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APPLICATIONS OF THE THEORY OF ORLICZ SPACES TO VECTOR MEASURES

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Abstract. Let $(\Omega, \Sigma, \lambda)$ be a finite complete measure space, (E, ξ) be a sequentially complete locally convex Hausdorff space and E' be its topological dual. Let $\mathrm{ca}_\lambda(\Sigma, E)$ stand for the space of all λ-absolutely continuous measures $m\colon \Sigma \to E$. We show that a uniformly bounded subset $\mathcal{M}$ of $\mathrm{ca}_\lambda(\Sigma, E)$ is uniformly λ-absolutely continuous if and only if for every equicontinuous subset D of E', there exists a submultiplicative Young function φ such that the set $\{\frac{d(e'\circ m)}{d\lambda} : m \in \mathcal{M}, e' \in D\}$ is relatively weakly compact in the Orlicz space $L^\varphi(\lambda)$. As a consequence, we present a generalized Vitali–Hahn–Saks theorem on the setwise limit of a sequence of λ-absolutely continuous vector measures in terms of Orlicz spaces.

1. Introduction and preliminaries

Let $(\Omega, \Sigma, \lambda)$ be a finite complete measure space. By $L^o(\lambda)$ we denote the space of λ-equivalence classes of all complex-valued λ-measurable functions defined on Ω. Then $L^o(\lambda)$ can be equipped with the F-norm topology $\mathcal{T}_o$ of convergence in measure λ.

Let $L^\infty(\lambda)$ denote the space of all $v \in L^o(\lambda)$ such that

$$\|v\|_\infty := \operatorname*{ess\,sup}_{\omega\in\Omega} |v(\omega)| < \infty$$

and $\mathcal{T}_\infty$ denote the topology of the norm $\|\cdot\|_\infty$. By $\mathcal{T}_1$ we denote the topology on $L^1(\lambda)$ of the norm $\|v\|_1 := \int_\Omega |v|\, d\lambda$. Then $\mathcal{T}_o\,|_{L^\infty(\lambda)} \subset \mathcal{T}_1\,|_{L^\infty(\lambda)} \subset \mathcal{T}_\infty$.

From now on, we will assume (unless otherwise specified) that (E, ξ) is a sequentially complete locally convex Hausdorff space over the complex field. By E' we will denote the topological dual of (E, ξ). Let $\mathcal{E}$ be the family

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of all ξ-equicontinuous subsets of E'. Then ξ is generated by the family of seminorms $\{p_D : D \in \mathcal{E}\}$, where $p_D(e) := \sup\{|e'(e)| : e' \in D\}$ for $e \in E$ (see [2, Theorem 9.22]).

Let $\sigma(E, E')$ and $\tau(E, E')$ stand for the weak topology and the Mackey topology on E with respect to the dual pair $\langle E, E' \rangle$.

For terminology and basic facts concerning vector measures we refer the reader to [8,12].

For a finitely additive measure $m\colon \Sigma \to E$ we write $m \ll \lambda$ if $m(A) = 0$ whenever $\lambda(A) = 0$ for $A \in \Sigma$. By $\mathrm{ba}_\lambda(\Sigma, E)$ we denote the set of all bounded finitely additive measures $m\colon \Sigma \to E$ (i.e., the range of m is ξ-bounded in E) such that $m \ll \lambda$. Let $\mathrm{ba}_\lambda(\Sigma) := \mathrm{ba}_\lambda(\Sigma, \mathbb{C})$.

Let $m \in \mathrm{ba}_\lambda(\Sigma, E)$. For $e' \in E'$, let

$$m_{e'}(A) := e'(m(A)) \quad \text{for } A \in \Sigma.$$

For $D \in \mathcal{E}$ one can define the *semivariation* $\|m\|_D(A)$ of m on $A \in \Sigma$ by setting:

$$\|m\|_D(A) := \sup_{e' \in D} |m_{e'}|(A),$$

where $|m_{e'}|(A)$ stands for the variation of $m_{e'}$ on $A \in \Sigma$. In view of [8, Proposition 11, pp. 4–5] for $D \in \mathcal{E}$ and $A \in \Sigma$, we have

$$\begin{aligned} \sup\big\{p_D(m(B)) : B \in \Sigma,\ B \subset A\big\} &\le \|m\|_D(A) \\ &\le 4 \sup\big\{p_D(m(B)) : B \in \Sigma,\ B \subset A\big\} < \infty. \end{aligned}$$

It follows that $\|m\|_D(\Omega) < \infty$ for every $D \in \mathcal{E}$. Every $v \in L^\infty(\lambda)$ is m-integrable and one can define the integral $\int_\Omega v\, dm$ by

$$\int_\Omega v\, dm := \lim_n \int_\Omega s_n\, dm,$$

where (s_n) is a sequence of λ-simple functions on Ω such that $\|v - s_n\|_\infty \to 0$ (see [8,12]). The corresponding integration operator $T_m\colon L^\infty(\lambda) \to E$ defined by $T_m(v) := \int_\Omega v\, dm$ is $(\mathcal{T}_\infty, \xi)$-continuous and for $e' \in E'$,

$$e'(T_m(v)) = \int_\Omega v\, dm_{e'} \quad \text{for } v \in L^\infty(\lambda),$$

where $\|e' \circ T_m\| = |m_{e'}|(\Omega)$. Note that for $e' \in E'$, and $v \in L^\infty(\lambda)$,

$$|e'(T_m(v))| \le \|v\|_\infty\, |m_{e'}|(\Omega)$$

and hence for every $D \in \mathcal{E}$,

$$p_D(T_m(v)) \le \|v\|_\infty\, \|m\|_D(\Omega).$$

Recall that a subset $\mathcal{M}$ of $\mathrm{ba}_\lambda(\Sigma, E)$ is said to be:
(i) *uniformly bounded* if for every $D \in \mathcal{E}$, $\sup_{m\in\mathcal{M}} \|m\|_D(\Omega) < \infty$;
(ii) *uniformly countably additive* if for every $D \in \mathcal{E}$,

$$\sup_{m\in\mathcal{M}} p_D(m(A_n)) \to 0$$

whenever $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$;
(iii) *uniformly λ-absolutely continuous* if for every $D \in \mathcal{E}$,

$$\sup_{m\in\mathcal{M}} p_D(m(A_n)) \to 0$$

whenever $\lambda(A_n) \to 0$, $(A_n) \subset \Sigma$.

It is known that every countably additive measure $m\colon \Sigma \to E$ is bounded (see [9, Corollary 4.12]). Note that a countably additive measure $m\colon \Sigma \to E$ is λ-absolutely continuous if and only if $m_{e'} \ll \lambda$ for every $e' \in E'$ (see [8, Theorem 1, p. 10 and Corollary 5, p. 29]. By $\mathrm{ca}_\lambda(\Sigma, E)$ we denote the space of all λ-absolutely continuous measures $m\colon \Sigma \to E$. In particular, we write $\mathrm{ca}_\lambda(\Sigma) := \mathrm{ca}_\lambda(\Sigma, \mathbb{C})$.

In view of the Radon–Nikodym theorem for every $\mu \in \mathrm{ca}_\lambda(\Sigma)$ there exists a unique $u_\mu \in L^1(\lambda)$ such that $\mu(A) = \int_A u_\mu \, d\lambda$ for every $A \in \Sigma$, i.e., $u_\mu = \frac{d\mu}{d\lambda}$. Then for $m \in \mathrm{ca}_\lambda(\Sigma, E)$ and $e' \in E'$, $\int_\Omega v \, dm_{e'} = \int_\Omega v \, u_{m_{e'}} \, d\lambda$ for $v \in L^\infty(\lambda)$.

Now we recall basic concepts concerning Orlicz spaces (see [16] for more details). By a *Young function* we mean here a convex function $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\varphi(s) = 0$ if and only if $s = 0$ and $\varphi(s)/s \to 0$ as $s \to 0$ and $\varphi(s)/s \to \infty$ as $s \to \infty$. We denote by $L^\varphi(\lambda)$ the Orlicz space associated with φ, that is,

$$L^\varphi(\lambda) = \Big\{ v \in L^o(\lambda) : \int_\Omega \varphi(a|v|) \, d\lambda < \infty \text{ for some } a > 0 \Big\},$$

$L^\varphi(\lambda)$, equipped with the norm:

$$\|v\|_\varphi := \inf \Big\{ a > 0 : \int_\Omega \varphi(|v|/a) \, d\lambda \le 1 \Big\}$$

is a Banach space. Then $\|v\|_\varphi \le 1$ if and only if $\int_\Omega \varphi(|v|) \, d\lambda \le 1$.

The theory of Orlicz spaces has been used to characterize integrability of Banach space-valued functions on Ω (see [3,13,18]). Indeed, if E is a Banach space, then Uhl [18, Theorem] using the de la Vallée-Poussin's theorem on relative weak compactness in $L^1(\lambda)$, showed that a strongly measurable function $f\colon \Omega \to E$ is Pettis integrable with respect to λ if and only if there exists a Young function φ such that $e' \circ f \in L^\varphi(\lambda)$ for each $e' \in E'$ (see also

[13, Theorem 5.3]). Barcenas and Finol [3, Corollary 2.2] provided a characterization of the Pettis integral of a Dunford integrable function $f\colon \Omega \to E$ in terms of relative weak compactness in some Orlicz space $L^\varphi(\lambda)$.

In this paper we use the theory of Orlicz spaces to study λ-absolutely continuous measures $m\colon \Sigma \to E$, where (E, ξ) is a sequentially complete locally convex Hausdorff space.

In Section 2 we show that a uniformly bounded subset $\mathcal{M}$ of $\mathrm{ca}_\lambda(\Sigma, E)$ is uniformly λ-absolutely continuous if and only if for every $D \in \mathcal{E}$ there exists a submultiplicative Young function φ such that the set $\{u_{m_{e'}} : m \in \mathcal{M}, e' \in D\}$ is relatively weakly compact in $L^\varphi(\lambda)$ (see Theorem 2.3 below). If, in particular, E is a Banach space, we obtain that $m \in \mathrm{ba}_\lambda(\Sigma, E)$ is λ-absolutely continuous if and only if there exists a Young function φ such that $\{\int_\Omega v\, dm : v \in L^\infty(\lambda), \int_\Omega \varphi(|v|)\, d\lambda \le 1\}$ is a relatively weakly compact subset of E (see Corollary 2.6 below). As a consequence, in Section 3, we present a generalized Vitali–Hahn–Saks theorem on the setwise limit of a sequence of λ-absolutely continuous measures, in terms of Orlicz spaces.

2. Mixed topologies on $L^\infty(\lambda)$ and λ-absolutely continuous vector measures

By $\gamma[\mathcal{T}_\infty, \mathcal{T}_1]$ (briefly, γ) we denote the natural *mixed topology* on $L^\infty(\lambda)$ (see [5,6,19]) for more details). In view of [5, Proposition 1] γ is the finest locally convex Hausdorff topology on $L^\infty(\lambda)$ which agrees with $\mathcal{T}_1$ on $\mathcal{T}_\infty$-bounded sets. The topologies $\mathcal{T}_\infty$ and γ have the same bounded sets in $L^\infty(\lambda)$. This means that $(L^\infty(\lambda), \gamma)$ is a generalized DF-space (see [17] for more details).

Let Φ denote the family of all Young functions. Then $L^\infty(\lambda) \subset L^\varphi(\lambda)$ for $\varphi \in \Phi$.

Theorem 2.1. (i) *γ coincides with the Mackey topology $\tau(L^\infty(\lambda), L^1(\lambda))$.*
(ii) *γ is generated by the family of norms $\{\|\cdot\|_\varphi\,|_{L^\infty(\lambda)} \colon \varphi \in \Phi\}$.*
(iii) *For a sequence (v_n) in $L^\infty(\lambda)$ the following statements are equivalent:*
(a) *$v_n \to 0$ in γ.*
(b) *$\|v_n\|_\varphi \to 0$ for every $\varphi \in \Phi$.*
(c) *$\sup_n \|v_n\|_\infty < \infty$ and $\int_\Omega |v_n|\, d\lambda \to 0$.*
(d) *$\sup_n \|v_n\|_\infty < \infty$ and $v_n \to 0$ in measure λ.*

Proof. (i) See [6, p. 172].
(ii) It follows from (i) and [14, Theorem 2.2].
(iii) From (ii) it follows that (a)$\Leftrightarrow$(b). By [5, Proposition 1] we have that (a) $\Leftrightarrow$ (c). Since $\mathcal{T}_0\,|_{L^\infty(\lambda)} \subset \mathcal{T}_1\,|_{L^\infty(\lambda)}$, we get (c) $\Rightarrow$ (d). In view of the Lebesgue dominated convergence theorem, we have that (d) $\Rightarrow$ (c). $\square$

Combining the de la Vallée-Poussin type characterization of relatively weakly compact sets in $L^1(\lambda)$ (see [1, Theorem 2.5]) and the classical Dunford–Pettis theorem (see [7, Theorem, p. 93]), we have:

Theorem 2.2. *For a bounded subset $\mathcal{A}$ of $L^1(\lambda)$ the following statements are equivalent*:

(i) *$\mathcal{A}$ is relatively weakly compact in $L^1(\lambda)$.*

(ii) *The set of the indefinite integrals of members of $\mathcal{A}$, $\{\int_{(\cdot)} u\,d\lambda : u \in \mathcal{A}\}$ is uniformly countably additive.*

(iii) *$\mathcal{A}$ is uniformly integrable.*

(iv) *There exists a submultiplicative Young function φ such that $\mathcal{A}$ is relatively weakly compact in $L^\varphi(\lambda)$.*

The following theorem gives a characterization of uniformly λ-absolutely continuous subsets $\mathcal{M}$ of $\mathrm{ba}_\lambda(\Sigma, E)$.

Theorem 2.3. *For a uniformly bounded subset $\mathcal{M}$ of $\mathrm{ba}_\lambda(\Sigma, E)$ the following statements are equivalent*:

(i) *$\{T_m : m \in \mathcal{M}\}$ is (γ, ξ)-equicontinuous.*

(ii) *$\int_\Omega v_n\,dm \to 0$ in ξ uniformly with respect to $m \in \mathcal{M}$ whenever (v_n) is a sequence in $L^\infty(\lambda)$ such that $\|v_n\|_\varphi \to 0$ for every Young function φ.*

(iii) *$\int_\Omega v_n\,dm \to 0$ in ξ uniformly with respect to $m \in \mathcal{M}$ whenever (v_n) is a sequence in $L^\infty(\lambda)$ such that $\sup_n \|v_n\|_\infty < \infty$ and $v_n \to 0$ in measure λ.*

(iv) *$\mathcal{M}$ is uniformly λ-absolutely continuous.*

(v) *$\mathcal{M}$ is uniformly countably additive.*

If, in particular, $\mathcal{M}$ is a uniformly bounded subset of $\mathrm{ca}_\lambda(\Sigma, E)$, then each of the statements (i)–(v) *is equivalent to the following*:

(vi) *For every $D \in \mathcal{E}$, there exists a submultiplicative Young function φ such that the set $\{u_{m_{e'}} : m \in \mathcal{M}, e' \in D\}$ is relatively weakly compact in $L^\varphi(\lambda)$.*

Proof. (i) $\Rightarrow$ (ii) Assume that (i) holds and (v_n) is a sequence in $L^\infty(\lambda)$ such that $\|v_n\|_\varphi \to 0$ for every $\varphi \in \Phi$. Then by Theorem 2.1 $v_n \to 0$ in γ. Let $D \in \mathcal{E}$ and $\varepsilon > 0$ be given. Hence there exists a γ-neighborhood V of 0 such that $\sup_{m\in\mathcal{M}} p_D(T_m(v)) \le \varepsilon$ for $v \in V$. Choose $n_o \in \mathbb{N}$ such that $v_n \in V$ for $n \ge n_o$. Then for $n \ge n_o$, we get $\sup_{m\in\mathcal{M}} p_D(T_m(v_n)) \le \varepsilon$.

(ii) $\Leftrightarrow$ (iii) This follows from Theorem 2.1.

(iii) $\Rightarrow$ (iv) Assume that (iii) holds and $\lambda(A_n) \to 0$, $(A_n) \subset \Sigma$. Then $\mathbb{1}_{A_n} \to 0$ in measure λ and $\sup_n \|\mathbb{1}_{A_n}\|_\infty \le 1$. It follows that for every $D \in \mathcal{E}$,

$$\sup_{m\in\mathcal{M}} p_D(m(A_n)) = \sup_{m\in\mathcal{M}} p_D(T_m(\mathbb{1}_{A_n})) \to 0.$$

(iv) $\Rightarrow$(v) Obvious.

(v) $\Rightarrow$ (i) Assume that (v) holds and $D \in \mathcal{E}$ and $\varepsilon > 0$ are given. Then the set $\mathcal{M}_D := \{m_{e'} : m \in \mathcal{M},\ e' \in D\}$ is uniformly countably additive and

$\sup\{|m_{e'}|(\Omega) : m \in \mathcal{M},\ e' \in D\} < \infty$, where $m_{e'} \in \mathrm{ca}_\lambda(\Sigma)$ for every $m \in \mathcal{M}$, $e' \in D$. Hence by the Dunford–Pettis theorem (see [7, Theorem, p. 93]) $\mathcal{A} = \{u_{m_{e'}} : m \in \mathcal{M},\ e' \in D\}$ is a relatively $\sigma(L^1(\lambda), L^\infty(\lambda))$-compact subset of $L^1(\lambda)$. By the Krein–Smulian theorem (see [2, Theorem 10.15]) abs conv $\mathcal{A}$ is relatively $\sigma(L^1(\lambda), L^\infty(\lambda))$-compact. Hence the polar $\mathcal{A}^0$ of $\mathcal{A}$ with respect to the dual pair $\langle L^1(\lambda), L^\infty(\lambda)\rangle$ is a $\tau(L^\infty(\lambda), L^1(\lambda))$-neighborhood of 0 in $L^\infty(\lambda)$. Then for every $m \in \mathcal{M}$, $e' \in D$ and $v \in \varepsilon\mathcal{A}^0$, we get $|\int_\Omega v\, dm_{e'}| = |\int_\Omega v\, u_{m_{e'}}\, d\lambda| \le \varepsilon$. It follows that $\sup_{m\in\mathcal{M}} p_D(T_m(v)) \le \varepsilon$ for $v \in \varepsilon\mathcal{A}^0$. Since $\gamma = \tau(L^\infty(\lambda), L^1(\lambda))$, the proof is complete.

Assume that $\mathcal{M}$ is a uniformly bounded subset of $\mathrm{ca}_\lambda(\Sigma, E)$.

(v) $\Rightarrow$ (vi) Assume that (v) holds and $D \in \mathcal{E}$. Then the family $\{m_{e'} : m \in \mathcal{M},\ e' \in D\}$ is uniformly countably additive, where $m_{e'}$ is absolutely λ-continuous for $m \in \mathcal{M}$, $e' \in D$. Moreover, $\sup\{|m_{e'}|(\Omega) : m \in \mathcal{M},\ e' \in D\} < \infty$. Hence by the Theorem 2.2 condition (vi) holds.

(vi) $\Rightarrow$ (v) Assume that (vi) holds and $D \in \mathcal{E}$. Then by Theorem 2.2 the family $\{m_{e'} : m \in \mathcal{M}$ and $e' \in D\}$ is uniformly countably additive, and this means that $\mathcal{M}$ is uniformly countably additive. □

As an application of Theorem 2.3, we have a characterization of λ-absolutely continuous measures $m\colon \Sigma \to E$.

Corollary 2.4. *For $m \in \mathrm{ba}_\lambda(\Sigma, E)$ the following statements are equivalent*:

(i) $m \in \mathrm{ca}_\lambda(\Sigma, E)$.

(ii) T_m *is* $(\sigma(L^\infty(\lambda), L^1(\lambda)), \sigma(E, E'))$*-continuous.*

(iii) T_m *is* $(\gamma, \tau(E, E'))$*-continuous.*

(iv) T_m *is* (γ, ξ)*-continuous.*

(v) $\int_\Omega v_n dm \to 0$ *in* ξ *whenever* (v_n) *is a sequence in* $L^\infty(\lambda)$ *such that* $\|v_n\|_\varphi \to 0$ *for every Young function* φ.

(vi) $\int_\Omega v_n dm \to 0$ *in* ξ *whenever* (v_n) *is a sequence in* $L^\infty(\lambda)$ *such that* $\sup_n \|v_n\|_\infty < \infty$ *and* $v_n \to 0$ *in measure* λ.

Proof. (i) $\Rightarrow$ (ii) Assume that $m \in \mathrm{ca}_\lambda(\Sigma, E)$ and (v_α) is a net in $L^\infty(\lambda)$ such that $v_\alpha \to 0$ in $\sigma(L^\infty(\lambda), L^1(\lambda))$. Then for every $e' \in E'$, we have

$$e'(T_m(v_\alpha)) = \int_\Omega v_\alpha\, dm_{e'} = \int_\Omega v_\alpha\, u_{m_{e'}}\, d\lambda \underset{\alpha}{\longrightarrow} 0.$$

(ii) $\Leftrightarrow$ (iii) This follows from [2, Exercise 11, p. 149] because $\gamma = \tau(L^\infty(\lambda), L^1(\lambda))$.

(iii) $\Rightarrow$ (iv) This is obvious.

(iv) $\Rightarrow$ (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) See Theorem 2.3. □

Corollary 2.5. *Assume that $m \in \mathrm{ca}_\lambda(\Sigma, E)$. Then T_m is weakly compact.*

Proof. It follows from Corollary 2.4 because the closed unit ball $B_\infty(1) = \{v \in L^\infty(\lambda) : \|v\|_\infty \le 1\}$ is $\sigma(L^\infty(\lambda), L^1(\lambda))$-compact. □

Corollary 2.6. *Assume that* $(E, \|\cdot\|_E)$ *is a Banach space and* $m \in \mathrm{ba}_\lambda(\Sigma, E)$. *Then the following statements are equivalent*:

(i) $m \in \mathrm{ca}_\lambda(\Sigma, E)$.

(ii) T_m *is* $(\gamma, \|\cdot\|_E)$*-weakly compact, i.e.,* $T_m(V)$ *is a relatively weakly compact subset of* E *for some* γ*-neighborhood* V *of* 0 *in* $L^\infty(\lambda)$.

(iii) *There exists a Young function* φ *such that*

$$\left\{ \int_\Omega v\, dm : v \in L^\infty(\lambda),\ \int_\Omega \varphi(|v|)\, d\lambda \le 1 \right\}$$

is a relatively weakly compact subset of E.

Proof. (i) ⇒ (ii) Assume that $m \in \mathrm{ca}_\lambda(\Sigma, E)$. Then by Corollary 2.5 T_m is $(\gamma, \|\cdot\|_E)$-continuous. $\mathcal{T}_\infty$ and γ have the same bounded sets, so by Corollary 2.5 T_m maps γ-bounded sets onto relatively weakly compact sets in E. Since $(L^\infty(\lambda), \gamma)$ is a generalized DF-space, in view of the Grothendieck type theorem (see [17, Theorem 3.1]) $T_m(V)$ is relatively weakly compact in E for some γ-neighborhood V of 0 in $L^\infty(\lambda)$.

(ii) ⇒ (i) Assume that (ii) holds. Then T_m is $(\gamma, \|\cdot\|_E)$-continuous and hence, by Corollary 2.4 $m \in \mathrm{ca}_\lambda(\Sigma, E)$.

(ii) ⇔ (iii) This follows from Theorem 2.1. □

Recall that a bounded linear operator T between two Banach spaces is a *Dunford–Pettis operator* if T maps weakly convergent sequences onto norm convergent sequences (see [2, Section 19]). It is well known that every weakly compact operator T from $L^1(\lambda)$ to a Banach space E is a Dunford–Pettis operator (see [10, Chapter 6, Section 8, Theorem 12]). Bourgain [4, Proposition 1] showed that a bounded linear operator T from $L^1(\lambda)$ to a Banach space E is a Dunford–Pettis operator if and only if T restricted to $L^p(\lambda)$ for some $p \in (1, \infty]$ is compact.

Assume that $(E, \|\cdot\|_E)$ is a Banach space and $T\colon L^1(\lambda) \to E$ is a bounded linear operator. Let $i_\infty\colon L^\infty(\lambda) \to L^1(\lambda)$ denote the inclusion map. Let $m(A) := T(\mathbb{1}_A)$ for $A \in \Sigma$. Then $\|m\|(\Omega) < \infty$ and

$$T(v) = \int_\Omega v\, dm \quad \text{for } v \in L^\infty(\lambda).$$

Now using [15, Corollary 2.5], we state a characterization of Dunford–Pettis operators $T\colon L^1(\lambda) \to E$ in terms of Orlicz spaces.

Corollary 2.7. *Assume that* $(E, \|\cdot\|_E)$ *is a Banach space. Then for a bounded linear operator* $T\colon L^1(\lambda) \to E$ *the following statements are equivalent*:

(i) *T is a Dunford–Pettis operator.*

(ii) *$T \circ i_\infty \colon L^\infty(\lambda) \to E$ is $(\gamma, \|\cdot\|_E)$-compact, i.e., $T(V)$ is a relatively compact subset of E for some γ-neighborhood V of 0 in $L^\infty(\lambda)$.*

(iii) *There exists a Young function φ such that*

$$\Big\{T(v) : v \in L^\infty(\lambda),\ \int_\Omega \varphi(|v|)\, d\lambda \le 1\Big\}$$

is a relatively compact set in E.

Proof. (i) $\Leftrightarrow$ (ii) This follows from [15, Corollary 2.5] because $\gamma = \tau(L^\infty(\lambda), L^1(\lambda))$.

(ii) $\Leftrightarrow$ (iii) This follows from Theorem 2.1. $\square$

3. A Vitali–Hahn–Saks theorem for vector measures

Recall that the general Vitali–Hahn–Saks theorem says that if (m_k) is a sequence of λ-absolutely continuous measures on a σ-algebra Σ taking values in a Banach space E, and $m(A) := \lim m_k(A)$ for each $A \in \Sigma$, then $m \colon \Sigma \to E$ is a λ-absolutely continuous measure and the family $\{m_k : k \in \mathbb{N}\}$ is uniformly λ-absolutely continuous (see [8, Corollary 10, p. 24–25]).

Now using Theorem 2.3 we present a generalized Vitali–Hahn–Saks theorem in case that (E, ξ) is a sequentially complete locally convex Hausdorff space.

Theorem 3.1. *Let $m_k \in \mathrm{ca}_\lambda(\Sigma, E)$ for $k \ge 1$ and assume that $m_o(A) := \xi - \lim m_k(A)$ exists in E for every $A \in \Sigma$. Then the following statements hold:*

(i) *$\{m_k : k \ge 0\}$ is uniformly λ-absolutely continuous.*

(ii) *For every $D \in \mathcal{E}$, there exists a submultiplicative Young function φ such that the set $\{u_{(m_k)_{e'}} : k \ge 0,\ e' \in D\}$ is relatively weakly compact in $L^\varphi(\lambda)$.*

(iii) *$\int_\Omega v_n\, dm_k \xrightarrow[n]{} 0$ in ξ uniformly with respect to $k \ge 0$ whenever (v_n) is a sequence in $L^\infty(\lambda)$ such that $\sup_n \|v_n\|_\infty < \infty$ and $v_n \to 0$ in measures λ.*

(iv) *$\int_\Omega v_n\, dm_k \xrightarrow[n]{} 0$ in ξ uniformly with respect to $k \ge 0$ whenever (v_n) is a sequence in $L^\infty(\lambda)$ such that $\|v_n\|_\varphi \to 0$ for every Young function φ.*

(v) *$\int_\Omega v\, dm_k \xrightarrow[k]{} \int_\Omega v\, dm_o$ in ξ for every $v \in L^\infty(\lambda)$.*

Proof. (i) In view of the Nikodym convergence theorem $m_o \colon \Sigma \to E$ is a countably additive measure and $\{m_k : k \ge 1\}$ is a uniformly countably additive subset of $\mathrm{ca}_\lambda(\Sigma, E)$ (see [11, Theorem 9]). According to the Nikodym boundedness theorem (see [11, Theorem 8]) $\sup_{k \ge 0} \|m_k\|_D(\Omega) < \infty$ holds for every $D \in \mathcal{E}$. By the Vitali–Hahn–Saks theorem (see [8, Corollary 10, p. 24–25]), we have $(m_o)_{e'} \in \mathrm{ca}_\lambda(\Sigma)$ for every $e' \in E'$ and it follows that $m_o \in \mathrm{ca}_\lambda(\Sigma, E)$. Moreover, by Theorem 2.3 $\{m_k : k \geq 0\}$ is uniformly λ-absolutely continuous.

(ii)–(iv) This follows from (i) and Theorem 2.3.

(v) Let $D \in \mathcal{E}$ and $\varepsilon > 0$ be given. Let $c_D = \sup_{k \geq 0} \|m_k\|_D(\Omega)$. Assume that $v \in L^\infty(\lambda)$. Choose a λ-simple function $s = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}$ such that $\|v - s\|_\infty \leq \frac{\varepsilon}{3c_D}$. Then for every $k \geq 0$,

$$p_D(T_{m_k}(v - s)) \leq \|v - s\|_\infty \|m_k\|_D(\Omega) \leq \frac{\varepsilon}{3}.$$

We can choose $k_0 \in \mathbb{N}$ such that for $k \geq k_0$,

$$|\alpha_i|\, p_D(m_k(A_i) - m_o(A_i)) \leq \frac{\varepsilon}{3n} \quad \text{for } i = 1, \ldots, n.$$

Then for $k \geq k_0$,

$$p_D(T_{m_k}(s) - T_{m_o}(s)) \leq \sum_{i=1}^n |\alpha_i|\, p_D(m_k(A_i) - m_o(A_i)) \leq n \cdot \frac{\varepsilon}{3n} = \frac{\varepsilon}{3}.$$

Hence for $k \geq k_0$, we get

$$p_D(T_{m_k}(v) - T_{m_o}(v)) \leq p_D\big(T_{m_k}(v) - T_{m_k}(s)\big)$$
$$+ p_D\big(T_{m_k}(s) - T_{m_o}(s)\big) + p_D\big(T_{m_o}(s) - T_{m_o}(v)\big) \leq \varepsilon. \quad \square$$

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