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# UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING VALUES WITH THEIR $n$TH ORDER EXACT DIFFERENCES

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**Abstract.** Let $f(z)$ be a transcendental meromorphic function in the complex plane of hyper-order strictly less than 1. It is shown that if $f(z)$ and its $n$th exact difference $\Delta^n f(z)$ $(\not\equiv 0)$ share three distinct periodic functions $a, b, c \in \hat{\mathcal{S}}(f)$ with period 1 CM, where $\hat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$ and $\mathcal{S}(f)$ denotes the set of all small functions of $f(z)$, then $\Delta^n f(z) \equiv f(z)$.

## 1. Introduction

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane $\mathbb{C}$, and $a$ be a value on $\mathbb{C} \cup \{\infty\}$. We say that $f$ and $g$ share $a$ CM (IM) provided that $f$ and $g$ have the same $a$-points counting multiplicities (ignoring multiplicities). It is well known that if two nonconstant meromorphic functions $f$ and $g$ share four distinct values CM, then $f$ is a Möbius transformation of $g$. Rubel and Yang [19] initiated the study of entire functions sharing values with their derivatives, and they proved that $f' \equiv f$ if a nonconstant entire function $f$ and its derivative $f'$ share two distinct finite values CM. Mues and Steinmetz [18] and Gundersen [4] extended this conclusion for meromorphic functions.

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THEOREM 1.1 (see [4,18]). *If a meromorphic function $f(z)$ and its derivative $f'(z)$ share two distinct values $a_1, a_2 \in \mathbb{C}$ CM, then $f' \equiv f$.*

Frank and Weissenborn [3] extended Theorem 1.1 by showing that the conclusion of Theorem 1.1 still holds when replacing $f'$ with the $n$th derivative $f^{(n)}$, $n \geq 2$. For the case that $f(z)$ and its $n$th derivative $f^{(n)}$ share two distinct small functions of $f(z)$, Li [12, Theorem 1] obtained: if a transcendental meromorphic function $f$ shares two distinct small functions CM with its $n$th derivative $f^{(n)}$, $n \geq 2$, then $f^{(n)} \equiv f$. Li [12] also showed that this is not valid generally when $n = 1$.

We assume that the readers are familiar with the standard notions and the basic results of Nevanlinna's theory (see, e.g., [10,20]), such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$ and the integrated counting function $N(r, f)$. In addition, we denote by $S(r, f)$ any quantity that satisfies the condition $S(r, f) = o(1)T(r, f)$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, and we say a meromorphic function $a(z)$ ($\not\equiv \infty$) is a *small* function of $f(z)$ if $T(r, a(z)) = S(r, f)$. Denote by $\mathcal{S}(f)$ the field of all small functions of $f(z)$ and set $\hat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$. We will use the notation $\sigma(f)$ to denote the order of growth of $f(z)$ and, if $f(z)$ has infinite order of growth, we use the notation $\varsigma(f)$ to denote the hyper-order of $f(z)$ which is defined by:

$$\varsigma(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

Letting $\eta \in \mathbb{C} \setminus \{0\}$, we define the shift of $f(z)$ by $f(z+\eta)$ and the exact differences of $f(z)$ by $\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $\Delta_\eta^n f(z) = \Delta_\eta(\Delta_\eta^{n-1} f(z))$ $= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(z + i\eta)$, where $n$ ($\geq 2$) is an integer. Moreover, we will use the usual notation $\Delta^n f(z)$ when $\eta = 1$.

For finite order meromorphic functions, Halburd and Korhonen [5] and Chiang and Feng [2] proved a difference analogue of the lemma on the logarithmic derivatives, independently. Using this new tool, Halburd and Korhonen [6] extended the usual Second Main Theorem for the exact difference $\Delta_\eta f(z)$. By the new type of Second Main Theorem [6, Theorem 2.4], Heittokangas et. al. [8] proved the following shift analogue of Theorem 1.1 for small periodic functions of $f(z)$.

THEOREM 1.2 [8]. *Let $f(z)$ be a meromorphic function of finite order, and let $\eta \in \mathbb{C}$. If $f(z)$ and $f(z + \eta)$ share three distinct functions $a, b, c \in \hat{\mathcal{S}}(f)$ with period $\eta$ CM, then $f(z) = f(z + \eta)$ for all $z \in \mathbb{C}$.*

Heittokangas et. al. [9] have improved the conclusion of Theorem 1.2 by replacing the condition 3 CM with 2 CM + 1 IM. Moreover, the conclusion of Theorem 1.2 still holds when $f$ has hyper-order $\varsigma(f) < 1$. This can be seen by applying an extension of the difference analogue of the lemma on the logarithmic derivatives for meromorphic functions of hyper-order strictly less than 1 in [7], and by following the proof of Theorem 1.2 in [8].

Theorem 1.2 and its generalizations give us a good idea on how the uniqueness theory of meromorphic functions works when comparing a meromorphic function with its shift. However, when looking for a difference analogue of the derivative, the exact difference operator is a more natural analogue than the shift operator. This raises the following question: does the conclusion of Theorem 1.2 still hold for meromorphic functions of hyper-order strictly less than 1 when replacing the shift $f(z+\eta)$ with the $n$th order difference $\Delta_\eta^n f(z)$? In papers [1,13–15,22], the authors have proved some uniqueness theorems related to this question on finite order meromorphic functions sharing values CM with their differences $\Delta_\eta^n f(z)$. We recall the following two theorems.

Theorem 1.3 [14]. *Let $f(z)$ be a nonconstant entire function of finite order, $\eta\in\mathbb{C}$ and $n$ be a positive integer. Suppose that $f(z)$ and $\Delta_\eta^n f(z)$ share two distinct finite values $a_1, a_2$ CM and one of the following cases is satisfied*: (i) $a_1a_2=0$; (ii) $a_1a_2\neq 0$ *and* $\sigma(f)\notin\mathbb{N}$. *Then* $\Delta_\eta^n f(z)\equiv f(z)$.

Theorem 1.4 [15]. *Let $f(z)$ be a nonconstant entire function of finite order, and let $\eta\in\mathbb{C}$ be a nonzero complex number. If $f(z)$ and $\Delta_\eta f(z)$ share $a_1, a_2, a_3$ CM, where $a_1$, $a_2$, $a_3$ are three distinct values in the extended complex plane, then $\Delta_\eta f(z)\equiv f(z)$.*

In this paper, we will give a positive answer to the question posed above by proving the following Theorem 1.5, which can be viewed as a difference analogue of a result by Li [12, Theorem 1] on the uniqueness of meromorphic functions sharing values with their $n$th order derivatives.

Theorem 1.5. *Let $f(z)$ be a transcendental meromorphic function of hyper-order strictly less than 1 such that $\Delta^n f(z)\not\equiv 0$. If $f(z)$ and $\Delta^n f(z)$ share three distinct periodic functions $a,b,c\in\hat{\mathcal{S}}(f)$ with period 1 CM, then $\Delta^n f(z)\equiv f(z)$.*

Let $f(z)$ be a meromorphic solution of the linear difference equation: $\Delta^n f(z)-1=\xi(f(z)-1)$, where $\xi\neq 0,1$ is a constant. Then $\Delta^n f(z)$ and $f(z)$ share the values $1,\infty$ CM, but $\Delta^n f(z)\not\equiv f(z)$. This counterexample shows that the 3 CM condition in Theorem 1.5 cannot be reduced to 2 CM condition in general.

In Theorem 1.5, we have assumed $\Delta^n f(z)\not\equiv 0$ since $\Delta^n f(z)=0$ is an $n$th order linear difference equation in $f(z)$ with constant coefficients and can be solved explicitly. Consider the meromorphic solution $f(z)$ of the linear difference equation: $\Delta^n f(z)=f(z)$. The distinct roots of $\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}\lambda^k=1$ are $\lambda_k=1+e^{\frac{2k\pi i}{n}}$, $k=0,\dots,n-1$, thus $f(z)$ can be expressed as $f(z)=\sum_{k=0}^{n-1}\pi_k(z)\lambda_k^z$, $k=0,\dots,n-1$, where $\pi_k(z)$ are arbitrary 1-periodic functions. We conjecture that the conclusion of Theorem 1.5 is still true when the assumption $\varsigma(f)<1$ is dropped.

## 2. Some lemmas

In this section, we present the auxiliary results needed to prove Theorem 1.5.

LEMMA 2.1 [7]. *Let $T\colon[0,\infty)\to[0,\infty)$ be a non-decreasing, continuous function and let $s\in(0,\infty)$. If the hyper-order of $T$ is strictly less than* 1, *i.e.*

$$\limsup_{r\to\infty}\frac{\log\log T(r)}{\log r}=\varsigma<1,$$

*and $\delta\in(0,1-\varsigma)$, then*

$$T(r+s)=T(r)+o\Big(\frac{T(r)}{r^{\delta}}\Big),$$

*where $r$ runs to infinity outside of a set of finite logarithmic measure.*

Denote by $\mathcal{P}_\eta^1$ the field of $\eta$-periodic meromorphic functions defined in $\mathbb{C}$ of hyper-order strictly less than 1. The following result is a difference analogue of the lemma on the logarithmic derivatives.

LEMMA 2.2 [6,7]. *Let $\eta\in\mathbb{C}$, $n\in\mathbb{N}$, $n\geq 1$, and let $f(z)$ be a meromorphic function of hyper-order strictly less than* 1. *If $a(z)\in\mathcal{P}_\eta^1\cap\mathcal{S}(f)$, then*

$$m\Big(r,\frac{f(z+\eta)}{f(z)}\Big)=S(r,f),\quad m\Big(r,\frac{\Delta_\eta^n f(z)}{f(z)-a(z)}\Big)=S(r,f).$$

The following Lemma 2.3 from [7] is a difference analogue of the classical Borel's lemma (see, e.g., [11]) for entire functions with no zeros. We say the zero $z_0$ of an entire function $g(z)$ with order $i\geq 1$ is *forward invariant* with respect to the translation $\tau(z)=z+\eta$ when $z_0$ is also a zero of $g(z+\eta)$ with order $j$ and $j\geq i$. For example, all the zeros of an entire function with period $\eta$ are forward invariant with respect to the translation $\tau(z)=z+\eta$.

LEMMA 2.3 [7]. *Let $\eta\in\mathbb{C}$, and $g_0,\dots,g_n$ be entire functions such that $\varsigma(g_i)<1$, $i=0,\dots,n$, and such that all zeros of $g_0,\dots,g_n$ are forward invariant with respect to the translation $\tau(z)=z+\eta$. If $g_i/g_j\notin\mathcal{P}_\eta^1$ for all $i,j\in\{0,\dots,n\}$ such that $i\neq j$, then $g_0,\dots,g_n$ are linearly independent over $\mathcal{P}_\eta^1$.*

We also need the following lemma due to Li and Wang [16] to prove Theorem 1.5.

Lemma 2.4 [16]. *Suppose that $h$ is a nonconstant meromorphic function satisfying $\overline{N}(r,h)+\overline{N}(r,1/h)=S(r,h)$. Let $f=a_ph^p+a_{p-1}h^{p-1}+\cdots+a_1h+a_0$ and $g=b_qh^q+b_{q-1}h^{q-1}+\cdots+b_1h+b_0$, where $a_0,\ \dots,\ a_p$ and $b_0,\ \dots,\ b_q$ are small functions of $h$ such that $a_0a_pb_q\not\equiv 0$. If $q\le p$, then we have $m(r,g/f)=S(r,h)$.*

Finally, we need a classical result introduced originally by Valiron [21], and generalized by Mohon′ko [17].

Lemma 2.5 [17,21] or [10]. *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f$,*

$$R(z,f)=\frac{P(z,f)}{Q(z,f)}=\frac{\sum_i^p a_i(z)f^i}{\sum_j^q b_j(z)f^j},$$

*such that the meromorphic coefficients $a_i(z)$, $b_j(z)$ satisfy*

$$\begin{cases} T(r,a_i(z))=S(r,f), & i=0,1,\dots,p,\\ T(r,b_i(z))=S(r,f), & i=0,1,\dots,q,\end{cases}$$

*we have*

$$T(r,R(z,f))=\max\{p,q\}T(r,f)+S(r,f).$$

## 3. Proof of Theorem 1.5

Without loss of generality, we may suppose that $a,b\in\mathcal{S}(f)$. Below we consider the two cases where $c\equiv\infty$ and $c\not\equiv\infty$, respectively.

*Case* 1: $c\equiv\infty$. By assumption, we have

$$\frac{\Delta^n f(z)-a}{f(z)-a}=e^{\alpha},\quad \frac{\Delta^n f(z)-b}{f(z)-b}=e^{\beta}, \tag{1}$$

where $\alpha$ and $\beta$ are two entire functions. By Lemma 2.2, we deduce from the first equation in (1) that

$$T(r,e^{\alpha})=m(r,e^{\alpha})$$
$$\le m\Big(r,\frac{\Delta^n f(z)}{f(z)-a}\Big)+m\Big(r,\frac{a}{f(z)-a}\Big)+O(1)\le T(r,f)+S(r,f).$$

By applying [10, Lemma 1.1.2] to the inequality $T(r,e^{\alpha})\le(1+\varepsilon)T(r,f)$, where $\varepsilon>0$, $r\notin E$ and $E$ is an exceptional set of finite logarithmic measure, we know that there is a constant $\tau\in(1,1/\varsigma(f))$ and a constant $r_0\ge 0$ such that $T(r,e^{\alpha})\le(1+\varepsilon)T(r^{\tau},f)$ for all $r\in[r_0,\infty)$, which yields $\varsigma(e^{\alpha})\le\tau\varsigma(f)<1$. Similarly, we also have $T(r,e^{\beta})\le T(r,f)+S(r,f)$ and

$\varsigma(e^\beta) < 1$. For convenience, in what follows we will use the notations: $\alpha_i = \alpha(z+i)$ and $A_i = \alpha_i - \alpha$, $i = 0, \dots, n$. By Lemma 2.2, we have

$$T(r, e^{A_i}) = T(r, e^{\alpha(z+i)-\alpha(z)}) = m(r, e^{\alpha(z+i)-\alpha(z)}) = S(r, e^\alpha), \;\; i = 0, 1, \dots, n.$$

Similarly, by denoting $\beta_i = \beta(z+i)$ and $B_i = \beta_i - \beta$, $i = 0, \dots, n$, we have $T(r, e^{B_i}) = S(r, e^\beta)$, $i = 0, 1, \dots, n$. Eliminating $\Delta^n f(z)$ from equations in (1) gives

$$(e^\beta - e^\alpha) f = b e^\beta - a e^\alpha - (b - a).$$

We claim that $e^\alpha \equiv e^\beta$. In this case, we have $(e^\alpha - 1)(b - a) = 0$ and so $e^\alpha \equiv 1$ since $a \not\equiv b$, which with (1) yields the conclusion $\Delta^n f(z) \equiv f(z)$. Otherwise, we have

$$f = \frac{b e^\beta - a e^\alpha - (b-a)}{e^\beta - e^\alpha} = \frac{(b-a)(e^\beta - 1)}{e^\beta - e^\alpha} + a. \tag{2}$$

Moreover, from (2) it follows that

$$\begin{aligned} T(r, f) &\le T(r, e^\beta) + T(r, e^\beta - e^\alpha) + S(r, f) \\ &\le T(r, e^\alpha) + 2T(r, e^\beta) + S(r, f) \le 3T(r, f) + S(r, f). \end{aligned} \tag{3}$$

From (2) and the first equation in (1) we get

$$e^\alpha = \frac{\Delta^n f}{f - a} - \frac{a}{f - a} = \frac{\Delta^n f}{f - a} - \frac{a e^\beta}{(b-a)(e^\beta - 1)} + \frac{a e^\alpha}{(b-a)(e^\beta - 1)}. \tag{4}$$

Suppose that $ab \not\equiv 0$. If $a \equiv (b-a)(e^\beta - 1)$, i.e. $e^\beta = \frac{b}{b-a}$, then from (2) and (4), we get

$$f - a = \frac{a(b-a)}{b - (b-a)e^\alpha}, \quad \Delta^n f = \frac{b}{b-a}(f - a) = \frac{ab}{b - (b-a)e^\alpha}.$$

It follows that $T(r, f) = T(r, e^\alpha) + S(r, f)$ and so $a, b$ are both small functions of $e^\alpha$. Note that $a, b$ are periodic functions of period 1. Now we have

$$\frac{ab}{b - (b-a)e^\alpha} = \Delta^n (f - a) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{a(b-a)}{b - (b-a)e^{A_i} e^\alpha}. \tag{5}$$

Multiplying by $\prod_{i=0}^{n} [b - (b-a)e^{A_i} e^\alpha]$ and $b - (b-a)e^\alpha$ on both sides of (5), we get a polynomial in $e^\alpha$ with small functions as coefficients of degree at most $n+1$. By Lemma 2.5, we conclude that the term free of $e^\alpha$ must vanish identically, which implies that

$$abb^{n+1} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a(b-a) b^{n+1} = 0,$$

a contradiction. Hence $a \not\equiv (b-a)(e^\beta - 1)$. Then it follows from (4) that

$$e^\alpha = \Big(1 + \frac{a}{(b-a)e^\beta - b}\Big)\Big(\frac{\Delta^n f}{f-a} - \frac{a}{(b-a)(e^\beta-1)} - \frac{a}{b-a}\Big). \tag{6}$$

Since $ab \not\equiv 0$, we deduce from (6) that

$$m(r,e^\alpha) \le m\Big(r, \frac{a}{(b-a)e^\beta - b}\Big) + m\Big(r, \frac{\Delta^n f}{f-a}\Big) + m\Big(r, \frac{1}{e^\beta - 1}\Big) + S(r,f). \tag{7}$$

By Lemma 2.2, we obtain from (7) $T(r,e^\alpha) = m(r,e^\alpha) \le 2T(r,e^\beta) + S(r,f)$. On the other hand, since $ab \not\equiv 0$, we also have $T(r,e^\beta) \le 2T(r,e^\alpha) + S(r,f)$ by similar arguments. These two inequalities together with (3) imply that both $a$, $b$ are small functions of $e^\alpha$ and $e^\beta$. But then by Lemma 2.2 and Lemma 2.4, we get from (7) that $T(r,e^\alpha) = S(r,e^\beta) + S(r,f) = S(r,f)$ and also $T(r,e^\beta) = S(r,f)$, which contradicts (3). This implies that one of $a$, $b$ is equal to 0. Without loss of generality, we may suppose that $b \equiv 0$. By Lemma 2.2, we obtain from the second equation of (1) that $T(r,e^\beta) = m(r,e^\beta) = S(r,f)$, which and (3) show that both $e^\beta$ and $a$ are small functions of $e^\alpha$. Now (4) becomes

$$e^\alpha = \frac{e^\beta - 1}{e^\beta}\Big(\frac{\Delta^n f}{f-a} + \frac{e^\beta}{e^\beta - 1}\Big). \tag{8}$$

By Lemma 2.2, we deduce from (8) that

$$T(r,e^\alpha) = m(r,e^\alpha) \le m\Big(r, \frac{\Delta^n f}{f-a}\Big) + S(r,e^\alpha) = S(r,e^\alpha),$$

a contradiction. Therefore, we have $e^\alpha \equiv e^\beta \equiv 1$ when $c = \infty$ and thus Case 1 is proved.

*Case* 2: $c \not\equiv \infty$. Without loss of generality, we may suppose that $ac \not\equiv 0$. Let

$$g_1(z) = \frac{f(z)-a}{f(z)-b} \cdot \frac{c-b}{c-a}, \quad g_2(z) = \frac{\Delta^n f(z) - a}{\Delta^n f(z) - b} \cdot \frac{c-b}{c-a}. \tag{9}$$

Then $g_1(z)$ and $g_2(z)$ share the values $0, 1, \infty$ CM. Therefore, there exist two entire functions $\alpha$ and $\beta$ such that

$$\frac{g_1(z)}{g_2(z)} = e^\alpha, \quad \frac{g_1(z)-1}{g_2(z)-1} = e^\beta. \tag{10}$$

By Lemma 2.1, we deduce from equations in (9) and (10) that

$$T(r,e^\alpha) \le T(r,g_1) + T(r,g_2) + O(1)$$
$$\le T(r,f) + T(r,\Delta^n f) + S(r,f) \le (n+2)T(r,f) + S(r,f).$$

Similarly, we also have $T(r,e^{\beta}) \le (n+2)T(r,f)+S(r,f)$. Therefore, by applying [10, Lemma 1.1.2] as in Case 1 we know that both $e^{\alpha}$ and $e^{\beta}$ are of hyper-order strictly less than 1. We claim that $e^{\alpha} \equiv e^{\beta}$. In this case, we get from equations in (10) that $g_1(z) \equiv g_2(z)$ and $e^{\alpha} \equiv 1$ and then from (9) that $\Delta^n f(z) \equiv f(z)$. Otherwise, from equations in (10), we obtain

$$g_1(z) = \frac{e^{\alpha}(1-e^{\beta})}{e^{\alpha}-e^{\beta}}, \quad g_2(z) = \frac{1-e^{\beta}}{e^{\alpha}-e^{\beta}}. \tag{11}$$

From the first equation in (9) we have $T(r,f) = T(r,g_1)+S(r,f)$ and since $g_1(z) = \frac{1-e^{\beta}}{1-e^{\beta-\alpha}}$ we have $T(r,g_1) \le T(r,e^{\beta}) + T(r,e^{\beta-\alpha}) + O(1)$. Therefore,

$$T(r,f) \le T(r,e^{\alpha}) + 2T(r,e^{\beta}) + S(r,f) \le (3n+6)T(r,f)+S(r,f). \tag{12}$$

For simplicity, denote $d = \frac{c-b}{c-a}$. Then $d \not\equiv 0,1$ since $a$, $b$, $c$ are three distinct functions. Combining equations in (9) and (11), we get

$$f(z) = \frac{be^{\alpha}g_2 - ad}{e^{\alpha}g_2 - d} = \frac{(b-a)d}{e^{\alpha}g_2-d} + b, \quad \Delta^n f(z) = \frac{bg_2-ad}{g_2-d} = \frac{(b-a)d}{g_2-d}+b. \tag{13}$$

Note that $b$ is a periodic function of period 1. Substituting $g_2(z) = \frac{1-e^{\beta}}{e^{\alpha}-e^{\beta}}$ into equations in (13) gives

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)d(e^{\alpha_i}-e^{\beta_i})}{(1-d)e^{\alpha_i}+de^{\beta_i}-e^{\alpha_i+\beta_i}} = \frac{b-ade^{\alpha}-(b-ad)e^{\beta}}{1-de^{\alpha}-(1-d)e^{\beta}}. \tag{14}$$

Multiplying by $\prod_{i=0}^{n}[(1-d)e^{\alpha_i}+de^{\beta_i}-e^{\alpha_i+\beta_i}]$ and $1-de^{\alpha}-(1-d)e^{\beta}$ on both sides of (14), we obtain

$$ade^{h_1} + (b-ad)e^{h_2} + \sum_{j=3}^{m} c_j e^{h_j} = 0, \tag{15}$$

where $c_3, \ldots, c_m$ ($m \in \mathbb{N}^+$) are nonzero periodic functions of period 1 and

$$h_1 = \sum_{i=0}^{n}(\alpha_i+\beta_i)+\alpha, \quad h_2 = \sum_{i=0}^{n}(\alpha_i+\beta_i)+\beta, \quad h_j = \sum_{i=0}^{n}(l_i\alpha_i + s_i\beta_i),$$

and $l_i$, $s_i$ are integers such that $0 \le l_i, s_i \le 2$. From (14), we see that the combined number of $\alpha_i$ and $\beta_i$ corresponding to $h_j$, $j \in \{3,\ldots,m\}$ is at least $n+1$ and at most $2n+2$. Now all the coefficients of (15) are nonzero periodic functions belonging to the set $\mathcal{P}_1^1$ and $e^{h_1}$, $e^{h_2}$, $\ldots$, $e^{h_m}$ all have no zeros. By Lemma 2.3, we conclude that there exist two integers $\mu,\nu \in \{1,\ldots,m\}$ such that $e^{h_\mu - h_\nu} \in \mathcal{P}_1^1$, which implies that $h_\mu - h_\nu = \kappa(z) + 2k_0\pi iz$, where

$\kappa(z)$ is some periodic entire function of period 1 and $k_0 \in \mathbb{Z}$. If $\kappa(z)$ is transcendental, then $T(r,\kappa(z)) \geq Kr$ for some $K>0$. But according to [20, Theorem 1.45], we have $\sigma(h_\mu - h_\nu) = \varsigma(e^{h_\mu - h_\nu}) < 1$, a contradiction. Hence $\kappa(z)$ must be a constant and it follows that $e^{h_\mu - h_\nu} = a_0 e^{2k_0\pi iz}$, where $a_0 \neq 0$ is a constant and $k_0 \in \mathbb{Z}$. If $\alpha$ and $\beta$ are both polynomials of degree $\leq 1$, then $a$, $b$, $c$ are all constants and in this case $k_0 = 0$; if one of $\alpha$ and $\beta$, say $\alpha$, is transcendental or a polynomial of degree $\geq 2$, then $T(r, e^{2k_0\pi iz}) = S(r, e^\alpha)$. Therefore, we always have $T(r, e^{h_\mu - h_\nu}) = S(r,f)$. If $\mu \neq 1, 2$, we combine the two terms $c_\mu e^{h_\mu}$ and $c_\nu e^{h_\nu}$ of (15) by writing $c_\mu e^{h_\mu} + c_\nu e^{h_\nu} = (a_0 e^{2k_0\pi iz} c_\mu + c_\nu) e^{h_\nu}$ and then apply Lemma 2.3 to the resulting equation again to obtain another two integers $\mu_1, \nu_1 \in \{1,\dots,m\}$ such that $e^{h_{\mu_1} - h_{\nu_1}} \in \mathcal{P}_1^1$. Note here that it is possible that $\nu = 1$ or $\nu = 2$ and the new coefficient $a_0 e^{2k_0\pi iz} c_\mu + c_\nu$ vanishes identically. By continuing this process, we finally get $\mu = 1$ or $\mu = 2$. Recalling that $A_i = \alpha_i - \alpha$ and $B_i = \beta_i - \beta$, we have $e^{h_\mu - h_\nu} = H_\mu e^{k_\mu \alpha} H_\nu e^{k_\nu \beta}$, where $k_\mu, k_\nu \in \mathbb{Z}$, $H_\mu \not\equiv 0$ is formulated in terms of $e^{A_i}$ and $H_\nu \not\equiv 0$ is formulated in terms of $e^{B_i}$, respectively. Moreover, $H_\mu$ and $H_\nu$ satisfy $T(r, H_\mu) = S(r, e^\alpha)$ and $T(r, H_\nu) = S(r, e^\beta)$. Since the combined number of $\alpha_i$ and $\beta_i$ in $h_1$ (also $h_2$) is $2n+3$ and the combined number of $\alpha_i$ and $\beta_i$ corresponding to $h_j$, $j \in \{3,\dots,m\}$, is at most $2n+2$ and since $e^{h_1 - h_2} = e^\alpha e^{-\beta}$, $k_\mu$, $k_\nu$ must satisfy one of the following: (1) $k_\mu \neq 0$, $k_\nu \neq 0$; (2) $k_\mu = 0$, $k_\nu \neq 0$; (3) $k_\mu \neq 0$, $k_\nu = 0$. If (1) holds, then there exits an entire function $H \not\equiv 0$ such that $e^\beta = He^{t\alpha}$ for a nonzero rational number $t = -k_\mu/k_\nu$ and $H$ satisfies $H^{k_\nu} = \frac{e^{h_\mu - h_\nu}}{H_\mu H_\nu}$ and $T(r,H) = S(r,f)$. If (2) holds, then we have

$$T(r, e^{k_\nu \beta}) = T\Big(r, \frac{e^{h_\mu - h_\nu}}{H_\mu H_\nu}\Big)$$
$$\leq T(r, e^{h_\mu - h_\nu}) + T(r, H_\mu) + T(r, H_\nu) + O(1) = S(r,f),$$

and so $T(r, e^\beta) = S(r,f)$. Similarly, if (3) holds, we have $T(r, e^\alpha) = S(r,f)$. Below we discuss these three cases separately.

*Subcase* 2.1: $e^\alpha$ and $e^\beta$ satisfy $e^\beta = He^{t\alpha}$ for some nonzero rational number $t$. Without loss of generality, we may suppose $|t| \leq 1$ for otherwise we may consider $e^\alpha = [e^\beta/H]^{1/t}$. In this case, both $\alpha$ and $\beta$ are transcendental or nonconstant polynomials with the same degrees and, moreover, we have $T(r, e^\beta) = |t| T(r, e^\alpha) + S(r,f)$. Substitution into (12) gives

$$T(r,f) \leq (1 + 2|t|) T(r, e^\alpha) + S(r,f) \leq (3n+6) T(r,f) + S(r,f).$$

Therefore, $a$, $b$, $c$ are all small functions of $e^\alpha$ and $e^\beta$ and we have $T(r, e^{B_i}) = S(r, e^\alpha)$, $i = 0,1,\dots,n$, and $T(r, e^{A_i}) = S(r, e^\beta)$, $i = 0,1,\dots,n$. Denoting $h = \sum_{i=0}^n (A_i + B_i)$, we have $T(r, e^h) = S(r,f)$.

Suppose first that $t = q/p > 0$, where $p, q$ are positive co-prime integers and $q \le p$. Denote $e^{\tilde{\alpha}} = e^{\alpha/p}$. Then $T(r, e^{\alpha}) = pT(r, e^{\hat{\alpha}})$ and substituting $e^{\beta} = He^{t\alpha} = He^{q\tilde{\alpha}}$ into (15) gives

$$(16)\quad H^{n+1}e^{h}\big[ade^{(n+1)(p+q)\tilde{\alpha}+p\tilde{\alpha}} + (b-ad)He^{(n+1)(p+q)\tilde{\alpha}+q\tilde{\alpha}}\big] + P_0(e^{\tilde{\alpha}}) = 0,$$

where $P_0(e^{\tilde{\alpha}})$ is a polynomial in $e^{\tilde{\alpha}}$ of degree at most $(n+2)p + nq$ with coefficients being small with respect to $e^{\tilde{\alpha}}$. If $q < p$, then we rewrite equation (16) as

$$ade^{(n+1)(p+q)\tilde{\alpha}+p\tilde{\alpha}} = (ad-b)He^{(n+1)(p+q)\tilde{\alpha}+q\tilde{\alpha}} - \frac{P_0(e^{\tilde{\alpha}})}{H^{n+1}e^{h}},$$

where the right-hand side of the above equation is a polynomial in $e^{\tilde{\alpha}}$ of degree at most $\max\big\{(n+1)p+(n+2)q, (n+2)p+nq\big\}$. By applying Lemma 2.5 to take the characteristic functions on both sides, we get

$$\begin{aligned}&\big[(n+1)(p+q)+p\big]T(r, e^{\tilde{\alpha}}) + S(r, e^{\tilde{\alpha}})\\ &\le \max\big\{(n+1)p+(n+2)q, (n+2)p+nq\big\}\cdot T(r, e^{\tilde{\alpha}}) + S(r, e^{\tilde{\alpha}}),\end{aligned}$$

a contradicion, which implies that $p = q$ and hence $t = 1$. Now (16) becomes

$$H^{n+1}e^{h}\big[ad + (b-ad)H\big]e^{(2n+3)\alpha} + P_0(e^{\alpha}) = 0,$$

where $P_0(e^{\alpha})$ is a polynomial in $e^{\alpha}$ of degree at most $2(n+1)$ with coefficients being small with respect to $e^{\alpha}$. Hence all the coefficients of the above equation must vanish identically. In particular, we have $ad + (b-ad)H \equiv 0$ and it follows that $ade^{\alpha} + (b-ad)e^{\beta} = 0$. We see that $b \not\equiv 0$ since $e^{\alpha} \not\equiv e^{\beta}$. Substituting $e^{\alpha} = \frac{ad-b}{ad}e^{\beta}$ into (14) gives

$$(17)\qquad \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{1}{D_0e^{\beta_i} - D_1} = \frac{D_2^2}{e^{\beta} - D_2},$$

where

$$D_0 = 1 - \frac{b}{ad}, \quad D_1 = 1 - \frac{b(1-d)}{ad}, \quad D_2 = \frac{a}{a-b}.$$

Note that $D_0D_2 \not\equiv 0$. Multiplying by $\prod_{i=0}^{n}(D_0e^{\beta_i} - D_1)$ and $e^{\beta} - D_2$ on both sides of (17), we get a polynomial in $e^{\alpha}$ with small functions as coefficients of degree at most $n+1$. By Lemma 2.5, we conclude that the term free of $e^{\beta}$ must vanish identically, which implies that

$$(-D_1)^{n+1}D_2^2 = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(-D_1)^{n}(-D_2) = 0.$$

Hence $D_1 \equiv 0$. For simplicity, denote $D_4 = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}e^{-B_i}$. Since $D_0 D_2 \not\equiv 0$, we see that $D_4 \not\equiv 0$. From (17), we get

$$(D_4 - D_0 D_2^2)e^{\beta} = D_4 D_2,$$

which yields $T(r, e^{\beta}) + S(r, e^{\beta}) = S(r, e^{\beta})$ since $D_4 - D_0 D_2^2 \not\equiv 0$, a contradiction.

Suppose now that $t = -q/p < 0$, where $p, q$ are positive co-prime integers and $q \le p$. Now, for $g(z) = f(z) - b$, $g(z+i)$, $i = 0, \dots, n$, can be rewritten as

$$g(z+i) = \frac{(b-a)d(e^{2\alpha} - e^{\alpha+\beta}e^{B_i - A_i})}{(1-d)e^{2\alpha} + de^{\alpha+\beta}e^{B_i-A_i} - e^{2\alpha+\beta}e^{B_i}}$$
$$= \frac{(b-a)d}{1-d} - \frac{(b-a)d}{1-d}\,\frac{e^{\alpha+\beta}e^{B_i-A_i} - e^{2\alpha+\beta}e^{B_i}}{(1-d)e^{2\alpha} + de^{\alpha+\beta}e^{B_i-A_i} - e^{2\alpha+\beta}e^{B_i}}.$$

Note that $a$, $b$, $c$ are all periodic functions of period 1. Then (14) becomes

$$\frac{P_2(e^{\tilde\alpha})}{e^{2p(n+1)\tilde\alpha} + P_1(e^{\tilde\alpha})} = \frac{e^{p\tilde\alpha}[b - ade^{p\tilde\alpha} - (b-ad)e^{-q\tilde\alpha}]}{e^{p\tilde\alpha}[1 - de^{p\tilde\alpha} - (1-d)e^{-q\tilde\alpha}]},$$

where $P_1(e^{\tilde\alpha})$ and $P_2(e^{\tilde\alpha})$ are two polynomials in $e^{\alpha}$ of degree at most $2p(n+1) - q$ with coefficients being small with respect to $e^{\tilde\alpha}$. It follows that

$$(18)\qquad ade^{2p(n+2)\tilde\alpha} = \big[be^{p\tilde\alpha} - ade^{2p\tilde\alpha} - (b-ad)e^{(p-q)\tilde\alpha}\big]P_1(e^{\tilde\alpha})$$
$$- (b-ad)e^{p(2n+3)\tilde\alpha - q\tilde\alpha} - \big[e^{p\tilde\alpha} - de^{2p\tilde\alpha} - (1-d)e^{(p-q)\tilde\alpha}\big]P_2(e^{\tilde\alpha}) + be^{p(2n+3)\tilde\alpha},$$

where the right-hand side is a polynomial in $e^{\tilde\alpha}$ of degree at most $2p(n+2) - q$. Since $ad \not\equiv 0$, by Lemma 2.5, we deduce that

$$2p(n+2)T(r, e^{\alpha}) + S(r, e^{\tilde\alpha}) \le [2p(n+2) - q]T(r, e^{\tilde\alpha}) + S(r, e^{\tilde\alpha}),$$

which yields $qT(r, e^{\tilde\alpha}) = S(r, e^{\tilde\alpha})$, a contradiction. Therefore, Subcase 2.1 cannot occur.

*Subcase* 2.2: $e^{\beta}$ satisfies $T(r, e^{\beta}) = S(r, f)$. In this case, if both $\alpha$ and $\beta$ are polynomials, then the degree of $\beta$ is less than the degree of $\alpha$. From (12) we see that $a$, $b$, $c$ are all small functions of $e^{\alpha}$ and $T(r, e^{\beta_i}) = S(r, e^{\alpha})$, $i = 0, 1, \dots, n$. We now show that there exists some $i$, $0 \le i \le n$ such that $1 - d - e^{\beta_i} \equiv 0$. To this end, we assume that $1 - d - e^{\beta_i} \not\equiv 0$ for each $i \in \{0, 1, \dots, n\}$ and rewrite $g(z+i) = f(z+i) - b$ as

$$(19)\qquad g(z+i) = \frac{(b-a)d}{1-d-e^{\beta_i}} - \frac{(b-a)de^{\beta_i}(1-e^{\beta_i})}{(1-d-e^{\beta_i})[(1-d-e^{\beta_i})e^{A_i}e^{\alpha} + de^{\beta_i}]}, \quad i = 0, \dots, n.$$

Combining (14) and (19), we get

$$\begin{aligned}
(20)\qquad & \left[ad-\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)d^2}{1-d-e^{\beta_i}}\right]e^{\alpha} \\
&+\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)d[1-(1-d)e^{\beta}]}{1-d-e^{\beta_i}} \\
&=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)de^{\beta_i}(1-e^{\beta_i})[1-de^{\alpha}-(1-d)e^{\beta}]}{(1-d-e^{\beta_i})[(1-d-e^{\beta_i})e^{A_i}e^{\alpha}+de^{\beta_i}]}+b-(b-ad)e^{\beta}.
\end{aligned}$$

If $ad-\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)d^2}{1-d-e^{\beta_i}}\not\equiv 0$, then by Lemma 2.4, we deduce from (20) that

$$\begin{aligned}
T(r,e^{\alpha}) &= m\left(r,\left[ad-\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)d^2}{1-d-e^{\beta_i}}\right]e^{\alpha}\right)+S(r,e^{\alpha}) \\
&\le \sum_{i=0}^{n} m\left(r,\frac{1-de^{\alpha}-(1-d)e^{\beta}}{(1-d-e^{\beta_i})e^{A_i}e^{\alpha}+de^{\beta_i}}\right)+S(r,e^{\alpha})=S(r,e^{\alpha}),
\end{aligned}$$

a contradiction. Hence $ad-\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)d^2}{1-d-e^{\beta_i}}\equiv 0$ and thus (20) becomes

$$\begin{aligned}
(21)\qquad & \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)de^{\beta_i}(1-e^{\beta_i})}{(1-d-e^{\beta_i})[(1-d-e^{\beta_i})e^{A_i}e^{\alpha}+de^{\beta_i}]} \\
&= \frac{(b-a)(e^{\beta}-1)}{1-de^{\alpha}-(1-d)e^{\beta}}.
\end{aligned}$$

If $1-(1-d)e^{\beta}\equiv 0$, then $e^{\beta}$ is a periodic function of period 1 and so $\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)d^2}{1-d-e^{\beta_i}}\equiv 0$. But then it follows that $ad\equiv 0$, a contradiction. Therefore, $1-(1-d)e^{\beta}\not\equiv 0$. Multiplying by $\prod_{i=0}^{n}[(1-d-e^{\beta_i})e^{A_i}e^{\alpha}+de^{\beta_i}]$ and $1-de^{\alpha}-(1-d)e^{\beta}$ on both sides of (21) and applying Lemma 2.5 to the resulting equation, we conclude that the term free of $e^{\alpha}$ must vanish identically. Hence we have

$$\begin{aligned}
&[1-(1-d)e^{\beta}]\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)de^{\beta_i}(1-e^{\beta_i})}{1-d-e^{\beta_i}}\prod_{j=0,j\neq i}^{n}(de^{\beta_j}) \\
&= (b-a)(e^{\beta}-1)\prod_{j=0}^{n}(de^{\beta_j}).
\end{aligned}$$

Combining this equation with $ad - \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)d^2}{1-d-e^{\beta_i}} \equiv 0$, we get $(b-ad)e^{\beta} = b$. Since $b - ad \not\equiv 0$, we see that $e^{\beta}$ is a periodic function of period 1. But this leads to equation $\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(b-a)d^2}{1-d-e^{\beta_i}} \equiv 0$ again, a contradiction. Therefore, there exists some $i$, $0 \le i \le n$ such that $1 - d - e^{\beta_i} \equiv 0$ and it follows that $1 - d - e^{\beta_i} \equiv 0$ for all $i \in \{0, \dots, n\}$ from the assumption that $d$ is a periodic function of period 1. Now (14) becomes

$$\frac{b-a}{1-d}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}e^{A_i}e^{\alpha} = \frac{b+a-ad-ae^{\alpha}}{2-d-e^{\alpha}}.$$

For simplicity, denote $U = \frac{b-a}{1-d}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}e^{A_i}$. Then $U \not\equiv 0$ and we have $T(r, U) = S(r, e^{\alpha})$. From the above equation, we get

$$Ue^{2\alpha} = [a - (d-2)U]e^{\alpha} + ad - b - a,$$

which yields $2T(r, e^{\alpha}) + S(r, e^{\alpha}) \le T(r, e^{\alpha}) + S(r, e^{\alpha})$, a contradiction. Hence Subcase 2.2 cannot occur.

*Subcase* 2.3: $e^{\alpha}$ satisfies $T(r, e^{\alpha}) = S(r, f)$. In this case, if both $\alpha$ and $\beta$ are polynomials, then the degree of $\alpha$ is less than the degree of $\beta$. From (12) we see that $a$, $b$, $c$ are all small functions of $e^{\beta}$ and $T(r, e^{\alpha_i}) = S(r, e^{\beta})$, $i = 0, 1, \dots, n$. By similar arguments as in Subcase 2.2, we easily obtain $e^{\alpha} \equiv d$ and then from (14) that

$$(a-b)\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}B_i e^{\beta}(e^{\beta} - d - 1) = (b - ad)e^{\beta} - ad^2 + b.$$

For simplicity, denote $V = (a-b)\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}B_i$. Since $b - ad \not\equiv 0$, we see that $V \not\equiv 0$ and we have $T(r, V) = S(r, e^{\beta})$. From the above equation, we get

$$Ve^{2\beta} = [(d+1)V + b - ad]e^{\beta} - ad^2 + b,$$

which yields $2T(r, e^{\beta}) + S(r, e^{\beta}) \le T(r, e^{\beta}) + S(r, e^{\beta})$, a contradiction. So this subcase cannot occur either.

From the above discussions, we know that $e^{\alpha} \equiv e^{\beta} \equiv 1$ when $c \not\equiv \infty$. This also completes the proof.

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