Analysis Mathematica, **41**(2015), 117–132
DOI: 10.1007/s10476-015-0109-2

Generalization of certain well-known inequalities for the derivative of polynomials

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Received August 26, 2014; in revised form May 18, 2015.

Abstract. For a polynomial $P(z)$ of degree n which has no zeros in $|z| < 1$, Dewan et al. [4] established the inequality

$$\Big|zP'(z) + \frac{n\beta}{2}P(z)\Big| \le \frac{n}{2}\Big\{\Big(\Big|\frac{\beta}{2}\Big| + \Big|1+\frac{\beta}{2}\Big|\Big)\max_{|z|=1}|P(z)| - \Big(\Big|1+\frac{\beta}{2}\Big| - \Big|\frac{\beta}{2}\Big|\Big)\min_{|z|=1}|P(z)|\Big\},$$

for any $|\beta| \le 1$ and $|z| = 1$. In this paper we improve the above inequality for the sth derivative of a polynomial which has no zeros in $|z| < k$, $k \le 1$. Our results generalize certain well-known polynomial inequalities.

1. Introduction and statement of results

According to a well-known result known as Bernstein's inequality on the derivative of a polynomial $P(z)$ of degree n, we have

$$\max_{|z|=1}|P'(z)| \le n\max_{|z|=1}|P(z)|. \tag{1.1}$$

The result is best possible and equality holds for the polynomials having all their zeros at the origin (see [12]).

The inequality (1.1) can be sharpened, if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, Erdős conjectured and later Lax [10] proved that if $P(z) \ne 0$ in $|z| < 1$, then (1.1) can be

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replaced by

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.2}$$

The inequality (1.2) is sharp and equality holds for polynomials having all their zeros on $|z| = 1$.

As an extension of (1.2) Malik [11] proved that if $P(z) \ne 0$ in $|z| < k$, $k \ge 1$, then

$$\max_{|z|=1}|P'(z)| \le \frac{n}{1+k}\max_{|z|=1}|P(z)|, \tag{1.3}$$

Aziz and Dawood [1] obtained a refinement of inequality (1.2) by demonstrating that if $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2}\Big\{\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\Big\}. \tag{1.4}$$

As an improvement of inequality (1.2) Jain [9] proved that if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$ then

$$\Big|zP'(z) + \frac{n\beta}{2}P(z)\Big| \le \frac{n}{2}\Big(\Big|\frac{\beta}{2}\Big| + \Big|1+\frac{\beta}{2}\Big|\Big)\max_{|z|=1}|P(z)|, \tag{1.5}$$

for every real or complex number β with $|\beta| \le 1$ and $|z| = 1$. The equality holds for $P(z) = az^n + b$, $|a| = |b|$.

As a refinement of (1.5), Dewan and Hans [4, Theorem 2] proved that if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$ then

$$\Big|zP'(z) + \frac{n\beta}{2}P(z)\Big| \le \tag{1.6}$$

$$\le \frac{n}{2}\Big\{\Big(\Big|\frac{\beta}{2}\Big| + \Big|1+\frac{\beta}{2}\Big|\Big)\max_{|z|=1}|P(z)| - \Big(\Big|1+\frac{\beta}{2}\Big| - \Big|\frac{\beta}{2}\Big|\Big)\min_{|z|=1}|P(z)|\Big\},$$

for every real or complex number β with $|\beta| \le 1$ and $|z| = 1$. In the same paper Dewan and Hans [4, Theorem 1] proved that if $P(z)$ has all its zeros in $|z| \le 1$, then for every real or complex number β with $|\beta| \le 1$,

$$\min_{|z|=1}\Big|zP'(z) + \frac{n\beta}{2}P(z)\Big| \ge n\Big|1+\frac{\beta}{2}\Big|\min_{|z|=1}|P(z)|. \tag{1.7}$$

In this paper, we first generalize inequality (1.7) for the sth derivative of polynomials with restricted zeros, as follows:

Theorem 1.1. *Let $P(z)$ be a polynomial of degree n, having all its zeros in $|z| \le k$, $k \le 1$. Then for every real or complex number β with $|\beta| \le 1$ and $1 \le s \le n$,*

$$\min_{|z|=1}\Big|z^sP^{(s)}(z) + \beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big| \ge \tag{1.8}$$

$$\geq n(n-1)\cdots(n-s+1)k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|\min_{|z|=k}|P(z)|,$$

The result is best possible and equality holds for the polynomials $P(z)=az^n$.

If we take $k=1$ in Theorem 1.1, then we have the following generalization of inequality (1.7) for sth derivative of polynomials.

Corollary 1.2. *If* $P(z)$ *is a polynomial of degree* n, *having all its zeros in* $|z|\leq 1$, *then for every real or complex number* β *with* $|\beta|\leq 1$ *and* $1\leq s\leq n$,

$$\min_{|z|=1}\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{2^s}P(z)\Big|\geq \tag{1.9}$$

$$\geq n(n-1)\cdots(n-s+1)|1+\frac{\beta}{2^s}|\min_{|z|=1}|P(z)|.$$

The result is best possible and equality holds for the polynomials $P(z)=az^n$.

For every β with $|\beta|\leq 1$ and $k>0$ we have

$$k|\beta|\leq k\leq k+(1-|\beta|)\leq(1+k)^s-|\beta|\leq|(1+k)^s+\beta|.$$

Hence $k|\beta|\leq|(1+k)^s+\beta|$, or $\frac{k}{(1+k)^s}|\beta|\leq|1+\frac{\beta}{(1+k)^s}|$, i.e.,

$$\Big|1+\frac{\beta}{(1+k)^s}\Big|\geq k\Big|\frac{\beta}{(1+k)^s}\Big|. \tag{1.10}$$

On the other hand, by applying Lemma 2.2 for the polynomial $P(z)$ of degree n, which has all its zeros in $|z|\leq k$, $k\leq 1$, we have

$$|z^sP^{(s)}(z)|\geq\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}|P(z)|,\qquad |z|=1.$$

Then for an appropriate choice of the argument of β we have

$$|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)|= \tag{1.11}$$

$$|z^sP^{(s)}(z)|-|\beta|\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}|P(z)|.$$

Therefore by combining (1.8), (1.10) and (1.11), we have

$$|z^sP^{(s)}(z)|-|\beta|\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}|P(z)|=$$

$$=\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big|\geq$$

$$\geq\min_{|z|=1}\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big|\geq$$

$$\geq n(n-1)\cdots(n-s+1)k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|\min_{|z|=k}|P(z)|\geq$$

$$\geq n(n-1)\cdots(n-s+1)k^{-n+1}\Big|\frac{\beta}{(1+k)^s}\Big|\min_{|z|=k}|P(z)|,$$

i.e.,

$$|P^{(s)}(z)|\geq|\beta|\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}\Big\{|P(z)|+\frac{1}{k^{n-1}}\min_{|z|=k}|P(z)|\Big\}, \tag{1.12}$$

where $|z|=1$.

Now taking $|\beta|\to 1$, we have

Corollary 1.3. *If $P(z)$ is a polynomial of degree n, having all its zeros in $|z|\leq k$, $k\leq 1$, then for $1\leq s\leq n$*

$$\max_{|z|=1}|P^{(s)}(z)|\geq\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}\Big\{\max_{|z|=1}|P(z)|+\frac{1}{k^{n-1}}\min_{|z|=k}|P(z)|\Big\}. \tag{1.13}$$

Remark 1.4. If we take $s=1$ in Corollary 1.3, inequality (1.13) reduces to a result proved by Govil [7].

Next, by using Theorem 1.1, we generalize inequality (1.6) for the sth derivative of polynomials with restricted zeros, more precisely:

Theorem 1.5. *If $P(z)$ is a polynomial of degree n, having no zeros in $|z|<k$, $k\leq 1$, then for every real or complex number β with $|\beta|\leq 1$, and $1\leq s\leq n$ we have*

$$\max_{|z|=1}\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big|\leq \tag{1.14}$$

$$\leq\frac{n(n-1)\cdots(n-s+1)}{2}\Big\{\Big(k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|+\Big|\frac{\beta}{(1+k)^s}\Big|\Big)\max_{|z|=k}|P(z)|-$$

$$-\Big(k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|-\Big|\frac{\beta}{(1+k)^s}\Big|\Big)\min_{|z|=k}|P(z)|\Big\}.$$

The result is best possible and equality holds in (1.14) for $P(z)=az^n+bk^n$, $|a|=|b|$ and $\beta\geq 0$.

If we take $k=1$ in Theorem 1.5, then inequality (1.14) reduces to a result which was recently proved by Hans and Lal [8].

If we take $\beta=0$ in Theorem 1.5, we have the following result which is a generalization of inequality (1.4) for sth derivative of a polynomial.

Corollary 1.6. *If $P(z)$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \le 1$ then for $1 \le s \le n$*

$$\max_{|z|=1} |P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{2k^n}\Big\{\max_{|z|=k}|P(z)| - \min_{|z|=k}|P(z)|\Big\}. \tag{1.15}$$

The result is best possible and equality holds in (1.20) *for* $P(z) = az^n + bk^n$, $|a| = |b|$.

Let $P(z)$ have all its zeros on $|z| = k$, $k \le 1$, then $\min_{|z|=k}|P(z)| = 0$. Now if we take $r = k$ in Lemma 2.7, we obtain

$$\max_{|z|=k}|P(z)| \le \Big(\frac{2k}{1+k}\Big)^n \max_{|z|=1}|P(z)|;$$

by using this in Corollary 1.6 we have

Corollary 1.7. *Let $P(z)$ be a polynomial of degree n, not vanishing in $|z| < k$, $k \le 1$. If $P(z)$ has all its zeros on $|z| = k$, then for $1 \le s \le n$*

$$\max_{|z|=1} |P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)2^{n-1}}{(1+k)^n} \max_{|z|=1}|P(z)|. \tag{1.16}$$

Obviously $\frac{2^{n-1}}{(1+k)^n} \le \frac{1}{k^{n-1}(1+k)} = \frac{1}{k^{n-1}+k^n}$, where $k \le 1$. Hence Corollary 1.7 is a refinement of the following result.

Corollary 1.8. *Let $P(z)$ be a polynomial of degree n, not vanishing in $|z| < k$, $k \le 1$. If $P(z)$ has all its zeros on $|z| = k$, then for $1 \le s \le n$*

$$\max_{|z|=1} |P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{k^{n-1}+k^n} \max_{|z|=1}|P(z)|. \tag{1.17}$$

Remark 1.9. If we take $s = 1$ in Corollary 1.8, inequality (1.17) reduces to a result proved by Govil [6].

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first lemma is due to Malik [11].

Lemma 2.1. *If $P(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z| \le k$, $k \le 1$, then*

$$|zP'(z)| \ge \frac{n}{1+k}|P(z)|, \tag{2.1}$$

where $|z| = 1$.

Lemma 2.2. *If $P(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z| \le k$, $k \le 1$, and $1 \le s \le n$ then*

$$|z^s P^{(s)}(z)| \ge \frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}|P(z)|, \tag{2.2}$$

where $|z| = 1$.

Proof. Since $P(z)$ has all its zeros in $|z| \le k$, by using the Gauss–Lucas theorem, the polynomial $P^{(s-1)}(z)$ has all its zeros in $|z| \le k$. Applying Lemma 2.1 to the polynomial $P^{(s-1)}(z)$ of degree $(n-s+1)$, we have

$$|zP^{(s)}(z)| \ge \frac{n-s+1}{1+k}|P^{(s-1)}(z)|, \quad \text{for } |z| = 1.$$

This implies

$$|z^2 P^{(s)}(z)| \ge \frac{n-s+1}{1+k}|zP^{(s-1)}(z)|, \quad \text{for } |z| = 1. \tag{2.3}$$

Further applying Lemma 2.1 to the polynomial $P^{(s-1)}(z)$ of degree $(n-s+2)$, we have

$$|zP^{(s-1)}(z)| \ge \frac{n-s+2}{1+k}|P^{(s-2)}(z)| \quad \text{for } |z| = 1. \tag{2.4}$$

Combining inequalities (2.3) and (2.4), we get

$$|z^2 P^{(s)}(z)| \ge \frac{(n-s+1)(n-s+2)}{(1+k)^2}|P^{(s-2)}(z)|, \quad \text{for } |z| = 1. \tag{2.5}$$

Similarly applying Lemma 2.1 again and again, we get the desired result. □

Lemma 2.3. *Let $F(z)$ be a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, and $P(z)$ a polynomial of degree not exceeding that of $F(z)$. If $|P(z)| \le |F(z)|$ for $|z| = k$, $k \le 1$, then for any $\beta \in \mathbb{C}$ with $|\beta| \le 1$ and $|z| = 1$, $1 \le s \le n$,*

$$\Big|z^s P^{(s)}(z) + \beta\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big| \le \tag{2.6}$$

$$\le \Big|z^s F^{(s)}(z) + \beta\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}F(z)\Big|.$$

Proof. By using the inequality $|P(z)| \le |F(z)|$ for $|z| = k$, any zero of $F(z)$ that lies on $|z| = k$ is a zero of $P(z)$. On the other hand, from Rouche's theorem, it is obvious that for α with $|\alpha| < 1$, $F(z) + \alpha P(z)$ has as many zeros in $|z| < k$ as $F(z)$, and so has all of its zeros in $|z| < k$.

Therefore $F(z)+\alpha P(z)$ has all its zeros in $|z|\le k$. By applying Lemma 2.2 we get for α with $|\alpha|<1$ and $|z|=1$,

$$\Big|z^sF^{(s)}(z)+\alpha z^sP^{(s)}(z)\Big|\ge\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}\Big|F(z)+\alpha P(z)\Big|.$$

Therefore, for any β with $|\beta|<1$, we have

$$\Big(z^sF^{(s)}(z)+\alpha z^sP^{(s)}(z)\Big)+\beta\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}\Big(F(z)+\alpha P(z)\Big)\ne 0.$$

This means that the polynomial

$$T(z)=\Big(z^sF^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}F(z)\Big)+ \\ +\alpha\Big(z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big) \tag{2.7}$$

will have no zeros on $|z|=1$.

Therefore, on $|z|=1$ we have

$$\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big|\le \\ \le\Big|z^sF^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}F(z)\Big|. \tag{2.8}$$

Indeed, if inequality (2.8) is not true, then there is a point $z=z_0$ with $|z_0|=1$ such that

$$\Big|z_0^sP^{(s)}(z_0)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z_0)\Big|> \\ >\Big|z_0^sF^{(s)}(z_0)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}F(z_0)\Big|.$$

Now take

$$\alpha=-\frac{z_0^sF^{(s)}(z_0)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}F(z_0)}{z_0^sP^{(s)}(z_0)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z_0)},$$

then $|\alpha|<1$ and with this choice of α, we have from (2.7) that $T(z_0)=0$ for $|z_0|=1$. But this contradicts the fact that $T(z)\ne 0$ for $|z|=1$. For β with $|\beta|=1$, (2.6) follows by continuity. This is equivalent to the desired result. □

If we take $F(z) = (\frac{z}{k})^n \max\limits_{|z|=k} |P(z)|$ in Lemma 2.3, we have the following

Lemma 2.4. *If $P(z)$ is a polynomial of degree n and $k \leq 1$, then for any β with $|\beta| \leq 1$, and $|z| = 1$, $1 \leq s \leq n$,*

$$\Big|z^s P^{(s)}(z) + \beta \frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} P(z)\Big| \leq$$

$$\leq n(n-1)\cdots(n-s+1)k^{-n}\Big|1 + \frac{\beta}{(1+k)^s}\Big| \max_{|z|=k} |P(z)|.$$

Lemma 2.5. *If $P(z)$ is a polynomial of degree n and $k \leq 1$, then for any β with $|\beta| \leq 1$ and $|z| = 1$,*

$$\Big|z^s P^{(s)}(z) + \beta \frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} P(z)\Big| +$$

$$+ \Big|z^s Q^{(s)}(z) + \beta \frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} Q(z)\Big| \leq$$

$$\leq n(n-1)\cdots(n-s+1)\Big\{k^{-n}\Big|1 + \frac{\beta}{(1+k)^s}\Big| + \Big|\frac{\beta}{(1+k)^s}\Big|\Big\} \max_{|z|=k} |P(z)|,$$

where

$$Q(z) = \Big(\frac{z}{k}\Big)^n \overline{P\Big(\frac{k^2}{\overline{z}}\Big)}.$$

Proof. Let $M = \max_{|z|=k} |P(z)|$. For α with $|\alpha| > 1$, it follows by Rouche's theorem that the polynomial $G(z) = P(z) - \alpha M$ has no zeros in $|z| \leq k$. Correspondingly the polynomial

$$H(z) = \Big(\frac{z}{k}\Big)^n \overline{G\Big(\frac{k^2}{\overline{z}}\Big)}$$

has all its zeros in $|z| < k$ and $|G(z)| = |H(z)|$ for $|z| = k$. Therefore, by Lemma 2.3, for $|\beta| \leq 1$ and $|z| = 1$, we have

$$\Big|z^s G^{(s)}(z) + \beta \frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} G(z)\Big| \leq \tag{2.9}$$

$$\leq \Big|z^s H^{(s)}(z) + \beta \frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} H(z)\Big|.$$

On the other hand

$$H(z) = \Big(\frac{z}{k}\Big)^n \overline{G\Big(\frac{k^2}{\overline{z}}\Big)} = \Big(\frac{z}{k}\Big)^n \overline{P\Big(\frac{k^2}{\overline{z}}\Big)} - \overline{\alpha} M \Big(\frac{z}{k}\Big)^n = Q(z) - \overline{\alpha} M \Big(\frac{z}{k}\Big)^n,$$

or

$$H(z) = Q(z) - \overline{\alpha} M \Big(\frac{z}{k}\Big)^n,$$

then by substituting this in (2.9), we have

$$\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}(P(z)-\alpha M)\Big|\le$$
$$\le\Big|z^sQ^{(s)}(z)-n(n-1)\cdots(n-s+1)\overline{\alpha}M\Big(\frac{z}{k}\Big)^n+$$
$$+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}\Big(Q(z)-\overline{\alpha}M\Big(\frac{z}{k}\Big)^n\Big)\Big|.$$

This implies

$$\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big|-\tag{2.10}$$
$$-\,n(n-1)\cdots(n-s+1)M\Big|\frac{\alpha\beta}{(1+k)^s}\Big|\le$$
$$\le\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)-$$
$$-\,n(n-1)\cdots(n-s+1)\overline{\alpha}M\Big(\frac{z}{k}\Big)^n\Big(1+\frac{\beta}{(1+k)^s}\Big)\Big|.$$

As $|P(z)|=|Q(z)|$ for $|z|=k$, we have

$$M=\max_{|z|=k}|P(z)|=\max_{|z|=k}|Q(z)|.$$

On applying Lemma 2.4 to the polynomial $Q(z)$, we have for $|z|=1$, $|\beta|\le 1$ and $|\alpha|>1$

$$\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)\Big|\le$$
$$\le n(n-1)\cdots(n-s+1)k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|\max_{|z|=k}|Q(z)|<$$
$$<n(n-1)\cdots(n-s+1)|\alpha|k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|M.$$

Hence by a suitable choice of the argument of α, we have

$$\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)-\tag{2.11}$$
$$-\,n(n-1)\cdots(n-s+1)\overline{\alpha}M\Big(\frac{z}{k}\Big)^n\Big(1+\frac{\beta}{(1+k)^s}\Big)\Big|=$$
$$=|\alpha|n(n-1)\cdots(n-s+1)k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|M-$$
$$-\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)\Big|.$$

By combining (2.10) and (2.11) we obtain

$$\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big|-$$

$$-|\alpha|n(n-1)\cdots(n-s+1)\Big|\frac{\beta}{(1+k)^s}\Big|M\le$$

$$\le|\alpha|n(n-1)\cdots(n-s+1)k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|M-$$

$$-\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)\Big|,$$

i.e.,

$$\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big|+$$

$$+\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)\Big|\le$$

$$\le|\alpha|n(n-1)\cdots(n-s+1)\Big(k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|+\Big|\frac{\beta}{(1+k)^s}\Big|\Big)M.$$

Taking $|\alpha|\to 1$, Lemma 2.5 follows. □

The following lemma is due to Aziz and Mohammad [2].

Lemma 2.6. *If $P(z)$ is a polynomial of degree n not vanishing in $|z|<k$, $k\ge 1$, then*

$$\max_{|z|=R}|P(z)|\le\Big(\frac{R+k}{1+k}\Big)^n\max_{|z|=1}|P(z)| \tag{2.12}$$

for $1\le R\le k^2$.

Applying this result to $q(z)=z^n\overline{P(1/\overline{z})}$ we obtain

Lemma 2.7. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, then*

$$\max_{|z|=r}|P(z)|\le\Big(\frac{r+k}{1+k}\Big)^n\max_{|z|=1}|P(z)|, \tag{2.13}$$

for $1\ge r\ge k^2$.

The following lemma is due to Gardner, Govil and Musukula [5].

Lemma 2.8. *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n, $p(z)\ne 0$ in $|z|<k$, $(k>0)$, then $m<|p(z)|$ for $|z|<k$, and in particular $m<|a_0|$, where $m=\min_{|z|=k}|p(z)|$.*

In the lines of Lemma 2.8, by using the Maximum Modulus Principle, one can easily prove the following

Lemma 2.9. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n, $p(z) \neq 0$ *in* $|z| < k$, $(k \geq 1)$, *then*

$$\min_{|z|=k} |p(z)| < \max_{|z|=1} |p(z)|, \tag{2.14}$$

and in particular $\min_{|z|=k} |p(z)| < |a_0|$.

Lemma 2.10. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all zeros in* $|z| \leq k$, $(k \leq 1)$, *then*

$$\min_{|z|=k} |p(z)| < k^n \max_{|z|=1} |p(z)|, \tag{2.15}$$

and in particular $\min_{|z|=k} |p(z)| < k^n |a_n|$.

Proof. Since the polynomial $p(z)$ has all zeros in $|z| \leq k$, $(k \leq 1)$, the polynomial

$$q(z) = z^n \overline{p\Big(\frac{1}{\overline{z}}\Big)} = \overline{a_n} + \overline{a_{n-1}} z + \cdots + \overline{a_1} z^{n-1} + \overline{a_0} z^n$$

has no zero in $|z| < \frac{1}{k}$, $(\frac{1}{k} \geq 1)$. Thus by applying Lemma 2.9 to the polynomial $q(z)$, we get

$$\min_{|z|=1/k} |q(z)| < \max_{|z|=1} |q(z)|, \qquad \min_{|z|=1/k} |q(z)| < |a_n|. \tag{2.16}$$

Since

$$\min_{|z|=\frac{1}{k}} |q(z)| = \frac{1}{k^n} \min_{|z|=k} |p(z)| \quad \text{and} \quad \max_{|z|=1} |q(z)| = \max_{|z|=1} |p(z)|,$$

(2.16) implies that

$$\frac{1}{k^n} \min_{|z|=k} |p(z)| < \max_{|z|=1} |p(z)| \quad \text{and} \quad \frac{1}{k^n} \min_{|z|=k} |p(z)| < |a_n|. \qquad \square$$

3. Proofs of the theorems

Proof of Theorem 1.1. If $P(z)$ has a zero on $|z| = k$, then the inequality is trivial. Therefore we assume that $P(z)$ has all its zeros in $|z| < k$. If $m = \min_{|z|=k} |P(z)|$, then $m > 0$ and $|P(z)| \geq m$ for $|z| = k$. Therefore, if $|\lambda| < 1$ then it follows by Rouché's theorem that the polynomial $P(z) - \lambda m (\frac{z}{k})^n$ has all its zeros in $|z| < k$. Also by using Lemma 2.10 the polynomial $G(z) = P(z) - \lambda m(\frac{z}{k})^n$ is of degree n. On applying Lemma 2.2 to the polynomial $G(z)$ of degree n, we get

$$|z^s G^{(s)}(z)| \geq \frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}|G(z)|,$$

i.e.,

$$\Big|z^s P^{(s)}(z) - \lambda mn(n-1)\cdots(n-s+1)\Big(\frac{z}{k}\Big)^n\Big| \geq$$
$$\geq \frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}\Big|P(z) - \lambda m\Big(\frac{z}{k}\Big)^n\Big|,$$

where $|z| = 1$.

Therefore, for β with $|\beta| < 1$, it can be easily verified that the polynomial

$$z^s P^{(s)}(z) - n(n-1)\cdots(n-s+1)\lambda m\Big(\frac{z}{k}\Big)^n +$$
$$+\beta\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}\Big\{P(z) - \lambda m\Big(\frac{z}{k}\Big)^n\Big\} =$$
$$= \Big(z^s P^{(s)}(z) + \beta\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big) -$$
$$- n(n-1)\cdots(n-s+1)\lambda m\Big(\frac{z}{k}\Big)^n\Big(1 + \frac{\beta}{(1+k)^s}\Big)$$

will have no zeros on $|z| = 1$. As $|\lambda| < 1$, we have for β with $|\beta| < 1$ and $|z| = 1$

$$\Big|z^s P^{(s)}(z) + \beta\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big| \geq$$
$$\geq n(n-1)\cdots(n-s+1)\Big|1 + \frac{\beta}{(1+k)^s}\Big|\,\Big|\frac{z}{k}\Big|^n m,$$

i.e.,

(3.1)
$$\Big|z^s P^{(s)}(z) + \beta\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big| \geq$$
$$\geq n(n-1)\cdots(n-s+1)\Big|1 + \frac{\beta}{(1+k)^s}\Big|k^{-n}m.$$

For β with $|\beta| = 1$, (3.1) follows by continuity. This completes the proof of Theorem 1.1. □

P r o o f o f T h e o r e m 1.5. Let $m = \min_{|z|=k}|P(z)|$, then $m \leq |P(z)|$ for $|z| \leq k$. Now for λ with $|\lambda| < 1$, we have

$$|\lambda m| < m \leq |P(z)|,$$

where $|z| = k$.

Hence by Rouché's theorem the polynomial $G(z) = P(z) - \lambda m$ has no zero in $|z| < k$. Therefore the polynomial

$$H(z) = \Big(\frac{z}{k}\Big)^n \overline{G(k^2/\overline{z})} = Q(z) - \overline{\lambda} m \Big(\frac{z}{k}\Big)^n$$

will have all its zeros in $|z| \le k$, where $Q(z) = (\frac{z}{k})^n \overline{P(\frac{k^2}{\overline{z}})}$. Also $|G(z)| = |H(z)|$ for $|z| = k$. On the other hand, by using Lemma 2.10, the polynomial $Q(z) - \overline{\lambda} m (\frac{z}{k})^n$ is of degree n. On applying Lemma 2.3 to the polynomial $H(z)$ of degree n, we have for $|\beta| \le 1$ and $|z| = 1$ that

$$\Big|z^s G^{(s)}(z) + \beta \,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} G(z)\Big| \le$$

$$\le \Big|z^s H^{(s)}(z) + \beta \,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} H(z)\Big|,$$

i.e.,

$$\Big|z^s P^{(s)}(z) + \beta \,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} (P(z) - \lambda m)\Big| \le$$

$$\le \Big|z^s Q^{(s)}(z) - n(n-1)\cdots(n-s+1)\overline{\lambda} m \Big(\frac{z}{k}\Big)^n +$$

$$+ \beta \,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} \Big(Q(z) - \overline{\lambda} m \Big(\frac{z}{k}\Big)^n\Big)\Big|.$$

This can be rewritten as

$$\Big|z^s P^{(s)}(z) + \beta \,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} P(z) - \tag{3.2}$$

$$- \beta \,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} \lambda m\Big| \le$$

$$\le \Big|z^s Q^{(s)}(z) + \beta \,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} Q(z) -$$

$$- n(n-1)\cdots(n-s+1)\overline{\lambda} m (\frac{z}{k})^n \Big(1 + \frac{\beta}{(1+k)^s}\Big)\Big|.$$

Since all the zeros of $Q(z)$ lie in $|z| \le k \le 1$, we have $|P(z)| = |Q(z)|$ for $|z| = k$. On applying Theorem 1.1 to the polynomial $Q(z)$, we have

$$\Big|z^s Q^{(s)}(z) + \beta \,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} Q(z)| \ge$$

$$\ge n(n-1)\cdots(n-s+1) k^{-n} \Big|1 + \frac{\beta}{(1+k)^s}\Big| m,$$

where $|z| = 1$ and $|\beta| \le 1$.

Then, for an appropriate choice of the argument of λ, we have

$$\begin{aligned}(3.3)\qquad &\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)-\\ &-n(n-1)\cdots(n-s+1)\overline{\lambda}m\Big(\frac{z}{k}\Big)^n\Big(1+\frac{\beta}{(1+k)^s}\Big)\Big|=\\ &=\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)\Big|-\\ &-|\lambda|n(n-1)\cdots(n-s+1)k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|m.\end{aligned}$$

By combining (3.2) and (3.3), we get for $|z|=1$ and $|\beta|\le 1$

$$\begin{aligned}&\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big|-\\ &-n(n-1)\cdots(n-s+1)\Big|\frac{\beta}{(1+k)^s}\lambda m\Big|\le\\ &\le\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)\Big|-\\ &-n(n-1)\cdots(n-s+1)k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big||\lambda|m.\end{aligned}$$

Equivalently,

$$\begin{aligned}&\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big|\le\\ &\le\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)\Big|-\\ &-n(n-1)\cdots(n-s+1)\Big(k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|-\Big|\frac{\beta}{(1+k)^s}\Big|\Big)|\lambda|m.\end{aligned}$$

As $|\lambda|\to 1$, we have

$$\begin{aligned}&\Big|z^sP^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big|\le\\ &\le\Big|z^sQ^{(s)}(z)+\beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)\Big|-\\ &-n(n-1)\cdots(n-s+1)\Big(k^{-n}\Big|1+\frac{\beta}{(1+k)^s}\Big|-\Big|\frac{\beta}{(1+k)^s}\Big|\Big)m,\end{aligned}$$

which implies for every real or complex number β with $|\beta| \le 1$ and $|z| = 1$

$$2\Big|z^s P^{(s)}(z) + \beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big| \le$$

$$\le \Big|z^s P^{(s)}(z) + \beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big| +$$

$$+ \Big|z^s Q^{(s)}(z) + \beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}Q(z)\Big| -$$

$$- n(n-1)\cdots(n-s+1)\Big(k^{-n}\Big|1 + \frac{\beta}{(1+k)^s}\Big| - \Big|\frac{\beta}{(1+k)^s}\Big|\Big)m.$$

This in, conjunction with Lemma 2.5, gives for $|\beta| \le 1$ and $|z| = 1$

$$2\Big|z^s P^{(s)}(z) + \beta\,\frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}P(z)\Big| \le$$

$$\le n(n-1)\cdots(n-s+1)\Big\{\Big(k^{-n}\Big|1 + \frac{\beta}{(1+k)^s}\Big| + \Big|\frac{\beta}{(1+k)^s}\Big|\Big)\max_{|z|=k}|P(z)| -$$

$$- \Big(k^{-n}\Big|1 + \frac{\beta}{(1+k)^s}\Big| - \Big|\frac{\beta}{(1+k)^s}\Big|\Big)\min_{|z|=k}|P(z)|\Big\}.$$

This completes the proof of Theorem 1.5. □

Acknowledgment. The author thanks the referee for the careful reading of this paper and for the helpful suggestions and comments.

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Обобщение некоторых известных неравенств для производной многочленов

Ахмад Зирех

Для многочлена $P(z)$ степени n, не имеющего корней в круге $|z|<1$, Деван и др. [4] установили, что

$$\left|zP'(z)+\frac{n\beta}{2}P(z)\right|\le\frac{n}{2}\left\{\left(\left|\frac{\beta}{2}\right|+\left|1+\frac{\beta}{2}\right|\right)\max_{|z|=1}|P(z)|-\left(\left|1+\frac{\beta}{2}\right|-\left|\frac{\beta}{2}\right|\right)\min_{|z|=1}|P(z)|\right\},$$

для любых $|\beta|\le 1$ и $|z|=1$. В этой статье мы уточняем неравенство выше для s-й производной многочлена, не имеющего нулей в $|z|<k$, $k\le 1$. Наши результаты обобщают некоторые известные неравенства для многочленов.