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ON A CLASS OF ANALYTIC FUNCTIONS RELATED TO CONIC DOMAINS INVOLVING q-CALCULUS

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Abstract. The theory of q-analysis has many applications in various subfields of mathematics and quantum physics. Research work in connection with function theory and q-theory together was first introduced by Ismail et al. [6]. Till now only non-significant interest in this area was shown although it deserves more attention. Exploiting this, we aim to introduce a new class of analytic functions that are closely related to the domains bounded by conic sections. The authors hope this article will motivate future researchers to work in the area of q-calculus which can find many applications in the theory of hypergeometric series and quantum theory.

1. Introduction and definitions

The theory of q-analysis in the recent past has been applied in many areas of mathematics and physics, for example, in the areas of ordinary fractional calculus, optimal control problems, q-difference and q-integral equations and in q-transform analysis. The q-theory has wide applications in special functions and quantum physics which makes the study interesting and pertinent in this field. Note that the q-difference operator plays an important role in the theory of hypergeometric series and quantum theory, number theory, statistical mechanics, etc. At the beginning of the last century studies on q-difference equations appeared in intensive works especially by Jackson [7], Carmichael [4], Mason [20], Adams [1] and Trjitzinsky [27]. Research work

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in connection with function theory and q-theory together was first introduced by Ismail et al. [6]. Till now only non-significant interest in this area was shown although it deserves more attention. The q-difference operator related to the q-calculus was introduced by Andrews et al. (see[3, Ch. 10]). The q-derivative operator is defined by

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q))}, \quad z \in \mathbb{U}, \text{ where } \mathbb{U} = \{z \in \mathbb{C} \text{ and } |z| < 1\}. \tag{1.1}$$

One can easily verify that $\partial_q f(z) \to f'(z)$ as $q \to 1^-$. For any non-negative integer n, the *q-integer number n* denoted by $[n]$, is defined by

$$[n] = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad [0] = 0. \tag{1.2}$$

The q-number shifted factorial is defined by $[0]! = 1$ and $[n]! = [1][2][3] \cdots [n]$. Clearly,

$$\lim_{q \to 1^-} [n] = n \quad \text{and} \quad \lim_{q \to\ 1^-} [n]! = n!\,.$$

In general, for a non-integer number t, $[t]$ is defined by $[t] = \frac{1-q^t}{1-q}$. Throughout this paper, we will assume q to be a fixed number between 0 and 1. Let $\mathcal{A}$ denote the class of all functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.3}$$

which are analytic in the open unit disk $\mathbb{U}$ and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{A}$ is said to be *starlike* of order α, $0 \le \alpha < 1$, written as $f \in \mathcal{S}^*(\alpha)$, if $\operatorname{Re}\big(\frac{zf'(z)}{f(z)}\big) > \alpha$, $z \in \mathbb{U}$. A function $f \in \mathcal{A}$ is said to be *convex* order of α order, $0 \le \alpha < 1$, written as $f \in \mathcal{K}(\alpha)$ if $\operatorname{Re}\big(1 + \frac{zf''(z)}{f'(z)}\big) > \alpha$, $z \in \mathbb{U}$. The class of all *k-starlike* functions and k-uniformly *convex* functions were introduced and studied by Kanas and Wiśniowska [9] and [13] as follows. A function $f \in \mathcal{S}$ is said to be in the class k-$\mathcal{ST}$, if

$$\operatorname{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > k\Big|\frac{zf'(z)}{f(z)} - 1\Big| \quad (0 \le k < \infty,\ z \in \mathbb{U}). \tag{1.4}$$

A function $f \in \mathcal{S}$ is said to be in the class k-$\mathcal{UCV}$, if

$$\operatorname{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > k\Big|\frac{zf''(z)}{f'(z)}\Big| \quad (0 \le k < \infty,\ z \in \mathbb{U}). \tag{1.5}$$

In particular, when $k=1$, we obtain k-$\mathcal{UCV} \equiv \mathcal{UCV}$ and k-$\mathcal{ST} \equiv \mathcal{SP}$, where $\mathcal{UCV}$ and $\mathcal{SP}$ are the familiar classes of uniformly *convex* functions and parabolic *starlike* functions in $\mathbb{U}$ respectively (see for details, [5]). Many works related to k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ are investigated by Kanas in many of her papers (for related works of her, one may refer to [8], [10], [12], [14] and [22]). A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}(k,\alpha)$ that are *k-starlike* functions of order α, $0 \le \alpha < 1$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha \quad (0 \le k < \infty,\ z \in \mathbb{U}). \tag{1.6}$$

Applying the well-known Alexander relation in the above inequality, we obtain

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| + \alpha \quad (0 \le k < \infty,\ z \in \mathbb{U}). \tag{1.7}$$

These classes have been studied recently by Kanas and Răducanu [17] by applying the theory of q-calculus. We note that $\mathcal{UCV}(k,0) \equiv k - \mathcal{UCV}$ and $\mathcal{ST}(k,0) \equiv k - \mathcal{ST}$. The classes $\mathcal{UCV}(1,\alpha)$ and $\mathcal{ST}(1,\alpha)$ were investigated in [2], [16] and [25]. The aim of this work is to adopt a notion of uniform convexity in some classes defined by the q-calculus. In recent times, the univalent function theorists have shown good affinity towards q-calculus by relating it with the area of geometric function theory. One such work is by Kanas and Răducanu [17] where the q-calculus version of the Ruscheweyh derivative was defined. Therefore it is natural to see the q-calculus version of the Sãlãgean derivative which is introduced in this article. For $f \in \mathcal{A}$, let the Sãlãgean q-differential operator be defined by

$$\mathcal{S}_q^0 f(z) = f(z), \quad \mathcal{S}_q^1 f(z) = z\partial_q f(z), \quad \dots, \quad \mathcal{S}_q^m f(z) = z\partial_q(\mathcal{S}_q^{m-1} f(z)).$$

A simple calculation implies

$$\mathcal{S}_q^m f(z) = f(z) * G_{q,m}(z) \quad (z \in \mathbb{U},\ m \in \mathbb{N} \cup \{0\} = \mathbb{N}_0) \tag{1.8}$$

where

$$G_{q,m}(z) = z + \sum_{n=2}^{\infty} [n]^m z^n \quad (z \in \mathbb{U},\ m \in \mathbb{N}_0). \tag{1.9}$$

Making use of (1.8) and (1.9), the power series of $\mathcal{S}_q^m f(z)$ for f of the form (1.3) is given by

$$\mathcal{S}_q^m f(z) = z + \sum_{n=2}^{\infty} [n]^m a_n z^n \quad (z \in \mathbb{U}). \tag{1.10}$$

Note that

$$\lim_{q\to 1^-} G_{q,m}(z) = z + \sum_{n=2}^{\infty} n^m z^n$$

and

$$\lim_{q\to 1^-} \mathcal{S}_q^m f(z) = f(z) * (z + \sum_{n=2}^{\infty} n^m z^n)$$

which is the familiar Sãlãgean derivative [26].

Motivated by the works of Kanas and Răducanu [17] and Kanas and Yaguchi [15], we define the following class of functions with the theory of q-calculus.

DEFINITION 1.1. Let $0 \le k < \infty$, $0 \le \alpha < 1$, $q \in (0,1)$ and $m \in \mathbb{N}_0$. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_q(k,\alpha,m)$ if it satisfies the condition

$$\text{Re}\left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)}\right) > k\left|\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1\right| + \alpha, \quad (z \in \mathbb{U}). \tag{1.11}$$

Note that as $q \to 1^-$, the class $\mathcal{S}_q(k,0,m)$ reduces to the class $\mathcal{T}(k,m)$ investigated by Kanas and Yaguchi [15] and the class $\mathcal{S}_q(k,\alpha,0)$ reduces to the class $\mathcal{ST}(k,\alpha)$ investigated by Kanas and Răducanu [17] respectively.

Motivated by the aforementioned works, the aim of this paper is to investigate the new class of functions $\mathcal{S}_q(k,\alpha,m)$. We remark here that these classes of function are related to the domains bounded by conical sections.

2. The class $\mathcal{S}_q(k,\alpha,m)$

Consider $p(z) = \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)}$. Then the condition (1.11) may be rewritten into the form

$$\text{Re}(p(z)) > k|p(z) - 1| + \alpha \quad (z \in \mathbb{U}).$$

It follows that the range of the expression $p(z)$, $z \in \mathbb{U}$, is a conical domain

$$\Omega_{k,\alpha} = \big\{\omega \in \mathbb{C} : \text{Re}(\omega) > k|\omega - 1| + \alpha\big\},$$

or

$$\Omega_{k,\alpha} = \big\{\omega = u + iv : u > k\sqrt{(u-1)^2 + v^2} + \alpha\big\},$$

where $0 \le k < \infty$ and $0 \le \alpha < 1$. Note that $\Omega_{k,\alpha}$ is such that $1 \in \Omega_{k,\alpha}$ and $\partial\Omega_{k,\alpha}$ is a curve defined by

$$\partial\Omega_{k,\alpha} = \big\{\omega = u + iv : (u-\alpha)^2 = k^2(u-1)^2 + k^2v^2\big\}. \tag{2.1}$$

Elementary computations show that $\Omega_{k,\alpha}$ represents a conic section symmetric about the real axis. It follows that the domain $\Omega_{k,\alpha}$ is bounded by an ellipse for $k > 1$, by a parabola for $k = 1$ and by a hyperbola if $0 < k < 1$. Finally, for $k = 0$, $\Omega_{k,\alpha}$ is the right half plane $\mathrm{Re}(\omega) > \alpha$. From (1.11), we obtain that $f \in \Omega_{k,\alpha}$ if and only if, for $z \in \mathbb{U}$,

$$\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} \in \Omega_{k,\alpha}. \tag{2.2}$$

Making use of the properties of the domain $\Omega_{k,\alpha}$ and (2.2), it follows that if $f \in \Omega_{k,\alpha}$, then

$$\mathrm{Re}\left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)}\right) > \frac{k+\alpha}{k+1} \quad (z \in \mathbb{U}),$$

and

$$\left|\mathrm{Arg}\, \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)}\right| \leq \begin{cases} \arctan \frac{1-\alpha}{\sqrt{|k^2-\alpha^2|}}, & \text{if } 0 \leq \alpha < 1,\ k > 0, \\ \frac{\pi}{2}, & \text{if } k = 0. \end{cases}$$

Denote by $\mathcal{P}$ the class of *analytic* and normalized Carathéodory functions and by $p_{k,\alpha} \in \mathcal{P}$, the function such that $p_{k,\alpha}(\mathbb{U}) = \Omega_{k,\alpha}$. Following the notation applied by Ma and Minda [19], for $0 \leq k < \infty$ and $0 \leq \alpha < 1$, let $\mathcal{P}(p_{k,\alpha})$ denote the following class of functions:

$$\mathcal{P}(p_{k,\alpha}) = \{p \in \mathcal{P} : p(\mathbb{U}) \subset \Omega_{k,\alpha}\} = \{p \in \mathcal{P} : p \prec p_{k,\alpha} \text{ in } \mathbb{U}\}.$$

The functions which play the role of extremal functions for the class $\mathcal{P}(p_{k,\alpha})$ may be obtained by a simple modification of related functions described in [9] (see also [21], [22]) and are defined by

$$p_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z}, & \text{if } k = 0, \\ 1 + \frac{2(1-\alpha)}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \text{if } k = 1, \\ \frac{1-\alpha}{1-k^2} \cos\left(A(k) i \log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right) - \frac{k^2-\alpha}{1-k^2}, & \text{if } 0 < k < 1, \\ \frac{1-\alpha}{k^2-1} \sin\left(\frac{\pi}{2\kappa(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right) + \frac{k^2-\alpha}{k^2-1}, & \text{if } k > 1, \end{cases} \tag{2.3}$$

with $A(k) = \frac{2}{\pi} \arccos k$,

$$u(z) = \frac{z - \sqrt{t}}{1 - \sqrt{tz}} \quad (0 < t < 1,\ z \in \mathbb{U}),$$

where t is so such that $k = \cosh \frac{\pi\kappa'(t)}{4\kappa(t)}$, and $\kappa(t)$ is Legendre's complete elliptic integral of the first kind and $\kappa'(t)$ the complementary integral of $\kappa(t)$. Obviously, if $k = 0$ then

$$p_{0,\alpha} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \cdots .$$

For $k = 1$, we have (see [18] and also [24])

$$p_{1,\alpha} = 1 + \frac{8}{\pi^2}(1-\alpha)z + \frac{16}{3\pi^2}(1-\alpha)z^2 + \cdots .$$

Using the Taylor expansion in [8] and [9], for $0 < k < 1$, we have

$$p_{k,\alpha} = 1 + \frac{(1-\alpha)}{1-k^2}\sum_{n=1}^{\infty}\left[\sum_{l=1}^{2n} 2^l \binom{A(k)}{l}\binom{2n-1}{2n-l}\right] z^n.$$

Finally, when $k > 1$

$$p_{k,\alpha} = 1 + \frac{\pi^2(1-\alpha)}{4\sqrt{t}(k^2-1)\kappa^2(t)(1+t)} \times \left\{ z + \frac{4\kappa^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}\kappa^2(t)(1+t)} z^2 + \cdots \right\},$$

so that, denoting $p_{k,\alpha}(z) = 1 + P_1 z + P_2 z^2 + \cdots$ $(P_j = P_j(k,\alpha),\ j = 1,2,\ldots)$, we get

$$P_1 = \begin{cases} \frac{8(1-\alpha)(\arccos k)^2}{\pi^2(1-k^2)}, & \text{if } 0 \le k < 1, \\ \frac{8(1-\alpha)}{\pi^2}, & \text{if } k = 1, \\ \frac{\pi^2(1-\alpha)}{4\sqrt{t}(k^2-1)\kappa^2(t)(1+t)}, & \text{if } k > 1. \end{cases} \tag{2.4}$$

Let $f_{k,\alpha}(z) = z + A_2 z^2 + A_3 z^3 + \cdots$ be the extremal function in the class $\mathcal{S}_q(k,\alpha,m)$. Then, the relation between the extremal functions in the classes $\mathcal{P}(p_{k,\alpha})$ and $\mathcal{S}_q(k,\alpha,m)$ is given by

$$p_{k,\alpha}(z) = \frac{\mathcal{S}_q^{m+1} f_{k,\alpha}(z)}{\mathcal{S}_q^m f_{k,\alpha}(z)} \quad (z \in \mathbb{U}). \tag{2.5}$$

Making use of (1.10), (1.11) and (2.5), we obtain the following coefficient relation for $p_{k,\alpha}(z)$:

$$[n]^m([n]-1)A_n = \sum_{\mu=1}^{n-1} [\mu]^m A_\mu P_{n-\mu}, \quad A_1 = 1. \tag{2.6}$$

In particular, by a straightforward computation, we get

$$A_2 = \frac{P_1}{q(1+q)^m} \tag{2.7}$$

and

$$A_3 = \frac{qP_2 + P_1^2}{q^2(1+q)(1+q+q^2)^m}\,. \tag{2.8}$$

Since $m \in \mathbb{N}_0$, $q \in (0,1)$ and the P_n's are nonnegative, it follows that the A_n's are nonnegative. The following propositions follow directly from the definition and by making use of the geometric properties of the domains $\Omega_{k,\alpha}$.

PROPOSITION 2.1. *$\mathcal{S}_q(k_1,\alpha,m) \subset \mathcal{S}_q(k_2,\alpha,m)$, when $k_1 \geq k_2$.*

PROPOSITION 2.2. *$\mathcal{S}_q(k,\alpha_1,m) \subset \mathcal{S}_q(k,\alpha_2,m)$, when $\alpha_1 \geq \alpha_2$.*

PROPOSITION 2.3. *Let $0 \leq k < \infty$, $0 \leq \alpha < 1$, $q \in (0,1)$ and $m \in \mathbb{N}_0$ and $f_{k,\alpha}(z)$ and $h_{k,\alpha}(z)$ be the extremal functions of the classes $\mathcal{S}_q(k,\alpha,m)$ and $k-\mathcal{UCV}$ respectively. Moreover, let $f_{k,\alpha}(z) = z + A_2z^2 + A_3z^3 + \cdots$ and $h_{k,\alpha}(z) = z + B_2z^2 + B_3z^3 + \cdots$. Then*

$$B_n = [n]^{m-1}A_n, \quad n = 2,3,\ldots. \tag{2.9}$$

THEOREM 2.1. *Let $0 \leq k < \infty$, $0 \leq \alpha < 1$, $q \in (0,1)$ and $m \in \mathbb{N}_0$. If f of the form (1.3) belongs to the class $\mathcal{S}_q(k,\alpha,m)$, then*

$$|a_2| \leq A_2 \quad \text{and} \quad |a_3| \leq A_3. \tag{2.10}$$

PROOF. Let $p(z) = \frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^{m}f(z)}$. Using the relation (1.10) for $p(z) = 1 + p_1z + p_2z^2 + \cdots$, we have

$$[n]^m([n]-1)a_n = \sum_{\mu=1}^{n-1}[\mu]^m a_\mu p_{n-\mu}, \quad a_1 = 1. \tag{2.11}$$

Since $p_{k,\alpha}$ is univalent in $\mathbb{U}$, the function

$$q(z) = \frac{1 + p_{k,\alpha}^{-1}(p(z))}{1 - p_{k,\alpha}^{-1}(p(z))} = 1 + c_1z + c_2z^2 + \cdots$$

is analytic in $\mathbb{U}$ and $\mathrm{Re}(q(z)) > 0$. From

$$p(z) = p_{k,\alpha}\Big(\frac{q(z)-1}{q(z)+1}\Big) = 1 + \frac{1}{2}c_1P_1z + \Big(\frac{1}{2}c_2P_1 + \frac{1}{4}c_1^2(P_2 - P_1)\Big)z^2 + \cdots,$$

we have

$$|a_2| = \frac{1}{q(1+q)^m}|p_1| = \frac{1}{2q(1+q)^m}|c_1P_1| \leq \frac{P_1}{q(1+q)^m} = A_2, \tag{2.12}$$

where we have used the inequality $|c_n| \le 2$ and (2.6). In view of the relation $|p_1|^2 + |p_2| \le P_1^2 + P_2$ (see [8]) and (2.7), we obtain

$$(2.13)\qquad |a_3| = \frac{|qp_2 + p_1^2|}{q^2(1+q)(1+q+q^2)^m} \le \frac{q(|p_2| + |p_1|^2) + (1-q)|p_1|^2}{q^2(1+q)(1+q+q^2)^m}$$
$$\le \frac{q(|P_2| + |P_1|^2) + (1-q)|P_1|^2}{q^2(1+q)(1+q+q^2)^m} = \frac{q|P_2| + |P_1|^2}{q^2(1+q)(1+q+q^2)^m} = A_3.$$

Thus, the proof of the theorem is completed. □

THEOREM 2.2. *Let* $0 \le k < \infty$, $0 \le \alpha < 1$, $q \in (0,1)$ *and* $m \in \mathbb{N}_0$. *If* f *of the form* (1.3) *belongs to the class* $\mathcal{S}_q(k, \alpha, m)$, *then*

$$(2.14)\quad |a_n| \le \frac{P_1(P_1+q)(P_1+[2]q)\cdots(P_1+[n-2]q)}{q^{n-1}(1+q+\cdots+q^{n-1})^m \prod_{k=1}^{n-1}(1+q+\cdots+q^{k-1})}, \ n \ge 2.$$

PROOF. The result is clearly true for $n = 2$. Let n be an integer with $n \ge 2$, and assume that the inequality is true for all $m \le n-1$. Making use of (2.6), we have

$$|a_n| = \frac{1}{[n]^m([n]-1)}\left|p_{n-1} + \sum_{\mu=2}^{n-1}[\mu]^m a_\mu p_{n-\mu}\right|$$
$$\le \frac{1}{[n]^m([n]-1)}\left\{P_1 + \sum_{\mu=2}^{n-1}[\mu]^m |a_\mu| P_1\right\} \le \frac{1}{[n]^m([n]-1)}$$
$$\times P_1\left\{1 + \sum_{\mu=2}^{n-1}[\mu]^m \frac{P_1(P_1+q)(P_1+[2]q)\cdots(P_1+[\mu-2]q)}{q^{\mu-1}(1+q+\cdots+q^{n-1})^m \prod_{k=1}^{\mu-1}(1+q+\cdots+q^{k-1})}\right\},$$

where we applied the induction hypothesis to $|a_m|$ and the Rogosinski result $|p_n| \le P_1$ (see [23]). Therefore,

$$|a_n| \le \frac{1}{[n]^m([n]-1)} P_1\left\{1 + \sum_{\mu=2}^{n-1}\frac{P_1(P_1+q)(P_1+[2]q)\cdots(P_1+[\mu-2]q)}{q^{\mu-1}\prod_{k=1}^{\mu-1}(1+q+q^2+\cdots+q^{k-1})}\right\}.$$

Applying the principle of mathematical induction, we find that

$$1 + \sum_{\mu=2}^{n-1}\frac{P_1(P_1+q)(P_1+[2]q)\cdots(P_1+[\mu-2]q)}{q^{\mu-1}\prod_{k=1}^{\mu-1}(1+q+q^2+\cdots+q^{k-1})}$$
$$= \frac{(P_1+q)(P_1+[2]q)\cdots(P_1+[n-2]q)}{q^{n-2}\prod_{k=1}^{\mu-1}(1+q+q^2+\cdots+q^{k-2})},$$

from which the inequality (2.14) follows. □

3. Properties of the class $\mathcal{S}_q(k,\alpha,m)$

In this section, we discuss certain sufficient condition for a class of functions f to be in the class $\mathcal{S}_q(k,\alpha,m)$.

THEOREM 3.1. *Let $f\in\mathcal{A}$ be given by (1.3). If the inequality*

$$\sum_{n=2}^{\infty}[n]^m\big([n](k+1)-k-\alpha\big)|a_n|<1-\alpha \tag{3.1}$$

holds true for some k $(0\le k<\infty)$, $m\in\mathbb{N}_0$ and α $(0\le\alpha<1)$, then $f\in\mathcal{S}_q(k,\alpha,m)$. The result is sharp for the function

$$f_n(z)=z-\frac{(1-\alpha)}{[n]^m([n](k+1)-k-\alpha)}z^n\,.$$

PROOF. Making use of the definition (1.11) it suffices to prove that

$$k\left|\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^{m}f(z)}-1\right|-\mathrm{Re}\left\{\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^{m}f(z)}-1\right\}<1-\alpha.$$

Observe that

$$k\left|\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^{m}f(z)}-1\right|-\mathrm{Re}\left\{\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^{m}f(z)}-1\right\}\le(k+1)\left|\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^{m}f(z)}-1\right|$$

$$=(k+1)\left|\frac{\sum_{n=2}^{\infty}[n]^m([n]-1)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}[n]^ma_nz^{n-1}}\right|<(k+1)\frac{\sum_{n=2}^{\infty}[n]^m([n]-1)|a_n|}{1-\sum_{n=2}^{\infty}[n]^m|a_n|}\,.$$

The last expression is bounded by $1-\alpha$ if inequality (3.1) holds. It is obvious that the function f_n satisfies inequality (3.1) and thus the number $1-\alpha$ cannot be replaced by a larger number. Therefore, we only need to show that $f_n\in\mathcal{S}_q(k,\alpha,m)$. Since

$$k\left|\frac{\mathcal{S}_q^{m+1}f_n(z)}{\mathcal{S}_q^{m}f_n(z)}-1\right|=k\left|\frac{(1-\alpha)(1-[n])z^{n-1}}{([n](k+1)-k-\alpha)-(1-\alpha)z^{n-1}}\right|<\frac{k(1-\alpha)}{k+1}\,,$$

and

$$\mathrm{Re}\left\{\frac{\mathcal{S}_q^{m+1}f_n(z)}{\mathcal{S}_q^{m}f_n(z)}\right\}=\mathrm{Re}\left\{\frac{[n](k+1)-k-\alpha-[n](1-\alpha)z^{n-1}}{[n](k+1)-k-\alpha-(1-\alpha)z^{n-1}}\right\}>\frac{k+\alpha}{k+1}\,,$$

condition (1.11) holds true for $f_n(z)$. Thus, $f_n\in\mathcal{S}_q(k,\alpha,m)$. □

The next few corollaries can be easily obtained from Theorem 3.1.

Corollary 3.1. *Let $f(z) = z + a_n z^n$. If*

$$|a_n| \leq \frac{(1-\alpha)}{[n]^m\big([n](k+1)-k-\alpha\big)} z^n \quad (n \geq 2),$$

then $f \in \mathcal{S}_q(k,\alpha,m)$.

For the choices of $m = 0$ and $\alpha = 0$, Theorem 3.1 reduces to the following.

Corollary 3.2. *A function $f \in \mathcal{A}$ of the form (1.3) is in the class $\mathcal{S}_q(k,0,0)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}([n](k+1)-k)|a_n| < 1 \quad (0 \leq k < \infty). \tag{3.2}$$

For the choice of $m = 0$, Theorem 3.1 reduces to the following.

Corollary 3.3. *A function $f \in \mathcal{A}$ of the form (1.3) is in the class $\mathcal{S}_q(k,\alpha,0)$,if it satisfies the condition*

$$\sum_{n=2}^{\infty}([n](k+1)-(k+\alpha))|a_n| < 1-\alpha \quad (0 \leq k < \infty, 0 \leq \alpha < 1). \tag{3.3}$$

For the choices of $m = 0$ and $k = 0$ Theorem 3.1 reduces to the following.

Corollary 3.4. *A function $f \in \mathcal{A}$ of the form (1.3) is in the class $\mathcal{S}_q(0,\alpha,0)$,if it satisfies the condition*

$$\sum_{n=2}^{\infty}([n]-\alpha)|a_n| < 1-\alpha \quad (0 \leq \alpha < 1). \tag{3.4}$$

4. A coefficient inequality for the class $\mathcal{S}_q(k,\alpha,m)$

To obtain the coefficient inequality over the class $\mathcal{S}_q(k,\alpha,m)$, we need the following lemma.

Lemma 4.1 ([19]). *If $q(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with positive real part in $\mathbb{U}$, then*

$$|c_2 - vc_1^2| \leq 2\max\{1; |2v-1|\}. \tag{4.1}$$

In particular, if v is a real number, then

$$|c_2 - vc_1^2| \leq \begin{cases} -4v+2, & \text{if } v \leq 0, \\ 2, & \text{if } 0 \leq v \leq 1, \\ 4v-2, & \text{if } v \geq 1. \end{cases} \tag{4.2}$$

When $v<0$ or $v>1$, the equality holds true if and only if $q(z)$ is equal to the function $\frac{1+z}{1-z}$ or one of its rotations. If $0<v<1$, then the equality holds true if and only if $q(z)$ is equal to the function $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $v=0$, the the equality holds true if and only if $g(z)$ is equal to the function

$$\Big(\frac{1}{2}+\frac{\lambda}{2}\Big)\frac{1+z}{1-z}+\Big(\frac{1}{2}-\frac{\lambda}{2}\Big)\frac{1-z}{1+z}\quad (0\le\lambda\le 1)$$

or one of its rotations. If $v=1$, then the equality holds true if $q(z)$ is a reciprocal of one of the functions such that the equality holds true in the case when $v=0$.

THEOREM 4.1. *Let $0\le k<\infty$, $0\le\alpha<1$, $q\in(0,1)$ and $m\in\mathbb{N}_0$. Suppose that the function f of the form (1.3) belongs to the class $\mathcal{S}_q(k,\alpha,m)$. Then, for a complex number μ,*

$$|a_3-\mu a_2^2|\le\frac{P_1}{q(1+q)(1+q+q^2)^m}\times\max\Big\{1;\Big|\frac{\mu P_1(1+q)(1+q+q^2)^m}{q(1+q)^{2m}}-\frac{P_2}{P_1}-\frac{P_1}{q}\Big|\Big\}. \tag{4.3}$$

PROOF. Let us consider the function $q(z)$ given by $q(z)=\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^{m}f(z)}$. Since $p_{k,\alpha}$ is univalent in $\mathbb{U}$, the function

$$q(z)=\frac{1+p_{k,\alpha}^{-1}(p(z))}{1-p_{k,\alpha}^{-1}(p(z))}=1+c_1z+c_2z^2+\cdots$$

is analytic in $\mathbb{U}$ and $\mathrm{Re}(q(z))>0$. From

$$p(z)=p_{k,\alpha}\Big(\frac{q(z)-1}{q(z)+1}\Big)=1+\frac{1}{2}c_1P_1z+\Big(\frac{1}{2}c_2P_1+\frac{1}{4}c_1^2(P_2-P_1)\Big)z^2+\cdots,$$

we have

$$a_2=\frac{c_1P_1}{2q(1+q)^m}$$

$$a_3=\frac{1}{2q(1+q)(1+q+q^2)^m}\Big(c_2P_1+\frac{c_1^2P_2}{2}-\frac{c_1^2P_1}{2}+\frac{c_1^2P_1^2}{2q}\Big),$$

which together imply that

$$a_3-\mu a_2^2=\frac{P_1}{2q(1+q)(1+q+q^2)^m}\big[c_2-vc_1^2\big],$$

where

$$v = \frac{1}{2}\Big(1 + \frac{\mu P_1(1+q)(1+q+q^2)^m}{q(1+q)^{2m}} - \frac{P_2}{P_1} - \frac{P_1}{q}\Big). \qquad \square$$

It is easy to see that Theorem 4.2 directly follows from (4.2), hence we omit the details.

Theorem 4.2. *Let $0 \le k < \infty$, $0 \le \alpha < 1$, $q \in (0,1)$ and $m \in \mathbb{N}_0$. Suppose that the function f of the form (1.3) belongs to the class $\mathcal{S}_q(k,\alpha,m)$. Then, for a real number μ,*

$$|a_3 - \mu a_2^2| \le \frac{1}{q(1+q)(1+q+q^2)^m} \tag{4.4}$$
$$\times \begin{cases} P_2 + \frac{P_1^2}{q} - \frac{\mu P_1^2(1+q)(1+q+q^2)^m}{q(1+q)^{2m}}, & \text{if } \mu \le \sigma_1, \\ P_1, & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ -P_2 - \frac{P_1^2}{q} + \frac{\mu P_1^2(1+q)(1+q+q^2)^m}{q(1+q)^{2m}}, & \text{if } \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{q(1+q)^{2m}}{P_1^2(1+q)(1+q+q^2)^m}\Big(P_2 + \frac{P_1^2}{q} - P_1\Big),$$

and

$$\sigma_2 = \frac{q(1+q)^{2m}}{P_1^2(1+q)(1+q+q^2)^m}\Big(P_2 + \frac{P_1^2}{q} + P_1\Big).$$

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