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APPROXIMATION BY Θ-MEANS OF WALSH–FOURIER SERIES

I. BLAHOTA* and K. NAGY

Institute of Mathematics and Computer Sciences, University of Nyíregyháza, P.O. Box 166, H-4400 Nyíregyháza, Hungary
e-mails: blahota.istvan@nye.hu, nagy.karoly@nye.hu

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Abstract. We discuss the behaviour of the Θ-means of Walsh series of a function in L^p $(1 \le p \le \infty)$. We investigate the rate of the approximation by this means, in particular, in $\operatorname{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \le p \le \infty$. In case $p = \infty$ by L^p we mean C, the class of the continuous functions.

Our main theorems give a common generalization of two results of Móricz and Siddiqi on Nörlund means [11] and Móricz and Rhoades on weighted means [9].

1. Walsh–Fourier series and summation methods

Now, we give a brief introduction to the Walsh–Fourier analysis [1,14].

Let $\mathbb{P}$ be the set of positive natural numbers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition. Let be every subset open. The normalized Haar measure μ on $\mathbb{Z}_2$ is given such that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. $G := \times_{k=0}^{\infty} \mathbb{Z}_2$, G is called the Walsh group. The elements of Walsh group G are sequences of numbers 0 and 1, that is $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$ $(k \in \mathbb{N})$.

The group operation on G is the coordinate-wise modulo 2 addition (denoted by $+$), the normalized Haar measure μ is the product measure and the topology is the product topology. Dyadic intervals are defined in the usual way

$$I_0(x) := G, \quad I_n(x) := \big\{ y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots) \big\}$$

for $x \in G$, $n \in \mathbb{P}$. They form a base for the neighbourhoods of G. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbb{N}$. Set $e_i := (0, \dots, 0, 1, 0, \dots)$, where the ith coordinate is 1 and the rest are 0 $(i \in \mathbb{N})$.

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* Corresponding author.

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Let L^p denote the usual Lebesgue spaces on G (with the corresponding norm $\|\cdot\|_p$). For the sake of brevity in notation, we agree to write L^∞ instead of C and set $\|f\|_\infty := \sup\{|f(x)| : x \in G\}$.

Next, we define the modulus of continuity in L^p, $1 \le p \le \infty$, of a function $f \in L^p$ by

$$\omega_p(f,\delta) := \sup_{|t|<\delta} \|f(\cdot + t) - f(\cdot)\|_p, \quad \delta > 0,$$

with the notation

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \quad \text{for all } x \in G.$$

The Lipschitz classes in L^p for each $\alpha > 0$ are defined by

$$\operatorname{Lip}(\alpha, p) := \big\{ f \in L^p : \omega_p(f,\delta) = O(\delta^\alpha) \text{ as } \delta \to 0 \big\}.$$

Now, we introduce some concepts of Walsh–Fourier analysis. The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G,\ k \in \mathbb{N}).$$

The Walsh–Paley functions are the products of the Rademacher functions. Namely, each natural number n can be uniquely expressed in the number system based 2, in the form

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0,1\}\ (i \in \mathbb{N}),$$

where only a finite number of n_i's are different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \ne 0\}$. Walsh–Paley functions are $w_0 = 1$ and for $n \ge 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

Let $\mathcal{P}_n$ be the collection of Walsh polynomials of order less than n, that is, functions of the form

$$P(x) = \sum_{k=0}^{n-1} a_k w_k(x),$$

where $n \ge 1$ and $\{a_k\}$ is a finite sequence of real numbers.

The Dirichlet kernels are defined by

$$D_n := \sum_{k=0}^{n-1} w_k,$$

where $n \in \mathbb{P}$, $D_0 := 0$. It is known that

$$D_{2^A+j}(x) = D_{2^A}(x) + r_A(x)D_j(x), \quad j = 0, 1, \ldots, 2^A - 1 \tag{1}$$

(see [14]). The 2^nth Dirichlet kernels have a closed form (see e.g. [14])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{otherwise} \end{cases} \quad (n \in \mathbb{N}). \tag{2}$$

The nth Fejér mean of Walsh–Fourier series of a function f and the Fejér kernel are defined by

$$\sigma_n(f; x) := \frac{1}{n} \sum_{i=0}^{n-1} S_i(f; x), \quad K_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} D_i(x)$$

The next important lemma was proved by Yano.

Lemma 1 (Yano [19]). *Let $n \in \mathbb{P}$, then*

$$\int_G |K_n(x)| \le 2.$$

Let $\{q_k : k \ge 0\}$ be a sequence of nonnegative numbers. The nth Nörlund mean of the Walsh–Fourier series are defined by

$$t_n(f; x) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k(f; x), \tag{3}$$

where $Q_n := \sum_{k=0}^{n-1} q_k$ $(n \ge 1)$. It is always assumed that $q_0 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty. \tag{4}$$

In this case, the summability method generated by $\{q_k\}$ is regular (see [11]) if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0. \tag{5}$$

In the paper [11] the rate of the approximation by Nörlund means $t_n(f)$ of Walsh–Fourier series of a function f in L^p (in particular, in $\mathrm{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \le p \le \infty$) was studied. As special cases Móricz and Siddiqi obtained the earlier results given by Yano [20], Jastrebova [7] and Skvortsov [15] on the rate of the approximation by Cesàro means. The approximation properties of the Cesàro means of negative order was studied by

Goginava [6]. In 2008 Fridli, Manchanda and Siddiqi [5] generalized the result of Móricz and Siddiqi for homogeneous Banach spaces and dyadic Hardy spaces. The approximation properties of the Nörlund means with respect to other rearrangement of the Walsh system was studied by the second author [13]. Recently, the first author, Baramidze, Memić, Persson, Tephnadze and Wall have results with respect to this topic [2,3,8].

Let $\{p_k : k \geq 1\}$ be a sequence of nonnegative numbers. The nth weighted mean T_n of Walsh–Fourier series is defined by

$$T_n(f;x) := \frac{1}{P_n} \sum_{k=1}^{n} p_k S_k(f;x), \tag{6}$$

where $P_n := \sum_{k=1}^{n} p_k$ $(n \geq 1)$. In the particular case when $p_k = 1$ for all k, T_n are the Walsh–Fejér means. It is always assumed that $p_1 > 0$ and

$$\lim_{n\to\infty} P_n = \infty, \tag{7}$$

which is the condition for regularity.

In the paper [9] the rate of the approximation by weighted means of Walsh–Fourier series of a function in L^p (in particular, in $\mathrm{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \leq p \leq \infty$) was studied. As special cases Móricz and Rhoades obtained the earlier results given by Yano [20], Jastrebova [7] on the rate of the approximation by Cesàro means. The approximation properties of the weighted means with respect to other rearrangement of the Walsh system was studied by the second author [12].

Now, let us set the sequence of matrices Θ_n in the following form:

$$\Theta_n := \begin{pmatrix} \theta_{0,1} & 0 & 0 & \dots & 0 \\ \theta_{0,2} & \theta_{1,2} & 0 & \dots & 0 \\ \theta_{0,3} & \theta_{1,3} & \theta_{2,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{0,n} & \theta_{1,n} & \theta_{2,n} & \dots & \theta_{n-1,n} \end{pmatrix}.$$

We always suppose that $\theta_{0,k} = 1$ for all $k \in \{1, \dots, n\}$.

Let the nth Θ-mean and kernel be defined by

$$\sigma_n^{\Theta}(f,x) := \sum_{k=0}^{n-1} \theta_{k,n} \widehat{f}(k) w_k(x), \quad K_n^{\Theta}(x) := \sum_{k=0}^{n-1} \theta_{k,n} w_k(x)$$

(see [4,18]). It is easily seen that

$$\sigma_n^{\Theta}(f,x) := \int_G f(t) K_n^{\Theta}(t+x)\, d\mu(t).$$

EXAMPLE 1. Let $\{q_n : n \geq 0\}$ be a nonnegative sequence. If we choose $\theta_{k,n} = \frac{1}{Q_n}\sum_{i=0}^{n-k-1} q_i$ $(0 \leq k \leq n-1)$, taking in account the equality (8), we immediately have $K_n^{\Theta} = \sum_{k=1}^{n} \frac{q_{n-k}}{Q_n} D_k$. It means that Nörlund-mean is a special Θ-mean.

EXAMPLE 2. Let $\{p_n : n \geq 1\}$ be a sequence as in (6) it is given. If we choose $\theta_{k,n} = \frac{1}{P_n}\sum_{i=k+1}^{n} p_i$ $(0 \leq k \leq n-1)$, taking in account the equality (8), we get $K_n^{\Theta} = \frac{1}{P_n}\sum_{k=1}^{n} p_k D_k$. It means that weighted mean is a special Θ-mean.

Our paper is motivated by the work of Móricz and Siddiqi [11] on Nörlund mean method and the result of Móricz and Rhoades [9] on weighted mean method. It is important to note that some idea is coming from the paper of Chripkó [4], in which the author studied the order of convergence of Θ-mean with respect to Jacobi–Fourier series. Our main aim is to investigate the rate of the approximation of Θ-mean by the help of modulus of continuity under some general conditions. Our main theorem (Theorem 1) and Lemma 2 give a common generalization of the two results of Móricz and Siddiqi on Nörlund means [11] and Móricz, Rhoades on weighted means [9] (see Examples 1 and 2). Moreover, we present some new results under general conditions for Θ-summability.

Other aspects of Θ-summability methods with respect to Walsh–Fourier series are treated in [16–18].

2. Auxiliary results

We introduce the notation

$$\Delta\theta_{k,n} := \theta_{k+1,n} - \theta_{k,n}, \quad \Delta^2\theta_{k,n} := \Delta\theta_{k+1,n} - \Delta\theta_{k,n},$$

where $k \in \{0, \dots, n-1\}$ and $\theta_{n,n} := \theta_{n+1,n} := 0$ (it is natural, see the matrix Θ_{n+2}).

In the following lemma we give a decomposition of the nth Θ-kernel of Walsh series. This lemma is a common generalization of two lemmas proved by Móricz and Siddiqi in [11, Lemma 3] and Móricz and Rhoades in [9, Lemma 2].

LEMMA 2. *Let $n > 2$ be a positive integer, then we have*

$$K_n^{\Theta}(x) = -\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n} D_{2^j}(x) - \sum_{k=0}^{n-2^{|n|}} \Delta\theta_{2^{|n|}+k-1,n} D_{2^{|n|}}(x)$$
$$+ \sum_{j=0}^{|n|-1} r_j(x) \sum_{k=0}^{2^j-2} \Delta^2\theta_{2^j+k-1,n}(k+1)K_{k+1}(x)$$

$$- \sum_{j=0}^{|n|-1} r_j(x)\Delta\theta_{2^{j+1}-2,n}2^j K_{2^j}(x) - r_{|n|}(x)R_n(x),$$

with the notation $R_n(x) = \sum_{k=0}^{n-2^{|n|}} \Delta\theta_{2^{|n|}+k-1,n} D_k(x)$.

Proof. An Abel transformation immediately gives

$$K_n^{\Theta}(x) = \sum_{k=0}^{n-1} \theta_{k,n}\omega_k(x) = \sum_{k=0}^{n-2}(\theta_{k,n} - \theta_{k+1,n}) \sum_{j=0}^{k} \omega_j(x) + \theta_{n-1,n} \sum_{k=0}^{n-1} \omega_k(x) \tag{8}$$

$$= -\Big(\sum_{k=0}^{n-2} \Delta\theta_{k,n} D_{k+1}(x) + \Delta\theta_{n-1,n} D_n(x) \Big) = -\Big(\sum_{l=1}^{n} \Delta\theta_{l-1,n} D_l(x) \Big).$$

Now, we use equality (1)

$$K_n^{\Theta}(x) = -\sum_{j=0}^{|n|-1} \sum_{l=2^j}^{2^{j+1}-1} \Delta\theta_{l-1,n} D_l(x) - \sum_{l=2^{|n|}}^{n} \Delta\theta_{l-1,n} D_l(x)$$

$$= -\sum_{j=0}^{|n|-1} \sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n} D_{2^j+k}(x) - \sum_{k=0}^{n-2^{|n|}} \Delta\theta_{2^{|n|}+k-1,n} D_{2^{|n|}+k}(x)$$

$$= -\sum_{j=0}^{|n|-1} \sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n} D_{2^j}(x) - \sum_{j=0}^{|n|-1} \sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n} r_j(x) D_k(x)$$

$$- \sum_{k=0}^{n-2^{|n|}} \Delta\theta_{2^{|n|}+k-1,n} D_{2^{|n|}}(x) - r_{|n|}(x) \sum_{k=0}^{n-2^{|n|}} \Delta\theta_{2^{|n|}+k-1,n} D_k(x)$$

$$=: K_n^{\Theta,1}(x) + K_n^{\Theta,2}(x) + K_n^{\Theta,3}(x) + K_n^{\Theta,4}(x).$$

For the expression $K_n^{\Theta,2}(x)$ we use Abel transformation again

$$K_n^{\Theta,2}(x) = -\sum_{j=0}^{|n|-1} \sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n} r_j(x) D_k(x)$$

$$= -\sum_{j=0}^{|n|-1} r_j(x) \Big(\sum_{k=0}^{2^j-2} \big(\Delta\theta_{2^j+k-1,n} - \Delta\theta_{2^j+k,n}\big) \sum_{i=0}^{k} D_i(x)$$

$$+ \Delta\theta_{2^{j+1}-2,n} \sum_{k=0}^{2^j-1} D_k(x) \Big)$$

$$= \sum_{j=0}^{|n|-1} r_j(x)\Big(\sum_{k=0}^{2^j-2} \Delta^2\theta_{2^j+k-1,n}(k+1)K_{k+1}(x) - \Delta\theta_{2^{j+1}-2,n}2^j K_{2^j}(x)\Big).$$

Summarizing our results on the expressions $K_n^{\Theta,1}(x)$, $K_n^{\Theta,2}(x)$, $K_n^{\Theta,3}(x)$, $K_n^{\Theta,4}(x)$ completes the proof. □

During the proof of our main theorem we need the following lemmas.

Lemma 3 (Móricz and Siddiqi [11]). *If $g \in \mathcal{P}_{2^m}$, $f \in L^p$, where $m \geq 0$ and $1 \leq p \leq \infty$, then there exists a positive constant c such that*

$$\Big\| \int_G r_m(t)g(t)|f(\cdot + t) - f(\cdot)|\, d\mu(t)\Big\|_p \leq c\|g\|_1 \omega_p(f, 2^{-m}).$$

A Sidon type inequality follows in the next lemma [10, Lemma 1], we will apply it to the kernels R_n later.

Lemma 4 (Móricz and Schipp [10]). *For every $1 < p \leq 2$, sequence $\{a_k\}$ of real numbers and integer $n \geq 1$ we have*

$$\Big\| \sum_{k=1}^{n} a_k D_k \Big\|_1 \leq \frac{2p}{p-1} n^{1-1/p} \Big[\sum_{k=1}^{n} |a_k|^p\Big]^{1/p}.$$

3. The rate of the approximation by Θ-means

Next our main theorem follows.

Theorem 1. *Let $f \in L^p$ $(1 \leq p \leq \infty)$. Let the finite sequences $\{\theta_{k,n} : 0 \leq k \leq n-1\}$ of nonnegative numbers be nonincreasing (in sign: $\theta_{k,n} \downarrow$).*

(a) *Let the finite sequence of differences $\{\Delta\theta_{k,n} : 0 \leq k \leq n-1\}$ be nonincreasing (in sign: $\Delta\theta_{k,n} \downarrow$). Moreover, we suppose that*

$$\theta_{n-1,n} = O\Big(\frac{1}{n}\Big). \tag{9}$$

Then

$$\big\|\sigma_n^{\Theta}(f) - f\big\|_p \leq 5 \sum_{j=0}^{|n|-1} 2^j \big|\Delta\theta_{2^{j+1}-1,n}\big| \omega_p(f, 2^{-j}) + O\big(\omega_p(f, 2^{-|n|})\big). \tag{10}$$

(b) *Let the finite sequences of differences $\{\Delta\theta_{k,n} : 0 \leq k \leq n-1\}$ be nondecreasing (in sign: $\Delta\theta_{k,n} \uparrow$). Then*

$$\big\|\sigma_n^{\Theta}(f) - f\big\|_p \leq 5 \sum_{j=0}^{|n|-1} 2^j \big|\Delta\theta_{2^j-1,n}\big| \omega_p(f, 2^{-j}) + O\big(\omega_p(f, 2^{-|n|})\big). \tag{11}$$

Remark 1. The condition $0 \le \theta_{k,n} \le 1$ for all $k \in \{0,\dots,n-1\}$ is not a strong restriction, since in Examples 1 and 2 it is satisfied. Both Nörlund and weighted mean summation cover a wide class of summation methods.

In case (a) the finite sequence $\{\theta_{k,n} : 0 \le k < n\}$ for a fix n is concave. In case (b) the finite sequence $\{\theta_{k,n} : 0 \le k < n\}$ for a fix n is convex.

In case (a) we could reach a little bit sharper form of inequality (10), that is, the expression $|\Delta\theta_{2^{j+1}-1,n}|$ could be changed by $|\Delta\theta_{2^{j+1}-2,n}|$ (see Remark 3).

Remark 2. For Example 1, a simple calculation yields that $\Delta\theta_{2^j-1,n} = -\frac{q_{n-2^j}}{Q_n}$ and $\Delta\theta_{2^{j+1}-1,n} = -\frac{q_{n-2^{j+1}}}{Q_n}$. Thus, as consequence of our main theorem we get back an analogical form of the result of Móricz and Siddiqi [11]. For Example 2, $\Delta\theta_{2^j-1,n} = -\frac{p_{2^j}}{P_n}$ and $\Delta\theta_{2^{j+1}-1,n} = -\frac{p_{2^{j+1}}}{P_n}$ hold. Thus, as consequence of our main theorem we get back an analogical form of the result of Móricz and Rhoades [9].

Proof of Theorem 1. We present the proof for $1 \le p < \infty$; for $p = \infty$ the proof is similar (where $L^\infty = C$). Keeping in mind that $\theta_{0,k} = 1$ for all k, applying Lemma 2 and the usual Minkowski inequality gives that

$$\left\|\sigma_n^\Theta(f) - f\right\|_p = \left(\int_G |\sigma_n^\Theta(f,x) - f(x)|^p \, d\mu(x)\right)^{1/p}$$

$$= \left(\int_G \left|\int_G K_n^\Theta(t)(f(x+t) - f(x))\, d\mu(t)\right|^p d\mu(x)\right)^{1/p}$$

$$\le \sum_{j=0}^{|n|-1} \left|\sum_{k=0}^{2^j-1} \Delta\theta_{2^j+k-1,n}\right| \left(\int_G \left|\int_G D_{2^j}(t)(f(x+t) - f(x))\, d\mu(t)\right|^p d\mu(x)\right)^{1/p}$$

$$+ \left|\sum_{k=0}^{n-2^{|n|}} \Delta\theta_{2^{|n|}+k-1,n}\right| \left(\int_G \left|\int_G D_{2^{|n|}}(t)(f(x+t) - f(x))\, d\mu(t)\right|^p d\mu(x)\right)^{1/p}$$

$$+ \sum_{j=0}^{|n|-1} \sum_{k=0}^{2^j-2} \left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1)$$

$$\times \left(\int_G \left|\int_G r_j(t) K_{k+1}(t)(f(x+t) - f(x))\, d\mu(t)\right|^p d\mu(x)\right)^{1/p}$$

$$+ \sum_{j=0}^{|n|-1} |\Delta\theta_{2^{j+1}-2,n}| 2^j \left(\int_G \left|\int_G r_j(t) K_{2^j}(t)(f(x+t) - f(x))\, d\mu(t)\right|^p d\mu(x)\right)^{1/p}$$

$$+\Big(\int_G\Big|\int_G r_{|n|}(t)R_n(t)(f(x+t)-f(x))\,d\mu(t)\Big|^p\,d\mu(x)\Big)^{1/p}$$
$$=: I_{1,n}+I_{2,n}+I_{3,n}+I_{4,n}+I_{5,n}.$$

Using generalized Minkowski inequality for the expressions $I_{1,n}$ and $I_{2,n}$, we obtain

(12)
$$I_{1,n}\le\sum_{j=0}^{|n|-1}\Big|\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}\Big|\int_G D_{2^j}(t)\Big(\int_G|f(x+t)-f(x)|^p\,d\mu(x)\Big)^{1/p}d\mu(t)$$
$$\le\sum_{j=0}^{|n|-1}\Big|\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}\Big|\omega_p(f,2^{-j}),$$

(13)
$$I_{2,n}\le\Big|\sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}\Big|\int_G D_{2^{|n|}}(t)\Big(\int_G|f(x+t)-f(x)|^p\,d\mu(x)\Big)^{1/p}d\mu(t)$$
$$\le\Big|\sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}\Big|\omega_p(f,2^{-|n|}).$$

In case (a) ($\Delta\theta_{k,n}\downarrow$) we immediately get $\big|\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}\big|\le-2^j\Delta\theta_{2^{j+1}-1,n}$ and

$$I_{1,n}\le-\sum_{j=0}^{|n|-1}2^j\Delta\theta_{2^{j+1}-1,n}\omega_p(f,2^{-j}).$$

In case (b) ($\Delta\theta_{k,n}\uparrow$) we calculate $\big|\sum_{k=0}^{2^j-1}\Delta\theta_{2^j+k-1,n}\big|\le-2^j\Delta\theta_{2^j-1,n}$ and

$$I_{1,n}\le-\sum_{j=0}^{|n|-1}2^j\Delta\theta_{2^j-1,n}\omega_p(f,2^{-j}).$$

Since $\big|\sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}\big|=\theta_{2^{|n|}-1,n}-\theta_{n,n}\le1$, so in both cases (a) and (b) we write

$$I_{2,n}\le c\omega_p(f,2^{-|n|}).$$

Now, by Lemma 3 and Lemma 1 we get

$$I_{3,n}\le\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-2}\big|\Delta^2\theta_{2^j+k-1,n}\big|(k+1)\omega_p(f,2^{-j})\|K_{k+1}\|_1 \tag{14}$$

$$\leq 2\sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1)\omega_p(f,2^{-j}),$$

In case (a) ($\Delta\theta_{k,n}\downarrow$), we have

$$\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1)=\sum_{k=0}^{2^j-2}(\Delta\theta_{2^j+k-1,n}-\Delta\theta_{2^j+k,n})(k+1)$$
$$=\sum_{k=0}^{2^j-2}\Delta\theta_{2^j+k-1,n}-(2^j-1)\Delta\theta_{2^{j+1}-2,n}\leq -2^j\Delta\theta_{2^{j+1}-1,n}$$

and

$$I_{3,n}\leq -2\sum_{j=0}^{|n|-1}2^j\Delta\theta_{2^{j+1}-1,n}\omega_p(f,2^{-j}).$$

In case (b) ($\Delta\theta_{k,n}\uparrow$) we have

$$\sum_{k=0}^{2^j-2}\left|\Delta^2\theta_{2^j+k-1,n}\right|(k+1)=(2^j-1)\Delta\theta_{2^{j+1}-2,n}-\sum_{k=0}^{2^j-2}\Delta\theta_{2^j+k-1,n}$$
$$\leq -\sum_{k=0}^{2^j-2}\Delta\theta_{2^j+k-1,n}\leq -2^j\Delta\theta_{2^j-1,n}$$

and

$$I_{3,n}\leq -2\sum_{j=0}^{|n|-1}2^j\Delta\theta_{2^j-1,n}\omega_p(f,2^{-j}).$$

Now, we discuss the expression $I_{4,n}$.

$$I_{4,n}\leq\sum_{j=0}^{|n|-1}\left|\Delta\theta_{2^{j+1}-2,n}\right|2^j\omega_p(f,2^{-j})\|K_{2^j}\|_1 \tag{15}$$
$$\leq -2\sum_{j=0}^{|n|-1}\Delta\theta_{2^{j+1}-2,n}2^j\omega_p(f,2^{-j}).$$

In case (a) ($\Delta\theta_{k,n}\downarrow$), we immediately write

$$I_{4,n} \le -2\sum_{j=0}^{|n|-1} \Delta\theta_{2^{j+1}-1,n} 2^j \omega_p(f,2^{-j}).$$

In case (b) ($\Delta\theta_{k,n}\uparrow$) we have

$$I_{4,n} \le -2\sum_{j=0}^{|n|-1} \Delta\theta_{2^{j}-1,n} 2^j \omega_p(f,2^{-j}).$$

By Lemma 3 we write for the last expression $I_{5,n}$

$$I_{5,n} \le c\omega_p(f,2^{-|n|})\|R_n\|_1. \tag{16}$$

The Sidon type inequality in Lemma 4 implies that

$$\|R_n\|_1 \le c \quad \text{for all } n\in\mathbb{P} \tag{17}$$

in both cases (a) and (b). Namely, for $p=2$ we apply Lemma 4 and we have

$$\|R_n\|_1 \le 4(n-2^{|n|})^{1/2}\left[\sum_{k=0}^{n-2^{|n|}} |\Delta\theta_{2^{|n|}+k-1,n}|^2\right]^{1/2}. \tag{18}$$

In case (a) ($\Delta\theta_{k,n}\downarrow$), using condition (9) gives that

$$\|R_n\|_1 \le 4(n-2^{|n|}+1)|\Delta\theta_{n-1,n}| \le 4n\theta_{n-1,n} \le c.$$

In case (b) ($\Delta\theta_{k,n}\uparrow$) we have

$$\sum_{k=0}^{n-2^{|n|}} |\Delta\theta_{2^{|n|}+k-1,n}|^2 \le (n-2^{|n|}+1)|\Delta\theta_{2^{|n|}-1,n}|^2.$$

Since $n-2^{|n|}+1\le 2^{|n|}$ we write

$$\|R_n\|_1 \le 4(n-2^{|n|}+1)|\Delta\theta_{2^{|n|}-1,n}|$$
$$\le 4\big(|\Delta\theta_{0,n}|+\ldots+|\Delta\theta_{2^{|n|}-1,n}|\big) \le 4\big((\theta_{0,n}-\theta_{2^{|n|},n}\big) \le 4.$$

This yields that the inequality (17) is proved for all n. From inequality (16) we immediately get

$$I_{5,n} \le c\omega_p\left(f,2^{-|n|}\right) \qquad \text{for all } n.$$

By summarizing our results on the expressions $I_{1,n}$, $I_{2,n}$, $I_{3,n}$, $I_{4,n}$, $I_{5,n}$ we complete the proof of our main theorem. □

Now, we present a statement for Lipschitz functions (see [9,11] for analogical theorems).

THEOREM 2. *Let $f \in \mathrm{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \le p \le \infty$. For Θ-mean σ_n^Θ, we suppose the conditions in Theorem 1.*

In case (a), *the estimate*

$$\|\sigma_n^\Theta(f) - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if } 0 < \alpha < 1, \\ O(\log n/n), & \text{if } \alpha = 1, \\ O(1/n), & \text{if } \alpha > 1. \end{cases}$$

holds. In case (b), *the estimate*

$$\|\sigma_n^\Theta(f) - f\|_p = O\Big(\sum_{j=0}^{|n|-1} |\Delta\theta_{2^j-1,n}| 2^{j(1-\alpha)} + 2^{-|n|\alpha}\Big)$$

holds.

PROOF. Theorem 1(a) with condition (9) yields that

$$\big\|\sigma_n^\Theta(f) - f\big\|_p = O\Big(\sum_{j=0}^{|n|-1} 2^j |\Delta\theta_{2^{j+1}-1,n}| 2^{-j\alpha} + 2^{-|n|\alpha}\Big)$$

$$= O\Big(|\Delta\theta_{n-1,n}| \sum_{j=0}^{|n|-1} 2^{j(1-\alpha)} + 2^{-|n|\alpha}\Big) = O\Big(\frac{1}{n} \sum_{j=0}^{|n|-1} 2^{j(1-\alpha)} + 2^{-|n|\alpha}\Big).$$

Hence our statement follows easily.

Part (b) immediately follows from Theorem 1. □

In the next theorem we allow that the finite sequence $\{\theta_{k,n} : 0 \le k \le n-1\}$ take some negative values.

THEOREM 3. *Let $f \in L^p$, $(1 \le p \le \infty)$. Let the finite sequence $\{\theta_{k,n} : 0 \le k \le n-1\}$ be nonincreasing $(\theta_{k,n} \downarrow)$ and $\theta_{n-1,n} < 0$.*

(a) *Let the finite sequences of differences $\{\Delta\theta_{k,n} : 0 \le k \le n-2\}$ be nonincreasing $(\Delta\theta_{k,n} \downarrow)$. Moreover, we suppose that*

$$|\theta_{n-1,n}| = O\Big(\frac{1}{\sqrt{n}}\Big) \quad \text{and} \quad |\Delta\theta_{n-2,n}| = O\Big(\frac{1}{n}\Big). \tag{19}$$

Then

$$\big\|\sigma_n^{\Theta}(f)-f\big\|_p \le 5\sum_{j=0}^{|n|-1} 2^j\big|\Delta\theta_{2^{j+1}-1,n}\big|\omega_p(f,2^{-j})+O\big(\omega_p(f,2^{-|n|})\big). \tag{20}$$

(b) *Let the finite sequences of differences* $\{\Delta\theta_{k,n}: 0\le k\le n-2\}$ *be non-decreasing* ($\Delta\theta_{k,n}\uparrow$). *Moreover, we suppose that there exists a negative constant* c_* *such that* $\theta_{n-1,n}\ge c_*$ *for all* n. *Then*

$$\big\|\sigma_n^{\Theta}(f)-f\big\|_p \le 5\sum_{j=0}^{|n|-1} 2^j\big|\Delta\theta_{2^{j}-1,n}\big|\omega_p(f,2^{-j})+O\big(\omega_p(f,2^{-|n|})\big) \tag{21}$$

holds.

Proof of Theorem 3. We make the proof for such a finite sequence $\{\theta_{k,n}: 0\le k\le n-1\}$ for which at least the last member $\theta_{n-1,n}$ is negative. We use the method and notations of the proof of Theorem 1.

$$\|\sigma_n^{\Theta}(f)-f\|_p\le I_{1,n}+I_{2,n}+I_{3,n}+I_{4,n}+I_{5,n}.$$

Since the most of the proof runs along the same line as that just given above, we give details about the necessary changes.

For the expression $I_{2,n}$ we have inequality (13). Since

$$\Big|\sum_{k=0}^{n-2^{|n|}}\Delta\theta_{2^{|n|}+k-1,n}\Big| = |\theta_{2^{|n|}-1,n}-\theta_{n,n}|\le 1+|c'|$$

(where c' is coming from condition (19)), in case (a) and $|\theta_{2^{|n|}-1,n}-\theta_{n,n}|$ $\le 1+|c_*|$ in case (b), we write

$$I_{2,n}\le c\omega_p(f,2^{-|n|}).$$

Let us turn our attention to the expression $I_{5,n}$. In case (a) ($\Delta\theta_{k,n}\downarrow$) we have

$$\sum_{k=0}^{n-2^{|n|}}|\Delta\theta_{2^{|n|}+k-1,n}|^2\le (n-2^{|n|})|\Delta\theta_{n-2,n}|^2+|\Delta\theta_{n-1,n}|^2$$

Using condition (19) and estimate (18),

$$\begin{aligned}\|R_n\|_1 &\le 4(n-2^{|n|})|\Delta\theta_{n-2,n}|+4(n-2^{|n|})^{1/2}|\theta_{n-1,n}|\\ &\le 4n|\Delta\theta_{n-2,n}|+4n^{1/2}|\theta_{n-1,n}|\le c.\end{aligned}$$

In case (b) ($\Delta\theta_{k,n}\uparrow$) we have

$$\|R_n\|_1 \leq 4(\theta_{0,n} - \theta_{2^{|n|},n}) \leq 4(1 - c_*) \leq c.$$

This yields

$$I_{5,n} \leq c\omega_p(f, 2^{-|n|}) \quad \text{for all } n.$$

This completes the proof of our theorem. □

Remark 3. Under the conditions of Theorem 3, the statement of Theorem 2 for Lipschitz functions holds again. We note that during the proof of case (a) we have to use a little bit sharper form of inequality (20) (see the last note in Remark 1).

Remark 4. Let us suppose that the finite sequence of $\{\theta_{k,n} : 0 \leq k < n-1\}$ is nondecreasing ($\theta_{k,n}\uparrow$) and bounded by a positive constant. Then the Sidon type inequality does not guarantee the uniform boundedness of the L^1-norm of kernels R_n, in both cases $\Delta\theta_{k,n}\uparrow$ and $\Delta\theta_{k,n}\downarrow$. So, we do not discuss this case.

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