption, there exists $u \in X$ such that

$$\frac{T(h)x-x}{h} \xrightarrow{ru} Ax, \quad \frac{S(h)x-x}{h} \xrightarrow{ru} Bx,$$
$$T(h)(B-A)x \xrightarrow{ru} (B-A)x, \quad S(h)(B-A)x \xrightarrow{ru} (B-A)x$$

with respect to the regulator u as $h \searrow 0$. By [5, Proposition 3.4], there exists $v_t, w_t \in X$ such that $S(s)u \leq v_t$ and $T(s)v_t \leq w_t$ holds for all $s \in [0,t]$.

Fix $\tau \in (0,t)$ and $\varepsilon > 0$. Then for some $\delta > 0$ we have

$$\begin{aligned}
&\Big|\frac{f(\tau+h)-f(\tau)}{h} - f'(\tau)\Big| \\
&= \Big|\frac{T(t-\tau-h)S(\tau+h)x - T(t-\tau)S(\tau)x}{h} - T(t-\tau)S(\tau)(B-A)x\Big| \\
&\leq T(t-\tau-h)S(\tau)\Big|\frac{S(h)x-T(h)x}{h} - T(h)(B-A)x\Big| \\
&\leq T(t-\tau-h)S(\tau)\bigg(\Big|\frac{S(h)x-x}{h} - Bx\Big| + \Big|\frac{T(h)x-x}{h} - Ax\Big| \\
&\quad + \big|(B-A)x - T(h)(B-A)x\big|\bigg) \leq \varepsilon \cdot T(t-\tau-h)S(\tau)\,3u \leq \varepsilon \cdot 3w_t
\end{aligned}$$

for all $h \in [0, \min\{\delta, t-\tau\}]$. Similarly,

$$\Big|\frac{f(\tau-h)-f(\tau)}{h} - f'(\tau)\Big| \leq \varepsilon \cdot 3w_t$$

holds for all $h \in [0, \min\{\delta, \tau\}]$. This proves that f is ru-differentiable on $[0,t]$ with ru-derivative f' and that $\tau \mapsto 3w_t$ is a differentiation regulator of f.

Furthermore, by using similar arguments, we obtain

$$\big|f'(\tau+h) - f'(\tau)\big| = \big|T(t-\tau-h)S(\tau+h)(B-A)x - T(t-\tau)S(\tau)(B-A)x\big|$$

$$\leq T(t-\tau-h)S(\tau)\big(|S(h)(B-A)x-(B-A)x|+|T(h)(B-A)x-(B-A)x|\big)$$

$$\leq \varepsilon \cdot 2w_t \leq \varepsilon \cdot 3w_t$$

for some $\delta > 0$ and all $h \in [0, \min\{\delta, t-\tau\}]$, and

$$\big|f'(\tau - h) - f'(\tau)\big|$$

$$= \big|T(t-\tau+h)S(\tau-h)(B-A)x - T(t-\tau)S(\tau)(B-A)x\big| \leq \varepsilon \cdot 3w_t$$

for all $h \in [0, \min\{\delta, \tau\}]$. This proves that f' is ru-continuous on $[0,t]$ with continuity regulator $\tau \mapsto 3w_t$ and hence, we conclude the result. □

It is well-known that every strongly continuous semigroup on a Banach space is exponentially bounded, see e.g. [3, Proposition I.5.5]. We now define an analogous property for semigroups on vector lattices.

We call a semigroup $(T(t))_{t\geq 0}$ on X *exponentially order bounded* if there exists some $w \in \mathbb{R}$ such that for each $x \in X$ there exists $u \in X$ such that for all $t \geq 0$ we have

$$|T(t)x| \leq e^{w\cdot t}u.$$

We call such an $w \in \mathbb{R}$ an *order exponent* of $(T(t))_{t\geq 0}$.

Example 4.9. The multiplication semigroup $(T_q(t))_{t\geq 0}$, defined by

$$T_q(t)f = e^{q(\cdot)t}f, \quad q \in \mathrm{C}_b(\mathbb{R})$$

for each $f\colon \mathbb{R} \to \mathbb{C}$ and $t \geq 0$, is an exponentially order bounded semigroup on $\mathrm{C}_c(\mathbb{R})$, $\mathrm{Lip}(\mathbb{R})$, $\mathrm{UC}(\mathbb{R})$, $\mathrm{UCB}(\mathbb{R})$, $\mathrm{C}_b(\mathbb{R})$, $\mathrm{C}(\mathbb{R})$, and $L^p(\mathbb{R})$ $(0 < p < \infty)$ with order exponent $\|q\|_\infty$, since

$$|T(t)f| \leq e^{\|q\|_\infty t}|f|.$$

In general, a relatively uniformly continuous semigroup is exponentially order bounded only under some additional assumptions.

Proposition 4.10. *If a vector lattice X has an order unit $u \in X$, then every relatively uniformly continuous positive semigroup $(T(t))_{t\geq 0}$ is exponentially order bounded.*

Proof. First, by assumption, there exists $\lambda > 1$ such that $T(1)u \leq \lambda u$ hold. Fix $x \in X$. By [5, Proposition 3.4], there exists $v \in X$ such that $|T(s)x| \leq v$ holds for all $s \in [0,1]$. Fix $t \geq 0$, $N \in \mathbb{N}_0$, $0 \leq s < 1$ such that $t = N + s$ and pick $\mu > 0$ such that $v \leq \mu u$. Then for $w := \ln(\lambda)$ we have

$$|T(t)x| \leq T(N)|T(s)x| \leq T(N)v \leq \mu \cdot T(1)^N u \leq \lambda^N \cdot (\mu u) \leq e^{w\cdot t}(\mu u). \quad \square$$

Example 4.11. By [5, Lemma 2.4], the vector lattices $\mathrm{Lip}(\mathbb{R})$ and $\mathrm{UC}(\mathbb{R})$ have an order unit and, by [5, Proposition 6.3], the (left) translation semigroup $(T_l(t))_{t\geq 0}$ is a relatively uniformly continuous positive semigroup on $\mathrm{Lip}(\mathbb{R})$ and $\mathrm{UC}(\mathbb{R})$. Hence, by Proposition 4.10, $(T_l(t))_{t\geq 0}$ is exponentially order bounded on $\mathrm{Lip}(\mathbb{R})$ and $\mathrm{UC}(\mathbb{R})$.

By [5, Proposition 3.11] and [5, Proposition 6.2], the (left) translation semigroup is relatively uniformly continuous on $\mathrm{C}_c(\mathbb{R})$ and $\mathrm{C}(\mathbb{R})$, but it is not exponentially order bounded on these lattices as the next example shows.

Example 4.12. The (left) translation semigroup $(T_l(t))_{t\geq 0}$ is not exponentially order bounded on the following spaces.

(a) On $\mathrm{C}_c(\mathbb{R})$. Fix a positive function $f \in \mathrm{C}_c(\mathbb{R})$ with $f(0) = 1$ and assume that there exists $w \in \mathbb{R}$ and $u \in \mathrm{C}_c(\mathbb{R})$ such that $T_l(t)f \leq e^{w\cdot t}u$ holds for all $t \geq 0$. Then $1 = f(0) = (T_l(t)f)(-t) \leq e^{w\cdot t}u(-t)$ and, hence $u(-t) \geq e^{-w\cdot t} > 0$ for all $t \geq 0$ which contradicts $u \in \mathrm{C}_c(\mathbb{R})$.

(b) On $\mathrm{C}(\mathbb{R})$. Consider the function $f\colon x \mapsto e^{x^2}$ and assume that there exist $w \in \mathbb{R}$ and $u \in \mathrm{C}(\mathbb{R})$ such that $T_l(t)f \leq e^{w\cdot t}u$ holds for all $t \geq 0$. Then $e^{t^2 - w\cdot t} \leq u(0)$ for all $t \geq 0$ which is a contradiction.

(c) On $L^p(\mathbb{R})$ for $0 < p < \infty$. Consider the function

$$f\colon x \mapsto \left|\frac{1}{x-\frac{1}{2}}\right|^{\frac{1}{2p}} \cdot \chi_{[0,1]}(x)$$

in $L^p(\mathbb{R})$. Assume that there exist $w \in \mathbb{R}$ and $u \in L^p(\mathbb{R})$ such that $T_l(t)f \leq e^{w\cdot t}u$, i.e.,

$$e^{-w\cdot t}\left|\frac{1}{x+t-\frac{1}{2}}\right|^{\frac{1}{2p}} \cdot \chi_{[0,1]}(x+t) \leq u(x)$$

holds for all $t \geq 0$ and almost every $x \in \mathbb{R}$. Furthermore, for each $x \in [0, \frac{1}{2}]$ there exists $t \in [0, \frac{1}{2}]$ such that $x + t - \frac{1}{2} = 0$ and hence, u attains infinity a.e. on $[0, \frac{1}{2}]$ which contradicts $u \in L^p(\mathbb{R})$.

For further studies we need to define the resolvent set and the resolvent operator in our setting. In order to do that it is necessary to consider vector lattices over complex fields. A *complex vector lattice* $X_{\mathbb{C}}$ is a complexification of an ru-complete vector lattice X endowed with the modulus function

$$|z| := \sup_{0\leq\theta<2\pi} |\cos(\theta)x + \sin(\theta)y|$$

which, by [7, Lemma 3.1], exists for each $z := x + iy \in X_{\mathbb{C}}$. It is well-known that all complex vector lattices are ru-complete. For a better understanding of complex vector lattices we refer to [7].

Motivated by the fact that a strongly continuous semigroup on a Banach lattice is positive iff its generator is a resolvent positive operator (see [2, Corollary 11.4]), we introduce the following notion. For an operator A on $X_{\mathbb{C}}$ we define its *positive resolvent set* by

$$\rho_+(A) := \big\{\lambda \in \mathbb{C} : R(\lambda, A) := (\lambda - A)^{-1} \text{ exists} \\ \text{and is a positive operator on } X_{\mathbb{C}}\big\}.$$

For each $w \in \mathbb{R}$ set $\mathbb{C}_{>w} := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > w\}$.

In [5, Section 4] it was shown, that rescaling does not change the ru-continuity of the semigroup. One can show even more.

LEMMA 4.13. *Let $(T(t))_{t\geq 0}$ be a positive ruc-semigroup on $X_{\mathbb{C}}$ with generator A. Let $\mu \in \mathbb{R}$ and $\alpha > 0$. The rescaled semigroup $(S(t))_{t\geq 0}$ defined by*

$$S(t) := e^{\mu t} T(\alpha t)$$

is again a positive ruc-semigroup with generator $B = \alpha A + \mu I_X$, $D(B) = D(A)$ and resolvent $R(\lambda, B) = \frac{1}{\alpha} R(\frac{\lambda-\mu}{\alpha}, A)$ for $\lambda \in \rho_+(B)$. Moreover, if $(T(t))_{t\geq 0}$ is exponentially order bounded with order exponent w, then $(S(t))_{t\geq 0}$, is also exponentially order bounded with order exponent $w + \mu$.

PROOF. The claim that $(S(t))_{t\geq 0}$ is a positive ruc-semigroup follows directly from [5, Section 4]. To see that B is the generator of $(S(t))_{t\geq 0}$, fix $x \in D(A)$. For each $h > 0$ we have

$$\frac{e^{\mu h} T(\alpha h)x - x}{h} = \alpha e^{\mu h} \frac{T(\alpha h)x - x}{\alpha h} + \frac{e^{\mu h} - 1}{h} \cdot x$$

and hence, by assumption, the right-hand side converges relatively uniformly to $\alpha Ax + \mu x$ as $h \searrow 0$. It is clear that $D(B) = D(A)$. For $\lambda \in \rho_+(B)$ we have

$$\lambda - B = \alpha \cdot \Big(\frac{\lambda - \mu}{\alpha} - A\Big)$$

and hence, $R(\lambda, B) = \frac{1}{\alpha} R(\frac{\lambda-\mu}{\alpha}, A)$ follows. Now, if there exists $w \in \mathbb{R}$ such that for each $x \in X_{\mathbb{C}}$ there exists $u \in X_{\mathbb{C}}$ such that $|T(t)x| \leq e^{wt} u$ holds for all $t \geq 0$, then $|S(t)x| \leq e^{(\mu + w\cdot\alpha)t} u$ holds for all $t \geq 0$. $\square$

5. A Hille–Yosida-type generation theorem

In this section we present and prove the main result of the paper, Theorem 5.4, which is an analogue to the classical Hille–Yosida Theorem (see [3, II.3.5 Generation Theorem]) for ruc-semigroups. It provides a characterisation of those linear operators that are the generators of some exponentially order bounded, relatively uniformly continuous, positive semigroups. More precisely, the generators are characterised via the behaviour of their resolvents. Throughout this section X denotes an ru-complete complex vector lattice.

The following result allows us to work with the resolvents of the generators of exponentially order bounded positive ruc-semigroups. It shows that these resolvents are the Laplace transforms of the corresponding semigroups.

PROPOSITION 5.1. *Let $(T(t))_{t\geq 0}$ be a positive exponentially order bounded ruc-semigroup on X with order exponent $w \in \mathbb{R}$ and generator A. Then the following assertions hold.*

(i) *For each $\lambda \in \mathbb{C}_{>w}$ the mapping*

$$x \mapsto R(\lambda)x := \int_0^\infty e^{-\lambda\cdot t} T(t)x \,\mathrm{d}t$$

defines a positive linear operator on X which is positive whenever $\lambda \in (w,\infty)$.

(ii) *For each $x \in X$ there exists $u \in X$ such that*

$$|R(\lambda)^k x| \leq (\operatorname{Re}\lambda - w)^{-k} \cdot u$$

holds for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}_{>w}$.

(iii) *The positive resolvent set $\rho_+(A)$ contains (w,∞) and $R(\lambda) = R(\lambda, A)$ holds for each $\lambda \in \mathbb{C}_{>w}$.*

PROOF. By assumption, there exists $w \in \mathbb{R}$ such that for each $x \in X$, there exists $u \in X$ such that

$$|T(t)x| \leq e^{tw} u$$

holds for all $t \geq 0$. Hence, by Proposition 3.3.(ii)–(iii), for each $\operatorname{Re}\lambda > w$, $S > s \geq 0$ and fixed $x \in X$ we obtain

$$\left| \int_0^S e^{-\lambda\cdot t} T(t)x \,\mathrm{d}t - \int_0^s e^{-\lambda\cdot t} T(t)x \,\mathrm{d}t \right| = \left| \int_0^{S-s} e^{-\lambda\cdot(s+t)} T(s+t)x \,\mathrm{d}t \right|$$
$$\leq \int_0^{S-s} e^{-(\operatorname{Re}\lambda - w)\cdot(s+t)} \cdot u \,\mathrm{d}t \leq (\operatorname{Re}\lambda - w)^{-1} e^{-(\operatorname{Re}\lambda - w)\cdot s} \cdot u.$$

Since X is relatively uniformly complete, the improper ru-integral defining $R(\lambda)x$ exists. Furthermore, by Lemma 2.1.(ii) and the fact that $T(t)$ is linear and positive for each $t \geq 0$, the operator $R(\lambda)$ is also linear and positive whenever $\lambda \in (w,\infty)$. This proves (i).

To prove (ii) we use the assumption that $(T(t))_{t\geq 0}$ is exponentially order bounded with order exponent w and Proposition 3.3.(iii)–(iv) $(n-1)$ times to estimate

$$|R(\lambda)^k x| = \left| \int_0^\infty \dots \int_0^\infty e^{-\lambda\cdot(\sum_{l=1}^k t_l)} T\left(\sum_{l=1}^k t_l\right) x \,\mathrm{d}t_1 \dots \mathrm{d}t_k \right|$$
$$\leq \int_0^\infty \dots \int_0^\infty e^{-(\operatorname{Re}\lambda - w)\cdot(\sum_{l=1}^k t_l)} u \,\mathrm{d}t_1 \dots \mathrm{d}t_k$$
$$= \left(\int_0^\infty e^{-(\operatorname{Re}\lambda - w)\cdot t} \,\mathrm{d}t \right)^k \cdot u = (\operatorname{Re}\lambda - w)^{-k} \cdot u.$$

Next, we show (iii). By a simple rescaling argument, see Lemma 4.13, we may assume that $\lambda = 0$. We need to show that $R(0, A)$ exists and equals $R(0)$. By Proposition 3.3.(ii), for each $h > 0$ and $x \in X$ we have

$$\frac{T(h) - I}{h} R(0)x = \frac{T(h) - I}{h} \int_0^\infty T(t)x \,\mathrm{d}t$$
$$= \frac{1}{h} \int_0^\infty T(t+h)x \,\mathrm{d}t - \frac{1}{h} \int_0^\infty T(t)x \,\mathrm{d}t = -\frac{1}{h} \int_0^h T(t)x \,\mathrm{d}t.$$

By Lemma 4.2.(iii), the right-hand side converges relatively uniformly to $-x$ as $h \searrow 0$ and therefore $R(0)x \in D(A)$ with $AR(0)x = -x$ for all $x \in X$. On the other hand, for $x \in D(A)$, we obtain by Lemma 4.6.(iv) that

$$AR(0)x = A \int_0^\infty T(t)x \,\mathrm{d}t \; = \int_0^\infty T(t)Ax \,\mathrm{d}t \; = R(0)Ax.$$

This proves (iii). $\square$

The following property was introduced in [5, Section 5]. It provides a substitute for the Principle of Uniform Boundedness, which is an essential assumption in the rest of this paper.

Definition 5.2. A vector lattice X has the property (D) if for each net of linear operators $(T_\alpha)_\alpha$ on X the following two assertions imply $T_\alpha x \xrightarrow{ru} 0$ for each $x \in X$.

(a) There exists an ru-dense subset $D \subset X$ such that $T_\alpha y \xrightarrow{ru} 0$ for each $y \in D$.

(b) For each sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \xrightarrow{ru} 0$ there exists $u \in X_+$ such that for each $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and α_ε such that

$$|T_\alpha x_n| \leq \varepsilon \cdot u$$

holds for all $n \geq N_\varepsilon$ and $\alpha \geq \alpha_\varepsilon$.

By [5, Section 5], the class of vector lattices which have the property (D) contains all Banach lattices as well as $L^p(\mathbb{R})$ $(0 < p < 1)$, $\mathrm{C}(\mathbb{R})$, $\mathrm{C}_c(\mathbb{R})$, $\mathrm{Lip}(\mathbb{R})$, $\mathrm{UC}(\mathbb{R})$, $\mathrm{UCB}(\mathbb{R})$ and $\mathrm{C}_b(\mathbb{R})$.

Let us immediately state a very useful criterion for the ru-continuity of an exponentially order bounded positive semigroup on an ru-complete vector lattice with the property (D). It follows directly from [5, Theorem 5.7].

LEMMA 5.3. *Let X have the property (D) and let $(T(t))_{t\geq 0}$ be an exponentially order bounded positive semigroup on X. If there exists an ru-dense set $D \subset X$ such that $T(h)y \xrightarrow{ru} y$ as $h \searrow 0$ holds for each $y \in D$, then $(T(t))_{t\geq 0}$ is relatively uniformly continuous on X.*

We are now ready to state our main generation result which is motivated by the classical theorems of Hille and Yosida.

THEOREM 5.4. *Let X be an ru-complete vector lattice with the property (D) and A a linear operator on X. Then the following assertions are equivalent.*

(i) *Operator A is the generator of an exponentially order bounded relatively uniformly continuous positive semigroup with order exponent* 0.

(ii) *Operator A is ru-closed, ru-densely defined, $(0, \infty) \subset \rho_+(A)$ and for each $x \in X$ there exists $u \in X$ such that*

$$|R(\lambda, A)^k x| \leq (\operatorname{Re} \lambda)^{-k} \cdot u \tag{5.1}$$

holds for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}_{>0}$.

While one of the implications in this theorem follows directly from Proposition 4.7 and Proposition 5.1.(ii), more effort is needed for the proof of the other one. We start by showing a couple of lemmas. The operators $\lambda A R(\lambda, A)$ appearing in the following lemmas are known as *Yosida approximants.*

LEMMA 5.5. *Let X have the property (D) and let A be an ru-closed and ru-densely defined operator on X with $(0, \infty) \subset \rho_+(A)$. Suppose that for each $x \in X$ there exists $u \in X$ such that*

$$|R(\lambda, A)x| \leq \lambda^{-1} \cdot u$$

holds for all $\lambda > 0$. Then the following assertions hold.

(i) *For each relatively uniformly convergent sequence $(x_n)_{n\in\mathbb{N}} \subset X$ there exists $u \in X$ such that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\big|\lambda R(\lambda, A)x_n - x_n\big| \leq \varepsilon \cdot u$$

holds for all $\lambda, n \geq N$.

(ii) *For each relatively uniformly convergent sequence* $(x_n)_{n\in\mathbb{N}} \subset D(A)$ *there exists* $u \in X$ *such that for each* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$\big|\lambda AR(\lambda, A)x_n - Ax_n\big| \le \varepsilon \cdot u$$

holds for all $\lambda, n \ge N$.

Proof. To show (i) we prove first that $\lambda R(\lambda, A)x \xrightarrow{ru} x$ as $\lambda \to \infty$ for each $x \in X$. Set $T_\lambda := \lambda R(\lambda, A) - I_X$ for each $\lambda > 0$. Since X has the property (D), it suffices to verify that $(T_\lambda)_\lambda$ satisfies assertions (a) and (b) from Definition 5.2.

(a) By assumption, the set $D := D(A)$ is ru-dense in X. For $x \in D$ we have $T_\lambda x = R(\lambda, A)Ax$ and hence, by assumption, there exists $u \in X$ such that $|T_\lambda x| \le \lambda^{-1}u$ holds for all $\lambda > 0$ which yields $T_\lambda x \xrightarrow{ru} 0$ as $\lambda \to \infty$.

(b) Pick a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \xrightarrow{ru} 0$ with respect to regulator $v \in X$. Fix $\varepsilon > 0$. Then there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n| \le \varepsilon \cdot v$ holds for all $n \ge N_\varepsilon$. By assumption, there exists $u \in X$ such that $R(\lambda, A)v \le \lambda^{-1} \cdot u$ holds for all $\lambda > 0$ and since $R(\lambda, A)$ is positive for each $\lambda > 0$ we estimate

$$\begin{aligned}|T_\lambda x_n| = |\lambda R(\lambda, A)x_n - x_n| &\le \lambda \cdot R(\lambda, A)|x_n| + |x_n| \\ &\le \varepsilon \cdot (\lambda R(\lambda, A)v + v) \le \varepsilon \cdot (u + v)\end{aligned}$$

for all $\lambda > 0$ and $n \ge N_\varepsilon$.

By property (D), we conclude that $\lambda R(\lambda, A)x \xrightarrow{ru} x$ as $\lambda \to \infty$ for each $x \in X$.

To finish the proof of (i) pick a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$ with $x_n \xrightarrow{ru} x$ with respect to regulator $u \in X$ as $n \to \infty$ and find regulators $v_1, v_2 \in X$ such that $\lambda R(\lambda, A)x \xrightarrow{ru} x$ with respect to v_1 and $\lambda R(\lambda, A)u \xrightarrow{ru} u$ with respect to v_2 as $\lambda \to \infty$. Then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x| \le \varepsilon \cdot u, \quad |\lambda R(\lambda, A)x - x| \le \varepsilon \cdot v_1, \quad \text{and} \quad |\lambda R(\lambda, A)u - u| \le \varepsilon \cdot v_2$$

hold for all $\lambda, n \ge N$ and hence,

$$\begin{aligned}|\lambda R(\lambda, A)x_n - x_n| &\le \big|\lambda R(\lambda, A)(x_n - x)\big| + \big|\lambda R(\lambda, A)x - x\big| + |x - x_n| \\ &\le \varepsilon \cdot \lambda R(\lambda, A)u + \varepsilon \cdot v_1 + \varepsilon \cdot u \le \varepsilon \cdot (v_1 + \varepsilon v_2 + 2u).\end{aligned}$$

This proves (i). The second statement is an immediate consequence of the first one. □

By using Yosida approximants we construct a sequence of exponentially order bounded, relatively uniformly continuous, positive semigroups which will play a crucial role in the proofs of Theorem 5.4 and Proposition 5.8.

LEMMA 5.6. *Let X be an ru-complete vector lattice with the property (D) and A be as in Lemma 5.5. Then for each $n \in \mathbb{N}$ the operator*

$$A_n := n^2 R(n, A) - nI_X = nAR(n, A)$$

on X is the generator of an exponentially order bounded, relatively uniformly continuous, positive semigroup $(T_n(t))_{t\geq 0}$ with order exponent 0. Furthermore, these semigroups satisfy the following assertions.

(i) *For each $x \in X$ there exists $u \in X$ such that*

$$|T_n(t)x| \leq u$$

holds for all $n \in \mathbb{N}$ and $t \geq 0$.

(ii) *For each $x \in X$ there exists $u \in X$ such that for each $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left|\frac{T_n(h)x - x}{h} - A_n x\right| \leq \varepsilon \cdot u$$

holds for all $h \in [0, \delta]$.

(iii) *The operators $T_n(t)$ and $T_m(s)$ commute for all $n, m \in \mathbb{N}$ and $t, s \geq 0$.*

PROOF. For each $n \in \mathbb{N}$ we first define the operator $T_n(t)$ and show assertion (i). From this immediately follows that $(T_n(t))_{t\geq 0}$ is an exponentially order bounded positive semigroup with order exponent 0. Then we show assertion (ii) which also yields that A_n is the generator of the ruc-semigroup $(T_n(t))_{t\geq 0}$. At the end we verify assertion (iii).

Fix $x \in X$. By assumption, there exists $u \in X$ such that

$$|(nR(n, A))^k x| \leq u \tag{5.2}$$

holds for all $n, k \in \mathbb{N}$. Then for each $t \geq 0$, $n \in \mathbb{N}$ and all $\ell, m \in \mathbb{N}$ with $\ell \geq m$ we estimate

$$\begin{aligned} &\left|\sum_{k=0}^{\ell} \frac{(tn)^k}{k!}(nR(n, A))^k x - \sum_{k=0}^{m} \frac{(tn)^k}{k!}(nR(n, A))^k x\right| \\ &= \left|\sum_{k=m+1}^{\ell} \frac{(tn)^k}{k!}(nR(n, A))^k x\right| \leq \sum_{k=m+1}^{\ell} \frac{(tn)^k}{k!} \cdot u. \end{aligned}$$

This shows that $\left(\sum_{k=0}^{\ell} \frac{t^k}{k!}(n^2 R(n, A))^k x\right)_{\ell\in\mathbb{N}}$ is a relatively uniform Cauchy sequence in X and hence, it has a unique limit which we denote by $\sum_{k=0}^{\infty} \frac{t^k}{k!}(n^2 R(n, A))^k x$ for each $t \geq 0$ and $n \in \mathbb{N}$. Since $n^2 R(n, A)$ is a positive linear operator the mapping

$$T_n(t)\colon y \mapsto e^{-nt} \sum_{k=0}^{\infty} \frac{t^k}{k!}(n^2 R(n, A))^k y$$

defines a positive linear operator on X for each $n \in \mathbb{N}$ and $t \geq 0$. Furthermore, using (5.2) we estimate

$$|T_n(t)x| \leq e^{-nt} \sum_{k=0}^{\infty} \frac{(tn)^k}{k!} \big|(nR(n,A))^k x\big| \leq e^{-nt} \sum_{k=0}^{\infty} \frac{(tn)^k}{k!} \cdot u = u \tag{5.3}$$

for all $t \geq 0$ and $n \in \mathbb{N}$. This proves (i). Moreover, it follows that $T_n(t)x$ is an element of the principal ideal $I_u \subset X$ generated by u for all $t \geq 0$, $n \in \mathbb{N}$ and since X is ru-complete, I_u, endowed with the norm

$$\|y\|_u := \sup \big\{\lambda > 0 : |y| \leq \lambda u\big\},$$

is a Banach lattice and, by (5.3), both series

$$\sum_{j=0}^{\infty} \frac{(tn)^j}{j!} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{t^k}{k!} (n^2 \|R(n,A)x\|_u)^k$$

converge absolutely. Hence, one can show, as for the Cauchy product for scalar series, that for each $n \in \mathbb{N}$ and $t \geq 0$, it holds

$$\begin{aligned} T_n(t)x &= \sum_{k=0}^{\infty} \frac{t^k(-n)^k}{k!} \cdot \sum_{k=0}^{\infty} \frac{t^k}{k!} (n^2 R(n,A))^k x \\ &= \sum_{k=0}^{\infty} \bigg(\sum_{j=0}^{k} \frac{t^{k-j}(-n)^{k-j}}{(k-j)!} \cdot \frac{t^j}{j!} (n^2 R(n,A))^j x \bigg) \\ &= \sum_{k=0}^{\infty} \bigg(\sum_{j=0}^{k} \binom{k}{j} (-n)^{k-j} (n^2 R(n,A))^j x \bigg) \cdot \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(n^2 R(n,A) - n)^k x}{k!} \cdot t^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} A_n^k x. \end{aligned} \tag{5.4}$$

Furthermore, using (5.4) and similar arguments as in [3, Proposition I.2.3], it is easy to see that $(T_n(t))_{t \geq 0}$ defines a positive semigroup on X for each $n \in \mathbb{N}$.

We now show (ii). Using the binomial formula and (5.2), we obtain

$$\begin{aligned} |A_n^k x| &= \bigg| \sum_{j=0}^{k} \binom{k}{j} (-n)^j (n^2 R(n,A))^{k-j} x \bigg| \\ &\leq \bigg(\sum_{j=0}^{k} \binom{k}{j} n^j n^{k-j} \bigg) \cdot u = (2n)^k \cdot u. \end{aligned} \tag{5.5}$$

Now, fix $n \in \mathbb{N}$ and $0 < \varepsilon < 1$. Then, by using (5.4) and (5.5), for all $h \in [0, \varepsilon \cdot e^{-2n}]$ we estimate

$$\left|\frac{T_n(h)x - x}{h} - A_n x\right| = h \cdot \left|\sum_{k=2}^{\infty} \frac{h^{k-2} \cdot A_n^k x}{k!}\right|$$

$$\leq h \cdot \sum_{k=2}^{\infty} \frac{|A_n^k x|}{k!} \leq h \cdot \left(\sum_{k=2}^{\infty} \frac{(2n)^k}{k!}\right) \cdot u \leq \varepsilon \cdot u.$$

This proves (ii) and shows that the orbit map $t \mapsto T_n(t)x$ is ru-differentiable and hence, by Proposition 3.2, it is also ru-continuous, i.e., $(T_n(t))_{t\geq 0}$ is an ruc-semigroup on X.

Finally, assertion (iii) follows from formula (5.4) and the fact that A_n and A_m commute for all $n, m \in \mathbb{N}$. □

We now proceed with the proof of the left implication of Theorem 5.4 which we divide into three steps:

Step 1. By using the semigroups $(T_n(t))_{t\geq 0}$ defined in Lemma 5.6, for each $y \in D(A)$ and $t \geq 0$ we define $T(t)y$ as the ru-limit of $T_n(t)y$ as $n \to \infty$ and extend this definition to X.

Step 2. We show that $(T(t))_{t\geq 0}$ is an exponentially order bounded relatively uniformly continuous positive semigroup with order exponent 0.

Step 3. We prove that A is the generator of $(T(t))_{t\geq 0}$.

Proof of Theorem 5.4, Step 1. Consider the Yosida approximants A_n and the corresponding semigroups $(T_n(t))_{t\geq 0}$ as defined in Lemma 5.6. Fix $x \in X$. Since A is ru-densely defined, there exists a sequence $(x_k)_{k\in\mathbb{N}} \subset D(A)$ such that $x_k \xrightarrow{ru} x$. Take any such sequence.

We show first that for each $t \geq 0$ and $k \in \mathbb{N}$ the sequence $(T_n(t)x_k)_{n\in\mathbb{N}}$ is a relatively uniform Cauchy sequence. By Lemma 5.5.(ii), there exists $\tilde{u} \in X$ such that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left|A_n x_k - A_m x_k\right| \leq \left|nAR(n,A)x_k - Ax_k\right| + \left|Ax_k - mAR(m,A)x_k\right| \leq \varepsilon \cdot (2\tilde{u})$$

holds for all $n, m, k \geq N$. Furthermore, by Lemma 5.6.(i), there exist $\tilde{w}, v \in X$ such that for all $n \in \mathbb{N}$, $t \geq 0$ we have $T_n(t)(2\tilde{u}) \leq \tilde{w}$ and $T_n(t)\tilde{w} \leq v$. Since $T_n(t)$ and $T_m(t)$ are positive operators, we estimate

$$\left|T_m(t-\tau)T_n(\tau)(A_n x_k - A_m x_k)\right| \leq \varepsilon \cdot v \tag{5.6}$$

for all $n, m, k \geq N$ and $\tau \in [0, t]$. Moreover, by Lemma 5.6.(iii), for each $n, m \in \mathbb{N}$ and $t, s \geq 0$ the operators $T_n(t), T_m(s)$ commute and hence, by Lemma 4.8, (5.6), and Proposition 3.3.(iii), for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|T_n(t)x_k - T_m(t)x_k| = \left|\int_0^t T_m(t-\tau)T_n(\tau)(A_n x_k - A_m x_k)\,\mathrm{d}\tau\right| \leq t \cdot \varepsilon \cdot v$$

holds for all $n, m, k \geq N$ and $t \geq 0$. This proves that for each $k \geq N$ and $t \geq 0$ the sequence $(T_n(t)x_k)_{n\in\mathbb{N}}$ is a relatively uniform Cauchy sequence and hence, it has a limit which we denote by $T(t)x_k$. Furthermore, there exists $v \in X$ such that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\big|T_n(t)x_k - T(t)x_k\big| \leq t \cdot \varepsilon \cdot v \tag{5.7}$$

holds for all $n, k \geq N$ and $t \geq 0$. In particular, for each $t > 0$ and $\varepsilon > 0$ there exists $\tilde{N} \in \mathbb{N}$ such that

$$\big|T_n(t)x_k - T(t)x_k\big| \leq t \cdot \frac{\varepsilon}{t} \cdot v = \varepsilon \cdot v \tag{5.8}$$

holds for all $n, k \geq \tilde{N}$.

Next, we prove that $(T(t)x_k)_{k\in\mathbb{N}}$ is a relatively uniform Cauchy sequence for each $t \geq 0$. Assume that $x_k \xrightarrow{ru} x$ with respect to a regulator u. By Lemma 5.6.(i), there exists $\tilde{v} \in X$ such that $T_n(t)u \leq \tilde{v}$ holds for all $n \in \mathbb{N}$, $t \geq 0$ and hence, by (5.8), for each $\varepsilon > 0$ and $t \geq 0$ there exists $\tilde{N} \in \mathbb{N}$ such that

$$\begin{aligned} \big|T(t)x_k - T(t)x_m\big| &\leq \big|T(t)x_k - T_n(t)x_k\big| + T_n(t)|x_k - x_m| \\ &+ \big|T_n(t)x_m - T(t)x_m\big| \leq \varepsilon \cdot \big(v + T_n(t)u + v\big) \leq \varepsilon \cdot (2v + \tilde{v}) \end{aligned}$$

holds for all $k, m \geq \tilde{N}$. Hence, for each $t \geq 0$ the sequence $(T(t)x_k)_{k\in\mathbb{N}}$ is a relatively uniform Cauchy sequence and it has a limit which we denote by $T(t)x$. Furthermore, there exists $\tilde{w} \in X$ such that for each $t \geq 0$ and $\varepsilon > 0$ there exists $\tilde{N} \in \mathbb{N}$ such that

$$\big|T(t)x_k - T(t)x\big| \leq \varepsilon \cdot \tilde{w} \tag{5.9}$$

holds for all $k \geq \tilde{N}$. As in the Banach space case, it is not difficult to verify that the limit $T(t)x$ is independent of the choice of $(x_k)_{k\in\mathbb{N}}$. □

Proof of Theorem 5.4, Step 2. Since positivity and semigroup property are preserved under ru-limits, $(T(t))_{t\geq 0}$ is a positive semigroup. We now show that it is exponentially order bounded with order exponent 0. To this end, fix $x \in X$ and pick any sequence $(x_k)_{k\in\mathbb{N}} \subset D(A)$ such that $x_k \xrightarrow{ru} x$ with respect to a regulator $u \in X$. Then, by (5.9), (5.8) and Lemma 5.6.(i), there exists $v_1, v_2, v_3 \in X$ such that for each $t \geq 0$ there exists $N \in \mathbb{N}$ such that

$$\begin{gathered} |T(t)x - T(t)x_N| \leq v_1, \quad |T(t)x_N - T_N(t)x_N| \leq v_2, \\ |x_N| \leq u + |x|, \quad T_N(t)(u + |x|) \leq v_3 \end{gathered}$$

hold and hence,

$$\begin{aligned}|T(t)x| &\le \big|T(t)x - T(t)x_N\big| + \big|T(t)x_N - T_N(t)x_N\big| + T_N(t)|x_N| \\ &\le v_1 + v_2 + T_N(t)(u + |x|) \le v_1 + v_2 + v_3.\end{aligned}$$

This proves that $(T(t))_{t\ge 0}$ is exponentially order bounded with order exponent 0.

It remains to prove that $(T(t))_{t\ge 0}$ is ru-continuous. By Lemma 5.3, it suffices to check that $T(h)y \xrightarrow{ru} y$ as $h \searrow 0$ for each $y \in D(A)$. By the same reasoning as in the proof of (5.7), we derive that there exists $w_1 \in X$ such that for fixed $0 < \tilde{\varepsilon} \le 1$ there exists $\tilde{N} \in \mathbb{N}$ such that $|T(h)y - T_{\tilde{N}}(h)y| \le h \cdot \tilde{\varepsilon} \cdot w_1$ holds for all $h \ge 0$. Furthermore, since the semigroup $(T_{\tilde{N}}(t))_{t\ge 0}$ is ru-continuous there exists $w_2 \in X$ such that for each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that $|T_{\tilde{N}}(h)y - y| \le \varepsilon \cdot w_2$ holds for all $h \in [0, \delta]$ and hence,

$$\begin{aligned}\big|T(h)y - y\big| &\le \big|T(h)y - T_{\tilde{N}}(h)y\big| + \big|T_{\tilde{N}}(h)y - y\big| \\ &\le h \cdot \tilde{\varepsilon} \cdot w_1 + \varepsilon \cdot w_2 \le \varepsilon \cdot (w_1 + w_2). \qquad \square\end{aligned}$$

Proof of Theorem 5.4, Step 3. Let B denote the generator of $(T(t))_{t\ge 0}$. We show that A and B coincide on $D(A)$ and that $D(A) = D(B)$ which will conclude the proof.

Fix $y \in D(A)$. As we mentioned in Step 2, from the proof of (5.7) one can deduce that there exists $u_1 \in X$ such that for each $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that

$$|T(h)y - T_N(h)y| \le h \cdot \varepsilon \cdot u_1$$

holds for all $h \ge 0$ and $n \ge N$. Furthermore, by Lemma 5.6.(ii) and Lemma 5.5.(ii), there exist $u_2, u_3 \in X$ such that for each $\varepsilon > 0$ there exist $M \ge N$ and $\delta > 0$ such that

$$\Big|\frac{T_M(h)y - y}{h} - A_M y\Big| \le \varepsilon \cdot u_2, \quad |A_M y - Ay| \le \varepsilon \cdot u_3$$

hold for all $h \in [0, \delta]$ and hence, we obtain

$$\begin{aligned}\Big|\frac{T(h)y - y}{h} - Ay\Big| &\le \Big|\frac{T(h)y - T_M(h)y}{h}\Big| + \Big|\frac{T_M(h)y - y}{h} - A_M y\Big| + |A_M y - Ay| \\ &\le \frac{h \cdot \varepsilon \cdot u_1}{h} + \varepsilon \cdot u_2 + \varepsilon \cdot u_3 \le \varepsilon \cdot (u_1 + u_2 + u_3).\end{aligned}$$

This proves that $D(A) \subset D(B)$ and that A coincides with B on $D(A)$.

To prove $D(B) \subset D(A)$ fix $x \in D(B)$. Since B is the generator of an exponentially order bounded semigroup with order exponent 0, by

Proposition 5.1.(iii), we have $1 \in \rho_+(A) \cap \rho_+(B)$ and hence, $(I_X - A)$ and $(I_X - B)$ are bijective operators. Thus, there exists $y \in D(A)$ such that $(I_X - B)x = (I_X - A)y$. Since $(I_X - A)$ and $(I_X - B)$ coincide on $D(A)$ we obtain $(I_X - B)x = (I_X - B)y$ and hence, we have $x = y$. This proves that $x \in D(A)$. $\square$

By applying Lemma 4.13 we directly obtain a generalization of Theorem 5.4 for exponentially order bounded ruc-semigroups of any order exponent.

Corollary 5.7. *Let X be an ru-complete vector lattice with the property (D). For $w \in \mathbb{R}$ the following assertions are equivalent.*

(i) *The operator A is the generator of an exponentially order bounded, relatively uniformly continuous, positive semigroup with order exponent w.*

(ii) *The operator A is ru-closed, ru-densely defined, $(w, \infty) \subset \rho_+(A)$ and for each $x \in X$ there exists $u \in X$ such that*

$$|R(\lambda, A)^k x| \leq (\operatorname{Re} \lambda - w)^{-k} \cdot u$$

holds for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}_{>w}$.

We conclude by showing that every exponentially order bounded positive ruc-semigroup is uniquely determined by its generator.

Proposition 5.8. *Let X be an ru-complete vector lattice with the property (D). Every exponentially order bounded relatively uniformly continuous positive semigroup on X is uniquely determined by its generator.*

Proof. By a simple rescaling argument, see Lemma 4.13, we may assume that $(S(t))_{t\geq 0}$ is an exponentially order bounded positive ruc-semigroup with order exponent 0. We will prove that $(S(t))_{t\geq 0}$ coincides with the semigroup $(T(t))_{t\geq 0}$ which was constructed in Step 1 of the proof of the backward implication in Theorem 5.4.

Assume that A is the generator of $(S(t))_{t\geq 0}$. By Proposition 5.1, the resolvent set $\rho_+(A)$ contains $(0, \infty)$ and we have

$$R(n, A)x = \int_0^\infty e^{-n \cdot t} S(t)x \, \mathrm{d}t \tag{5.10}$$

for each $n \in \mathbb{N}$ and $x \in X$. Furthermore, by Proposition 5.1.(ii) and Proposition 4.7, A satisfies the assumptions of Lemma 5.5 and hence, by Lemma 5.6, for each $n \in \mathbb{N}$ the operator

$$A_n := n^2 R(n, A) - nI_X = nAR(n, A)$$

on X is the generator of the exponentially order bounded positive ruc-semigroup $(T_n(t))_{t\geq 0}$ with order exponent 0.

Fix $y \in D(A)$. By Lemma 5.5.(ii), there exists $u \in X$ such that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|A_n y - Ay| \leq \varepsilon \cdot u$$

for all $n \geq N$. Furthermore, by assumption and Lemma 5.6.(i), there exist $w, v \in X$ such that $S(t)w \leq v$ and $T_n(t)u \leq w$ for all $n \in \mathbb{N}$, $t \geq 0$. Hence, for each $t \geq 0$ we have

$$|S(t-\tau)T_n(\tau)(A_n y - Ay)| \leq \varepsilon \cdot v \tag{5.11}$$

for all $n \geq N$ and $\tau \in [0, t]$.

By identity (5.10), the operators $S(t)$ and A_n commute for each $n \in \mathbb{N}$, $t \geq 0$ and hence, by (5.4), the operators $S(t)$ and $T_n(s)$ commute for each $t, s \geq 0$ and $n \in \mathbb{N}$. Therefore, by Lemma 4.8, (5.11), and Proposition 3.3.(iii), for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|S(t)y - T_n(t)y| = \left| \int_0^t S(t-\tau)T_n(\tau)(A_n y - Ay)\, \mathrm{d}\tau \right| \leq t \cdot \varepsilon \cdot v$$

holds for all $n \geq N$ and $t \geq 0$. This proves that $T_n(t)y \xrightarrow{ru} S(t)y$ as $n \to \infty$ and hence, $S(t)y = T(t)y$ for each $t \geq 0$ and $y \in D(A)$. Since $D(A)$ is ru-dense in X and $S(t)$ and $T(t)$ preserve ru-convergence we obtain $S(t)x = T(t)x$ for every $x \in X$ and $t \geq 0$. ☐

Acknowledgements. We would like to express our gratitude to Marko Kandić for insightful comments and meticulous proofreading which improved the manuscript. The first author would also like to thank Marko for professional guidance. Special thanks belongs to Jochen Glück for his valuable remarks.

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