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ℓ^p-IMPROVING INEQUALITIES FOR DISCRETE SPHERICAL AVERAGES

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Abstract. Let $\lambda^2 \in \mathbb{N}$, and in dimensions $d \geq 5$, let $A_\lambda f(x)$ denote the average of $f : \mathbb{Z}^d \to \mathbb{R}$ over the lattice points on the sphere of radius λ centered at x. We prove ℓ^p improving properties of A_λ:

$$\|A_\lambda\|_{\ell^p \to \ell^{p'}} \leq C_{d,p,\omega(\lambda^2)} \lambda^{d(1-\frac{2}{p})}, \quad \tfrac{d-1}{d+1} < p \leq \tfrac{d}{d-2}.$$

It holds in dimension $d = 4$ for odd λ^2. The dependence is in terms of $\omega(\lambda^2)$, the number of distinct prime factors of λ^2. These inequalities are discrete versions of a classical inequality of Littman and Strichartz on the L^p improving property of spherical averages on $\mathbb{R}^d$. In particular they are scale free, in a natural sense. The proof uses the decomposition of the corresponding multiplier whose properties were established by Magyar–Stein–Wainger, and Magyar. We then use a proof strategy of Bourgain, which dominates each part of the decomposition by an endpoint estimate.

1. Introduction

The subject of this paper is in discrete harmonic analysis, in which continuous objects are studied in the setting of the integer lattice. Relevant norm properties are much more intricate, with novel distinctions with the continuous case arising.

In the continuous setting, L^p-improving properties of averages over lower dimensional surfaces are widely recognized as an essential property of such averages [3,14,25,26]. It continues to be very active subject of investigation. In the discrete setting, these questions are largely undeveloped. They are implicit in work on discrete fractional integrals by several authors [20–22,

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24], as well as two recent papers [5,11] on sparse bounds for discrete singular integrals.

Our main results concern ℓ^p improving estimates for averages over discrete spheres, in dimensions $d \geq 5$, and in dimension $d = 4$, for certain radii.

We recall the continuous case. For dimensions $d \geq 2$, let $d\sigma$ denote Haar measure on the sphere of radius one, and set $\mathcal{A}_1 f = \sigma * f$ be convolution with respect to σ. The classical result of Littman [14] and Strichartz [25] gives the sharp L^p improving property of this average. Here, we are stating the result in a restrictive way, but the full strength is obtained by interpolating with the obvious $L^1 \to L^1$ bound.

Theorem A [14,25]. *For dimensions $d \geq 2$, we have $\|\mathcal{A}_1\|_{\frac{d+1}{d} \to d+1}$.*

We study the discrete analog of $\mathcal{A}_1 f$ in higher dimensions. For $\lambda^2 \in \mathbb{N}$, let $\mathbb{S}^d_\lambda := \{n \in \mathbb{Z}^d : |n| = \lambda\}$. For a function f on $\mathbb{Z}^d$, define

$$A_\lambda f(x) = |\mathbb{S}^d_\lambda|^{-1} \sum_{n \in \mathbb{S}^d_\lambda} f(x-n).$$

The study of the harmonic analytic properties of these averages was initiated by Magyar [15], with Magyar, Stein and Wainger [18] proving a discrete variant of the Stein spherical maximal function theorem [23]. This result holds in dimensions $d \geq 5$, as irregularities in the number of lattice points on spheres presenting obstructions to a positive result in dimensions $d = 2, 3, 4$. In particular, they proved the result below. See Ionescu [9] for an endpoint result, and the work of several others which further explore this topic [1,4, 7,16,19].

Theorem B (Magyar, Stein, Wainger [18]). *For $d \geq 5$, there holds*

$$\Big\| \sup_\lambda |A_\lambda f| \Big\|_p \lesssim \|f\|_p, \quad \tfrac{d}{d-2} < p < \infty.$$

We will refer to $p_{\mathrm{MSW}} = \frac{d}{d-2}$ as the Magyar–Stein–Wainger index.

Our first main result is a discrete variant of the result of Littman and Strichartz above. First note that A_λ is clearly bounded from ℓ^p to ℓ^p, for all $1 \leq p \leq \infty$. Hence, it trivially improves any $f \in \ell^p(\mathbb{Z}^d)$ to an $\ell^\infty(\mathbb{Z}^d)$ function. But, proving a *scale-free version* of the inequality is not at all straightforward.

In dimensions $d = 4$, there is an arithmetical obstruction, namely for certain radii λ, the number of points on the sphere of radius λ can be very small. To address this, let $\Lambda_d := \{0 < \lambda < \infty : \lambda^2 \in \mathbb{N}\}$, for $d \geq 5$, and for $d = 4$,

$$\Lambda_4 := \big\{0 < \lambda < \infty : \lambda^2 \in \mathbb{N} \setminus 4\mathbb{N}\big\}$$

Following the work of Magyar [16], we will address the case of dimension $d = 4$ below. And, we will prove results *below the Magyar–Stein–Wainger index.*

THEOREM 1.1. *In dimensions $d \geq 4$, the inequality below holds for all $\lambda \in \Lambda_d$:*

$$\|A_\lambda\|_{p\to p'} \leq C_{d,p,\omega(\lambda^2)}\lambda^{d(1-\frac{2}{p})}, \quad \tfrac{d+1}{d-1} < p \leq 2. \tag{1.2}$$

Above, $\omega(\lambda^2)$ is the number of distinct prime factors of λ^2. In order that (1.2) hold, it is necessary that $p \geq \frac{d+1}{d}$, for $d \geq 5$.

This Theorem was independently proved by Hughes [8]. The proof herein uses the same elements, but optimizes the interpolation part of the argument. It is short, and simple enough that one can give concrete estimates for the dependence on λ, which we indicate below.

To explain our use of the phrase 'scale free' we make this definition. For a cube $Q \subset \mathbb{R}^d$ of volume at least one, we set localized and normalized norms to be

$$\langle f\rangle_{Q,p} := \Bigg[|Q|^{-1}\sum_{n\in Q\cap\mathbb{Z}^d}|f(n)|^p\Bigg]^{1/p}, \quad 0 < p \leq \infty.$$

An equivalent way to phrase our theorem above is the following corollary. Note that in this language, the inequalities in (1.4) are uniform in the choice of λ.

COROLLARY 1.3. *Let $d \geq 4$, and set $\mathbf{I}_d$ to be the open triangle with vertices $(0,1)$, $(1,0)$, and $(\frac{d-1}{d+1}, \frac{d-1}{d+1})$ (see Fig. 1). For $(1/p_1, 1/p_2) \in \mathbf{I}_d$, there is a finite constant $C = C_{d,p_1,p_2,\omega(\lambda^2)}$ so that*

$$\langle A_\lambda f_1, f_2\rangle \leq C\langle f_1\rangle_{Q,p_1}\langle f_2\rangle_{Q,p_2}|Q|, \quad \lambda \in \Lambda_d. \tag{1.4}$$

Our main inequality is only of interest for $\frac{d+1}{d-1} < p < \frac{d}{d-2} = p_{\mathrm{MSW}}$, in the case of $d \geq 5$. Indeed, at p_{MSW}, we know a substantially better result. For indicators functions $f = \mathbf{1}_F$ and $g = \mathbf{1}_G$ supported in a cube E of side length λ_0, we have [12] this restricted maximal estimate at the index p_{MSW}.

$$\Big\langle \sup_{\lambda_0/2<\lambda<\lambda_0} A_\lambda f, g\Big\rangle \lesssim \langle f\rangle_{E,\frac{d}{d-2}}\langle g\rangle_{E,\frac{d}{d-2}}|E|. \tag{1.5}$$

The proof of (1.2) requires a circle method decomposition of A_λ in terms of its Fourier multiplier. The key elements here were developed by Magyar, Stein and Wainger [18], with additional observations of Magyar [17]. We recall this in Section 2. The short proof in Section 3 uses indicator functions, following work of Bourgain [2], and in the discrete setting Ionescu [10], and Hughes [6]. We comment briefly on sharpness in the last section of the paper.

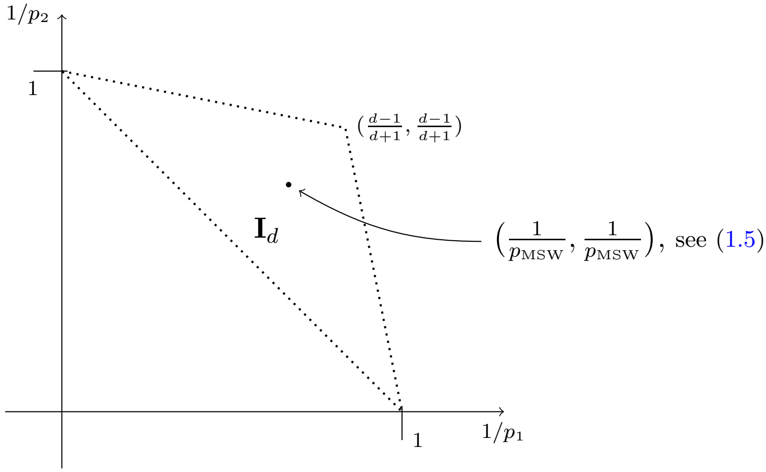

Fig. 1: The triangle $\mathbf{I}_d$ *of Theorem* 1.3 *for the* ℓ^p *improving inequality* (1.2) *is the dotted triangle with corners* $(0,1)$ *to* P_1 *to* $(1,0)$. *The diagram above is for the case of dimension* $d \geq 5$. *The point closest to the diagonal corresponds to the Magyar–Stein–Wainger index. At this point the maximal inequality* (1.5) *holds.*

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2. Decomposition

Throughout $e(x) = e^{2\pi i x}$. The Fourier transform on $\mathbb{Z}^d$ is given by

$$\widehat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} e(-\xi \cdot x) f(x), \quad \xi \in \mathbb{T}^d \equiv [0,1]^d.$$

We will write $\check{\phi}$ for the inverse Fourier transform. The Fourier transform on $\mathbb{R}^d$ is

$$\widetilde{\phi}(\xi) = \int_{\mathbb{R}^d} e(-\xi \cdot x) f(x)\, dx.$$

We work exclusively with convolution operators

$$K \colon f \mapsto \int_{\mathbb{T}^d} k(\xi) \widehat{f}(\xi) e(\xi \cdot x)\, d\xi.$$

In this notation, k is the multiplier, and the convolution is $\check{k} * f$. Lower case letters are frequently, but not exclusively, used for the multipliers, and capital letters for the corresponding convolution operators.

The following estimate for the number of lattice points on a sphere holds.

$$|\mathbb{S}^d_n| = \big|\big\{n \in \mathbb{Z}^d : |n| = \lambda\big\}\big| \simeq \lambda^{d-2}, \quad \lambda \in \Lambda_d.$$

Redefine the discrete spherical averages $A_\lambda f$ to be

$$A_\lambda f(x) = \lambda^{-d+2} \sum_{n\in\mathbb{Z}^d:|n|=\lambda} f(x-n) = \int_{\mathbb{T}^d} a_\lambda(\xi)\widehat{f}(\xi)e(\xi\cdot x)\,d\xi$$

where

$$a_\lambda(\xi) = \lambda^{-d+2} \sum_{n\in\mathbb{Z}^d:|n|=\lambda} e(\xi\cdot n).$$

The decomposition of a_λ into a 'main' term c_λ and an 'residual' term $r_\lambda = a_\lambda - c_\lambda$ follows development of Magyar, Stein and Wainger [18, §5], Magyar [17, §4] and Hughes [7, §4]. We will be very brief.

For integers q, set $\mathbb{Z}^d_q = (\mathbb{Z}/q\mathbb{Z})^d$. Set $\mathbb{Z}^\times_q = \{a \in \mathbb{Z}_q : (a,q) = 1\}$ to be the multiplicative group. We have

$$c_\lambda(\xi) = \sum_{1\le q\le\lambda} c_{\lambda,q}(\xi), \tag{2.1}$$

$$c_{\lambda,q}(\xi) = \sum_{\ell\in\mathbb{Z}^d_q} K(\lambda,q,\ell)\Phi_q(\xi-\ell/q)\widetilde{d\sigma_\lambda}(\xi-\ell/q),$$

$$K(\lambda,q,\ell) = q^{-d} \sum_{a\in\mathbb{Z}^\times_q} \sum_{n\in\mathbb{Z}^d_q} e_q\big(-a\lambda^2 + |n|^2 a + n\cdot\ell\big). \tag{2.2}$$

Above, Φ is a smooth non-negative radial bump function, $\mathbf{1}_{[-1/8,1/8]^d} \le \Phi \le \mathbf{1}_{[-1/4,1/4]^d}$. Further, $\Phi_q(\xi) = \Phi(q\xi)$. Throughout we use $e_q(x) = e(x/q) = e^{2\pi i x/q}$. The term in (2.2) is a Kloosterman sum, a fact that is hidden in the expression above, but becomes clear after exact summation of the quadratic Gauss sums. In addition, $d\sigma_\lambda$ is the continuous unit Haar measure on the sphere of radius λ in $\mathbb{R}^d$. Recall the stationary phase estimate

$$|\widetilde{d\sigma_\lambda}(\xi)| \lesssim |\lambda\xi|^{-\frac{d-1}{2}}. \tag{2.3}$$

Essential here is the *Kloosterman refinement*. The estimate below goes back to the work of Kloosterman [13] and Weil [27]. Magyar [17, §4] used it in this kind of setting. (It is essential to the proof of Lemma 2.9.)

LEMMA 2.4. [17, Proposition 7] *For all* $\eta > 0$, *and all* $1 \le q \le \lambda$, $\lambda \in \Lambda_d$,

$$\sup_\ell |K(\lambda,q,\ell)| \lesssim_\eta q^{-\frac{d-1}{2}+\eta}\rho(q,\lambda), \tag{2.5}$$

where we write $q = q_1 2^r$*, with* q_1 *odd, so that* $\rho(q,\lambda) = \sqrt{(q_1,\lambda^2)2^r}$*, where* (q_1,λ^2) *is the greatest common divisor of* q_1 *and* λ^2*. The implied constant only depends upon* $\eta > 0$*.*

Concerning the terms $\rho(q,\lambda)$, we need this Proposition.

PROPOSITION 2.6. *We have for* $N < \lambda$ *and* $a > 1$*, and all* $\eta > 0$

$$\sum_{q:N\le q} q^{-a}\rho(\lambda,q) \lesssim N^{1-a}\sigma_{-1/2}(\lambda^2), \tag{2.7}$$

$$\sum_{1\le q\le N} q^{\eta}\rho(\lambda,q) \lesssim N^{1+\eta}\sigma_{-1/2}(\lambda^2). \tag{2.8}$$

Above $\sigma_b(n) = \sum_{d:d\,|\,n} d^b$ *is the generalized sum of divisors function.*

PROOF. Write $q = 2^r st$, where s and t are odd, $r \ge 0$ and $(s,\lambda^2) = 1$. With this notation, $\rho(\lambda,q) = t2^r$. For (2.7), the sum we need to estimate is

$$\sum_{t:t\,|\,\lambda^2} \sum_{s=1}^{\infty} \sum_{\substack{r=0\\ 2^r st\ge N}}^{\infty} \frac{[t2^r]^{\frac12}}{[ts2^r]^a}.$$

We will sum over r first. There is first the cases in which $st \le N$:

$$\sum_{t:t|\lambda^2} \sum_{s=1}^{\infty} \sum_{\substack{r=0\\ 2^r st\ge N,\ st\le N}}^{\infty} \frac{[t2^r]^{\frac12}}{[st2^r]^a} \lesssim \sum_{t:t\,|\,\lambda^2} \sum_{\substack{s=1\\ st\le N}}^{\infty} \Big(\frac{st}{N}\Big)^{a-1/2} \frac{1}{s^a t^{a-1/2}}$$

$$\lesssim N^{1/2-a} \sum_{t:t\,|\,\lambda^2} (N/t)^{1/2} = N^{1-a} \sum_{t:t\,|\,\lambda^2} t^{-1/2} \lesssim N^{1-a}\sigma_{-1/2}(\lambda^2).$$

The second case of $st > N$ imposes no restriction on r. The sum over $r \ge 0$ is just a geometric series, therefore we have to bound

$$\sum_{t:t|\lambda^2} \sum_{\substack{s=1\\ st>N}}^{\infty} \frac{1}{s^a t^{a-1/2}} \lesssim \sum_{t:t|\lambda^2} \Big(\frac{t}{N}\Big)^{a-1} \frac{1}{t^{a-1/2}} \lesssim N^{1-a} \sum_{t:t|\lambda^2} \frac{1}{\sqrt{t}} \lesssim N^{1-a}\sigma_{-1/2}(\lambda^2).$$

We turn to (2.8) using the notation above. We estimate

$$\sum_{t:t\,|\,\lambda^2} \sum_{s=1}^{\infty} \sum_{\substack{r=0\\ 2^r st\le N}}^{\infty} [2^r st]^{\eta} [2^r t]^{\frac12} \lesssim \sum_{t:t\,|\,\lambda^2} \sum_{\substack{s=1\\ st\le N}}^{\infty} [st]^{\eta} t^{\frac12} (N/st)^{\frac12+\eta}$$

$$\lesssim N^{\frac12+\eta} \sum_{t:t\,|\,\lambda^2} \sum_{1\le s\le N/t} s^{-\frac12} \lesssim N^{1+\eta} \sum_{t:t\,|\,\lambda^2} t^{-\frac12} \lesssim N^{1+\eta}\sigma_{-1/2}(\lambda^2). \qquad \square$$

The 'main' term is $C_\lambda f$, and the 'residual' term is $R_\lambda = A_\lambda - C_\lambda$. This is a foundational estimate for us. (The reader should note that the normalizations here and in [17] are different.)

LEMMA 2.9 [17, Lemma 1, page 71]. *We have, for all* $\varepsilon > 0$, *uniformly in* $\lambda \in \Lambda_d$, $\|R_\lambda\|_{2\to 2} \lesssim_\varepsilon \lambda^{\frac{1-d}{2}+\varepsilon}$.

For a multiplier m on $\mathbb{T}^d$, define a family of related multipliers by

$$m_{\lambda,q} = \sum_{\ell \in \mathbb{Z}_q^d} K(\lambda, q, \ell) m(\xi - \ell/q). \tag{2.10}$$

We estimate the Fourier transform here.

PROPOSITION 2.11. *For a multiplier* $m_{\lambda,q}$ *as in* (2.10), *we have*

$$|\check{m}_{\lambda,q}(n)| \le q|\check{m}(n)|.$$

We include a proof for convenience.

PROOF. Our needs here are no different than those of [10,18]. See for instance the argument after [10, (2.9)]. Rewrite the Kloosterman sum in (2.2) in terms of Gauss sums, namely

$$K(\lambda, q, \ell) = \sum_{a \in \mathbb{Z}_q^\times} e_q(-a\lambda^2) G(a/q, \ell),$$

where

$$G(a/q, \ell) := q^{-d} \sum_{n \in \mathbb{Z}_q^d} e_q\big(|n|^2 a + n \cdot \ell\big).$$

Observe that $G(a/, \cdot)$ is a Fourier transform on the group $\mathbb{Z}_q^d$. Namely, if $\phi(\ell) = e(|\ell|^2 a/q)$ is the function on $\mathbb{Z}_q^d$, we have $\widehat{\phi}(-\ell) = \widehat{\phi}(\ell) = G(a/q, \ell)$. Using the version formula on that group we have

$$\sum_{\ell \in \mathbb{Z}_q^d} G(a/q, \ell) e_q(y \cdot \ell) = e_q(|y|^2 a), \quad y \in \mathbb{Z}_q^d. \tag{2.12}$$

Define

$$m^{a/q}(\xi) = e_q(-\lambda^2 a) \sum_{\ell \in \mathbb{Z}_q^d} G(a/q, \ell) m(\xi - \ell/q), \quad a \in \mathbb{Z}_q^\times.$$

By (2.12), we have

$$\widecheck{m^{a/q}}(n) = \int_{\mathbb{T}^d} m^{a/q}(\xi) e(-\xi \cdot n)\, d\xi$$

$$= e_q(-\lambda^2 a) \int_{\mathbb{T}^d} \sum_{\ell \in \mathbb{Z}_q^d} G(a/q,\ell) m(\xi - \ell/q) e(-\xi \cdot n)\, d\xi = e_q((|n|^2 - \lambda^2)a) \check{m}(n)\,.$$

Take the absolute value, and sum over $q \in \mathbb{Z}_q^\times$ to conclude the Proposition. □

3. Proof

It suffices to show the following statement: For $f = \mathbf{1}_F \subset E = [0,\lambda]^d \cap \mathbb{Z}^d$, choices of $0 < \varepsilon < 1$, and integers N we can write

$$A_\lambda f \le M_1 + M_2,$$

where

$$\langle M_1 \rangle_{E,\infty} \lesssim N^2 \langle f \rangle_E\,, \tag{3.1}$$

$$\langle M_2 \rangle_{E,2} \lesssim_\varepsilon N^{\varepsilon + \frac{3-d}{2}} \sigma_{-1/2}(\lambda^2) \cdot \langle f \rangle_E^{1/2}. \tag{3.2}$$

Above, $\sigma_{-1/2}(\lambda^2)$ is the generalized sum of divisors function, as in Proposition 2.6.

A straightforward argument concludes the proof from here, by optimizing over N. Indeed, for $g = \mathbf{1}_G$ with $G \subset E$, we have for any integer N,

$$|E|^{-1} \langle A_\lambda f, g \rangle \lesssim_\varepsilon N^2 \langle f \rangle_E \langle g \rangle_E + N^{\varepsilon + \frac{3-d}{2}} \sigma_{-1/2}(\lambda^2) \big[\langle f \rangle_E \langle g \rangle_E \big]^{1/2}.$$

Minimizing over N, we see that we should take

$$N^{\frac{d+1}{2} - \varepsilon} \simeq \sigma_{-1/2}(\lambda^2) \big[\langle f \rangle_E \langle g \rangle_E \big]^{-\frac{1}{2}}.$$

With this choice of N, we see that

$$|E|^{-1} \langle A_\lambda f, g \rangle \lesssim_\varepsilon \sigma_{-1/2}(\lambda^2)^{\frac{4}{d+1} + \varepsilon'} \big[\langle f \rangle_E \langle g \rangle_E \big]^{\frac{d-1}{d+1} + \varepsilon'}.$$

Above, $\varepsilon' = \varepsilon'(\varepsilon)$ tends to zero as ε does. This is a restricted weak type inequality. Interpolation with the obvious ℓ^2 bound completes the proof of our Theorem. We remark that this gives a concrete estimate of the dependence on λ^2. We have

$$\sigma_{-1/2}(n) \le \prod_{j=1}^{\omega(n)} \Big(1 - \frac{1}{\sqrt{p_j}} \Big)^{-1} \lesssim e^{c \frac{\sqrt{\omega(n)}}{\log \omega(n)}},$$

where $2 = p_1 < p_2 < \cdots$ is the increasing ordering of the primes. This is at most a constant depending upon $\omega(n)$, the number of distinct prime factors of n.

We turn to the construction of M_1 and M_2. If $\lambda \le N$, we set $M_1 = A_\lambda f$. Since we normalize the spherical averages by λ^{d-2}, (3.1) is immediate.

Proceed under the assumption that $N < \lambda$, and write $A_\lambda = C_\lambda + R_\lambda$, with c_λ defined in (2.1). The first contribution to M_2 is $M_{2,1} = R_\lambda f$. By Lemma 2.9, this satisfies (3.2). (We do not need the arithmetic function $\sigma_{-1/2}(\lambda^2)$ in this case.) Turn to C_λ. The second contribution to M_2 is the 'large q' term

$$M_{2,2} = \sum_{N \le q \le \lambda} C_{\lambda,q} f. \tag{3.3}$$

By the Weil estimates for Kloosterman sums (2.5), and Plancherel, we have

$$\langle M_{2,2} \rangle_{E,2} \lesssim_\varepsilon \langle f \rangle_E^{1/2} \sum_{N \le q \le \lambda} q^{\frac{1-d}{2}+\varepsilon} \rho(q,\lambda) \lesssim_\varepsilon \langle f \rangle_E^{1/2} N^{\varepsilon + \frac{3-d}{2}} \sigma_{-1/2}(\lambda^2).$$

The last estimate uses (2.7).

Turn to the 'small q' term. This requires additional contributions to the M_1 and M_2 terms. Write $c_{\lambda,q} = c^1_{\lambda,q} + c^2_{\lambda,q}$, where

$$c^1_{\lambda,q}(\xi) = \sum_{\ell \in \mathbb{Z}_q^d} K(\lambda, q, \ell) \Phi_{\lambda q/N}(\xi - \ell/q) \widetilde{d\sigma_\lambda}(\xi - \ell/q).$$

We have inserted an additional cutoff term $\Phi_{\lambda q/N}$ above. Then, our third contribution to M_2 is the high frequency term $M_{2,3} = \sum_{q \le N} C^2_{\lambda,q} f$. Using the stationary decay estimate (2.3) and the Kloosterman refinement (2.5) to see that

$$\begin{aligned} \langle M_{2,3} \rangle_{E,2} &\lesssim_\varepsilon \langle f \rangle_E^{1/2} \sum_{q \le N} (q/N)^{\frac{d-1}{2}} q^{\varepsilon + \frac{1-d}{2}} \rho(\lambda^2, q) \\ &\lesssim_\varepsilon \langle f \rangle_E^{1/2} N^{\frac{1-d}{2}} \sum_{q \le N} q^\varepsilon \rho(\lambda^2, q) \lesssim_\varepsilon \langle f \rangle_E^{1/2} N^{\varepsilon + \frac{3-d}{2}} \sigma_{-1/2}(\lambda^2). \end{aligned}$$

The last estimate follows from (2.8).

Then the main point is the last contributions to M_1 below. The definition of $M_{1,2}$ is of the form to which (2.11) applies.

$$M_{1,2}(n) \le \sum_{q \le N} q \cdot \check{\Phi}_{\lambda q/N} * d\sigma_\lambda * f(n) \lesssim N \langle f \rangle_E \sum_{q \le N} 1 \lesssim N^2 \langle f \rangle_E.$$

Observe that $\Phi_{\lambda q/N} * d\sigma_\lambda * f$ is an average of f over an annulus of radius λ, and width $\lambda q/N$. This is compared to $\langle f \rangle_E$, with loss of N/q. Our proof of (3.1) and (3.2) is complete.

4. Complements to the main theorems

Concerning sharpness of the ℓ^p improving estimates in Theorem 1.1, the best counterexample we have been able to find shows that if one has the inequality below,

$$\|A_\lambda f\|_{p'} \lesssim \lambda^{d(1-\frac{2}{p})} \|f\|_p,$$

valid for all λ, then necessarily $p \geq \frac{d+2}{d}$. provided $d \geq 5$.

Indeed, take λ^2 to be odd, and let f be the indicator of the sphere of radius λ. Use the fact that $A_\lambda f(0) \simeq 1$.

But, in the case of $d \geq 5$, also take g to be the indicator of the set $G_\lambda = \{A_\lambda f > c/\lambda\}$, for appropriate choice of constant c. That is, G_λ is the set of x's for which $\mathbb{S}_\lambda \cap x + \mathbb{S}_\lambda$ has about the expected cardinality of λ^{d-3}.

We claim that $|G_\lambda| \gtrsim \lambda$. For an choice of $0 < x_1 < \lambda/2$ divisible by 4, note that there are about λ^{d-3} points $(x_2, \dots, x_d) \in \mathbb{Z}^{d-1}$ of magnitude $\sqrt{\lambda^2 - (x_1/2)^2}$. From this, we see that

$$\big\|(x_1, 0, \dots, 0) - (x_1/2, y_2, \dots, y_d)\big\| = \lambda,$$

that is, $(x_1, 0, \dots, 0) \in G_\lambda$.

We also have an upper bound for G. Apply the ℓ^p improving inequality (1.2) exp to $f = \mathbf{1}_{S_\lambda}$ to see that for $0 < \varepsilon < 1$,

$$|G| = \big|\{A_\lambda \mathbf{1}_{\mathbb{S}_\lambda} > c/\lambda\}\big| \lesssim \lambda^{\frac{d+3}{2}+\varepsilon}, \quad \lambda^2 \in \mathbb{N}.$$

Is this estimate sharp? Notice that this estimate concerns the set of solutions n to a *pair of* quadratic equations below in which $x = (x_1, \dots, x_d)$ is fixed:

$$n_1^2 + \cdots + n_d^2 = \lambda^2, \quad (n_1 - x_1)^2 + \cdots + (n_d - x_d)^2 = \lambda^2.$$

Moreover, we require of x that the set of possible solutions n should be of about the expected cardinality. We could not find this estimate in the literature.

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