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A THEOREM OF PICCARD'S TYPE IN ABELIAN POLISH GROUPS

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Abstract. In the paper we prove a theorem of Piccard's type which generalizes [9, Theorem 2]. More precisely, we show that in an abelian Polish group X the set

$$\Big\{(x_1,\dots,x_N)\in X^N : A\cap\bigcap_{i=1}^N (A+x_i) \text{ is not Haar meager in } X\Big\}$$

is a neighbourhood of 0 for every $N\in\mathbb{N}$ and every Borel non-Haar meager set $A\subset X$. The paper refers to the paper [3].

1. Introduction

In 2013 Darji [4] introduced a new family of "small" sets. More precisely, in an abelian Polish group X he called a set $A\subset X$ *Haar meager* if there is a Borel set $B\subset X$ with $A\subset B$, a compact metric space K and a continuous function $f\colon K\to X$ such that

$$(*)\qquad f^{-1}(B+x) \text{ is meager in } K \text{ for every } x\in X.$$

He proved that the family of all Haar meager sets is a σ-ideal included in the σ-ideal of meager sets and, moreover, these two families are equivalent if and only if X is locally compact (see [4, Theorem 2.2, Example 2.6]).

This notion seems to be a topological analog to the concept of *Haar null* sets introduced by Christensen [2] in 1972 and next extended to nonabelian groups by Mycielski [11] (also rediscovered by Hunt, Sauer and Yorke [7,8] in a topological abelian group with a complete metric).

In [9], among others, a generalized Piccard's theorem was proved; i.e. $0\in\operatorname{int}(A-A)$ for each Borel non-Haar meager set. It is an analogous result

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to the well-known Christensen's theorem [2, Theorem 2], which has been generalized by Christensen and Fischer in [3]. They proved that in an abelian Polish group X the set

$$\left\{(x_1,\dots,x_N)\in X^N : A\cap\bigcap_{i=1}^{N}(A+x_i) \text{ is not Haar null in } X\right\}$$

is a neighbourhood of 0 for every $N\in\mathbb{N}$ and every universally measurable non-Haar null set $A\subset X$, thereby they answered a question formulated by Gajda in [6].

In this paper we generalize [9, Theorem 2] in such a way to obtain an analogous result to Christensen's and Fischer's theorem [3, Theorem 2], so in fact we present another theorem of Piccard's type.

2. The main result

To prove the announced theorem we need the following

PROPOSITION 2.1. *If* $(X,+)$ *is a topological group and* $A\subset X$ *is a non-meager set with the Baire property, then* $0\in\operatorname{int}F_n^*$ *for every* $n\in\mathbb{N}$, *where*

$$F_n^* := \left\{(x_1,\dots,x_n)\in X^n:\ A\cap\bigcap_{i=1}^{n}(A+x_i) \text{ is non-meager in } X\right\}.$$

PROOF. Let $A=G\bigtriangleup M$, where G is nonempty and open and M is meager in X. Put

$$A_{x_1\dots x_n} := A\cap\bigcap_{i=1}^{n}(A+x_i),\quad G_{x_1\dots x_n} := G\cap\bigcap_{i=1}^{n}(G+x_i),$$

$$M_{x_1\dots x_n} := M\cup\bigcup_{i=1}^{n}(M+x_i)$$

for every $x_1,\dots,x_n\in X$. It is easy to check that

$$A_{x_1\dots x_n} = (G_{x_1\dots x_n}\setminus M_{x_1\dots x_n})\cup I,$$

where $I\subset M_{x_1\dots x_n}\setminus G_{x_1\dots x_n}$. Hence $G_{x_1\dots x_n}\setminus A_{x_1\dots x_n}$ is a subset of $M_{x_1\dots x_n}$, which means that it is meager.

Clearly, $G_{x_1\dots x_n}$ is a nonempty open set for every $x_1,\dots,x_n$ from an open neighbourhood of 0 in X. Hence $A_{x_1\dots x_n}$ is non-meager in X for such $x_1\dots x_n$, what ends the proof. □

Since in an abelian Polish group each Haar meager set is meager, now we generalize the above result. In fact we prove an analogous result to Christensen's and Fischer's theorem.

THEOREM 2.2. *Let $(X,+,d)$ be an abelian Polish group. Let $A \subset X$ be a Borel set which is not Haar meager. Then*

$$F_N(A) := \left\{ (x_1,\dots,x_N) \in X^N : A \cap \bigcap_{i=1}^{N} (A + x_i) \text{ is not Haar meager in } X \right\}$$

is a neighbourhood of 0 *in* X^N *for every* $N \in \mathbb{N}$.

PROOF. We base on Christensen's and Fischer's idea from [3].

Since in each abelian Polish group there exists an equivalent invariant complete metric (see [1, p. 90]), we can assume that d is an invariant metric. Suppose that the set $F_N(A)$ is not the neighbourhood of zero for some N. Then we can choose

$$(x_{1,1}, x_{2,1}, \dots, x_{N,1}) \in \left[B\Big(0, \frac{1}{2}\Big) \times B\Big(0, \frac{1}{2^2}\Big) \times \dots \times B\Big(0, \frac{1}{2^N}\Big) \right] \setminus F_N(A),$$

$$(x_{1,2}, x_{2,2}, \dots, x_{N,2}) \in \left[B\Big(0, \frac{1}{2^{N+1}}\Big) \times \dots \times B\Big(0, \frac{1}{2^{2N}}\Big) \right] \setminus F_N(A)$$

and, consequently, by induction with respect to n, for $p := (n-1)N + i$ where $i \in \{1,\dots,N\}$ and $n \in \mathbb{N}$, we obtain a sequence $(x_p)_{p\in\mathbb{N}}$ with

$$d(0, x_p) < \frac{1}{2^p} \quad \text{for } p \in \mathbb{N}$$

and such that

$$S_n := A \cap \bigcap_{i=1}^{N} (A + x_{(n-1)N+i}) \quad \text{for } n \in \mathbb{N}$$

are Haar meager.

Let $A_0 := A \setminus \bigcup_{n=1}^{\infty} S_n$. Clearly,

$$A_0 \cap \bigcap_{i=1}^{N} (A + x_{(n-1)N+i}) = \emptyset \quad \text{for } n \in \mathbb{N}. \tag{2.1}$$

Since the family of all Haar meager sets is a σ-ideal in X (see [4, Theorem 2.9]), A_0 is not Haar meager. Let $K := \{0,1\}^{\omega}$ be the countable Cantor cube. It is well known that it is a compact metric group with the operation $\oplus\colon K\times K\to K$ given by

$$(k_i)_{i\in\mathbb{N}}\oplus(l_i)_{i\in\mathbb{N}}:=(k_i+_2 l_i)_{i\in\mathbb{N}}\quad\text{for }(k_i)_{i\in\mathbb{N}},\ (l_i)_{i\in\mathbb{N}}\in K$$

(where $+_2$ denotes the operation modulo 2) and with the product metric

$$d\big((k_i)_{i\in\mathbb{N}},(l_i)_{i\in\mathbb{N}}\big):=\sum_{i=1}^{\infty}2^{-i}\overline{d}(k_i,l_i)\quad\text{for }(k_i)_{i\in\mathbb{N}},\ (l_i)_{i\in\mathbb{N}}\in K,$$

where $\overline{d}$ is the discrete metric in $\{0,1\}$. Define a function $g\colon K\to X$ as follows:

$$g\big((k_i)_{i\in\mathbb{N}}\big)=\sum_{i=1}^{\infty}k_ix_i\quad\text{for }(k_i)_{i\in\mathbb{N}}\in K. \tag{2.2}$$

As in the proof of [9, Theorem 2] we obtain that g is continuous. Since A_0 is a Borel non-Haar meager set, there exists a $y\in X$ such that $B_0:=g^{-1}(y+A_0)$ is not meager and has the Baire property in K. Now, in view of Proposition 2.1 (with $n=2^N$), there is an $\varepsilon>0$ such that

$$U^{2^N}\subset\Big\{\big(\alpha^{(1)},\dots,\alpha^{(2^N)}\big)\in K^{2^N}:B_0\cap\bigcap_{i=1}^{2^N}\big(B_0\oplus\alpha^{(i)}\big)\text{ is non-meager in }K\Big\}, \tag{2.3}$$

where

$$U:=\Big\{(\beta_i)_{i\in\mathbb{N}}\in K:\sum_{k=1}^{\infty}\frac{1}{2^k}\overline{d}(0,\beta_k)<\varepsilon\Big\}.$$

Moreover, there exists $M_0\in\mathbb{N}$ with $\sum_{k=M+1}^{\infty}\frac{1}{2^k}<\varepsilon$ for each $M\ge M_0$. Now, let

$$S:=\Big\{(0,\dots,0,\beta_{M+1},\dots,\beta_{M+N},0,\dots):\beta_{M+j}\in\{0,1\}\text{ for }j\in\{1,\dots,N\}\Big\}.$$

Clearly, $S\subset U$ and $\operatorname{card}S=2^N$. So, take $\beta^{(1)},\dots,\beta^{(2^N)}\in S$ such that $\beta^{(i)}\ne\beta^{(j)}$ for each $i,j\in\{1,\dots,2^N\}$, $i\ne j$. Hence, $(\beta^{(1)},\dots,\beta^{(2^N)})\in U^{2^N}$ and, by (2.3),

$$B_0\cap\bigcap_{i=1}^{2^N}\big(B_0\oplus\beta^{(i)}\big)\ne\emptyset.$$

Now, take $h\in B_0\cap\bigcap_{i=1}^{2^N}(B_0\oplus\beta^{(i)})$. Then

$$g(h),g(h\ominus\beta^{(i)})\in y+A_0\quad\text{for each }i\in\{1,\dots,2^N\}. \tag{2.4}$$

There exists $i_0 \in \{1, \dots, 2^N\}$ such that

$$h_k \neq \beta_k^{(i_0)} \quad \text{for } k \in \{M+1, \dots, M+N\}.$$

Take any $p \in \{M+1, \dots, M+N\}$. Then we can choose $i_1 \in \{1, \dots, 2^N\}$ with $\beta_k^{(i_1)} = \beta_k^{(i_0)}$ for $k \in \{M+1, \dots, M+N\} \setminus \{p\}$ and $\beta_p^{(i_1)} \neq \beta_p^{(i_0)}$. Let $l := h \ominus \beta^{(i_0)}$. Clearly $l_k = 1$ for $k \in \{M+1, \dots, M+N\}$ and $l_k = h_k$ for other numbers k. Hence

$$g(l) - x_p = \sum_{i=1}^{M} l_i x_i + \sum_{i=M+1}^{M+N} x_i + \sum_{i=M+N+1}^{\infty} l_i x_i - x_p$$
$$= \sum_{i=1}^{M} h_i x_i + \sum_{i=M+1,\, i\neq p}^{M+N} x_i + \sum_{i=M+N+1}^{\infty} h_i x_i = g(h \ominus \beta^{(i_1)})$$

and, by (2.4), $g(l) - x_p \in y + A_0$ for each $p \in \{M+1, \dots, M+N\}$. Thus

$$g(l) \in (y + A_0) \cap \bigcap_{p=M+1}^{M+N} \left[y + (A_0 + x_p) \right],$$

which implies that

$$z := g(l) - y \in A_0 \cap \bigcap_{p=M+1}^{M+N} (A_0 + x_p)$$

for every $M \geq M_0$. Taking $M := (n-1)N$ with a large enough n, this contradicts (2.1). $\square$

3. A remark on analogies between Haar null sets and Haar meager sets

Matoušková and Zelený [10] constructed closed sets A, B in a non-locally compact abelian Polish group such that A, as well as B, includes a translation of each compact set and $(A+x) \cap B$ is compact for each $x \in X$. Thanks to this Dodos [5] showed that the invariance under bigger subgroups is not sufficient to establish a dichotomy. More precisely, he proved the following

PROPOSITION 3.1 [5, Proposition 12]. *If X is a non-locally compact abelian Polish group and G is a σ-compact subgroup of X, then there exists a G-invariant F_σ subset F of X such that F is neither prevalent nor Haar null.*

Since every set containing a translation of each compact set is neither Haar null, nor Haar meager, in the same way as Dodos we can prove that an another type of dichotomy doesn't hold.

PROPOSITION 3.2 [5, Proposition 12]. *Let X be a non-locally compact abelian Polish group and G be a σ-compact subgroup of X. Then there exists a G-invariant F_σ subset F of X such that neither F nor $X \setminus F$ is Haar meager.*

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