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ALMOST EVERYWHERE CONVERGENCE OF SUBSEQUENCE OF QUADRATIC PARTIAL SUMS OF TWO-DIMENSIONAL WALSH–FOURIER SERIES

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Abstract. For a non-negative integer n let us denote the dyadic variation of a natural number n by

$$V(n) := \sum_{j=0}^{\infty} |n_{j-} n_{j+1}| + n_0,$$

where $n := \sum_{i=0}^{\infty} n_i 2^i$, $n_i \in \{0,1\}$. In this paper we prove that for a function $f \in L \log L(\mathbb{I}^2)$ under the condition $\sup_A V(n_A) < \infty$, the subsequence of quadratic partial sums $S_{n_A}^{\square}(f)$ of two-dimensional Walsh–Fourier series converges to the function f almost everywhere. We also prove sharpness of this result. Namely, we prove that for all monotone increasing function $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\varphi(u) = o(u \log u)$ as $u \to \infty$ there exists a sequence $\{n_A : A \geq 1\}$ with the condition $\sup_A V(n_A) < \infty$ and a function $f \in \varphi(L)(\mathbb{I}^2)$ for which $\sup_A |S_{n_A}^{\square}(x^1, x^2; f)| = \infty$ for almost all $(x^1, x^2) \in \mathbb{I}^2$.

1. Introduction

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval

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$\mathbb{I} := [0,1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \le p < 2^n$.

Denote the dyadic expansion of $n \in \mathbb{N}$ and $x \in \mathbb{I}$ by

$$n = \sum_{j=0}^{\infty} n_j 2^j, \quad n_j = 0,1 \quad \text{and} \quad x = \sum_{j=0}^{\infty} \frac{x_j}{2^{j+1}}, \quad x_j = 0,1$$

(in the case of $x \in \mathbb{Q}$ chose the expansion which terminates in zeros). Define the dyadic addition $+$ as

$$x + y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$

The sets $I_n(x) := \{y \in \mathbb{I} : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$ for $x \in \mathbb{I}$, $I_n := I_n(0)$ for $0 < n \in \mathbb{N}$ and $I_0(x) := \mathbb{I}$ are the dyadic intervals of $\mathbb{I}$. For $0 < n \in \mathbb{N}$ denote by $|n| := \max\{j \in \mathbb{N} : n_j \ne 0\}$, that is, $2^{|n|} \le n < 2^{|n|+1}$.

The Rademacher system is defined by

$$r_n(x) := (-1)^{x_n} \quad (x \in \mathbb{I},\ n \in \mathbb{N}).$$

The Walsh–Paley system is defined as the sequence of the Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \quad (x \in \mathbb{I},\ n \in \mathbb{N}).$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [9, p. 28])

(1) $$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \in \mathbb{I} \backslash I_n(0). \end{cases}$$

For a non-negative integer n let us denote

$$V(n) := \sum_{i=0}^{\infty} |n_i - n_{i+1}| + n_0.$$

It is well known (see [9, p. 35]) that the following representation is valid:

(2) $$|D_n| = \left| \sum_{k=1}^{\infty} (n_{k-1} - n_k) D_{2^k} - n_0 \right|.$$

We consider the double system $\{w_n(x) \times w_m(y) : n, m \in \mathbb{N}\}$ on the unit square $\mathbb{I}^2 = [0,1) \times [0,1)$.

Let $Q(L) = Q(L)(\mathbb{I}^2)$ be the Orlicz space [5] generated by Young function Q, i.e. Q is a convex continuous even function such that $Q(0) = 0$ and

$$\lim_{u \to +\infty} \frac{Q(u)}{u} = +\infty, \quad \lim_{u \to 0} \frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{Q(L)(\mathbb{I}^2)} = \inf\Big\{k > 0 : \int_{\mathbb{I}^2} Q(|f|/k) \le 1\Big\}.$$

In particular, if $Q(u) = u \log(e+u)$, $u > 0$ and $Q(u) = u^p$, $p \ge 1$ ($e = \exp(1)$), then the corresponding space will be denoted by $L \log^+ L(\mathbb{I}^2)$ and $L_p(\mathbb{I}^2)$, respectively.

If $f \in L_1(\mathbb{I}^2)$, then

$$\widehat{f}(n^1, n^2) = \int_{\mathbb{I}^2} f(y^1, y^2) w_{n^1}(y^1) w_{n^2}(y^2)\, dy^1\, dy^2$$

is the (n^1, n^2)-th Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{N^1,N^2}(x^1, x^2; f) = \sum_{n^1=0}^{N^1-1} \sum_{n^2=0}^{N^2-1} \widehat{f}(n^1, n^2) w_{n^1}(x^1) w_{n^2}(x^2).$$

The quadratic partial sums are defined as

$$S_k^{\square}(x^1, x^2; f) = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \widehat{f}(i, j) w_i(x^1) w_j(x^2).$$

Let $f \in L_1(\mathbb{I}^2)$. Then the dyadic maximal function is given by

$$M(x^1, x^2; f) := \sup_{n,m \in \mathbb{N}} 2^{n+m} \int_{I_n \times I_m} \big|f(x^1 + s^1, x^2 + s^2)\big|\, ds^1\, ds^2.$$

There exists a Walsh–Fourier series such that it diverges at all dyadic irrational points. On the other hand, it is known that if $n_A < n_{A+1}$, $\sup_A V(n_A) < \infty$, then subsequences $S_{n_A}(f)$ of partial sums of one-dimensional Walsh–Fourier series of the function $f \in L_1(\mathbb{I})$ converges a.e. to the function f. Balashov has posed the following problem. Let a sequence $\{n_A\}$ be such that $\sup_A V(n_A) = \infty$, which is equivalent to the unboundedness of the Lebesgue constant. Does the existence of a function $f \in L_1(\mathbb{I})$ with a sequence $\{S_{n_A}(f)\}$ diverges a.e. follow from the latter? Konyagin [4] gave a negative answer to this question. In particular, the following is proved.

THEOREM A (Konyagin). *Suppose* $\{n_A\}$ *is an increasing sequence of natural numbers,* $k_A := [\log_2 n_A] + 1$, *and* 2^{k_A} *is a divisor of* n_{A+1} *for all* A. *Then* $S_{n_A}(f) \to f$ *a.e. for any function* $f \in L_1(\mathbb{I})$.

For instance, for the sequence (n_A), where $n_A := 2^{A^2} \sum_{i=0}^{A} 4^i$, we have that $\sup_A V(n_A) = \infty$ and it satisfies the hypotheses of the theorem. Lukomskii [8] note that if $f \in L_1(\mathbb{I}^2)$, $n_A < n_{A+1}$, $\sup_A V(n_A) < \infty$, then $S_{n_A}(f) \to f$ in L_1, and at the same time (see [6]) there exists a sequence $\{n_A\}$ and a function $f \in L_1(\mathbb{I}^2)$such that $\sup_A V(n_A) < \infty$ and $\{S^{\square}_{n_A}(f)\}$ diverges a.e. In [7] Lukomskii found necessary and sufficient conditions on subsequences $\{n_A\}$ which provide a.e. convergence of $\{S^{\square}_{n_A}(f)\}$ for all function $f \in L_1(\mathbb{I}^2)$.

In this paper we will find the maximal space which guarantee a.e. convergence of subsequence $\{S^{\square}_{n_A}(f)\}$ under the condition $\sup_A V(n_A) < \infty$. In particular, the following is true.

THEOREM 1. *Let* $\{N_A\}$ *be a subsequence of the natural numbers,* μ *is the Lebesgue measure on* $[0,1)^2$, $f \in L\log^+ L(\mathbb{I}^2)$ *and*

$$\sup_A V(N_A) < \infty. \tag{3}$$

Then

$$\mu\Big\{\sup_A |S^{\square}_{N_A}(f)| > \lambda\Big\} \lesssim \frac{1}{\lambda}\Big(1 + \int_{\mathbb{I}^2} |f| \log^+ |f|\Big).$$

The weak type $(L\log^+ L, L_1)$ inequality and the usual density argument of Marcinkiewicz and Zygmund imply (for details see Zygmund [12, Chapter 7])

THEOREM 2. *Let* $f \in L\log^+ L(\mathbb{I}^2)$ *and suppose that condition* (3) *holds. Then we have a.e. convergence* $S^{\square}_{N_A}(f) \to f$.

On the other hand, we also prove that $L\log^+ L$ is the maximal convergence space. That is,

THEOREM 3. *Let* $\varphi\colon [0,\infty) \to [0,\infty)$ *be a monotone increasing function such that* $\varphi(u) = o(u \log u)$ *as* $u \to \infty$. *Then there exists a sequence* $\{N_A : A \geq 1\} \subset \mathbb{N}$ *with condition* (3) *and a function* $f \in L_1(\mathbb{I}^2)$ *for which*

$$\int_{\mathbb{I}^2} \varphi\big(|f(x^1,x^2)|\big)\, dx^1\, dx^2 < \infty \quad \textit{and} \quad \sup_A \big|S^{\square}_{N_A}(x^1,x^2;f)\big| = +\infty$$

for a.e. $(x^1,x^2) \in \mathbb{I}^2$.

We note that Lukomskii [8] investigates convergence in norm and convergence in measure of subsequences of quadratic partial sums of multiple Walsh–Fourier series. In particular, Lukomskii proved that the condition (3) is necessary and sufficient for the L_1-convergence and convergence in measure of $S_{N_A}^{\square}(x^1,x^2;f)$ for all functions $f\in L_1(\mathbb{I}^2)$. Almost everywhere divergence of a subsequence of partial sums of Fourier series with respect to trigonometric system were investigated by Gosselin [3] and Totik [10]. In particular, Gosselin proved that for any increasing sequence $\{m_j\}$ there is $f\in L_1([-\pi,\pi])$ such that

$$\sup_j |S_{m_j}(x;f)|=\infty \tag{4}$$

for almost all $x\in[-\pi,\pi]$. Totik established the existence of f such that (4) holds for all $x\in[-\pi,\pi]$.

2. Proof of main results

Proof of Theorem 1. Since by (2) $(n_{-1}=0)$

$$|D_n|\le\sum_{k=0}^{\infty}|n_{k-1}-n_k|D_{2^k},$$

and then we have

$$\begin{aligned}
(5)\qquad \left|S_n^{\square}(x^1,x^2;f)\right| &= \left|\int_{\mathbb{I}^2} f(t^1,t^2)D_n(x^1+t^1)D_n(x^2+t^2)\,dt^1\,dt^2\right| \\
\le \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}|n_{k-1}-n_k|\,|n_{j-1}-n_j| &\int_{\mathbb{I}^2}|f(t^1,t^2)|D_{2^k}(x^1+t^1)D_{2^j}(x^2+t^2)\,dt^1\,dt^2 \\
&=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}|n_{k-1}-n_k|\,|n_{j-1}-n_j|2^{k+j}\int_{I_k(x^1)\times I_j(x^2)}|f(t^1,t^2)|\,dt^1\,dt^2 \\
&\le cV^2(n)M(x^1,x^2;f).
\end{aligned}$$

Since (see Weisz [11, p. 15])

$$\mu\{M(f)>\lambda\}\lesssim\frac{1}{\lambda}\left(1+\int_{\mathbb{I}^2}|f|\log^+|f|\right),$$

from (5) we conclude the proof of Theorem 1. □

Next, we turn our attention to the way of the proof of Theorem 3. We need some preliminary assumptions. The method of the construction of the counterexample function is similar to the ones can be found in papers [1] and [2]. Set

$$F_{b,a}(x^1,x^2) := \{I_{b+j}(x^1) \times I_{b+2a-j}(x^2) : j = 0,1,\dots,a\}.$$

Then it is easy to see that

$$\bigcap F_{b,a}(x^1,x^2) := \bigcap_{I\in F_{b,a}(x^1,x^2)} I = I_{b+a}(x^1) \times I_{b+2a}(x^2). \tag{6}$$

On the other hand, let $\bigcup F_{b,a}(x^1,x^2) := \bigcup_{I\in F_{b,a}(x^1,x^2)} I$. Since

$$\bigcup F_{b,a}(x^1,x^2)$$
$$= \Big(\bigcup_{j=1}^{a} I_{b+j}(x^1) \times \big(I_{b+2a-j}(x^2)\backslash I_{b+2a-j+1}(x^2)\big)\Big) \cup \big(I_b(x^1) \times I_{b+2a}(x^2)\big),$$

we can write

$$\mu\Big(\bigcup F_{b,a}(x^1,x^2)\Big) \tag{7}$$
$$= \frac{1}{2^{2b+2a}} + \sum_{j=1}^{a} \Big(\frac{1}{2^{b+j}}\frac{1}{2^{b+2a-j}} - \frac{1}{2^{b+j}}\frac{1}{2^{b+2a-j+1}}\Big) = \frac{1}{2^{2b+2a}}\Big(1+\frac{a}{2}\Big).$$

Set

$$J^0_{a,b}(t^1,t^2) := \{(t^1,t^2) \in \mathbb{I}^2\},$$

$$\Omega^0_{a,b}(t^1,t^2) := \bigcup_{j=0}^{a} I_{b+j}(t^1) \times I_{b+2a-j}(t^2) = \bigcup F_{b,a}(t^1,t^2),$$

$$\widetilde{\Omega}^0_{a,b}(t^1,t^2) := \Omega^0_{a,b}(t^1,t^2)\backslash\Big(I_{b+a}(t^1)\times I_{b+a}(t^2)\cup(I_b(t^1)\backslash I_{b+1}(t^1))\times I_{b+2a}(t^2)\Big).$$

Suppose that the sets $J^j_{a,b}(t^1,t^2)$, $\Omega^j_{a,b}(t^1,t^2)$ and $\widetilde{\Omega}^j_{a,b}(t^1,t^2)$ are defined for $j<k$. Then decompose

$$I_b(t^1) \times I_{b+a}(t^2)\backslash \bigcup_{j=0}^{k-1} \Omega^j_{a,b}(t^1,t^2)$$

as the disjoint union of dyadic rectangles of the form $I_{b+ka}(x^1)\times I_{b+(k+1)a}(x^2)$. Take from each dyadic rectangles an element to represent. The set of (x^1,x^2)'s corresponding to these rectangles is $J^k_{a,b}(t^1,t^2)$. In this process the number k is called the "depth" of the decomposition of the rectangle $I_b(t^1) \times I_{b+a}(t^2)$.

Set

$$\Omega^k_{a,b}(t^1,t^2) := \bigcup_{(x^1,x^2)\in J^k_{a,b}(t^1,t^2)} \bigcup_{j=0}^{a} I_{b+ka+j}(x^1) \times I_{b+(k+2)a-j}(x^2)$$

and

$$\widetilde{\Omega}^k_{a,b}(t^1,t^2) := \Omega^k_{a,b}(t^1,t^2) \setminus \bigcup_{(x^1,x^2)\in J^k_{a,b}(t^1,t^2)} \Big(I_{b+(k+1)a}(x^1) \times I_{b+(k+1)a}(x^2)$$
$$\cup \big(I_{b+ka}(x^1) \setminus I_{b+ka+1}(x^1)\big) \times I_{b+(k+2)a}(x^2)\Big).$$

Equality (7) means that the measure of $\Omega^0_{a,b}(t^1,t^2) \subset I_b(t^1) \times I_{b+a}(t^2)$ is $(1+a/2)2^{-2b-2a}$. That is, the ratio of the measure of $\Omega^0_{a,b}(t^1,t^2)$ and $I_b(t^1) \times I_{b+a}(t^2)$ is $(1+a/2)2^{-a}$. By the construction of $\widetilde{\Omega}^0_{a,b}(t^1,t^2)$ and $\Omega^0_{a,b}(t^1,t^2)$ we also have that the ratio of the measures of these two sets is $(a/2-1/2)/(a/2+1) = 1 - \frac{3}{a+2}$. Henceforth, by construction it also follows

$$\frac{\mu\big(\bigcup_{k=0}^k \Omega^k_{a,b}(t^1,t^2)\big)}{\mu\left(I_b(t^1) \times I_{b+a}(t^2)\right)} = 1 - \Big(1 - \frac{a/2+1}{2^a}\Big)^{k+1}, \tag{8}$$

$$\frac{\mu\big(\bigcup_{k=0}^k \widetilde{\Omega}^k_{a,b}(t^1,t^2)\big)}{\mu(I_b(t^1) \times I_{b+a}(t^2))} = \Big(1 - \frac{3}{a+2}\Big)\Big(1 - \Big(1 - \frac{a/2+1}{2^a}\Big)^{k+1}\Big).$$

For any finite zero-one sequence $t^1_i \in \{0,1\}$, $t^2_s \in \{0,1\}$ $(i<b,\ s<a+b)$ set

$$t = (t^1,t^2) = \Big(\sum_{i=0}^{b-1} t^1_i/2^{i+1}, \sum_{s=0}^{a+b-1} t^2_s/2^{s+1}\Big).$$

Alternatively, one can also say that t^1 and t^2 are zero-one vectors of length b and $a+b$. Now, for every $d \in \mathbb{N}$ we define the function

$$f^d_{a,b}(x^1,x^2)$$
$$:= 2^{2a} \sum_{\substack{t^1_i\in\{0,1\}\\ i=0,\dots,b-1}} \sum_{\substack{t^2_s\in\{0,1\}\\ s=0,\dots,a+b-1}} \sum_{k=0}^{d} \sum_{(y^1,y^2)\in J^k_{a,b}(t^1,t^2)} \mathbb{I}_{I_{b+(k+2)a}(y^1)\times I_{b+(k+2)a}(y^2)}(x^1,x^2),$$

where $\mathbb{I}_E$ denotes the characteristic function of the set E. In the definition of $f^d_{a,b}$ the number d is the "depth" of the decomposition of the rectangles

$I_b(t^1)\times I_{b+a}(t^2)$. These rectangles "parquets" the unit square as t_i^1 and t_s^2 take 0 or 1 ($i<b$, $s<a+b$). The number d has to be a finite one, since we want to keep the function $f_{a,b}^d$ to be locally constant on "some" rectangles.

In order to prove Theorem 3 we need the following lemma.

LEMMA 1. *Let* $a,b,d\in\mathbb{N}\backslash\{0\}$ *and*

$$(x^1,x^2)\in\bigcup_{\substack{t_i^1\in\{0,1\}\\ i=0,\dots,b-1}}\ \bigcup_{\substack{t_s^2\in\{0,1\}\\ s=0,\dots,a+b-1}}\ \bigcup_{k=0}^{d}\widetilde{\Omega}_{a,b}^k(t^1,t^2).$$

Then there exist unique $i<b$, $s<a+b$, $k\le d$ *for which*

$$(x^1,x^2)\in\big(I_{b+ka+j}(y^1)\backslash I_{b+(k+1)a}(y^1)\big)\times I_{b+(k+2)a-j}(y^2)$$

for some $(y^1,y^2)\in J_{a,b}^k(t^1,t^2)$ *and some* $j=1,\dots,a-1$. *Setting*

$$N_1:=b+ka+j,\quad N_2:=b+(k+2)a-j$$

we have

$$\big|S_{2^{N_1}+2^{N_2}}^{\square}(x^1,x^2;f_{a,b}^d)\big|\ge\frac{3}{4}\,.$$

PROOF OF LEMMA 1. We start with the proof of the first assertion of Lemma 1. Since (x^1,x^2) is an element of a union, then there exist $t_0^1,\dots,t_{b-1}^1\in\{0,1\}$ and $t_0^2,\dots,t_{b+a-1}^2\in\{0,1\}$ such that for

$$t=(t^1,t^2)=\left(\sum_{i=0}^{b-1}t_i^1/2^{i+1},\ \sum_{s=0}^{a+b-1}t_s^2/2^{s+1}\right)$$

we have $(x^1,x^2)\in\bigcup_{k=0}^d\widetilde{\Omega}_{a,b}^k(t^1,t^2)$ and consequently there is a $k\le d$ such that

$$(x^1,x^2)\in\widetilde{\Omega}_{a,b}^k(t^1,t^2)\subset\Omega_{a,b}^k(t^1,t^2).$$

Then there exits a $(y^1,y^2)\in J_{a,b}^k(t^1,t^2)$ such that $(x^1,x^2)\in\bigcup_{j=0}^k I_{b+ka+j}(y^1)\times I_{b+(k+2)a-j}(y^2)$. Consequently, $(x^1,x^2)\in I_{b+ka+j}(y^1)\times I_{b+(k+2)a-j}(y^2)$ for some $j\le k$. Since

$$\begin{aligned}(x^1,x^2)\notin&\big(I_{b+(k+1)a}(y^1)\times I_{b+(k+1)a}(y^2)\big)\\ \cup&\big(I_{b+ka}(y^1)\backslash I_{b+ka+1}(y^1)\times I_{b+(k+1)2}(y^2)\big)\end{aligned}$$

(as it comes from the definition of $\widetilde{\Omega}_{a,b}^k(t^1,t^2)$), then the first assertion of Lemma 1 follows.

Since

$$D_{2^{N_1}+2^{N_2}} = D_{2^{N_2}} + w_{2^{N_2}} D_{2^{N_1}}, \quad N_2 > N_1$$

we can write

(9)
$$S^{\square}_{2^{N_1}+2^{N_2}}(x^1, x^2; f^d_{a,b})$$
$$= \int_{\mathbb{I}^2} f^d_{a,b}(t^1, t^2) D_{2^{N_1}+2^{N_2}}(x^1 + t^1) D_{2^{N_1}+2^{N_2}}(x^2 + t^2)\, dt^1\, dt^2$$
$$= \int_{\mathbb{I}^2} f^d_{a,b}(t^1, t^2) D_{2^{N_2}}(x^1 + t^1) D_{2^{N_2}}(x^2 + t^2)\, dt^1\, dt^2$$
$$+ \int_{\mathbb{I}^2} f^d_{a,b}(t^1, t^2) D_{2^{N_2}}(x^1 + t^1) w_{2^{N_2}}(x^2 + t^2) D_{2^{N_1}}(x^2 + t^2)\, dt^1\, dt^2$$
$$+ \int_{\mathbb{I}^2} f^d_{a,b}(t^1, t^2) w_{2^{N_2}}(x^1 + t^1) D_{2^{N_1}}(x^1 + t^1) D_{2^{N_2}}(x^2 + t^2)\, dt^1\, dt^2$$
$$+ \int_{\mathbb{I}^2} f^d_{a,b}(t^1, t^2) w_{2^{N_2}}(x^1 + t^1) D_{2^{N_1}}(x^1 + t^1) w_{2^{N_2}}(x^2 + t^2) D_{2^{N_1}}(x^2 + t^2)\, dt^1\, dt^2$$
$$= J_1 + J_2 + J_3 + J_4.$$

By (1) we have

$$\int_{\mathbb{I}^2} f^d_{a,b}(t^1, t^2) D_{2^{N_2}}(x^1 + t^1) D_{2^{N_2}}(x^2 + t^2)\, dt^1\, dt^2$$
$$= \int_{I_{N_2(x^1)} \times I_{N_2(x^2)}} f^d_{a,b}(t^1, t^2) 2^{N_2} 2^{N_2}\, dt^1\, dt^2.$$

By $N_2 = b + (k+2)a - j$ and from the definition of the function $f^d_{a,b}$ we obtain

$$J_1 = 2^{2N_2} \int_{I_{b+(k+2)a-j}(x^1) \times I_{b+(k+2)a-j}(x^2)} f^d_{a,b}(t^1, t^2)\, dt^1\, dt^2$$
$$= 2^{2N_2+2a} \int_{I_{b+(k+2)a-j}(x^1) \times I_{b+(k+2)a-j}(x^2)} \mathbb{I}_{I_{b+(k+2)a}(y^1) \times I_{b+(k+2)a}(y^2)}(t^1, t^2)\, dt^1\, dt^2.$$

The condition in the statement of Lemma 1, $x^1 \in \big(I_{b+ka+j}(y^1) \backslash I_{b+(k+1)a}(y^1)\big)$ gives

$$x^1 \notin I_{b+(k+1)a}(y^1)$$

which implies that (two dyadic intervals are disjoint or one contains the other)

$$I_{b+(k+2)a-j}(x^1) \cap I_{b+(k+1)a}(y^1) = \varnothing.$$

Hence we integrate on the emptyset. That is,

$$J_1 = 0. \tag{10}$$

Analogously, we can prove that

$$J_2 = 0. \tag{11}$$

Next, we discuss J_3. By (1) we have that

$$w_{2^{N_2}}(x^1+t^1)D_{2^{N_1}}(x^1+t^1)D_{2^{N_2}}(x^2+t^2)$$

is either 0 or $w_{2^{N_2}}(x^1+t^1)2^{N_1+N_2}$ which occurs in the case of $(t^1,t^2) \in I_{N_1}(x^1)\times I_{N_2}(x^2)$. This fact and $N_1 = b+ka+j$, $N_2 = b+(k+2)a-j$ give for J_3

$$\begin{aligned} |J_3| &= \left| 2^{N_1+N_2} \int_{I_{N_1}(x^1)\times I_{N_2}(x^2)} f^d_{a,b}(t^1,t^2) w_{2^{N_2}}(t^1)\, dt^1\, dt^2 \right| \\ &= 2^{N_1+N_2+2a} \left| \int_{I_{b+ka+j}(x^1)\times I_{b+(k+2)a-j}(x^2)} \mathbb{I}_{I_{b+(k+2)a}(y^1)\times I_{b+(k+2)a}(y^2)}(t^1,t^2) \right. \\ &\qquad \left. \times\, w_{2^{b+(k+2)a-j}}(t^1)\, dt^1\, dt^2 \right| \\ &= 2^{N_1+N_2+2a} \left| \int_{I_{b+(k+2)a}(x^1)\times I_{b+(k+2)a}(x^2)} w_{2^{b+(k+2)a-j}}(t^1)\, dt^1\, dt^2 \right| \\ &= \frac{2^{N_1+N_2+2a}}{2^{2b+2a(k+2)}} = 1, \quad j \neq 0, \end{aligned} \tag{12}$$

where the last but one equality is given by the definition of $f^d_{a,b}$. For J_4 by the help of properties of the Walsh function and by the definition of function $f^d_{a,b}$ we can write $(j<a)$

$$\begin{aligned} |J_4| &= 2^{2N_1} \left| \int_{I_{N_1}(x^1)\times I_{N_1}(x^2)} w_{2^{N_2}}(t^1+t^2) f^d_{a,b}(t^1,t^2)\, dt^1\, dt^2 \right| \\ &= 2^{2N_1+2a} \left| \int_{I_{b+ka+j}(x^1)\times I_{b+ka+j}(x^2)} w_{2^{b+(k+2)a-j}}(t^1+t^2) \right. \\ &\qquad \left. \times\, \mathbb{I}_{I_{b+(k+2)a}(y^1)\times I_{b+(k+2)a}(y^2)}(t^1,t^2)\, dt^1\, dt^2 \right| \\ &\le \frac{2^{2N_1+2a}}{2^{2b+2ka+4a}} = \frac{2^{2j}}{2^{2a}} \le \frac{2^{2(a-1)}}{2^{2a}} = \frac{1}{4}. \end{aligned} \tag{13}$$

Combining (9)–(13) completes the proof of Lemma 1. □

PROOF OF THEOREM 3. Set

$$G_{a,b,d} := \bigcup_{\substack{t_i^1\in\{0,1\}\\ i=0,\dots,b-1}} \bigcup_{\substack{t_s^2\in\{0,1\}\\ s=0,\dots,a+b-1}} \bigcup_{k=0}^{d} \widetilde{\Omega}_{a,b}^{k}(t^1,t^2).$$

Next, we verify that

$$\mu(G_{a,b,d}) \to 1 \tag{14}$$

as $a,b,d\to\infty$ with the restriction that d tends to infinity faster than a. More precisely, $da/2^a\to\infty$.

To prove (14) we use (8). For each fixed t_i^1, t_s^2 ($i<b$, $s<a+b$), that is, for each dyadic rectangle of the form $I_b(t^1)\times I_{b+a}(t^2)$ (they "parquets" the unit square) we use (8). That is,

$$1\geq \mu(G_{a,b,d}) \geq \Big(1-\frac{3}{a+2}\Big)\Big(1-\Big(1-\frac{a/2+1}{2^a}\Big)^{d+1}\Big).$$

Then (14) is verified.

Define the sequences $\{a_n, n\in\mathbb{N}\}$, $\{b_n, n\in\mathbb{N}\}$ and $\{d_n, n\in\mathbb{N}\}$ in the following way: Let be $a_n, b_n, d_n\uparrow\infty$ and

$$\frac{2^{3a_0}}{\varphi(2^{3a_0})}\leq 2^{2b_0}. \tag{15}$$

(Condition (15) is a "technical" one and it plays a minor role in the estimation of the norm of the counterexample function at (19) and below.) Moreover, let

$$\frac{2^{3a_n}a_n}{\varphi(2^{3a_n})}\geq n4^n 2^{2b_{n-1}+2a_{n-1}(d_{n-1}+2)} \tag{16}$$

and

$$4\sum_{j=1}^{n-1}2^{2a_j}<\frac{1}{4}\,\frac{2^{3a_n}a_n}{4^n\varphi(2^{3a_n})2^{2b_{n-1}+2a_{n-1}(d_{n-1}+2)}}\,. \tag{17}$$

Define

$$G:=\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}G_{a_n,b_n,d_n}.$$

From (14) we can write $\mu(G)=1$.

Let $(x^1,x^2)\in G$. Then there is an infinite number $n\in\mathbb{N}$ for which $(x^1,x^2)\in G_{a_n,b_n,d_n}$. Then Lemma 1 gives there exist a $(t^1,t^2)\in\mathbb{I}^2$ and $k\leq d_n$

for which $(x^1,x^2)\in\widetilde{\Omega}^k_{a_n,b_n}(t^1,t^2)$. Hence, there is a $(y^1,y^2)\in J^k_{a_n,b_n}(t^1,t^2)$, and a unique $j\in\{0,1,\dots,a_n-1\}$ for which

$$(x^1,x^2)\in\big(I_{b_n+ka_n+j}(y^1)\backslash I_{b_n+(k+1)a_n}(y^1)\big)\times I_{b_n+(k+2)a_n-j}(y^2).$$

Set

$$N_1:=b_n+ka_n+j,\quad N_2:=b_n+(k+2)a_n-j$$

and

$$c_n:=\frac{2^{3a_n}a_n}{4^n\varphi(2^{3a_n})2^{2b_{n-1}+2a_{n-1}(d_{n-1}+2)}}\,.$$

Define the function

$$f(x^1,x^2):=\sum_{n=1}^{\infty}c_nf^{d_n}_{a_n,b_n}(x^1,x^2).$$

Now, we prove that

$$\int_{\mathbb{I}^2}\varphi(|f(x^1,x^2)|)\,dx^1\,dx^2<\infty. \tag{18}$$

First, we suppose that $\varphi(u)/u$ is nondecreasing. Since

$$\sum_{k=1}^{\infty}u_k\le\sup_{k\ge1}2^ku_k,$$

we write

$$\varphi\Big(\sum_{k=1}^{\infty}u_k\Big)\le\varphi\Big(\sup_{k\ge1}2^ku_k\Big)\le\sum_{k=1}^{\infty}\varphi(2^ku_k).$$

Consequently,

$$\int_{\mathbb{I}^2}\varphi\big(|f(x^1,x^2)|\big)\,dx^1\,dx^2\le\sum_{n=1}^{\infty}\int_{\mathbb{I}^2}\varphi\big(2^nc_nf^{d_n}_{a_n,b_n}(x^1,x^2)\big)\,dx^1\,dx^2. \tag{19}$$

Since $\varphi(u)/u$ is nondecreasing from (15) we have

$$\frac{2^nc_n}{a_n}=\frac{2^{3a_n}}{2^n\varphi(2^{3a_n})2^{2b_{n-1}+2a_{n-1}(d_{n-1}+2)}}\le\frac{2^{3a_0}}{\varphi(2^{3a_0})2^{2b_0}}\le1,$$

hence, to use again the fact that $\varphi(u)/u$ is nondecreasing and the function $\big\|f^{d_n}_{a_n,b_n}\big\|_\infty\le2^{2a_n}$ we have

$$\int_{\mathbb{I}^2}\varphi\big(2^nc_nf^{d_n}_{a_n,b_n}(x^1,x^2)\big)\,dx^1\,dx^2 \tag{20}$$

$$= \int_{\mathbb{I}^2} \varphi\Big(\frac{2^n c_n}{a_n} a_n f^{d_n}_{a_n,b_n}(x^1,x^2)\Big)\,dx^1\,dx^2 \le \frac{2^n c_n}{a_n}\int_{\mathbb{I}^2}\varphi\big(a_n f^{d_n}_{a_n,b_n}(x^1,x^2)\big)\,dx^1\,dx^2$$

$$\le \frac{2^n c_n}{a_n}\int_{\mathbb{I}^2}\frac{\varphi(a_n 2^{2a_n})}{a_n 2^{2a_n}} a_n f^{d_n}_{a_n,b_n}(x^1,x^2)\,dx^1\,dx^2$$

$$\le \frac{2^n c_n}{a_n}\frac{\varphi(2^{3a_n})}{2^{3a_n}}\int_{\mathbb{I}^2} a_n f^{d_n}_{a_n,b_n}(x^1,x^2)\,dx^1\,dx^2.$$

From the definition of the function $f^{d_n}_{a_n,b_n}$ we can write

$$\int_{\mathbb{I}^2} f^{d_n}_{a_n,b_n}(x^1,x^2)\,dx^1\,dx^2 = 2^{2a_n}\sum_{\substack{t^1_i\in\{0,1\}\\ i=0,\dots,b_n-1}}\ \sum_{\substack{t^2_s\in\{0,1\}\\ s=0,\dots,a_n+b_n-1}}\ \sum_{k=0}^{d_n}\ \sum_{(y^1,y^2)\in J^k_{a_n,b_n}(t^1,t^2)} \mu\big(I_{b_n+(k+2)a_n}(y^1)\times I_{b_n+(k+2)a_n}(y^2)\big)$$

$$= 2^{2a_n}\sum_{\substack{t^1_i\in\{0,1\}\\ i=0,\dots,b_n-1}}\ \sum_{\substack{t^2_s\in\{0,1\}\\ s=0,\dots,a_n+b_n-1}}\ \sum_{k=0}^{d_n}\ \sum_{(y^1,y^2)\in J^k_{a_n,b_n}(t^1,t^2)} \frac{1}{2^{2b_n+2(k+2)a_n}}.$$

Since by (6) and (7)

$$\frac{1}{2^{2b_n+2ka_n+4a_n}} = \frac{1}{2^{a_n}}\mu\Big(\bigcap F_{b_n+ka_n,a_n}\Big) = \frac{\mu\big(\bigcup F_{b_n+ka_n,a_n}\big)}{2^{2a_n}(1+a_n/2)}$$

and, consequently,

$$\int_{\mathbb{I}^2} f^{d_n}_{a_n,b_n}(x^1,x^2)\,dx^1\,dx^2 \tag{21}$$

$$= \sum_{\substack{t^1_i\in\{0,1\}\\ i=0,\dots,b_n-1}}\ \sum_{\substack{t^2_s\in\{0,1\}\\ s=0,\dots,a_n+b_n-1}}\ \sum_{k=0}^{d_n}\ \sum_{(y^1,y^2)\in J^k_{a_n,b_n}(t^1,t^2)} \frac{\mu\big(\bigcup F_{b_n+ka_n,a_n}\big)}{(1+a_n/2)} \le \frac{2}{a_n}.$$

Hence, by (19), (20) and (21) we conclude that

$$\int_{\mathbb{I}^2}\varphi(|f(x^1,x^2)|)\,dx^1\,dx^2$$

$$\le \sum_{n=1}^{\infty}\frac{2^n c_n}{a_n}\frac{\varphi(2^{3a_n})}{2^{3a_n}}\int_{\mathbb{I}^2} a_n f^{d_n}_{a_n,b_n}(x^1,x^2)\,dx^1\,dx^2 \le 2\sum_{n=1}^{\infty}\frac{2^n}{a_n}c_n\frac{\varphi(2^{3a_n})}{2^{3a_n}}$$

$$= 2\sum_{n=1}^{\infty}\frac{2^n}{a_n}\frac{2^{3a_n}a_n}{4^n\varphi(2^{3a_n})2^{2b_{n-1}+2a_{n-1}(d_{n-1}+2)}}\frac{\varphi(2^{3a_n})}{2^{3a_n}} \le 2\sum_{n=1}^{\infty}\frac{1}{2^n} = 2.$$

This implies (18) when $\varphi(u)/u$ is nondecreasing.

We have

$$S^{\square}_{2^{N_1}+2^{N_2}}(x^1,x^2;f) \tag{22}$$

$$=\sum_{j=1}^{n-1} c_j S^{\square}_{2^{N_1}+2^{N_2}}\left(x^1,x^2;f^{d_j}_{a_j,b_j}\right)+c_n S^{\square}_{2^{N_1}+2^{N_2}}\left(x^1,x^2;f^{d_n}_{a_n,b_n}\right)$$

$$+\sum_{j=n+1}^{\infty} c_j S^{\square}_{2^{N_1}+2^{N_2}}\left(x^1,x^2;f^{d_j}_{a_j,b_j}\right)=Q_1+Q_2+Q_3.$$

From Lemma 1 we can write

$$Q_2 \ge \frac{3}{4}c_n. \tag{23}$$

Since by (1) and (2)

$$\left|S^{\square}_{2^{N_1}+2^{N_2}}\left(x^1,x^2;f^{d_j}_{a_j,b_j}\right)\right| \le 4\|f^{d_j}_{a_j,b_j}\|_\infty \le 4\cdot 2^{2a_j},$$

from (17) we get

$$|Q_1| \le 4\sum_{j=1}^{n-1} 2^{2a_j} < \frac{c_n}{4}. \tag{24}$$

Since

$$\left|S^{\square}_{2^{N_1}+2^{N_2}}\left(x^1,x^2;f^{d_j}_{a_j,b_j}\right)\right| \le \sum_{v_1=0}^{2^{N_1}+2^{N_2}-1}\ \sum_{v_2=0}^{2^{N_1}+2^{N_2}-1} \left|\widehat{f}^{d_j}_{a_j,b_j}(v_1,v_2)\right|$$

$$\le (2^{N_1}+2^{N_2})^2\left\|f^{d_j}_{a_j,b_j}\right\|_1 \le 2^{2N_2}\frac{8}{a_j}, \quad N_2 \ge N_1,$$

we have

$$|Q_3| \le 8\sum_{j=n+1}^{\infty}\frac{c_j}{a_j}\cdot 2^{2b_n+2a_n(d_n+2)} \tag{25}$$

$$=8\sum_{j=n+1}^{\infty}\frac{2^{3a_j}a_j}{4^j\varphi(2^{3a_j})2^{2b_{j-1}+2a_{j-1}(d_{j-1}+2)}}\,\frac{1}{a_j}\cdot 2^{2b_n+2a_n(d_n+2)}$$

$$\le 8\frac{2^{2a_0}}{\varphi(2^{2a_0})}\sum_{j=n+1}^{\infty}\frac{1}{4^j} \le c < \infty.$$

Combining (22)–(25) and (16) we obtain

$$\left|S^{\square}_{2^{N_1}+2^{N_2}}(x^1,x^2;f)\right| \geq \frac{c_n}{2} - c \geq cn \to \infty \quad \text{as } n \to \infty.$$

Now, let $\varphi\colon [0,\infty) \to [0,\infty)$ is an arbitrary function. Define

$$\widetilde{\varphi}(u) := \begin{cases} \varphi(1)u, & 0 \leq u \leq 1, \\ \sup_{1\leq v\leq u} \frac{\varphi(v)}{v} u, & u > 1. \end{cases}$$

Then it is evident that

$$\widetilde{\varphi}(u) \geq \varphi(u), \quad u > 1. \tag{26}$$

On the other hand $(u > 1)$,

$$\frac{\widetilde{\varphi}(u)}{u\log u} = \frac{\sup_{1\leq v\leq u} \frac{\varphi(v)}{v} u}{u \log u} = \frac{\sup_{1\leq v\leq u} \frac{\varphi(v)}{v}}{\log u}. \tag{27}$$

Let $\varepsilon > 0$. Then, from the condition of Theorem 3, there exists an $u(\varepsilon)$ such that for every $u > u(\varepsilon)$ we have

$$\varphi(u) \leq \varepsilon u \log u.$$

Then from (27) we have

$$\frac{\widetilde{\varphi}(u)}{u\log u} \leq \frac{\sup_{1\leq v\leq u(\varepsilon)} \frac{\varphi(v)}{v}}{\log u} + \frac{\varepsilon \sup_{u(\varepsilon)\leq v\leq u} \log v}{\log u}.$$

Hence $\widetilde{\varphi}(u) = o(u\log u)$. Since $\widetilde{\varphi}(u)/u$ is nondecreasing there exists a function f such that

$$\int_{\mathbb{I}^2} \widetilde{\varphi}(|f(x^1,x^2)|)\, dx^1\, dx^2 < \infty$$

and

$$\sup_{N_1,N_2} |S^{\square}_{2^{N_1}+2^{N_2}}(x^1,x^2;f)| = \infty \quad \text{for a.e. } (x^1,x^2) \in \mathbb{I}^2.$$

On the other hand, from (26) we write

$$\int_{\mathbb{I}^2} \varphi(|f(x^1,x^2)|)\, dx^1\, dx^2 \leq \int_{\mathbb{I}^2} \widetilde{\varphi}(|f(x^1,x^2)|)\, dx^1\, dx^2 < \infty.$$

Theorem 3 is proved. $\square$

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