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Almost everywhere convergence of some subsequences of the Nörlund logarithmic means of Walsh–Fourier series

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Abstract. In this work we study the maximal operator for a class of subsequences of Nörlund logarithmic means of Walsh–Fourier series. For such a class we prove the almost everywhere convergence of $(t_{m_n}f)_n$ for every integrable function f. Besides, we establish a divergence theorem for other classes of subsequences.

1. Introduction

Almost everywhere convergence of $(t_{m_n}f)_n$ for some sequences $(m_n)_n$ with respect to the Walsh–Paley system was studied by Goginava in [5]. Nagy in [6] established a similar result for the Walsh–Kaczmarz system. However, a divergence result for the whole sequence $(t_nf)_n$ was proved in [3]. Uniform convergence and convergence in norm were considered in [4] and convergence results on the double Walsh–Fourier series can be found in [2].

In our study we enlarge the convergence class of subsequences given in [5] and prove divergence for some other class.

Let $\mathbb{Z}_2$ denote the discrete cyclic group $\mathbb{Z}_2 = \{0, 1\}$, where the group operation is addition modulo 2. If $|E|$ denotes the measure of the subset $E \subset \mathbb{Z}_2$, then $|\{0\}| = |\{1\}| = \frac{1}{2}$.

The dyadic group G is obtained from $G = \prod_{i=0}^{\infty} \mathbb{Z}_2$, where topology and measure are obtained by the product.

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Let $x = (x_n)_{n\geq 0} \in G$. The sets

$$I_n(x) := \Big\{ y \in G : y_0 = x_0,\ \ldots,\ y_{n-1} = x_{n-1} \Big\},\quad n \geq 1 \ \text{ and }\ I_0(x) := G$$

are dyadic intervals of G.

The Walsh–Paley system $(\omega_n)_n$ is defined as the set of Walsh–Paley functions

$$\omega_i(x) = \prod_{k=0}^{\infty} (r_k(x))^{i_k}, \qquad i \in \mathbb{N},\ x \in G,$$

where

$$i = \sum_{k=0}^{\infty} i_k 2^k \qquad \text{and} \qquad r_k(x) = (-1)^{x_k}.$$

The Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels and Fejér kernels are respectively defined as follows:

$$\widehat{f}(n) := \int f(x)\omega_n(x)\,dx, \qquad S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\omega_k,$$

$$D_n := \sum_{k=0}^{n-1} \omega_k, \qquad K_n := \frac{1}{n}\sum_{k=1}^{n} D_k.$$

It is clear that

$$S_n f(y) = \int D_n(y-x) f(x)\,dx,$$

and

$$D_{2^n}(x) = 2^n 1_{I_n}(x).$$

The Nörlund logarithmic means are defined by

$$t_n f := \frac{1}{l_n}\sum_{k=1}^{n-1} \frac{S_k f}{n-k}\,, \qquad l_n := \sum_{k=1}^{n-1} \frac{1}{k}\,.$$

The functions F_n, $n \in \mathbb{N}$, are defined by

$$F_n := \frac{1}{l_n}\sum_{k=1}^{n-1} \frac{D_k}{n-k}\,;$$

it is clear that

$$t_n f = F_n * f.$$

We also use the notation

$$f^* = \sup_n S_{2^n}|f|.$$

Define the function φ: $\mathbb{N}\setminus\{0\} \to \mathbb{N}$ by $\varphi(n) = n - 2^{[\log_2 n]}$. Set $\varphi^1(n) = \varphi(n)$, $\varphi^0(n) = n$ and $\varphi^i(n) = \varphi\circ\varphi^{i-1}(n)$ when $i \geq 2$. For every $n \in \mathbb{N}\setminus\{0\}$,

$i \geq 0$, such that $\varphi^i(n) > 0$, define the functions $\alpha_i(n) = [\log_2(\varphi^i(n))]$ and $\beta_i(n) = l_{\varphi^i(n)}$.

The notation C will be used for independent positive constants.

2. Main results

We first formulate Goginava's result [5, Lemma 4]:

Lemma (Goginava 2005). *Let* $2^n \leq m < 2^{n+1}$. *Then*

$$l_mF_m = l_mD_{2^n}(x) - \omega_{2^n-1}(x)\sum_{j=1}^{2^n-2}\Big(\frac{1}{m-2^n+j} - \frac{1}{m-2^n+j+1}\Big)jK_j(x) -$$

$$-\frac{2^n-1}{m-1}\omega_{2^n-1}(x)K_{2^n-1}(x) + \omega_{2^n}(x)l_{m-2^n}F_{m-2^n}(x).$$

Applying [5, Lemma 4], we easily obtain the following result.

Lemma 2.1. *Let n be a natural number and $s > 0$ be the largest integer such that $\varphi^{s+1}(n) > 0$. Then*

$$l_nF_n = \beta_0(n)D_{2^{\alpha_0(n)}} + \sum_{i=1}^{s} r_{\alpha_0(n)}\cdots r_{\alpha_{i-1}(n)}\beta_i(n)D_{2^{\alpha_i(n)}} -$$

$$-\omega_{2^{\alpha_0(n)}-1}\sum_{j=1}^{2^{\alpha_0(n)-2}}\Big(\frac{1}{\varphi^1(n)+j} - \frac{1}{\varphi^1(n)+j+1}\Big)jK_j -$$

$$-\sum_{i=1}^{s} r_{\alpha_0(n)}\cdots r_{\alpha_{i-1}(n)}\omega_{2^{\alpha_i(n)}-1} \times$$

$$\times \sum_{j=1}^{2^{\alpha_i(n)-2}}\Big(\frac{1}{\varphi^{i+1}(n)+j} - \frac{1}{\varphi^{i+1}(n)+j+1}\Big)jK_j -$$

$$-\frac{2^{\alpha_0(n)}-1}{\varphi^0(n)-1}\omega_{2^{\alpha_0(n)}-1}K_{2^{\alpha_0(n)}-1} -$$

$$-\sum_{i=1}^{s} r_{\alpha_0(n)}\cdots r_{\alpha_{i-1}(n)}\frac{2^{\alpha_i(n)}-1}{\varphi^i(n)-1}\omega_{2^{\alpha_i(n)}-1}K_{2^{\alpha_i(n)}-1} +$$

$$+ r_{\alpha_0(n)}\cdots r_{\alpha_s(n)}l_{\varphi^{s+1}(n)}F_{\varphi^{s+1}(n)}.$$

This can be written as

$$F_n = D_{2^{\alpha_0(n)}} + R_n + A_n + B_n + C_n,$$

where

$$A_n = -\frac{1}{\beta_0(n)}\omega_{2^{\alpha_0(n)}-1}\sum_{j=1}^{2^{\alpha_0(n)-2}}\Big(\frac{1}{\varphi^1(n)+j}-\frac{1}{\varphi^1(n)+j+1}\Big)jK_j -$$

$$-\frac{1}{\beta_0(n)}\sum_{i=1}^{s} r_{\alpha_0(n)}\cdots r_{\alpha_{i-1}(n)}\omega_{2^{\alpha_i(n)}-1}\times$$

$$\times\sum_{j=1}^{2^{\alpha_i(n)-2}}\Big(\frac{1}{\varphi^{i+1}(n)+j}-\frac{1}{\varphi^{i+1}(n)+j+1}\Big)jK_j,$$

and

$$B_n = -\frac{1}{\beta_0(n)}\frac{2^{\alpha_0(n)}-1}{\varphi^0(n)-1}\omega_{2^{\alpha_0(n)}-1}K_{2^{\alpha_0(n)}-1} -$$

$$-\frac{1}{\beta_0(n)}\sum_{i=1}^{s} r_{\alpha_0(n)}\cdots r_{\alpha_{i-1}(n)}\frac{2^{\alpha_i(n)}-1}{\varphi^i(n)-1}\omega_{2^{\alpha_i(n)}-1}K_{2^{\alpha_i(n)}-1}.$$

Proof. First we can easily see that for all nonnegative integers i and j and every positive integer n we have

$$\alpha_i(\varphi^j(n)) = \alpha_{i+j}(n), \quad \text{and} \quad \beta_i(\varphi^j(n)) = \beta_{i+j}(n).$$

If we combine these formulae with [5, Lemma 4], we immediately obtain that for every n such that $\varphi^i(n) > 1$ we have

$$l_{\varphi^i(n)}F_{\varphi^i(n)} =$$

$$= \beta_i(n)D_{2^{\alpha_i(n)}} - \omega_{2^{\alpha_i(n)}-1}\sum_{j=1}^{2^{\alpha_i(n)}-2}\Big(\frac{1}{\varphi^{i+1}(n)+j}-\frac{1}{\varphi^{i+1}(n)+j+1}\Big)jK_j -$$

$$-\frac{2^{\alpha_i(n)}-1}{\varphi^i(n)-1}\omega_{2^{\alpha_i(n)}-1}K_{2^{\alpha_i(n)}-1} + r_{\alpha_i(n)}l_{\varphi^{i+1}(n)}F_{\varphi^{i+1}(n)}.$$

The result will follow by induction. □

Lemma 2.2. *For every $f \in L^1$, $A_n * f \to 0$ almost everywhere.*

Proof. Proceeding as in the proof of [5, Corollary 1], it suffices to prove that $A^*f := \sup_n |A_n * f|$ is a weak type $(1,1)$ operator, because $A_n * P \to 0$ pointwise for every Walsh polynomial P. This can be obtained by applying the method used in [5, Theorem 1] if we prove that A^* is bounded in L^∞ and quasi-local at the same time.

Every $n \in \mathbb{N}$ can be written in the form

$$n = \sum_{i:\varphi^i(n)>0} 2^{\alpha_i(n)}.$$

We have

$$|A_n| \leq \frac{1}{l_n} \sum_{i:\varphi^{i+1}(n)>0} \Big(\sum_{j=1}^{2^{\alpha_{i+1}(n)}} \frac{|K_j|}{2^{\alpha_{i+1}(n)}} + \sum_{2^{\alpha_{i+1}(n)}+1}^{2^{\alpha_i(n)}-2} \frac{|K_j|}{j} \Big).$$

From $\sup_n \|K_n\|_1 < C$, we get $\sup_n \|A_n\|_1 < C$, from which we conclude that A^* is bounded in L^∞.

According to [5, Lemma 6], in order to prove the quasi-locality of A^*, it suffices to prove that

$$\int_{G\setminus I_l} \sup_{n\geq 2^l} |A_n(x)|\,dx = O(1)$$

uniformly for $l \in \mathbb{N}$. We estimate as follows:

$$\int_{G\setminus I_l} \sup_{n\geq 2^l} |A_n(x)|\,dx \leq$$

$$\leq \int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n} \sum_{\substack{i:\varphi^{i+1}(n)>0\\ \alpha_i(n)\leq l-1}} \left(\sum_{j=1}^{2^{\alpha_{i+1}(n)}} \frac{|K_j(x)|}{2^{\alpha_{i+1}(n)}} + \sum_{j=2^{\alpha_{i+1}(n)}+1}^{2^{\alpha_i(n)}-2} \frac{|K_j(x)|}{j} \right) dx +$$

$$+ \int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n} \sum_{\substack{i:\varphi^{i+1}(n)>0\\ \alpha_i(n)\geq l>\alpha_{i+1}(n)}} \left(\sum_{j=1}^{2^{\alpha_{i+1}(n)}-1} \frac{|K_j(x)|}{2^{\alpha_{i+1}(n)}} + \sum_{j=2^{\alpha_{i+1}(n)}}^{2^l-1} \frac{|K_j(x)|}{j} + \right.$$

$$\left. + \sum_{j=2^l}^{n} \frac{|K_j(x)|}{j} \right) dx + \int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n} \sum_{\substack{i:\varphi^{i+1}(n)>0\\ l\leq\alpha_{i+1}(n)}} \left(\sum_{j=1}^{2^l-1} \frac{|K_j(x)|}{2^{\alpha_{i+1}(n)}} + \right.$$

$$\left. + \sum_{j=2^l}^{2^{\alpha_{i+1}(n)}} \frac{|K_j(x)|}{2^{\alpha_{i+1}(n)}} + \sum_{j=2^{\alpha_{i+1}(n)}+1}^{2^{\alpha_i(n)}-2} \frac{|K_j(x)|}{j} \right) dx.$$

We have

$$\int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n} \sum_{\substack{i:\varphi^{i+1}(n)>0\\ \alpha_i(n)\leq l-1}} \sum_{j=1}^{2^{\alpha_{i+1}(n)}} \frac{|K_j(x)|}{2^{\alpha_{i+1}(n)}}\,dx \leq$$

$$\leq \frac{C}{l} \sum_{k=0}^{l-1} \frac{1}{2^k} \sum_{j=1}^{2^k} \int_{G\setminus I_l} |K_j(x)|\,dx < C.$$

The constant C does not depend on the choice of l since

$$\int_{G\setminus I_l} |K_j(x)|\,dx \leq \|K_j\|_1 \leq \sup_j \|K_j\|_1 \leq C < +\infty.$$

The term

$$\int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n} \sum_{\substack{i:\varphi^{i+1}(n)>0\\ \alpha_i(n)\geq l>\alpha_{i+1}(n)}} \sum_{j=1}^{2^{\alpha_{i+1}(n)}} \frac{|K_j(x)|}{2^{\alpha_{i+1}(n)}}\,dx$$

can be estimated in the same way. Now,

$$\int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n} \sum_{\substack{i:\varphi^{i+1}(n)>0\\ \alpha_i(n)\leq l-1}} \sum_{2^{\alpha_{i+1}(n)}+1}^{2^{\alpha_i(n)}-2} \frac{|K_j(x)|}{j}\,dx \leq$$

$$\leq \frac{C}{l}\sum_{j=1}^{2^{l-1}} \frac{1}{j}\int_{G\setminus I_l} |K_j(x)|\,dx < C.$$

Similarly,

$$\int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n} \sum_{\substack{i:\varphi^{i+1}(n)>0\\ \alpha_i(n)\geq l>\alpha_{i+1}(n)}} \sum_{j=2^{\alpha_{i+1}(n)}}^{2^l-1} \frac{|K_j(x)|}{j} \leq$$

$$\leq \frac{C}{l}\sum_{j=1}^{2^l-1} \frac{1}{j}\int_{G\setminus I_l} |K_j(x)|\,dx < C.$$

The term

$$\int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n} \sum_{\substack{i:\varphi^{i+1}(n)>0\\ l\leq\alpha_{i+1}(n)}} \sum_{j=1}^{2^l-1} \frac{|K_j(x)|}{2^{\alpha_{i+1}(n)}}\,dx$$

can be estimated in the same way. Besides, we have

$$\int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n} \sum_{\substack{i:\varphi^{i+1}(n)>0\\ l\leq\alpha_{i+1}(n)}} \Big(\sum_{j=2^l}^{2^{\alpha_{i+1}(n)}} \frac{|K_j(x)|}{2^{\alpha_{i+1}(n)}} + \sum_{j=2^{\alpha_{i+1}(n)}+1}^{2^{\alpha_i(n)}-2} \frac{|K_j(x)|}{j}\Big)\,dx \leq$$

$$\leq C\int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n}\sum_{j=2^l}^{n} \frac{|K_j(x)|}{j} \leq C\int_{G\setminus I_l} \sup_{n\geq 2^l} \frac{1}{l_n}\sum_{j=2^l}^{n} \frac{1}{j}\sup_{i\geq 2^l}|K_i(x)|\,dx \leq$$

$$\leq C\sup_{n\geq 2^l} \frac{1}{l_n}\sum_{j=2^l}^{n} \frac{1}{j}\int_{G\setminus I_l}\sup_{i\geq 2^l}|K_i(x)|\,dx < C.$$

The constant C is independent of the choice of l because

$$\int_{G\setminus I_l}\sup_{i\geq 2^l}|K_i(x)|\,dx < C,$$

which is a direct consequence of Gát's result [1, Lemma 4]. □

Lemma 2.3. *For every $f \in L^1$, $B_n * f \to 0$ almost everywhere.*

Proof. As in Lemma 2.2, since $B_n * P \to 0$ pointwise for every Walsh polynomial P, it suffices to prove that the operator B^* defined by $B^* f := \sup_n |B_n * f|$ is bounded in L^∞ and quasi-local at the same time.

There exists a constant $C > 0$ so that for every $n \in \mathbb{N}$, whenever $\varphi^i(n) > 0$, we have $i < Cl_n$. Then, the boundedness of B^* in L^∞ follows immediately from $\sup_n \|K_n\|_1 < +\infty$.

From $2^{\alpha_j(n)} \leq \varphi^j(n)$, for every nonnegative integer j and every natural number n, it can be easily deduced that for all $x \in G$ we have

$$|B_n(x)| \leq \frac{1}{\beta_0(n)} \sum_{j=0}^{s} |K_{2^{\alpha_j(n)}-1}(x)| \leq$$

$$\leq \frac{1}{\beta_0(n)} \Big(|K_{2^i-1}(x)| + |K_{2^{i-1}-1}(x)| + \cdots + |K_{2^1-1}(x)| \Big),$$

where the notation i is used here for $i := \alpha_0(n) = \log_2 n$. Since $\beta_0(n) \sim \alpha_0(n) = i$, we get

$$|B_n(x)| \leq C \frac{|K_{2^i-1}(x)| + |K_{2^{i-1}-1}(x)| + \cdots + |K_{2^1-1}(x)|}{i}.$$

In order to obtain the quasi-locality of B^*, we write for arbitrary $l \in \mathbb{N}$

$$\int_{G \setminus I_l} \sup_{n \geq 2^l} |B_n(x)|\, dx \leq$$

$$\leq C \int_{G \setminus I_l} \sup_{i \geq l} \frac{|K_{2^i-1}(x)| + |K_{2^{i-1}-1}(x)| + \cdots + |K_{2^1-1}(x)|}{i}\, dx \leq$$

$$\leq C \int_{G \setminus I_l} \sup_{i \geq l} \frac{|K_{2^i-1}(x)| + |K_{2^{i-1}-1}(x)| + \cdots + |K_{2^{l+1}-1}(x)|}{i}\, dx +$$

$$+ C \int_{G \setminus I_l} \sup_{i \geq l} \frac{|K_{2^l-1}(x)| + |K_{2^{l-1}-1}(x)| + \cdots + |K_{2^1-1}(x)|}{i}\, dx \leq$$

$$\leq C \int_{G \setminus I_l} \sup_{i \geq l+1} |K_{2^i-1}(x)|\, dx + C \max_{i \in \{1,\ldots,l\}} \int_{G \setminus I_l} |K_{2^i-1}(x)|\, dx < C. \qquad \square$$

Theorem 2.4. *Let $(m_n)_n$ be an increasing sequence of positive integers. Suppose that*

$$\sum_{i:\varphi^i(m_n)>0} \frac{\beta_i(m_n)}{l_{m_n}} = O(1).$$

Then $F_{m_n} * f \to f$ almost everywhere.

The condition of [5, Theorem 1] from which Goginava proves that $F_{m_n} * f \to f$ a.e. in our notation looks as follows:

$$\sum_{n=1}^{\infty} \frac{\alpha_1^2(m_n)}{\alpha_0(m_n)} < +\infty.$$

Since $\#\{i : \varphi^i(n) > 0\} \leq \alpha_1(n)$, it follows that

$$\sum_{i:\varphi^i(m_n)>0} \frac{\alpha_i(m_n)}{\alpha_0(m_n)} < \frac{\alpha_1^2(m_n)}{\alpha_0(m_n)}\,.$$

If the sequence $(m_n)_n$ satisfies the condition of [5, Theorem 1], then

$$\alpha_1^2(m_n) = o(\alpha_0(m_n)),$$

which implies that

$$\sum_{i:\varphi^i(m_n)>0} \alpha_i(m_n) = o(\alpha_0(m_n)),$$

or, equivalently,

$$\sum_{i:\varphi^i(m_n)>0} \beta_i(m_n) = o(\beta_0(m_n)) = o(l_{m_n}).$$

Therefore, Theorem 2.4 is a generalization of [5, Theorem 1].

Proof. It suffices to prove that $R_{m_n} * f \to 0$ and $C_{m_n} * f \to 0$ almost everywhere when $n \to \infty$, for every integrable function f. Since $R_n * \varphi \to 0$ and $C_n * \varphi \to 0$ when $n \to \infty$ for every Walsh polynomial φ, we only have to prove that the operators

$$R^* f := \sup_n |R_{m_n} * f| \quad \text{and} \quad C^* f := \sup_n |C_{m_n} * f|$$

are of weak type $(1,1)$.

Since

$$R_{m_n} = \sum_{i>0:\varphi^i(m_n)>0} \frac{\beta_i(m_n)}{l_{m_n}} r_{\alpha_0(m_n)} \cdots r_{\alpha_{i-1}(m_n)} D_{2^{\alpha_i(m_n)}}\,,$$

therefore, under our assumption $R^* f \leq C f^*$. The boundedness of the operator C^* can be deduced directly from [5, Theorem 1], which claims that the operator $\sup_n |F_{m_n} * f|$ is of weak type $(1,1)$ if

$$\sum_{n=1}^{\infty} \frac{\log^2(\varphi(m_n))}{\log m_n} < \infty.$$

Let us test this condition for the sequence of operators $\left(\frac{l_{\varphi^{s+1}(m_n)}}{l_{m_n}} F_{\varphi^{s+1}(m_n)}\right)_n$. We have first that

$$\left|\frac{l_{\varphi^{s+1}(m_n)}}{l_{m_n}} F_{\varphi^{s+1}(m_n)}\right| \leq |F_{\varphi^{s+1}(m_n)}|.$$

Then it suffices to verify the condition of [5, Theorem 1] only for the sequence $(\varphi^{s+1}(m_n))_n$. Meanwhile, $\varphi(\varphi^{s+1}(m_n)) = 0$, which means that the series mentioned in the condition is actually composed of zero terms. $\square$

Theorem 2.5. *Let $(m_n)_n$ and $(s_n)_n$ be increasing sequences of positive integers for which*

(1) *the sequence $(\varphi^{s_n}(m_n))_n$ is increasing, and*

(2) *$l_{m_n} = o(\beta_{s_n}(m_n)\sqrt{s_n})$, when $n \to \infty$.*

Then there exists an integrable function f such that $F_{m_n} * f \not\to f$ on a subset of positive measure.

Proof. We can choose a subsequence $(k_n)_n$ of $(m_n)_n$ with a convenient subsequence of $(s_n)_n$, which we still denote by $(s_n)_n$, such that

$$(1) \quad \text{for all } n \in \mathbb{N},\ \varphi^{s_n}(k_n) > k_{n-1}, \quad (2) \quad \sum_n \frac{l_{k_n}}{\beta_{s_n}(k_n)\sqrt{s_n}} < \infty.$$

Let

$$f = \sum_n \sum_{i=1}^{s_n} r_{\alpha_i(k_n)} \frac{l_{k_n}}{\beta_i(k_n)\sqrt{s_n}}\,.$$

It is clear that

$$\|f\|_1 \le \sum_n \frac{1}{\sqrt{s_n}} \Big\| \sum_{i=1}^{s_n} r_{\alpha_i(k_n)} \frac{l_{k_n}}{\beta_i(k_n)} \Big\|_2 \le$$

$$\le \sum_n \Big(\sum_{i=1}^{s_n} \Big(\frac{l_{k_n}}{\beta_i(k_n)} \Big)^2 \Big)^{1/2} \frac{1}{\sqrt{s_n}} \le C \sum_n \frac{l_{k_n}}{\beta_{s_n}(k_n)} \frac{1}{\sqrt{s_n}} < +\infty.$$

It is easily seen that for every i such that $\varphi^{i+1}(n) > 0$,

$$R_n * r_{\alpha_i(n)} = \frac{\beta_i(n)}{l_n} r_{\alpha_i(n)},$$

and $R_n * r_s = 0$, if $s \neq \alpha_i(n)$ whenever $\varphi^{i+1}(n) > 0$.

We conclude that

$$R_{k_n} * f - R_{k_{n-1}} * f = \sum_{i=1}^{s_n} r_{\alpha_i(k_n)} \frac{1}{\sqrt{s_n}}.$$

Define the sets

$$E_n = \Big\{ x \in G : 0 \le \sum_{i=1}^{s_n} x_{\alpha_i(k_n)} \le \frac{s_n}{2} - \sqrt{s_n} \Big\}.$$

Then let

$$E = \bigcap_{k \ge 1} \bigcup_{n \ge k} E_n.$$

Notice that for each $x \in E_n$,

$$|R_{k_n} * f(x) - R_{k_{n-1}} * f(x)| \ge 2.$$

This means that for every $x \in E$, $|R_{k_n} * f(x) - R_{k_{n-1}} * f(x)| \geq 2$, for infinitely many elements from $(k_n)_n$. Therefore,

$$R_{k_n} * f(x) \not\to f(x), \quad \text{for all } x \in E.$$

On the other hand, for every $n \geq 1$, if C_n^k denotes the binomial coefficient, we have

$$|E_n| = \sum_{k=1}^{\frac{s_n}{2}-\sqrt{s_n}} C_{s_n}^k \frac{1}{2^{s_n}}.$$

Since, $C_{s_n}^k \sim C_{s_n}^l$, for every $k \in \{\frac{s_n}{2} - 2\sqrt{s_n}, \dots, \frac{s_n}{2} - \sqrt{s_n} - 1\}$, $l \in \{\frac{s_n}{2} - \sqrt{s_n}, \dots, \frac{s_n}{2}\}$, it is easily seen that $\min_n |E_n| > 0$, which implies that $|E| > 0$. □

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Сходимость почти всюду некоторых подпоследовательностей логарифимических средних Нёрлунда ряда Уолша-Фурье

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В этой работе мы изучаем максимальный оператор для некоторого класса подпоследовательностей логарифимических средних Нёрлунда ряда Уолша–Фурье. Для такого класса мы доказываем сходимость почти всюду $(t_{m_n} f)_n$ для любой интегрируемой функции f. Кроме того, мы устанавливаем теорему о расходимости для других классов подпоследовательностей.