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INTERPOLATION BETWEEN CONTINUOUS PARAMETER MARTINGALE HARDY–LORENTZ AND BMO SPACES

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Abstract. In this paper, we consider continuous parameter martingale Hardy–Lorentz spaces and describe their real interpolation spaces when we apply function parameter to Hardy–Lorentz and $\mathcal{BMO}$ spaces. Some new interpolation theorems concerning continuous parameter Hardy–Lorentz spaces are formulated. The results generalize some fundamental interpolation theorems in continuous parameter martingale Hardy spaces.

1. Introduction

In this paper we introduce and study two classes of martingales: the so–called martingale Hardy–Lorentz and $\mathcal{BMO}$ spaces. The family of martingale Hardy spaces is one of the important martingale function spaces. Martingales arise naturally in many branches of the theory of stochastic process. They play an important role in probability theory and in statistics [5,15]. Moreover, interpolation of martingale Hardy spaces is one of the main topics in martingale H_p theory, and its theory has been applied to Fourier analysis. Here the interpolation spaces with a function parameter between martingale Hardy–Lorentz spaces, between Hardy–Lorentz and $\mathcal{BMO}$ spaces are identified. Some results due to [21] are extended to interpolation with a function parameter. In [9,21,22] martingale spaces are extended to the martingale Hardy–Lorentz spaces. This paper is devoted to provide a further extension of the martingale Hardy spaces $H_p^{\langle\rangle}$ and $H^{[]}$ to the martingale Hardy–Lorentz spaces $\Lambda_p^{\langle\rangle}(\varphi)$ and $\Lambda_p^{[]}(\varphi)$. Let us recall that

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the martingale Hardy spaces $H_p^{\langle\rangle}$ and $H^{[]}$ are generated by the sharp and square bracket, respectively.

The interpolation spaces between the classical Hardy spaces were identified by Fefferman, Riviere and Sagher [4,18]:

$$(\mathcal{H}_{p_0,q_0},\mathcal{H}_{p_1,q_1})_{\theta,q}=\mathcal{H}_{p,q}\,,\quad \frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},$$

where $0<\theta<1$, $0<p_0<p_1\le\infty$ and $0<q_0,q_1,q\le\infty$; and on martingale Hardy spaces by Janson and Jones [8] and Milman [11]. These results are obtained by choosing $\varrho(t)=t^\theta$. Hanks [6], Bennett and Sharpley [1] have identified the interpolation spaces between the classical Hardy and $\mathcal{BMO}$ spaces:

$$(\mathcal{H}_{p_0,q_0},\mathcal{BMO})_{\theta,q}=\mathcal{H}_{p,q},\quad \frac{1}{p}=\frac{1-\theta}{p_0},$$

where $0<\theta<1$, $0<p_0<\infty$ and $0<q_0,q\le\infty$.

Analogous results were obtained by Weisz [21] in the setting of the continuous parameter martingale $H_p^{\langle\rangle}$, $H^{[]}$ and $\mathcal{BMO}$ spaces. In this paper these results will be extended to continuous parameter martingale Hardy–Lorentz spaces by replacing θ by a more general (parameter) function $\varrho=\varrho(t)$.

2. Preliminaries

Let $(\Omega,\mathcal{F},P)$ be a probability space. The distribution function λ_f of a measurable function f on Ω is given by

$$\lambda_f(t)=P\big(\{w\in\Omega:|f(w)|>t\}\big),\quad t\ge0.$$

The decreasing rearrangement of f is the function $\tilde f$ defined on $[0,\infty)$ by

$$\tilde f(s)=\inf\{t>0:\lambda_f(t)\le s\},\quad s\ge0.$$

Let φ be a non-negative and locally integrable function on $[0,\infty)$. The classical Lorentz spaces $\Lambda_q(\varphi)$ are defined by the collections of all measurable functions f for which the quantity

$$\|f\|_{\Lambda_q(\varphi)}:=\begin{cases}\big(\int_0^\infty(\tilde f(t)\varphi(t))^q\,\frac{dt}{t}\big)^{1/q} & \text{if } 0<q<\infty,\\ \sup_s\tilde f(s)\varphi(s) & \text{if } q=\infty\end{cases}$$

is finite.

Let a and b be real numbers with $a<b$. Following Persson's convention [16], we adopt the following notation: $\varphi(t)\in Q[a,b]$ means that $\varphi(t)t^{-a}$ is

non–decreasing and $\varphi(t)t^{-b}$ is non-increasing for all $t > 0$. Moreover, we say that $\varphi(t) \in Q(a,b)$, when $\varphi(t) \in Q[a+\varepsilon, b-\varepsilon]$ for some $\varepsilon > 0$. By $\varphi(t) \in Q(a,-)$ (or $\varphi(t) \in Q(-,b)$) we mean that $\varphi(t) \in Q(a,c)$ (or $\varphi(t) \in Q(c,b)$) for some real number c.

Let $\bar{A} = (A_0, A_1)$ denote a compatible quasi-Banach pair (i.e. A_0 and A_1 are quasi-Banach spaces, both of which are continuously embedded in some Hausdorff topological vector space). For every $f \in A_0 + A_1$ and any $0 < t < \infty$, the so-called Peetre K-functional is defined by

$$K(t,f,A_0,A_1) = K(t,f) := \inf_{f_0+f_1=f} \{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\},$$

where $f_i \in A_i$, $i = 0,1$.

For $0 < q \le \infty$ and each measurable function ϱ, the real interpolation space $(A_0, A_1)_{\varrho,q}$ consists of all elements of $f \in A_0 + A_1$ such that the quantity

$$\|f\|_{(A_0,A_1)_{\varrho,q}} := \begin{cases} \big(\int_0^\infty (\frac{K(t,f)}{\varrho(t)})^q \, \frac{dt}{t}\big)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{t>0} \frac{K(t,f)}{\varrho(t)} & \text{if } q = \infty \end{cases}$$

is finite.

By replacing the measurable function $\varrho = \varrho(t)$ by t^θ we obtain $(A_0, A_1)_{\theta,q}$. In this paper we shall consider the interpolation spaces $(A_0, A_1)_{\varrho,q}$ with a parameter function $\varrho = \varrho(t) \in Q(0,1)$, where A_0 and A_1 are $\Lambda_p^{\langle\rangle}(\varphi)$, $\Lambda_p^{[]}(\varphi)$ and $\mathcal{BMO}$ spaces. It is easy to see that $\varrho(t) = t^\theta (0 < \theta < 1)$ belongs to $Q(0,1)$.

Let $0 < p < \infty$, $0 < q \le \infty$ and $\varrho \in Q(0,1)$. It was proved by Persson [16, Lemma 6.1] that

$$(L_p, L_\infty)_{\varrho,q} = \Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}})). \tag{2.1}$$

Our notation and terminology are standard, see [1,16] for more details. In order to prove our main results, we collect some lemmata, which will be used in the next sections.

LEMMA 2.1 [16]. *Let* $\varphi(t) \in Q[a,b]$. *Then*

(1) $\varphi(t^\alpha) \in Q[a\alpha, b\alpha]$, $\alpha > 0$.

(2) $t^\alpha(\varphi(t))^\beta \in Q[\alpha + b\beta, \alpha + a\beta]$, $\alpha \in R$, $\beta < 0$.

LEMMA 2.2 [16]. *Let* $0 < q \le \infty$, $0 < p < \infty$ *and* $\psi(t) \in Q(-,-)$. *Let* $h(t)$ *be a positive and non-increasing function on* $(0,\infty)$.

(1) *If* $\varphi(t) \in Q(-,0)$, *then*

$$\bigg(\int_0^\infty (\varphi(t))^q \bigg(\int_0^t (h(u)\psi(u))^p \frac{du}{u}\bigg)^{\frac{q}{p}} \frac{dt}{t}\bigg)^{\frac{1}{q}} \le C\bigg(\int_0^\infty (\varphi(t)h(t)\psi(t))^q \frac{dt}{t}\bigg)^{\frac{1}{q}}.$$

(2) *If* $\varphi(t) \in Q(0,-)$, *then*

$$\left(\int_0^\infty (\varphi(t))^q \left(\int_t^\infty (h(u)\psi(u))^p \frac{du}{u}\right)^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (\varphi(t)h(t)\psi(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

Here C depends only on q and the constants involved in the definition of φ and ψ.

Lemma 2.3 [16]. *Let $\varrho_0(t)$, $\varrho_1(t)$ and $\varrho(t)$ be in the class $Q(0,1)$ and $0 < q_0, q_1, q \le \infty$. If*

$$\varrho_2(t) := \varrho_0(t)\varrho(\varrho_1(t)/\varrho_0(t)), \quad \varrho_3(t) := \varrho_0(t)\varrho(t/\varrho_0(t)), \quad \varrho_4(t) := \varrho(\varrho_1(t)),$$

then

(1) $\left(\bar{A}_{\varrho_0,q_0}, A_1\right)_{\varrho,q} = \bar{A}_{\varrho_3,q}$;

(2) $\left(A_0, \bar{A}_{\varrho_1,q_1}\right)_{\varrho,q} = \bar{A}_{\varrho_4,q}$;

(3) *if, in addition, $\varrho_1(t)/\varrho_0(t) \in Q(0,-)$ or $\varrho_0(t)/\varrho_1(t) \in Q(0,-)$, then*

$$\left(\bar{A}_{\varrho_0,q_0}, \bar{A}_{\varrho_1,q_1}\right)_{\varrho,q} = \bar{A}_{\varrho_2,q}.$$

Lemma 2.4 [7]. *Let $0 < q \le \infty$, $\varrho \in Q(0,1)$, and let (A_0, A_1) and (B_0, B_1) be compatible quasi-normed pairs. Assume that T is a bounded sublinear operator from A_i to B_i, with a bound M_i, $i = 0, 1$, respectively. Then T is a bounded operator from $\bar{A}_{\varrho,q}$ to $\bar{B}_{\varrho,q}$ with a bound $M \le M_0\bar{\varrho}(M_1/M_0)$, where $\bar{\varrho}(s) = \sup_{t>0} \varrho(st)/\varrho(t)$.*

Next we state some basic facts and remind notation related to continuous parameter stochastic processes.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration $(\mathcal{F}_t)_{t\in\mathbf{R}^+}$ is a collection of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $0 \le s \le t$. We put $\mathcal{F}_\infty = \sigma\big(\bigcup_{t\in\mathbf{R}^+} \mathcal{F}_t\big)$. Given a filtration $(\mathcal{F}_t)_{t\in\mathbf{R}^+}$ for $t \in \mathbf{R}^+$, the σ-algebras $\mathcal{F}_{t^+}$ and $\mathcal{F}_{t^-}$ for $t > 0$ are defined as

$$\mathcal{F}_{t^+} := \bigcap_{s>t} \mathcal{F}_s \quad \text{and} \quad \mathcal{F}_{t^-} := \bigvee_{s<t} \mathcal{F}_s.$$

For $t = 0$ we set $\mathcal{F}_{0^-} := \mathcal{F}_0$. We call a filtration $(\mathcal{F}_t)_{t\in\mathbf{R}^+}$ right-continuous if $\mathcal{F}_t = \mathcal{F}_{t^+}$ for every $t \in \mathbf{R}^+$, left-continuous if $\mathcal{F}_t = \mathcal{F}_{t^-}$ for every $t \in \mathbf{R}^+$ and continuous if it is both left- and right-continuous. A filtration is said to satisfy the usual conditions if it is right-continuous and if $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets. Throughout the paper we assume a filtration satisfy the usual conditions.

A stochastic process X is a map from $\mathbf{R} \times \Omega$ into $\mathbf{R}$ such that for all $t \in \mathbf{R}^+$ the map $(t, w) \mapsto X_t(w) := X(t, w)$ is $\mathcal{F}$-measurable. A stochastic process X defined on $(\Omega, \mathcal{F}, P)$ is called adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbf{R}^+}$ if for every $t \in \mathbf{R}^+$, the random variable X_t is $\mathcal{F}_t$-measurable. A process X is regular if X is adapted and all the functions $t \mapsto X(t, w)$ have a left and a right limit for every $w \in \Omega$. We use the notation X_{t^-} (X_{t^+}, respectively) for the left (right, respectively) limit at a point t. If these functions are right-continuous (left–continuous) then X is called right-continuous (left–continuous). A process X is said to be $\mathcal{F}$–predictable if X is $\mathcal{F}$–adapted and X is left–continuous.

Recall that a stopping time τ is a random variable taking values in $\mathbf{R}^+$ such that $\{\tau \le t\} \in \mathcal{F}_t$ for every $t \in \mathbf{R}^+$. For a stopping time τ we define two σ-algebras $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau^-}$:

$$\mathcal{F}_\tau := \big\{A \in \mathcal{F}_\infty : A \cap \{\tau \le t\} \in \mathcal{F}_t,\ t \in \mathbf{R}^+\big\},$$

and

$$\mathcal{F}_{\tau^-} := \big\{A \cap \{\tau < t\} : A \in \mathcal{F}_t,\ t \in \mathbf{R}^+\big\} \cup \mathcal{F}_0.$$

Since a constant random variable $\tau = t_0$ is a stopping time, one has for this stopping time $\mathcal{F}_\tau = \mathcal{F}_{t_0}$ and $\mathcal{F}_{\tau^-} = \mathcal{F}_{t_0^-}$.

A stochastic process $X = (X_t)_{t\in\mathbf{R}^+}$ is called a martingale if

- each X_t is $\mathcal{F}_t$-measurable, i.e., X is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbf{R}^+}$,
- $E|X_t| < \infty$ for every t,
- $E[X_t|\mathcal{F}_s] = X_s$, if $s \le t$.

A martingale X is said to be L_p-bounded if

$$\sup_{t\in\mathbf{R}^+} \|X_t\|_p < \infty.$$

An adapted process X is a local martingale if there exists a sequence of stopping time $\tau_n \uparrow \infty$ such that for every n the process $X_{t\wedge\tau_n}$ is a uniformly integrable martingale on $\{\tau_n > 0\}$. Throughout this paper, we suppose that any local martingale adapted to this filtration is regular, right-continuous and $X_0 = 0$. A local martingale is said to be locally L_p-bounded if there exists a sequence of stopping time $\tau_n \uparrow \infty$ such that for every n the process $X_{t\wedge\tau_n}$ is an L_p-bounded martingale.

For a local martingale $X = (X_t)_{t\in\mathbf{R}^+}$ relative to $(\Omega, \mathcal{F}, P)$, denote

$$\Delta X_t := X_t - X_{t^-}, \quad X_s^* := \sup_{t\le s} |X_t|, \quad X_\infty^* := \sup_{t\in\mathbf{R}^+} |X_t|.$$

Let X be a locally L_2-bounded local martingale. The sharp bracket or the conditional quadratic variation process $\langle X\rangle$ is the unique predictable, right-continuous and increasing process that makes $X^2 - \langle X\rangle$ a local martingale and vanishing at 0. Moreover, if X is a local martingale, the square bracket or the quadratic variation process $[X]$ is the unique right-continuous and increasing process that makes $X^2 - [X]$ a local martingale and $\Delta[X]_t = |\Delta X_t|^2 \quad ([X]_0 = 0)$. In [19] Weisz introduced the sharp functions $X_r^{\langle\rangle}$ and $X_r^{[]}$ as follows

$$X_r^{\langle\rangle} = \sup_{t\in R^+} \eta_t, \quad X_r^{[]} = \sup_{t\in R^+} \zeta_t\,, \quad 0 < r < \infty,$$

where

$$\eta_t := \left[E_t(\langle X\rangle_\infty - \langle X\rangle_t)^{r/2}\right]^{1/r}, \quad \zeta_t := \left[E_t([X]_\infty - [X]_{t^-})^{r/2}\right]^{1/r}.$$

Denote the operators $X \mapsto X_r^{\langle\rangle}$ and $X \mapsto X_r^{[]}$ by $T_r^{\langle\rangle}$ and $T_r^{[]}$, respectively.

Let X be a local martingale. For $0 < q \le \infty$, we define martingale Hardy–Lorentz and $\mathcal{BMO}$ spaces as follows:

$$\Lambda_q^*(\varphi) = \left\{X = (X_t)_{t\in\mathbf{R}^+} : \|X\|_{\Lambda_q^*(\varphi)} := \|X_\infty^*\|_{\Lambda_q(\varphi)} < \infty\right\},$$

$$\Lambda_q^{[]}(\varphi) = \left\{X = (X_t)_{t\in\mathbf{R}^+} : \|X\|_{\Lambda_q^{[]}(\varphi)} := \|[X]_\infty^{\frac{1}{2}}\|_{\Lambda_q(\varphi)} < \infty\right\},$$

$$\Lambda_q^{\langle\rangle}(\varphi) = \left\{X = (X_t)_{t\in\mathbf{R}^+} : \|X\|_{\Lambda_q^{\langle\rangle}(\varphi)} := \|\langle X\rangle_\infty^{\frac{1}{2}}\|_{\Lambda_q(\varphi)} < \infty\right\},$$

$$\mathcal{BMO}_2 = \Big\{X = (X_t)_{t\in\mathbf{R}^+} : \|X\|_{\mathcal{BMO}_2} := \sup_{t\in\mathbf{R}^+} \|(E_t[\langle X\rangle_\infty - \langle X\rangle_t])^{\frac{1}{2}}\|_\infty < \infty\Big\},$$

and

$$\mathcal{BMO}_2^- = \Big\{X = (X_t)_{t\in\mathbf{R}^+} : \|X\|_{\mathcal{BMO}_2} := \sup_{t\in\mathbf{R}^+} \|(E_t[\langle X\rangle_\infty - \langle X\rangle_{t^-}])^{\frac{1}{2}}\|_\infty < \infty\Big\}.$$

Note that if $\varphi(t) = t^{\frac{1}{p}}$, then $\Lambda_q(\varphi) = L_{p,q}$, $\Lambda_q^*(\varphi) = H_{p,q}^*$, $\Lambda_q^{[]}(\varphi) = H_{p,q}^{[]}$ and $\Lambda_q^{\langle\rangle}(\varphi) = H_{p,q}^{\langle\rangle}$. In particular, if $\varphi(t) = t^{\frac{1}{q}}$, then $\Lambda_q(\varphi) = L_q$, $\Lambda_q^*(\varphi) = H_q^*$, $\Lambda_q^{[]}(\varphi) = H_q^{[]}$ and $\Lambda_q^{\langle\rangle}(\varphi) = H_q^{\langle\rangle}$.

It was proved by Dellacherie and Meyer [2], and Pratelli [17] that the dual of $H_1^{[]}$ resp. $H_1^{\langle\rangle}$ is $\mathcal{BMO}_2^-$ resp. $\mathcal{BMO}_2$ (see also Weisz [20]).

It is a well-known statement in martingale theory that if $X \in H^*$, $H_p^{[]}$, $H_p^{\langle\rangle}$ $(p \ge 1)$, then there exists a function X_∞ such that $X_t \to X_\infty$ a.e. and so $X_\infty \in L_1$ as $t \to \infty$.

The following result, which is a generalization of an inequality due to Fefferman, Stein [3] and Garsia [5], is verified by Weisz [19].

LEMMA 2.5. *The* L_∞ *resp.* L_p *norm of* $T_r^{\langle\rangle}(X)$ *is equivalent to the* BMO_2 *resp.* $H_p^{\langle\rangle}$ *norm of* X *and, moreover, the* L_∞ *resp.* L_p *norm of* $T_r^{[]}(X)$ *is equivalent to the* BMO_2^- *resp.* $H_p^{[]}$ *norm of* X.

In the sequel, we assume that $(\Omega,\mathcal{F},P)$ is a probability space and C denotes constants, not necessary the same at different occurrences.

3. Interpolation of martingale Hardy–Lorentz spaces

In this section, some interpolation theorems for martingale Hardy spaces are formulated and these results will be extended to interpolation of continuous parameter martingale Hardy–Lorentz spaces.

THEOREM 3.1. *Let* $0<p\le 1$, $0<q\le\infty$ *and* $\varrho\in Q(0,1)$ *be a parameter function. Then*

$$\big(H_p^{\langle\rangle},H_\infty^{\langle\rangle}\big)_{\varrho,q}=\Lambda_q^{\langle\rangle}\big(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}})\big).$$

We need the following lemma for proving Theorem 3.1.

LEMMA 3.2 [21, Lemma 1]. *If* $0<p\le 1$ *then*

$$K(t,X,H_p^{\langle\rangle},H_\infty^{\langle\rangle})\le C\bigg(\int_0^{t^p}\big((\langle\tilde{X}\rangle_\infty^{\frac{1}{2}})(s)\big)^p\,ds\bigg)^{\frac{1}{p}},\quad t>0.$$

PROOF OF THEOREM 3.1. Let $0<q<\infty$. By Lemma 2.1 it is easy to see that $1/\varrho(t^{\frac{1}{p}})\in Q(-\frac{1}{p},0)$. Hence

$$\|X\|^q_{(H_p^{\langle\rangle},H_\infty^{\langle\rangle})_{\varrho,q}}=\int_0^\infty\bigg(\frac{K(t,X,H_p^{\langle\rangle},H_\infty^{\langle\rangle})}{\varrho(t)}\bigg)^q\frac{dt}{t}$$

$$\le C\int_0^\infty\Big(\frac{1}{\varrho(t)}\Big)^q\bigg(\int_0^{t^p}\big((\langle\tilde{X}\rangle_\infty^{\frac{1}{2}})(s)\big)^p\,ds\bigg)^{\frac{q}{p}}\frac{dt}{t}\quad\text{(by Lemma 3.2)}$$

$$\le C\int_0^\infty\bigg(\frac{1}{\varrho(t^{\frac{1}{p}})}\bigg)^q\bigg(\int_0^{t}\big((\langle\tilde{X}\rangle_\infty^{\frac{1}{2}})(s)\big)^p\,ds\bigg)^{\frac{q}{p}}\frac{dt}{t}$$

$$\le C\int_0^\infty\bigg(\frac{1}{\varrho(t^{\frac{1}{p}})}\bigg)^q t^{\frac{q}{p}}\big((\langle\tilde{X}\rangle_\infty^{\frac{1}{2}})(t)\big)^q\frac{dt}{t}\quad\text{(by Lemma 2.2)}$$

$$=C\big\|\langle X\rangle_\infty^{\frac{1}{2}}\big\|^q_{\Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))}:=C\big\|X\big\|^q_{\Lambda_q^{\langle\rangle}(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))}.$$

To prove the converse, we consider the operator $T\colon X\mapsto \langle X\rangle_{\infty}^{\frac12}$. The sublinear operators $T\colon H_{\infty}^{\langle\rangle}\to L_{\infty}$ and $T\colon H_{p}^{\langle\rangle}\to L_p$ are bounded. From (2.1) and Lemma 2.4, the operator

$$T\colon \big(H_p^{\langle\rangle},H_{\infty}^{\langle\rangle}\big)_{\varrho,q}\to (L_p,L_{\infty})_{\varrho,q}=\Lambda_q(t^{\frac1p}/\varrho(t^{\frac1p}))$$

is bounded. Therefore

$$\|X\|_{\Lambda_q^{\langle\rangle}(t^{\frac1p}/\varrho(t^{\frac1p}))}:=\|\langle X\rangle_{\infty}^{\frac12}\|_{\Lambda_q(t^{\frac1p}/\varrho(t^{\frac1p}))}\le C\|X\|_{(H_p^{\langle\rangle},H_{\infty}^{\langle\rangle})_{\varrho,q}}.$$

The proof is complete if $0<q<\infty$.

Let $q=\infty$. The function $\varrho(t)t^{-\varepsilon}$ is non–decreasing for some $\varepsilon>0$ since $\varrho\in Q(0,1)$. Hence, we get

$$\|X\|_{(H_p^{\langle\rangle},H_{\infty}^{\langle\rangle})_{\varrho,\infty}}=\sup_{t>0}\frac{K(t,X,H_p^{\langle\rangle},H_{\infty}^{\langle\rangle})}{\varrho(t)}$$

$$\le C\sup_{t>0}\frac{\big(\int_0^{t^p}\big((\langle\tilde X\rangle_{\infty}^{\frac12})(s)\big)^p\,ds\big)^{\frac1p}}{\varrho(t)}\quad\text{(by Lemma 3.2)}$$

$$=C\sup_{t>0}\frac{\big(\int_0^{t}\big((\langle\tilde X\rangle_{\infty}^{\frac12})(s^p)\big)^p s^{p-1}\,ds\big)^{\frac1p}}{\varrho(t)}$$

$$\le C\sup_{s>0}\frac{s(\langle\tilde X\rangle_{\infty}^{\frac12})(s^p)}{\varrho(s)}\cdot\sup_{t>0}\frac{\varrho(t)t^{-\varepsilon}(\int_0^t s^{p\varepsilon-1}\,ds)^{\frac1p}}{\varrho(t)}\le C\|X\|_{\Lambda_{\infty}^{\langle\rangle}(t^{\frac1p}/\varrho(t^{\frac1p}))}.$$

To prove the converse, we consider the operator $T\colon X\mapsto\langle X\rangle_{\infty}^{\frac12}$. The sublinear operators $T\colon H_{\infty}^{\langle\rangle}\to L_{\infty}$ and $T\colon H_p^{\langle\rangle}\to L_p$ are bounded. It follows from (2.1) and Lemma 2.4 that the operator

$$T\colon\big(H_p^{\langle\rangle},H_{\infty}^{\langle\rangle}\big)_{\varrho,\infty}\to(L_p,L_{\infty})_{\varrho,\infty}=\Lambda_{\infty}\big(t^{\frac1p}/\varrho(t^{\frac1p})\big)$$

is bounded. So we have

$$\|X\|_{\Lambda_{\infty}^{\langle\rangle}(t^{\frac1p}/\varrho(t^{\frac1p}))}:=\|\langle X\rangle_{\infty}^{\frac12}\|_{\Lambda_{\infty}(t^{\frac1p}/\varrho(t^{\frac1p}))}\le C\|X\|_{(H_p^{\langle\rangle},H_{\infty}^{\langle\rangle})_{\varrho,\infty}}.$$

The proof is complete. □

THEOREM 3.3. *Let* $\varphi_i(t)\in Q(0,-)$, $i=0,1$, $0<p\le1$, $0<q_0,q_1,q\le\infty$ *and* $\varrho\in Q(0,1)$. *Then*

(1) *we have*

$$\big(\Lambda_{q_0}^{\langle\rangle}(\varphi_0),H_{\infty}^{\langle\rangle}\big)_{\varrho,q}=\Lambda_q^{\langle\rangle}(\varphi),$$

where $\varphi(t)=\varphi_0(t)/\varrho(\varphi_0(t))$.

(2) *If, in addition,* $\varphi_1(t) \in Q(0, 1/p)$, *then*

$$\big(H_p^{\langle\rangle}, \Lambda_{q_1}^{\langle\rangle}(\varphi_1)\big)_{\varrho.q} = \Lambda_q^{\langle\rangle}(\varphi),$$

where $\varphi(t) = t^{1/p}/\varrho(t^{1/p}/\varphi_1(t))$.

(3) *If, in addition,* $\varphi_0(t)/\varphi_1(t) \in Q(0,-)$ *or* $\varphi_0(t)/\varphi_1(t) \in Q(-,0)$, *then*

$$\big(\Lambda_{q_0}^{\langle\rangle}(\varphi_0), \Lambda_{q_1}^{\langle\rangle}(\varphi_1)\big)_{\varrho,q} = \Lambda_q^{\langle\rangle}(\varphi),$$

where $\varphi(t) = \varphi_0(t)/\rho(\varphi_0(t)/\varphi_1(t))$.

Proof. First we prove (3). Put $\varrho_i(t) = t/\varphi_i(t^p)$ and choose p so small that $\varrho_i(t) \in Q(0,1), i = 0,1$. According to (3) in Lemma 2.3 and Theorem 3.1 we get

$$\begin{aligned}\big(\Lambda_{q_0}^{\langle\rangle}(\varphi_0), \Lambda_{q_1}^{\langle\rangle}(\varphi_1)\big)_{\varrho.q} &= \big((H_p^{\langle\rangle}, H_\infty^{\langle\rangle})_{\varrho_0,p_0}, (H_p^{\langle\rangle}, H_\infty^{\langle\rangle})_{\varrho_1,p_1}\big)_{\varrho,q}\\ &= \big(H_p^{\langle\rangle}, H_\infty^{\langle\rangle}\big)_{\varrho_0\varrho(\varrho_1/\varrho_0),q} = \Lambda_q^{\langle\rangle}(\varphi),\end{aligned}$$

where $\varphi(t) = \varphi_0(t)/\rho(\varphi_0(t)/\varphi_1(t))$. To prove (2), we first note that by Lemma 2.1 the condition $\varphi_1(t) \in Q(0, 1/p)$ implies that $\varrho_1(t) = t/\varphi_1(t^p) \in Q(0,1)$. So the proof follows as above by using Theorem 3.1 and Lemma 2.3(2). In a similar way we see that (1) is an easy consequence of Theorem 3.1 and Lemma 2.3(1). The proof is complete. □

The following result is a simple application of (3) in Theorem 3.3 by replacing the parameter function $\varrho = \varrho(t)$ by t^θ.

Corollary 3.4. *Under the hypothesis of* (3) *in Theorem* 3.3, *we have*

$$\big(\Lambda_{q_0}^{\langle\rangle}(\varphi_0), \Lambda_{q_1}^{\langle\rangle}(\varphi_1)\big)_{\theta,q} = \Lambda_q^{\langle\rangle}(\varphi_0^{1-\theta}\varphi_1^{\theta}).$$

According to Theorem 3.3 we have the following corollary.

Corollary 3.5. *Let* $0 < p_i < \infty$, $0 < q_i, q \le \infty$, $i = 0, 1$ *and* $\varrho \in Q(0,1)$. *If* $p_0 \ne p_1$, *then*

$$\big(H_{p_0,q_0}^{\langle\rangle}, H_{p_1,q_1}^{\langle\rangle}\big)_{\varrho,q} = \Lambda_q^{\langle\rangle}\big(t^{\frac{1}{p_0}}/\varrho\big(t^{\frac{1}{p_0}-\frac{1}{p_1}}\big)\big)$$

and

$$\big(H_{p_0}^{\langle\rangle}, H_{p_1}^{\langle\rangle}\big)_{\varrho,q} = \Lambda_q^{\langle\rangle}\big(t^{\frac{1}{p_0}}/\varrho\big(t^{\frac{1}{p_0}-\frac{1}{p_1}}\big)\big).$$

In particular, if $\varrho(t) = t^\theta$, *then*

$$\big(H_{p_0}^{\langle\rangle}, H_{p_1}^{\langle\rangle}\big)_{\theta,q} = H_{p,q}^{\langle\rangle}, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

We now formulate an interpolation theorem between martingale Hardy–Lorentz spaces $\Lambda_q^{[\,]}(\varphi)$ generated by quadratic variation.

THEOREM 3.6. *If* $0 < q \le \infty$ *and* $\varrho \in Q(0,1)$, *then*

$$\left(H_1^{[\,]}, H_\infty^{[\,]}\right)_{\varrho,q} = \Lambda_q^{[\,]}(t/\varrho(t)).$$

This result, as we have seen in Theorem 3.1, follows from the next lemma.

LEMMA 3.7 [21, Lemma 3]. *If* $t > 0$, *then*

$$K\left(t, X, H_1^{[\,]}, H_\infty^{[\,]}\right) \le C \int_0^t ([\tilde{X}]_\infty^{\frac{1}{2}})(s)\, ds.$$

Using Theorem 3.6 and Lemma 2.3 we get the following results.

THEOREM 3.8. *Let* $\varphi_i(t) \in Q(0,-)$, $0 < q_0, q_1, q \le \infty$ *and* $\varrho \in Q(0,1)$.

(1) *Then we have*

$$\left(\Lambda_{q_0}^{[\,]}(\varphi_0), H_\infty^{[\,]}\right)_{\varrho,q} = \Lambda_q^{[\,]}(\varphi),$$

where $\varphi(t) = \varphi_0(t)/\varrho(\varphi_0(t))$;

(2) *if, in addition* $\varphi_1(t) \in Q(0,1)$, *then*

$$\left(H_1^{[\,]}, \Lambda_{q_1}^{[\,]}(\varphi_1)\right)_{\varrho.q} = \Lambda_q^{[\,]}(\varphi),$$

where $\varphi(t) = t/\varrho(t/\varphi_1(t))$;

(3) *if, in addition* $\varphi_0(t)/\varphi_1(t) \in Q(0,-)$ *or* $\varphi_0(t)/\varphi_1(t) \in Q(-,0)$, *then*

$$\left(\Lambda_{q_0}^{[\,]}(\varphi_0), \Lambda_{q_1}^{[\,]}(\varphi_1)\right)_{\varrho,q} = \Lambda_q^{[\,]}(\varphi),$$

where $\varphi(t) = \varphi_0(t)/\rho(\varphi_0(t)/\varphi_1(t))$.

PROOF. The proof is similar to the proof of the Theorem 3.3. □

4. Interpolation between martingale Hardy–Lorentz and $\mathcal{BMO}$ spaces

In this section, we keep all notation introduced in the preliminaries and discuss interpolation with a function parameter between martingale Hardy–Lorentz and $\mathcal{BMO}$ spaces.

THEOREM 4.1. *If* $1 < p < \infty$, $0 < q \le \infty$ *and* $\varrho \in Q(0,1)$, *then*

$$(H_p^{\langle\rangle}, \mathcal{BMO}_2)_{\varrho,q} = \Lambda_q^{\langle\rangle}\left(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}})\right). \tag{4.1}$$

First we will prove the following lemma.

LEMMA 4.2. *Let $1 \le r < \infty$. Then for any martingale X,*

$$\tilde{M}(t) \le \tilde{M}(2t) + 4\tilde{X}_r^{\langle\rangle}\Big(\frac{t}{2}\Big) \quad (t > 0)$$

holds, where $M = \langle X\rangle_\infty^{\frac{1}{2}}$.

PROOF. Let us introduce the following stopping times:

$$\mu := \inf\big\{s \in R^+ : \langle X\rangle_s^{\frac{1}{2}} > \tilde{M}(2t)\big\},$$

$$\nu := \inf\Big\{s \in R^+ : \langle X\rangle_s^{\frac{1}{2}} > \tilde{M}(2t) + 4\tilde{X}_r^{\langle\rangle}\Big(\frac{t}{2}\Big)\Big\},$$

$$\tau := \inf\Big\{s \in R^+ : \eta_s > \tilde{X}_r^{\langle\rangle}\Big(\frac{t}{2}\Big)\Big\}.$$

Note that $\mu \le \nu$ and

$$\{\mu < \infty\} = \big\{M > \tilde{M}(2t)\big\}, \quad \{\nu < \infty\} = \Big\{M > \tilde{M}(2t) + 4\tilde{X}_r^{\langle\rangle}\Big(\frac{t}{2}\Big)\Big\},$$

$$\{\tau < \infty\} = \Big\{X_r^{\langle\rangle} > \tilde{X}_r^{\langle\rangle}\Big(\frac{t}{2}\Big)\Big\}, \quad P(\mu < \infty) \le 2t, \quad P(\tau < \infty) \le \frac{t}{2}.$$

Then

$$\{\nu<\infty\} = \{\nu<\infty, \mu<\tau\} \cup \{\nu<\infty, \mu\ge\tau\} \subseteq \{\tau<\infty\} \cup \{\nu<\infty, \mu<\tau\},$$

$$\{\nu < \infty, \mu < \tau\} \subseteq \Big\{\mu < \tau, M - \langle X\rangle_\mu^{\frac{1}{2}} > 4\tilde{X}_r^{\langle\rangle}\Big(\frac{t}{2}\Big)\Big\}.$$

Therefore,

$$P(\nu < \infty, \mu < \tau) \le \frac{1}{4\tilde{X}_r^{\langle\rangle}(\frac{t}{2})} \int_{\{\mu<\tau\}} \big(M - \langle X\rangle_\mu^{\frac{1}{2}}\big)\, dP$$

$$\le \frac{1}{4\tilde{X}_r^{\langle\rangle}(\frac{t}{2})} \int_{\{\mu<\tau\}} E\big[M - \langle X\rangle_\mu^{\frac{1}{2}} | F_\mu\big]\, dP$$

$$\le \frac{1}{4\tilde{X}_r^{\langle\rangle}(\frac{t}{2})} \int_{\{\mu<\tau\}} \big(E\big[(\langle X\rangle_\infty - \langle X\rangle_\mu)^{\frac{r}{2}} | F_\mu\big]\big)^{\frac{1}{r}}\, dP \le \frac{1}{4} P(\mu < \infty) \le \frac{t}{2}.$$

Consequently,

$$P\Big(M > \tilde{M}(2t) + 4\tilde{X}_r^{\langle\rangle}(\frac{t}{2})\Big) = P(\nu < \infty) \le P(\tau < \infty) + \frac{t}{2} \le t,$$

which shows that

$$\tilde{M}(t) \le \tilde{M}(2t) + 4\tilde{X}_r^{\langle\rangle}\Big(\frac{t}{2}\Big), \quad t > 0. \quad \square$$

LEMMA 4.3 [10]. *Let* (F, G) *be a pair of nonnegative measurable functions on* Ω. *If* (F, G) *satisfies the rearrangement inequality*:

$$\tilde{F}(t) \le \tilde{F}(2t) + C\tilde{G}\Big(\frac{t}{2}\Big), \quad t > 0,$$

then with the some C,

$$\tilde{F}(t) \le 2C\tilde{G}\Big(\frac{t}{2}\Big) + \frac{C}{\log 2}\int_t^\infty \frac{\tilde{G}(s)}{s}\, ds, \quad t > 0.$$

LEMMA 4.4. *Let* $0 < p < \infty, 0 < q \le \infty, 1 \le r < \infty$ *and* $\varrho \in Q(0,1)$. *Then*

$$\big\|\langle X\rangle_\infty^{\frac{1}{2}}\big\|_{\Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))} \le C\big\|X_r^{\langle\rangle}\big\|_{\Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))}$$

PROOF. Since $\varrho(t^{\frac{1}{p}}) \in Q(0, \frac{1}{p})$ by Lemma 2.1, then $\varrho(t^{\frac{1}{p}})t^{-\varepsilon}$ is non–decreasing for some $\varepsilon > 0$. Hence $\varrho(t^{\frac{1}{p}}) \le C\varrho((2t)^{\frac{1}{p}})$ for $t > 0$. It follows that

$$\|\langle X\rangle_\infty^{\frac{1}{2}}\|_{\Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))} = \bigg(\int_0^\infty \bigg(\frac{t^{\frac{1}{p}}\langle \tilde{X}\rangle_\infty^{\frac{1}{2}}(t)}{\varrho(t^{\frac{1}{p}})}\bigg)^q \frac{dt}{t}\bigg)^{\frac{1}{q}}$$

$$\le C\bigg(\int_0^\infty\bigg(\frac{t^{\frac{1}{p}}\tilde{X}_r^{\langle\rangle}(\frac{t}{2})}{\varrho(t^{\frac{1}{p}})}\bigg)^q\frac{dt}{t}\bigg)^{\frac{1}{q}} + C\bigg(\int_0^\infty\bigg(\frac{t^{\frac{1}{p}}}{\varrho(t^{\frac{1}{p}})}\bigg)^q\bigg(\int_t^\infty \frac{\tilde{X}_r^{\langle\rangle}(s)}{s}\, ds\bigg)^q\frac{dt}{t}\bigg)^{\frac{1}{q}}$$

$$\le C\bigg(\int_0^\infty \bigg(\frac{t^{\frac{1}{p}}\tilde{X}_r^{\langle\rangle}(t)}{\varrho(t^{\frac{1}{p}})}\bigg)^q \frac{dt}{t}\bigg)^{\frac{1}{q}} = C\big\|X_r^{\langle\rangle}\big\|_{\Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))}$$

by lemmata 4.2, 4.3 and 2.2. $\square$

PROOF OF THEOREM 4.1. It is easy to see that

$$\|X\|_{\mathcal{BMO}_2} := \sup_{t\in\mathbf{R}^+}\big\|(E_t[\langle X\rangle_\infty - \langle X\rangle_t])^{\frac{1}{2}}\big\|_\infty$$

$$\le \sup_{t\in\mathbf{R}^+}\big\|(E_t[\langle X\rangle_\infty])^{\frac{1}{2}}\big\|_\infty \le \big\|\langle X\rangle_\infty^{\frac{1}{2}}\big\|_\infty .$$

Thus

$$\|X\|_{(H_p^{\langle\rangle},\mathcal{BMO}_2)_{\varrho,q}} \le C\|X\|_{(H_p^{\langle\rangle},H_\infty^{\langle\rangle})_{\varrho,q}} := C\|X\|_{\Lambda_q^{\langle\rangle}(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))},$$

from which we get $\Lambda_q^{\langle\rangle}(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}})) \subseteq (H_p^{\langle\rangle},\mathcal{BMO}_2)_{\varrho,q}$. To prove the converse we consider the sublinear operator $T_r^{\langle\rangle}\colon X \mapsto X_r^{\langle\rangle}$ for a fixed $1 \le r < p$. By Lemma 2.5 the operators $T_r^{\langle\rangle}\colon H_p^{\langle\rangle} \to L_p$ and $T_r^{\langle\rangle}\colon \mathcal{BMO}_2 \to L_\infty$ are bounded. It follows from (2.1) and Lemma 2.4 that the operator

$$T_r^{\langle\rangle}\colon \big(H_p^{\langle\rangle},\mathcal{BMO}_2\big)_{\varrho,q} \to (L_p,L_\infty)_{\varrho,q} = \Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}})),$$

is bounded. Now if $X \in \big(H_p^{\langle\rangle},\mathcal{BMO}_2\big)_{\varrho,q}$, then by Lemma 4.4 we have

$$\|X\|_{\Lambda_q^{\langle\rangle}(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))} \le C\big\|X_r^{\langle\rangle}\big\|_{\Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))} \le C\big\|X\big\|_{(H_p^{\langle\rangle},\mathcal{BMO}_2)_{\varrho,q}},$$

from which we get $(H_p^{\langle\rangle},BMO_2)_{\varrho,q} \subseteq \Lambda_q^{\langle\rangle}(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))$. The proof is complete. □

REMARK 4.5. Let $0<\theta<1$, $1<p_0<\infty$ and $0<q\le\infty$. Putting $\varrho(t)=t^\theta$ in Theorem 4.1 we obtain

$$\big(H_{p_0}^{\langle\rangle},\mathcal{BMO}_2\big)_{\theta,q} = H_{p,q}^{\langle\rangle}, \quad \frac{1}{p} = \frac{1-\theta}{p_0}.$$

Similary, in view of (1) in Theorem 3.3 , we obtain the following result.

THEOREM 4.6. *Let* $0<q_0,q\le\infty$ *and* $\varphi_0(t),\varrho(t)\in Q(0,1)$. *Then*

$$\big(\Lambda_{q_0}^{\langle\rangle}(\varphi_0),BMO_2\big)_{\varrho,q} = \Lambda_q^{\langle\rangle}(\varphi),$$

where $\varphi(t)=\varphi_0(t)/\varrho(\varphi_0(t))$.

We can state the following corollaries that are the special cases from the above theorem.

COROLLARY 4.7. *Let* $0<p_0<\infty$, $0<q_0,q\le\infty$ *and* $\varrho\in Q(0,1)$. *Then*

$$\big(H_{p_0,q_0}^{\langle\rangle},\mathcal{BMO}_2\big)_{\varrho,q} = \Lambda_q^{\langle\rangle}(\varphi),$$

where $\varphi(t)=t^{\frac{1}{p_0}}/\varrho(t^{\frac{1}{p_0}})$.

COROLLARY 4.8. *Let* $0 < q_0, q \leq \infty$, $0 < \theta < 1$ *and* $\varphi_0(t) \in Q(0,1)$. *Then*

$$\big(\Lambda_{q_0}^{\langle\rangle}(\varphi_0), \mathcal{BMO}_2\big)_{\theta,q} = \Lambda_q^{\langle\rangle}(\varphi),$$

where $\varphi(t) = \varphi_0(t)^{1-\theta}$.

REMARK 4.9. In the same spirit as Theorem 4.1, the analogous results hold for martingale Hardy–Lorentz spaces generated by quadratic variation, too.

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