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SIGNED TOPOLOGICAL MEASURES ON LOCALLY COMPACT SPACES

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Abstract. In this paper we define and study signed deficient topological measures and signed topological measures (which generalize signed measures) on locally compact spaces. We prove that a signed deficient topological measure is τ-smooth on open sets and τ-smooth on compact sets. We show that the family of signed measures that are differences of two Radon measures is properly contained in the family of signed topological measures, which in turn is properly contained in the family of signed deficient topological measures. Extending known results for compact spaces, we prove that a signed topological measure is the difference of its positive and negative variations if at least one variation is finite; we also show that the total variation is the sum of the positive and negative variations. If the space is locally compact, connected, locally connected, and has the Alexandrov one-point compactification of genus 0, a signed topological measure of finite norm can be represented as a difference of two topological measures.

1. Introduction

The study of topological measures (initially called quasi-measures) began with papers by J. F. Aarnes [1], [2], and [3]. Nowadays there are several papers devoted to topological measures and corresponding non-linear functionals; their application to symplectic topology has been studied in numerous papers (beginning with [9]) and the monograph ([15]). The natural generalizations of topological measures are signed topological measures and deficient topological measures. Signed topological measures of finite norm on a compact space were introduced in [10] then studied and used in various works, including [11], [13], [16], and [18]. Deficient topological measures (as real-valued functions on a compact space) were first defined and used by A. Rustad and Ø. Johansen in [13] and later independently rediscovered and further developed by M. Svistula in [16] and [17]. In this paper we define

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and study signed deficient topological measures and signed topological measures on locally compact spaces. These set functions may assume ∞ or $-\infty$. We prove that a signed deficient topological measure is τ-smooth on open sets and τ-smooth on compact sets. We show that the family of signed measures that are differences of two Radon measures is properly contained in the family of signed topological measures, which in turn is properly contained in the family of signed deficient topological measures. Extending known results for compact spaces, we prove that a signed topological measure is the difference of its positive and negative variations if at least one variation is finite. We also show that the total variation is the sum of the positive and negative variations. If the space is locally compact, connected, locally connected, and has the Alexandrov one-point compactification of genus 0, we prove that a signed topological measure of finite norm can be represented as the difference of two topological measures. This representation is not unique.

In this paper X will be a locally compact space. By $\mathscr{O}(X)$ we denote the collection of open subsets of X; by $\mathscr{C}(X)$ the collection of closed subsets of X; by $\mathscr{K}(X)$ the collection of compact subsets of X. We denote by $\overline{E}$ the closure of a set E, and $\sqcup$ stands for a union of disjoint sets. We say that a signed set function is real-valued if its values are real numbers. When we consider set functions into extended real numbers they are not identically ∞ or $-\infty$.

Definition 1. Let X be a topological space and μ be a set function on $\mathcal{E}$, a family of subsets of X that contains $\mathscr{O}(X) \cup \mathscr{C}(X)$. We say that

- μ is compact-finite if $|\mu(K)| < \infty$ for any $K \in \mathscr{K}(X)$.
- μ is τ-smooth on compact sets if $K_\alpha \searrow K$, $K_\alpha, K \in \mathscr{K}(X)$ implies $\mu(K_\alpha) \to \mu(K)$.
- μ is τ-smooth on open sets if $U_\alpha \nearrow U$, $U_\alpha, U \in \mathscr{O}(X)$ implies $\mu(U_\alpha) \to \mu(U)$.
- μ is simple if it only assumes values 0 and 1.

We recall the following easy lemma which can be found, for example, in [12, Chapter X, par. 50, Theorem A].

Lemma 2. *If $C \subseteq U \cup V$, where C is compact, U, V are open, then there exist compact sets K and D such that $C = K \cup D$, $K \subseteq U$, $D \subseteq V$.*

Recall the following fact (see, for example, [7, Chapter XI, 6.2]):

Lemma 3. *Let $K \subseteq U$, $K \in \mathscr{K}(X)$, $U \in \mathscr{O}(X)$ in a locally compact space X. Then there exists a set $V \in \mathscr{O}(X)$ with compact closure such that $K \subseteq V \subseteq \overline{V} \subseteq U$.*

Remark 4. Here is an observation which follows, for example, from [8, Corollary 3.1.5].

(i) If $K_\alpha \searrow K$, $K \subseteq U$, where $U \in \mathscr{O}(X)$, $K, K_\alpha \in \mathscr{C}(X)$, and K and at least one of K_α are compact, then there exists α_0 such that $K_\alpha \subseteq U$ for all $\alpha \geq \alpha_0$.

(ii) If $U_\alpha \nearrow U$, $K \subseteq U$, where $K \in \mathscr{K}(X)$, $U, U_\alpha \in \mathscr{O}(X)$ then there exists α_0 such that $K \subseteq U_\alpha$ for all $\alpha \geq \alpha_0$.

Definition 5. A deficient topological measure on a locally compact space X is a set function $\nu \colon \mathscr{C}(X) \cup \mathscr{O}(X) \to [0, \infty]$ which is finitely additive on compact sets, inner compact regular, and outer regular, i.e.

(DTM1) if $C \cap K = \emptyset$, $C, K \in \mathscr{K}(X)$ then $\nu(C \sqcup K) = \nu(C) + \nu(K)$;

(DTM2) $\nu(U) = \sup\{\nu(C) : C \subseteq U,\, C \in \mathscr{K}(X)\}$ for $U \in \mathscr{O}(X)$;

(DTM3) $\nu(F) = \inf\{\nu(U) : F \subseteq U,\, U \in \mathscr{O}(X)\}$ for $F \in \mathscr{C}(X)$.

We denote by $DTM(X)$ the collection of all deficient topological measures on X. We say that a deficient topological measure ν is finite if $\nu(X) < \infty$.

Obviously, for a closed set F, $\nu(F) = \infty$ iff $\nu(U) = \infty$ for every open set U containing F.

Definition 6. A topological measure on X is a set function $\mu \colon \mathscr{C}(X) \cup \mathscr{O}(X) \to [0, \infty]$ satisfying the following conditions:

(TM1) if $A, B, A \sqcup B \in \mathscr{K}(X) \cup \mathscr{O}(X)$ then $\mu(A \sqcup B) = \mu(A) + \mu(B)$;

(TM2) $\mu(U) = \sup\{\mu(K) : K \in \mathscr{K}(X),\, K \subseteq U\}$ for $U \in \mathscr{O}(X)$;

(TM3) $\mu(F) = \inf\{\mu(U) : U \in \mathscr{O}(X),\, F \subseteq U\}$ for $F \in \mathscr{C}(X)$.

We denote by $TM(X)$ the collection of all topological measures on X.

Definition 7 [6, Section 2]. Given signed set function $\lambda \colon \mathscr{K}(X) \to [-\infty, \infty]$ which assumes at most one of ∞, $-\infty$ we define two set functions on $\mathscr{O}(X) \cup \mathscr{C}(X)$, the positive variation λ^+ and the total variation $|\lambda|$, as follows: for an open subset $U \subseteq X$ let

$$\lambda^+(U) = \sup\bigl\{\lambda(K) : K \subseteq U,\ K \in \mathscr{K}(X)\bigr\}; \tag{1}$$

$$|\lambda|(U) = \sup\left\{\sum_{i=1}^n |\lambda(K_i)| : \bigsqcup_{i=1}^n K_i \subseteq U,\ K_i \subseteq \mathscr{K}(X),\ n \in \mathbb{N}\right\}; \tag{2}$$

and for a closed subset $F \subseteq X$ let

$$\lambda^+(F) = \inf\bigl\{\lambda^+(U) : F \subseteq U,\ U \in \mathscr{O}(X)\bigr\}. \tag{3}$$

$$|\lambda|(F) = \inf\bigl\{|\lambda|(U) : F \subseteq U,\ U \in \mathscr{O}(X)\bigr\}. \tag{4}$$

We define the negative variation λ^- associated with a signed set function λ as a set function $\lambda^- = (-\lambda)^+$.

One may consult [6] for more properties of deficient topological measures on locally compact spaces, including monotonicity and superadditivity, as well as more information about λ^+, λ^- and $|\lambda|$.

2. Signed deficient topological measures

Definition 8. A signed deficient topological measure on a locally compact space X is a set function $\nu\colon \mathscr{C}(X) \cup \mathscr{O}(X) \to [-\infty, \infty]$ that assumes at most one of ∞, $-\infty$ and that is finitely additive on compact sets, inner compact regular on open sets, and outer regular on closed sets, i.e.

(SDTM1) If $C \cap K = \emptyset$, $C, K \in \mathscr{K}(X)$ then $\nu(C \sqcup K) = \nu(C) + \nu(K)$;

(SDTM2) $\nu(U) = \lim\{\nu(K) : K \in \mathscr{K}(X),\ K \subseteq U\}$ for $U \in \mathscr{O}(X)$;

(SDTM3) $\nu(F) = \lim\{\nu(U) : U \in \mathscr{O}(X),\ F \subseteq U\}$ for $F \in \mathscr{C}(X)$.

By $SDTM(X)$ we denote the collection of all signed deficient topological measures on X.

Remark 9. In condition (SDTM2) we mean the limit of the net $\nu(C)$ with the index set $\{C \in \mathscr{K}(X) : C \subseteq U\}$ ordered by inclusion. The limit exists and is equal to $\nu(U)$. Condition (SDTM3) is interpreted in a similar way, with the index set being $\{U \in \mathscr{O}(X) : U \supseteq C\}$ ordered by reverse inclusion.

Remark 10. Since we consider set-functions that are not identically ∞ or $-\infty$, we see that for a signed deficient topological measure $\nu(\emptyset) = 0$. If ν and μ are signed deficient topological measures that agree on $\mathscr{K}(X)$, then $\nu = \mu$; if $\nu \le \mu$ on $\mathscr{K}(X)$, then $\nu \le \mu$.

Remark 11. Any deficient topological measure is a signed deficient topological measure.

Lemma 12. *Let $\mu\colon \mathscr{K}(X) \cup \mathscr{O}(X) \to [-\infty, \infty]$ be a set function that assumes at most one of ∞, $-\infty$ and such that*

(a1) $\mu(U) = \lim\{\mu(K) : K \in \mathscr{K}(X),\ K \subseteq U\}$ *for* $U \in \mathscr{O}(X)$;

(a2) $\mu(K) = \lim\{\mu(U) : U \in \mathscr{O}(X),\ K \subseteq U\}$ *for* $K \in \mathscr{K}(X)$.

Then μ is finitely additive on compact sets iff it is finitely additive on open sets. In particular, this holds for a signed deficient topological measure.

Proof. Without loss of generality, assume that μ does not assume $-\infty$. Suppose μ is finitely additive on compact sets. Let $U_1 \sqcup U_2 = U$, where $U_1, U_2, U \in \mathscr{O}(X)$.

First, we shall show that if at least one of $\mu(U_1)$, $\mu(U_2)$ is ∞, then also $\mu(U) = \infty$; in this case the finite additivity on open sets trivially holds. So let $\mu(U_1) = \infty$. Suppose to the contrary that $\mu(U) < \infty$. For $\varepsilon = 1$ let $C \in \mathscr{K}(X)$ be such that $C \subseteq U$ and $|\mu(U) - \mu(K)| < 1$ for any compact set K satisfying $C \subseteq K \subseteq U$. Choose an $n \in \mathbb{N}$ such that $n > |\mu(U)|$ if

$\mu(U_2) = \infty$, or n such that $n + \mu(U_2) - 1 > \mu(U) + 1$ if $\mu(U_2) \in \mathbb{R}$. Pick a compact $C_1 \subseteq U_1$ such that $\mu(C_1) > n$. Pick a compact $C_2 \subseteq U_2$ such that $\mu(C_2) > n$ if $\mu(U_2) = \infty$, and $|\mu(C_2) - \mu(U_2)| < 1$ if $\mu(U_2) < \infty$. We may assume by Lemma 2 that $C \subseteq C_1 \sqcup C_2 \subseteq U$, so $\mu(C_1 \sqcup C_2) \leq \mu(U) + 1$. But also $\mu(C_1 \sqcup C_2) = \mu(C_1) + \mu(C_2) > \mu(U) + 1$, whether $\mu(U_2) = \infty$ or not. The contradiction shows that we must have $\mu(U) = \infty$.

Now we shall show that if $\mu(U_1), \mu(U_2) < \infty$ then also $\mu(U) < \infty$. Suppose to the contrary that $\mu(U) = \infty$. Pick a natural number $n > \mu(U_1) + \mu(U_2) + 2$. Choose compact $C \subseteq U$ such that $\mu(K) > n$ whenever $K \in \mathscr{K}(X)$, $C \subseteq K \subseteq U$. Also for $i = 1, 2$ pick compact sets $C_i \subseteq U_i$ such that $|\mu(U_i) - \mu(C_i)| < 1$. We may assume that $C \subseteq C_1 \sqcup C_2$, so $\mu(C_1 \sqcup C_2) > n$. But also

$$\mu(C_1 \sqcup C_2) = \mu(C_1) + \mu(C_2) \leq \mu(U_1) + \mu(U_2) + 2 < n.$$

Thus, we must have $\mu(U) < \infty$.

We are left to show that $\mu(U_1) + \mu(U_2) = \mu(U)$ when $\mu(U_1), \mu(U_2), \mu(U) < \infty$. Given $\varepsilon > 0$, we may choose compact sets K_1, K_2, K such that $K \subseteq U$, $K_i \subseteq U_i$, $K = K_1 \sqcup K_2$ and $|\mu(U) - \mu(K)| < \varepsilon$, $|\mu(U_i) - \mu(K_i)| < \varepsilon$ for $i = 1, 2$. Then

$$\begin{aligned}\mu(U_1) + \mu(U_2) &\leq \mu(K_1) + \mu(K_2) + 2\varepsilon = \mu(K) + 2\varepsilon \leq \mu(U) + 3\varepsilon \\ &\leq \mu(K) + 4\varepsilon = \mu(K_1) + \mu(K_2) + 4\varepsilon \leq \mu(U_1) + \mu(U_2) + 4\varepsilon.\end{aligned}$$

Finite additivity on open sets follows.

The fact that finite additivity on open sets implies finite additivity on compact sets can be proved in a similar way. □

Definition 13. For a signed deficient topological measure ν we define $\|\nu\| = \sup\{|\nu(K)| : K \in \mathscr{K}(X)\}$. We denote by $\mathbf{SDTM}(X)$ the collection of all signed deficient topological measures on X for which $\|\nu\| < \infty$.

Remark 14. Note that

$$\|\nu\| = \sup\big\{|\nu(A)| : A \in \mathscr{O}(X) \cup \mathscr{K}(X)\big\} = \sup\big\{|\nu(U)| : U \in \mathscr{O}(X)\big\}.$$

The collection of all real-valued signed deficient topological measures is a linear space. Any $\nu \in \mathbf{SDTM}(X)$ is real-valued, and $\|\nu\|$ is a norm on a linear space $\mathbf{SDTM}(X)$.

Studying a signed deficient topological measure it is beneficial to consider its positive, negative and total variation, defined in Definition 7.

Remark 15. Let ν be a signed deficient topological measure on a locally compact space X. By [6, Proposition 21 in Section 3] the set functions ν^+, ν^-, $|\nu|$ defined in Definition 7 are deficient topological measures, and $|\nu| \leq \nu^+ + \nu^-$. Also, ν^+ is the unique smallest deficient topological measure such that $\nu^+ \geq \nu$ and ν^- is the unique largest deficient topological measure such that $-\nu^- \leq \nu$. Note that $\|\nu\| < \infty$ if and only if ν^+ and ν^- are finite, i.e. $\nu^+(X), \nu^-(X) < \infty$. To define ν^+, ν^- and $|\nu|$ on $\mathscr{O}(X)$ one may use instead of compact sets open sets or sets from $\mathscr{O}(X) \cup \mathscr{K}(X)$.

Remark 16. Let X be locally compact and let ν be a signed deficient topological measure on X. From [6, Lemma 10 in Section 2] we have the following: (a) $|\nu(A)| \leq |\nu|(A)$ for any $A \in \mathscr{O}(X) \cup \mathscr{C}(X)$; (b) superadditivity: if $\bigsqcup_{t\in T} A_t \subseteq A$, where $A_t, A \in \mathscr{O}(X) \cup \mathscr{C}(X)$, and at most one of the closed sets is not compact, then $\sum_{t\in T} |\nu(A_t)| \leq \sum_{t\in T} |\nu|(A_t) \leq |\nu|(A)$.

Lemma 17. *Let X be locally compact. The following holds for a signed deficient topological measure ν:*

(d1) *Given $U \in \mathscr{O}(X)$ with $|\nu|(U) < \infty$ and $\varepsilon > 0$, there exists $C \subseteq U$, $C \in \mathscr{K}(X)$ such that $|\nu(A)| \leq |\nu|(A) < \varepsilon$ for any compact or open $A \subseteq U \setminus C$.*

(d2) *Given $C \in \mathscr{C}(X)$ with $|\nu|(C) < \infty$ and $\varepsilon > 0$, there exists $U \in \mathscr{O}(X)$, $C \subseteq U$ such that $|\nu(A)| \leq |\nu|(A) < \varepsilon$ for any compact or open $A \subseteq U \setminus C$.*

(d3) *Given $U \in \mathscr{O}(X)$ with $|\nu|(U) < \infty$ and $\varepsilon > 0$, there exists $C \subseteq U$, $C \in \mathscr{K}(X)$ such that for any sets $A, B \in \mathscr{O}(X) \cup \mathscr{K}(X)$, $C \subseteq A \subseteq B \subseteq U$ we have $|\nu(A) - \nu(B)| < \varepsilon$.*

Proof. Let $\varepsilon > 0$. In part (d1) given $U \in \mathscr{O}(X)$ choose $C \in \mathscr{K}(X)$, $C \subseteq U$ and in part (d2) given $C \in \mathscr{C}(X)$ choose $U \in \mathscr{O}(X)$, $C \subseteq U$ such that $|\nu|(U) - |\nu|(C) < \varepsilon$. Then by monotonicity and superadditivity of $|\nu|$, we have

$$|\nu(A)| \leq |\nu|(A) \leq |\nu|(U \setminus C) \leq |\nu|(U) - |\nu|(C) < \varepsilon.$$

Now we shall show part (d3). Since $|\nu|(U) < \infty$, we have $|\nu|(A) < \infty$ so $\nu(A) \in \mathbb{R}$ for any $A \subseteq U$, $A \in \mathscr{K}(X) \cup \mathscr{O}(X)$. For $\varepsilon > 0$ we may find $C \in \mathscr{K}(X)$, $C \subseteq U$ such that $|\nu(K) - \nu(U)| < \varepsilon/4$ whenever $K \in \mathscr{K}(X)$, $C \subseteq K \subseteq U$. If $A, B \in \mathscr{K}(X)$ then $|\nu(A) - \nu(B)| \leq |\nu(A) - \nu(U)| + |\nu(U) - \nu(B)| < \varepsilon/2 < \varepsilon$. If $A, B \in \mathscr{O}(X)$ then find compact sets K, D such that $C \subseteq K \subseteq A$, $C \subseteq D \subseteq B$ and $|\nu(A) - \nu(K)| < \varepsilon/4$, $|\nu(B) - \nu(D)| < \varepsilon/4$. Then we have $|\nu(A) - \nu(B)| \leq |\nu(A) - \nu(K)| + |\nu(K) - \nu(D)| + |\nu(D) - \nu(B)| < 3\varepsilon/4 < \varepsilon$. The remaining two cases can be proved similarly. □

Lemma 18. *Let X be locally compact. Suppose ν is a signed deficient topological measure on X. For each open set U define $\widehat{\nu}(U) = \sup\{|\nu(C)| : C \subseteq U, C \in \mathscr{K}(X)\}$. Then we have*

$$\widehat{\nu}(U) = \sup\{|\nu(V)| : V \subseteq U,\ V \in \mathscr{O}(X)\}$$
$$= \sup\{|\nu(A)| : A \subseteq U,\ A \in \mathscr{O}(X) \cup \mathscr{K}(X)\},$$

and $\widehat{\nu}(U) \leq |\nu|(U) \leq 2\widehat{\nu}(U)$.

PROOF. From the definition of a signed deficient topological measure we see that $\widehat{\nu}(U) = \sup\{|\nu(V)| : V \subseteq U,\ V \in \mathscr{O}(X)\} = \sup\{|\nu(A)| : A \subseteq U,\ A \in \mathscr{O}(X) \cup \mathscr{K}(X)\}$. From Remark 16, $\widehat{\nu}(U) \le |\nu|(U)$. For a finite disjoint collection $\{K_i : K_i \in \mathscr{K}(X),\ i \in I\}$ let $I^+ = \{i \in I : \nu(K_i) \ge 0\}$, $I^- = \{i \in I : \nu(K_i) < 0\}$, $K^+ = \bigsqcup_{i \in I^+} K_i$, and $K^- = \bigsqcup_{i \in I^-} K_i$. If $K_i \subseteq U$ for $i \in I$ then

$$\sum_{i \in I} |\nu(K_i)| = \sum_{i \in I^+} \nu(K_i) + \sum_{i \in I^-} -\nu(K_i) = \nu(K^+) - \nu(K^-)$$
$$= |\nu(K^+)| + |\nu(K^-)| \le 2\widehat{\nu}(U).$$

Thus, $|\nu|(U) \le 2\widehat{\nu}(U)$. □

THEOREM 19. *The space* $\mathbf{SDTM}(X)$ *is a normed linear space under either of the two equivalent norms*: $\|\nu\|_1 = \sup\{|\nu(K)| : K \in \mathscr{K}(X)\}$, $\|\nu\|_2 = \sup\{|\nu|(K) : K \in \mathscr{K}(X)\} = |\nu|(X)$.

PROOF. It is easy to see that $\sup\{|\nu|(K) : K \in \mathscr{K}(X)\} = |\nu|(X)$, and that it is a norm. From Lemma 18 we see that $\|\nu\|_1 \le \|\nu\|_2 \le 2\|\nu\|_1$, so these two norms are equivalent. □

THEOREM 20. *Suppose* ν *is a signed deficient topological measure such that at most one of* $\nu^+(X)$, $\nu^-(X)$ *is infinity, or* ν *is real-valued. Then* ν *is* τ*-smooth on open sets, and also* τ*-smooth on compact sets.*

PROOF. Suppose ν is a signed deficient topological measure such that at most one of $\nu^+(X)$, $\nu^-(X)$ is infinity. (The case where ν is real-valued is similar but simpler.) Without loss of generality, let $\nu^-(X) \le M$, where $M \in \mathbb{N}$. First we shall show that ν is τ-smooth on open sets. Supose $U_s \nearrow U$, $U_s, U \in \mathscr{O}(X)$, $s \in S$.

(i) Assume first that $\nu(U) < \infty$. Let $\varepsilon > 0$. There exists $K \in \mathscr{K}(X)$, $K \subseteq U$ such that $|\nu(D) - \nu(U)| < \varepsilon$ whenever $D \in \mathscr{K}(X)$, $K \subseteq D \subseteq U$. By Remark 4 let $t \in S$ be such that $K \subseteq U_s$ for any $s \ge t$. We claim that $\nu(U_s) \in \mathbb{R}$ for each $s \ge t$. (Indeed, let $s \ge t$, ε as above. By assumption, we can not have $\nu(U_s) = -\infty$. Suppose that $\nu(U_s) = \infty$. For $n > |\nu(U)| + \varepsilon$ pick compact $K_s \subseteq U_s$ such that $\nu(C) > n$ for any $C \in \mathscr{K}(X)$ satisfying $K_s \subseteq C \subseteq U_s$. Then for the compact $D = K \cup K_s \subseteq U$ we have $|\nu(D) - \nu(U)| < \varepsilon$ and $\nu(D) > n > |\nu(U)| + \varepsilon \ge \nu(U) + \varepsilon$, which gives a contradiction).

For each $s \ge t$, pick $C_s \subseteq U_s$, $C_s \in \mathscr{K}(X)$ such that $|\nu(U_s) - \nu(C)| < \varepsilon$ for any compact C satisfying $C_s \subseteq C \subseteq U_s$. Then

$$|\nu(U) - \nu(U_s)| \le |\nu(U) - \nu(K \cup C_s)| + |\nu(K \cup C_s)) - \nu(U_s)| < 2\varepsilon,$$

and $\nu(U_s) \to \nu(U)$.

(ii) Now assume that $\nu(U) = \infty$. For $n > M$ pick $K \in \mathscr{K}(X)$ such that $K \subseteq U$ and $\nu(D) > 2n$ whenever $D \in \mathscr{K}(X)$, $K \subseteq D \subseteq U$. By Remark 4 let $t \in S$ be such that $K \subseteq U_s$ for any $s \ge t$. Suppose $\nu(U_s) < \infty$. For

$0 < \varepsilon < n$, there exists $D_s \in \mathscr{K}(X)$, $D_s \subseteq U_s$ such that $|\nu(U_s) - \nu(D)| < \varepsilon$ for any $D \in \mathscr{K}(X)$, $D_s \subseteq D \subseteq U_s$. Then

$$|\nu(U_s)| \geq ||\nu(U_s) - \nu(D_s \cup K)| - | - \nu(D_s \cup K)|| \geq 2n - \varepsilon > n.$$

It follows that for any $s \geq t$, whether $\nu(U_s) < \infty$ or $\nu(U_s) = \infty$, we have $|\nu(U_s)| > n > M \geq \nu^-(X)$, so $\nu(U_s)$ can not be negative. Thus, for $s \geq t$ we have $\nu(U_s) = |\nu(U_s)| > n$, so $\nu(U_s) \to \infty$.

Thus, ν is τ-smooth on open sets. We may show that ν is τ-smooth on compact sets in a similar fashion. □

3. Signed topological measures on a locally compact space

Definition 21. A signed topological measure on a locally compact space X is a set function $\mu\colon \mathscr{O}(X) \cup \mathscr{C}(X) \to [-\infty, \infty]$ that assumes at most one of ∞, $-\infty$ and satisfies the following conditions:

(STM1) if $A, B, A \sqcup B \in \mathscr{K}(X) \cup \mathscr{O}(X)$ then $\mu(A \sqcup B) = \mu(A) + \mu(B)$;

(STM2) $\mu(U) = \lim\{\mu(K) : K \in \mathscr{K}(X),\ K \subseteq U\}$ for $U \in \mathscr{O}(X)$;

(STM3) $\mu(F) = \lim\{\mu(U) : U \in \mathscr{O}(X),\ F \subseteq U\}$ for $F \in \mathscr{C}(X)$.

By $STM(X)$ we denote the collection of all signed topological measures on X.

Lemma 22. *Suppose $\mu\colon \mathscr{O}(X) \cup \mathscr{C}(X) \to [-\infty, \infty]$ is a set function that assumes at most one of $\infty, -\infty$ and that satisfies the following conditions*:

(i) $\mu(U) = \lim\{\nu(K) : K \subseteq U,\ K \in \mathscr{K}(X)\}$ *for* $U \in \mathscr{O}(X)$;

(ii) $\mu(F) = \lim\{\mu(U) : F \subseteq U,\ U \in \mathscr{O}(X)\}$ *for* $F \in \mathscr{C}(X)$;

(iii) μ *is finitely additive on* $\mathscr{O}(X)$ *or* ν *is finitely additive on* $\mathscr{K}(X)$;

(iv) *if* $K \sqcup V = W$, $K \in \mathscr{K}(X)$, $V, W \in \mathscr{O}(X)$ *then* $\mu(K) + \mu(V) = \mu(W)$;

(v) *for each* $K \in \mathscr{K}(X)$ *there exists an open neighborhood* W *of* K *such that* $\mu(W \setminus K) \in \mathbb{R}$.

Then μ is a signed topological measure on X. In particular, any real-valued set function μ on $\mathscr{O}(X) \cup \mathscr{C}(X)$ that satisfies (i)–(iv) is a real-valued signed topological measure.

Proof. We need to check the condition (STM1) of Definition 21. By Lemma 12 μ is finitely additive on $\mathscr{O}(X)$ and on $\mathscr{K}(X)$, so we only need to show that if $K \sqcup V = C$ where $K, C \in \mathscr{K}(X)$, $V \in \mathscr{O}(X)$ then $\mu(K) + \mu(V) = \mu(C)$. Let $W \in \mathscr{O}(X)$ be such that $C \subseteq W$ and $\mu(W \setminus C) \in \mathbb{R}$. Then $K \sqcup V = C \subseteq W$, so $K \sqcup V \sqcup (W \setminus C) = W = C \sqcup (W \setminus C)$. By part (iv) and finite additivity of μ on open sets

$$\mu(K) + \mu(V) + \mu(W \setminus C) = \mu(W) = \mu(C) + \mu(W \setminus C),$$

so $\mu(K) + \mu(V) = \mu(C)$, and the statement is proved. □

When X is compact and μ is real-valued Definition 21 simplifies to the following:

Definition 23. A signed real-valued topological measure on a compact space X is a set function $\mu\colon \mathscr{O}(X) \cup \mathscr{C}(X) \to (-\infty, \infty)$ that satisfies the following conditions:

(c1) if $A, B, A \sqcup B \in \mathscr{K}(X) \cup \mathscr{O}(X)$ then $\mu(A \sqcup B) = \mu(A) + \mu(B)$;

(c2) $\mu(U) = \lim\{\mu(K) : K \in \mathscr{K}(X),\ K \subseteq U\}$ for $U \in \mathscr{O}(X)$.

Remark 24. Condition (c2) of Definition 23 is equivalent to the following:

$$\mu(F) = \lim\big\{\mu(U) : F \subseteq U,\ U \in \mathscr{O}(X)\big\} \quad \text{for } F \in \mathscr{C}(X).$$

As was noticed in [10], condition (c1) of Definition 23 is equivalent to the following three conditions:

(i) $\mu(U \sqcup V) = \mu(U) + \mu(V)$ for any two disjoint open sets U, V.

(ii) If $X = U \cup V$, $U, V \in \mathscr{O}(X)$ then $\mu(U) + \mu(V) = \mu(X) + \mu(U \cap V)$.

(iii) $\mu(X \setminus U) = \mu(X) - \mu(U)$ for any open set U.

Thus, when X is compact, a real-valued signed topological measure can be defined by its actions on open sets.

Lemma 25. *Let X be locally compact. Let $\mu\colon \mathscr{O}(X) \cup \mathscr{C}(X)$ be a real-valued signed set function such that $\mu(U) = \mu(K) + \mu(U \setminus K)$ for any $U \in \mathscr{O}(X)$ and any compact $K \subseteq U$. Consider the following conditions*:

(p1) *For any open set U, the limit of the net $\{\mu(K)\}$ with index set $\{K \in \mathscr{K}(X) : K \subseteq U\}$ ordered by inclusion exists and*

$$\mu(U) = \lim_{K \subseteq U,\, K \in \mathscr{K}(X)} \mu(K).$$

(p2) *Given $U \in \mathscr{O}(X)$ and $\varepsilon > 0$, there exists $K \in \mathscr{K}(X)$, $K \subseteq U$ such that $|\mu(E)| < \varepsilon$ for any $E \subseteq U \setminus K$, $E \in \mathscr{K}(X) \cup \mathscr{O}(X)$.*

(p3) *Given $U \in \mathscr{O}(X)$ and $\varepsilon > 0$, there exists $K \in \mathscr{K}(X)$, $K \subseteq U$ such that $|\mu(V)| < \varepsilon$ for any $V \subseteq U \setminus K$, $V \in \mathscr{O}(X)$.*

(p4) *Given $U \in \mathscr{O}(X)$ and $\varepsilon > 0$, there exists $K \in \mathscr{K}(X)$, $K \subseteq U$ such that $|\mu(C)| < \varepsilon$ for any $C \subseteq U \setminus K$, $C \in \mathscr{K}(X)$.*

(p5) *For any compact K, the limit of the net $\{\mu(U)\}$ with index set $\{U \in \mathscr{O}(X) : K \subseteq U\}$ ordered by reverse inclusion exists and*

$$\mu(K) = \lim_{U \supset K,\, U \in \mathscr{O}(X)} \mu(U).$$

(p6) *Given $F \in \mathscr{C}(X)$ and $\varepsilon > 0$, there exists $U \supset F$, $U \in \mathscr{O}(X)$ such that $|\mu(C)| < \varepsilon$ for any $C \subseteq U \setminus F$, $C \in \mathscr{K}(X)$.*

(p7) *Given* $F \in \mathscr{C}(X)$ *and* $\varepsilon > 0$, *there exists* $U \supset F$, $U \in \mathscr{O}(X)$ *such that* $|\mu(V)| < \varepsilon$ *for any* $V \subseteq U \setminus F$, $V \in \mathscr{O}(X)$.

(p8) *Given* $F \in \mathscr{C}(X)$ *and* $\varepsilon > 0$, *there exists* $U \supset F$, $U \in \mathscr{O}(X)$ *such that* $|\mu(E)| < \varepsilon$ *for any* $E \subseteq U \setminus F$, $E \in \mathscr{K}(X) \cup \mathscr{O}(X)$.

(p9) *For any closed set* F, *the limit of the net* $\{\mu(U)\}$ *with index set* $\{U \in \mathscr{O}(X) : F \subseteq U\}$ *ordered by reverse inclusion exists and*

$$\mu(F) = \lim_{U \supset F, U \in \mathscr{O}(X)} \mu(U).$$

Conditions (p1), (p2), *and* (p3) *are equivalent and imply* (p4) *and* (p5). *If* X *is compact, then* (p1), (p2), (p3), (p5), (p7), (p8), (p9) *are equivalent and imply equivalent conditions* (p4), (p6).

Proof. (p1) $\Rightarrow$ (p2): Let $U \in \mathscr{O}(X)$, $\varepsilon > 0$. By (p1) pick $K \in \mathscr{K}(X)$ such that $|\mu(U \setminus C)| = |\mu(U) - \mu(C)| < \varepsilon$ for any $C \in \mathscr{K}(X)$ satisfying $K \subseteq C \subseteq U$. For any compact $D \subseteq U \setminus K$ we have

$$\mu(D) = \mu(U \setminus K) - \mu((U \setminus K) \setminus D)$$

and so $|\mu(D)| < 2\varepsilon$. For any open set $V \subseteq U \setminus K$ we then see that $|\mu(V)| \leq 2\varepsilon$, and so (p2) follows.

Obviously, (p2) $\Rightarrow$ (p3). (p3) $\Rightarrow$ (p1): For $U \in \mathscr{O}(X)$ and $\varepsilon > 0$ pick $K \in \mathscr{K}(X)$ such that $|\mu(V)| < \varepsilon$ for any open set $V \subseteq U \setminus K$. Then for any compact C such that $K \subseteq C \subseteq U$ we have $U \setminus C \subseteq U \setminus K$, so $|\mu(U) - \mu(C)| = |\mu(U \setminus C)| < \varepsilon$, and (p1) follows. Thus (p1), (p2), and (p3) are equivalent.

Obviously, (p2) $\Rightarrow$(p4). (p3) $\Rightarrow$ (p5): Let $K \in \mathscr{K}(X), \varepsilon > 0$. By (p3) for the set $U = X \setminus K$ find compact $C \subseteq U$ such that $|\mu(V)| < \varepsilon$ for any open set $V \subseteq U \setminus C$. Setting $W_\varepsilon = X \setminus C$ we see that $K \subseteq W_\varepsilon$ and for any open W such that $K \subseteq W \subseteq W_\varepsilon$ we have

$$W \setminus K \subseteq W_\varepsilon \setminus K = (X \setminus C) \setminus K = (X \setminus K) \setminus C = U \setminus C,$$

so $|\mu(W) - \mu(K)| = |\mu(W \setminus K)| < \varepsilon$, which shows (p5).

Suppose X is compact. (p1), (p2), and (p3) are equivalent and imply (p5), which is equivalent to (p9). From the duality between open and closed sets it is easy to see that (p9) $\Rightarrow$ (p1), (p2) $\Leftrightarrow$ (p8), (p3) $\Leftrightarrow$ (p7), and (p4) $\Leftrightarrow$ (p6). This finishes the proof. $\square$

Remark 26. Condition (c2) of Definition 23 and condition (STM2) of Definition 21 (for a real-valued signed topological measure) could be replaced by any of the equivalent conditions in Lemma 25.

We denote by $\mathbf{STM}(X)$ the collection of all signed topological measures on X for which $\|\mu\| < \infty$. As in Theorem 19, we have

THEOREM 27. *The space* $\mathbf{STM}(X)$ *is a normed linear space under either of the two equivalent norms*: $\|\mu\|_1 = \sup\{|\mu(A)| : A \in \mathscr{O}(X) \cup \mathscr{C}(X)\}$, $\|\mu\|_2 = \sup\{|\mu|(A) : A \in \mathscr{O}(X) \cup \mathscr{C}(X)\} = |\mu|(X)$.

REMARK 28. Signed topological measures of finite norm on a compact space were introduced in [10] then studied or used in [11], [13], [16], [18]. In these papers different definitions of a signed topological measure were given, but their equivalence, as well as equivalence to our Definition 23, follows from Lemma 25 and Theorem 27.

REMARK 29. Since any signed topological measure is a signed deficient topological measure, we may use the definitions and results from section 2. Note that in general, when μ is a signed topological measure, μ^+, μ^- and $|\mu|$ are deficient topological measures, but not topological measures. See, for instance, [13, Example 25]. It is easy to see that if μ is a signed measure then μ^+, μ^- and $|\mu|$ are the classical positive, negative, and total variations of a signed measure.

LEMMA 30. *Suppose* X *is locally compact,* ν *is a deficient topological measure on* X *which is not a topological measure on* X, μ *is a topological measure on* X, *and* $\|\nu\|, \|\mu\| < \infty$. *Then* $\lambda = \nu - \mu$ *is a signed deficient topological measure on* X *which is not a signed topological measure.*

PROOF. By Remark 14 λ is a signed deficient topological measure on X. Since ν is not a topological measure, by [6, Theorem 29 in Section 4] there exist open U and compact $C \subseteq U$ such that $\nu(U) > \nu(C) + \nu(U \setminus C)$. Note that λ is not a signed topological measure, since otherwise ν is also a signed topological measure, and then $\nu(U) = \nu(C) + \nu(U \setminus C)$, which gives a contradiction. $\square$

The next few results allow us to use a smaller collection than $\mathscr{K}(X)$ to check the equality of two signed topological measures. Let X be a locally compact non-compact space. A set $A \subseteq X$ is called bounded if $\overline{A}$ is compact. A set A is called solid if A is connected, and $X \setminus A$ has only unbounded connected components. Let $\mathscr{K}_0(X)$, $\mathscr{K}_c(X)$, $\mathscr{K}_s(X)$ and $\mathscr{O}_s^*(X)$ denote, respectively, the family of finite unions of disjoint compact connected sets, the family of compact connected sets, the family of compact solid sets, and the family of bounded open solid sets.

When X is compact, a set is called solid if it and its complement are both connected. For a compact space X we define a certain topological characteristic, genus. See [3] for more information about genus g of the space. A compact space has genus 0 iff any finite union of disjoint closed solid sets has a connected complement. Intuitively, X does not have holes or loops. In the case where X is locally path connected, $g = 0$ if the fundamental group $\pi_1(X)$ is finite (in particular, if X is simply connected). Knudsen [14] was able to show that if $H^1(X) = 0$ then $g(X) = 0$, and in the case of CW-complexes the converse also holds.

EXAMPLE 31. Let $X = \mathbb{R}^2$. Consider $\mu = \nu - \frac{1}{2}\delta$, where ν is a simple topological measure which is not a measure on X, and δ is a point mass. (As ν one may take, for instance, a topological measure from Example 1 or Example 2 in [4]). Then μ is a signed topological measure which is not a signed measure: if μ is a signed measure, then ν is also a signed measure, and, in fact, is a measure, which contradicts the choice of ν.

REMARK 32. Let X be locally compact, and let $\mathcal{M}$ be the collection of all Borel measures on X that are inner regular on open sets and outer regular on all Borel sets. Thus, $\mathcal{M}$ includes regular Borel measures and Radon measures. We denote by $M(X)$ the restrictions to $\mathscr{O}(X) \cup \mathscr{C}(X)$ of measures from $\mathcal{M}$. Let $SM(X)$ be the family of signed measures that are differences of two measures from $M(X)$, one of which is finite. We have

$$M(X) \subsetneqq TM(X) \subsetneqq DTM(X) \tag{5}$$

and

$$SM(X) \subsetneqq STM(X) \subsetneqq SDTM(X). \tag{6}$$

The inclusions follow from the definitions. Inclusions in (6) are proper by Lemma 30 and Example 31. When X is compact, there are examples of topological measures that are not measures and of deficient topological measures that are not topological measures in numerous papers, beginning with [1], [13], and [16]. When X is locally compact, see [4], [6, Sections 5,6], and [5, Section 9] for more information on proper inclusion in (5), criteria for a deficient topological measure to be a measure from $M(X)$, and various examples.

REMARK 33. As in [6, Remark 14 in Section 3], we have the following. Let ν be a signed deficient topological measure on X. If X is locally compact and locally connected then for each open set U

$$\nu(U) = \lim \left\{ \nu(K) : K \subseteq U,\ K \in \mathscr{K}_0(X) \right\}.$$

If X is locally compact, connected, and locally connected, then

$$\nu(X) = \lim \left\{ \nu(K) : K \in \mathscr{K}_c(X) \right\},$$

and also

$$\nu(X) = \lim \left\{ \nu(K) : K \in \mathscr{K}_s(X) \right\}.$$

LEMMA 34. *Let ν be a signed topological measure on a locally compact, locally connected space X. If $U \in \mathscr{O}_s^*(X)$ then $\nu(U) = \lim\{\nu(C) : C \subseteq U$, $C \in \mathscr{K}_s(X)\}$.*

PROOF. Follows from [5, Lemma 21 in Section 3]. □

THEOREM 35. *Let X be a locally compact, connected, and locally connected space. If ν and μ are real-valued signed topological measures on X such that $\mu = \nu$ on compact solid sets then $\nu = \mu$.*

PROOF. Suppose that $\nu(K) = \mu(K)$ for every $K \in \mathscr{K}_s(X)$. By Remark 33 we have $\nu(X) = \mu(X)$. By Lemma 34, $\nu(V) = \mu(V)$ for every $V \in \mathscr{O}_s^*(X)$. For a compact connected set C by [5, Lemma 18 in Section 3] we have

$$X = \tilde{C} \sqcup \bigsqcup_{i=1}^{n} U_i = C \sqcup \bigsqcup_{s \in S} V_s \sqcup \bigsqcup_{i=1}^{n} U_i, \tag{7}$$

where U_i are unbounded connected components of $X \setminus C$, V_s are bounded connected components of $X \setminus C$, and $\tilde{C}$ is a solid hull of C. By [5, Lemma 20 in Section 3], $\tilde{C}$ is a compact solid set. Since $\mu = \nu$ on compact solid sets and $\nu(X) = \mu(X)$, from the first equality in (7) by finite additivity of signed topological measures on $\mathscr{O}(X) \cup \mathscr{K}(X)$ we see that $\nu\big(\bigsqcup_{i=1}^n U_i\big) = \mu\big(\bigsqcup_{i=1}^n U_i\big)$. By [5, Lemma 17 in Section 3] each $V_s \in \mathscr{O}_s^*(X)$. By Lemma 12 both μ and ν are additive on open sets, so from (7) we have $\nu(C) = \mu(C)$ for each $C \in \mathscr{K}_c(X)$. Then $\nu = \mu$ on $\mathscr{K}_0(X)$. Then by Remark 33 $\nu = \mu$ on $\mathscr{O}(X)$. Thus, $\nu = \mu$. □

REMARK 36. Theorem 35 suggests the possibility of obtaining a signed topological measure as the unique extension of a suitable set function defined only on bounded solid sets with methods similar to ones in [5] (where on a locally compact, connected, locally connected space a solid-set function is extended uniquely to a topological measure) and in [13, Section 8] (where on a connected, locally connected, compact Hausdorff space a signed solid-set function is extended uniquely to a finite signed topological measure).

4. Decomposition of signed topological measures into deficient topological measures

LEMMA 37. *Let X be locally compact. Let $\lambda\colon \mathscr{K}(X) \cup \mathscr{O}(X) \to [-\infty, \infty]$ be a signed set function that assumes at most one of ∞, $-\infty$. Suppose λ is finitely additive on $\mathscr{K}(X)$ and $\lambda(U) = \lambda(U \setminus K) + \lambda(K)$ for any open U and any compact $K \subseteq U$. Let λ^+, λ^- and $|\lambda|$ be as in Definition 7. Then*

(I) *$\lambda(U) = \lambda^+(U) - \lambda^-(U)$ for any open set U such that at least one of $\lambda^+(U)$, $\lambda^-(U)$ is finite.*

(II) $|\lambda| = \lambda^+ + \lambda^-$.

PROOF. (I) Suppose λ does not assume $-\infty$. Let $U \in \mathscr{O}(X)$. For any compact $K \subseteq U$ we have

$$\lambda(U) = \lambda(U \setminus K) + \lambda(K) \geq -\lambda^-(U \setminus K) + \lambda(K) \geq -\lambda^-(U) + \lambda(K). \tag{8}$$

Assume that $\lambda^-(U) < \infty$. There are two possibilities for $\lambda^+(U)$. Suppose that $\lambda^+(U) < \infty$. Given $\varepsilon > 0$, we choose a compact set $K \subseteq U$ such that $\lambda(K) > \lambda^+(U) - \varepsilon$. Then from (8)

$$\lambda(U) \geq -\lambda^-(U) + \lambda(K) \geq -\lambda^-(U) + \lambda^+(U) - \varepsilon,$$

so $\lambda(U) \geq \lambda^+(U) - \lambda^-(U)$. Since $\lambda^-(U) < \infty$, we may apply the same argument to $-\lambda$ to get $-\lambda(U) \geq (-\lambda)^+(U) - ((-\lambda)^-(U)) = \lambda^-(U) - \lambda^+(U)$, i.e. $\lambda(U) \leq \lambda^+(U) - \lambda^-(U)$. Therefore, $\lambda(U) = \lambda^+(U) - \lambda^-(U)$.

Now suppose $\lambda^+(U) = \infty$. For a natural number n choose $K \subseteq U$ such that $\lambda(K) > n$. Then from (8) we have

$$\lambda(U) \geq -\lambda^-(U) + \lambda(K) > n - \lambda^-(U).$$

Letting $n \to \infty$ we obtain $\lambda(U) = \infty$. We again have $\lambda(U) = \lambda^+(U) - \lambda^-(U)$.

(II) To prove that $|\lambda| = \lambda^+ + \lambda^-$, by [6, Lemma 10 in Section 2] we only need to show that $|\lambda| \geq \lambda^+ + \lambda^-$, and it is enough to check this inequality on open sets. Let U be open. By [6, Lemma 10 in Section 2] the equality is trivial if $\lambda^+(U) = 0$ or $\lambda^-(U) = 0$. So we assume that $\lambda^+(U) > 0$ and $\lambda^-(U) > 0$. If $\lambda^+(U) = \infty$ or $\lambda^-(U) = \infty$ the statement holds because $|\lambda| \geq \lambda^+$, $|\lambda| \geq \lambda^-$. Now assume that $0 < \lambda^+(U), \lambda^-(U) < \infty$. Given $\varepsilon > 0$, choose a compact K such that $\lambda^+(U) - \lambda(K) < \varepsilon$ and $\lambda(K) > 0$. Note that $\lambda(U \setminus K) + \lambda^-(U) = \lambda(U) - \lambda(K) + \lambda^-(U) = \lambda^+(U) - \lambda(K) < \varepsilon$, i.e. $-\lambda(U \setminus K) > \lambda^-(U) - \varepsilon$. By superadditivity of $|\lambda|$ we have

$$\begin{aligned} |\lambda|(U) &\geq |\lambda|(U \setminus K) + |\lambda|(K) \geq |\lambda(U \setminus K)| + |\lambda(K)| = |\lambda(U \setminus K)| + \lambda(K) \\ &\geq -\lambda(U \setminus K) + \lambda(K) \geq \lambda^-(U) - \varepsilon + \lambda^+(U) - \varepsilon, \end{aligned}$$

i.e. $|\lambda|(U) \geq \lambda^+(U) + \lambda^-(U)$. Thus, $|\lambda| \geq \lambda^+ + \lambda^-$. □

Remark 38. Lemma 37 is related to [13, Proposition 24, parts 7,8].

Corollary 39. *Suppose λ is a signed topological measure, $A \in \mathscr{O}(X) \cup \mathscr{C}(X)$.*

(i) $|\lambda(A)| < \infty$ *implies* $\lambda^+(A)$, $\lambda^-(A)$ *are both finite or both infinite.*

(ii) *If λ is such that at least one of $\lambda^+(X)$, $\lambda^-(X)$ is finite, then $|\lambda(A)| < \infty$ if and only if $\lambda^+(A), \lambda^-(A) < \infty$.*

Proof. (i) Let U be open, $|\lambda(U)| < \infty$. If exactly one of $\lambda^+(U)$, $\lambda^-(U)$ were finite, it would contradict part (I) of Lemma 37. So the corollary holds for open sets. Suppose $F \in \mathscr{C}(X)$ and $\lambda(F) \in \mathbb{R}$. There is an open set U containing F for which $\lambda(V) \in \mathbb{R}$ for any $V \in \mathscr{O}(X)$ with $F \subseteq V \subseteq U$. If there is V with $\lambda^+(V), \lambda^-(V) < \infty$ then $\lambda^+(F) \leq \lambda^+(V) < \infty$, and also $\lambda^-(F) < \infty$. If for all V both $\lambda^+(V)$, $\lambda^-(V)$ are infinite, then also $\lambda^+(F), \lambda^-(F)$ are infinite.

(ii) Since at least one of λ^+, λ^- is finite, the direction "$\Rightarrow$" follows from part (i). The direction "$\Leftarrow$" for an open set follows from part (I) of Lemma 37, and then is easily checked for a closed set. $\square$

THEOREM 40. *Suppose λ is a signed topological measure.*

(I) *The positive variation λ^+ is the unique smallest deficient topological measure such that $\lambda^+ \geq \lambda$, and the negative variation λ^- is the unique largest deficient topological measure such that $-\lambda^- \leq \lambda$; also, $|\lambda| = \lambda^+ + \lambda^-$.*

(II) *If at least one of λ^+, λ^- is finite (in particular, if $\|\lambda\| < \infty$) then also $\lambda = \lambda^+ - \lambda^-$.*

PROOF. (I) Follows from Remark 15 and Lemma 37.

(II) Assume that at least one of λ^+, λ^- is finite. We shall show that $\lambda = \lambda^+ - \lambda^-$. Without loss of generality we may assume that λ does not assume $-\infty$. Note that $\lambda^-(X) < \infty$, for otherwise we would have $\lambda^+(X) < \infty$, and by part (I) of Lemma 37 $\lambda(X) = -\infty$. Assume that $\lambda^+(X) = \infty$ (the case $\lambda^+(X) < \infty$ is similar but simpler). By part (I) of Lemma 37 the equality $\lambda = \lambda^+ - \lambda^-$ holds for open sets. Let $F \in \mathscr{C}(X)$. We have $\lambda^-(F) < \lambda^-(X) < \infty$. If $\lambda(F) = \infty$ then from part (ii) of Corollary 39 we see that $\lambda^+(F) = \infty$. Then $\lambda(F) = \lambda^+(F) - \lambda^-(F)$. Now suppose $\lambda(F) \in \mathbb{R}$. There exists $W \in \mathscr{O}(X)$ such that $\lambda(U) \in \mathbb{R}$ for all $U \in \mathscr{O}(X)$, $F \subseteq U \subseteq W$. By part (ii) of Corollary 39 $\lambda^+(F), \lambda^-(F), \lambda^+(U), \lambda^-(U) \in \mathbb{R}$. Then

$$\lambda(F) = \lim \lambda(U) = \lim(\lambda^+(U) - \lambda^+(U))$$
$$= \lim \lambda^+(U) - \lim \lambda^-(U) = \lambda^+(F) - \lambda^-(F) \quad \square$$

5. Decomposition of signed topological measures into topological measures

THEOREM 41 [4, Theorem 5]. *Let X be a locally compact, connected, locally connected space whose one-point compactification has genus 0. Let ν be a deficient topological measure on X such that $\nu(X) < \infty$ and let $p \in X$ be an arbitrary point. Define a set function $\nu_p\colon \mathscr{O}^*_s(X) \cup \mathscr{K}_s(X) \to [0, \infty)$ by*

$$\nu_p(A) = \begin{cases} \nu(A), & \text{if } p \notin A, \\ \nu(X) - \nu(X \setminus A), & \text{if } p \in A. \end{cases}$$

Then ν_p is a solid set function and, hence, extends to a topological measure on X.

THEOREM 42. *Suppose X is a connected, locally connected, locally compact (non-compact) space whose one-point compactification has genus 0. Let ν be a signed topological measure with finite norm on X. Then ν can be represented as a difference of two topological measures.*

Proof. By Theorem 40, ν can be represented as the difference of two deficient topological measures, $\nu = \nu^+ - \nu^-$. By Remark 15, $\nu^+(X) < \infty$ and $\nu^-(X) < \infty$. Let $p \in X$. From ν^+, ν^- by Theorem 41 we obtain $\nu_1 = \nu_p^+$ and $\nu_2 = \nu_p^-$. Then ν_1, ν_2 are topological measures, and $\mu = \nu_1 - \nu_2$ is a signed topological measure. We shall show that $\mu = \nu$. If A is a bounded open solid set or a compact solid set and $p \notin A$, then using Theorem 40, we have

$$\mu(A) = \nu_1(A) - \nu_2(A) = \nu^+(A) - \nu^-(A) = \nu(A).$$

Now let K be a compact solid set, $p \in K$. Since $\nu_1(K) = \nu_p^+(K) = \nu^+(X) - \nu^+(X \setminus K)$ and $\nu_2(K) = \nu^-(X) - \nu^-(X \setminus K)$, by Theorem 40 we have

$$\mu(K) = \nu_1(K) - \nu_2(K) = \nu(X) - \nu(X \setminus K) = \nu(K).$$

By Theorem 35, $\nu = \mu$. $\square$

Remark 43. Theorem 42 is a locally compact version of the decomposition of a signed topological measure with finite norm into a difference of two topological measures when the underlying space is compact, Hausdorff, connected, locally connected, and has genus 0. See [13, Section 7].

Remark 44. (a) From Theorem 42 we see that the decomposition of a signed topological measure into a difference of two topological measures is not unique. This non-uniqueness of decomposition of a signed topological measure into a difference of topological measures is also demonstrated in [13, Example 25], where a signed topological measure on a compact space is written as a difference of topological measures in two different ways.

(b) If X satisfies the conditions of Theorem 42, then the family of topological measures on X is a generating cone for the family of signed topological measures with finite norms.

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