Analysis Mathematica, **41**(2015), 3–15
DOI: 10.1007/s10476-015-0102-9

On certain old and new trigonometric and hyperbolic inequalities

BARKAT ALI BHAYO[1] and JÓZSEF SÁNDOR[2]

[1] *Department of Mathematical Information Technology, University of Jyväskylä, 40014 Jyväskylä, Finland, e-mail: bhayo.barkat@gmail.com*

[2] *Babeş-Bolyai University, Department of Mathematics, Str. Kogalniceanu nr. 1, 400084 Cluj-Napoca, Romania, e-mail: jsandor@math.ubbcluj.ro*

Received January 25, 2014.

Abstract. In this paper we study two-sided inequalities of trigonometric and hyperbolic functions.

1. Introduction

Since last one decate many authors got interest to study the following inequalities

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3} \quad \text{for} \quad 0 < |x| < \pi/2\,. \tag{1.1}$$

The first inequality is due to by Adamović and Mitrinovic [10, p. 238], and the second one was obtained by N. Cusa and C. Huygens (see [21]). The hyperbolic version of (1.1) is given as

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{\cosh x + 2}{3} \quad \text{for} \quad x \neq 0\,. \tag{1.2}$$

The first inequality in (1.2) was obtained by Lazarević (see [10, p. 270], and the second inequality is called sometime hyperbolic Cusa–Huygens inequality [15].

There are many results on the refinement of the inequalities (1.1) and (1.2) in the literature, e.g, see [3, 11, 12, 13, 15, 16, 19, 20, 21, 22] and

0133–3852/$ 20.00

the references therein. In this paper, we give simple proofs of some known results, and establish new inequalities, as well. The main results of this paper read as follows.

Theorem 1.3. *For $x \in (0, \pi/2)$, the following inequalities hold true:*

$$\frac{x^{3/2}}{2(\sin x)^{1/2}} < \tan\frac{x}{2} < \frac{x^{2/\alpha}}{2(\sin x)^{2/\alpha-1}}, \tag{1.4}$$

where $\alpha = 2\log(\pi/2)/\log(2) \approx 1.30299$.

Theorem 1.5. *For $x \in (0, \pi/2)$, the following inequalities hold true:*

$$\frac{x}{\tan(x/2)} - 1 < \exp\Big(\frac{1}{2}\Big(\frac{x}{\tan x} - 1\Big)\Big) < \frac{\sin x}{x} < \exp\Big(\Big(\log\frac{\pi}{2}\Big)\Big(\frac{x}{\tan x} - 1\Big)\Big).$$

Theorem 1.6. *For $x \in (0, \infty)$, the following inequalities hold true:*

$$\frac{x}{\tanh(x/2)} - 1 < \exp\Big(\frac{1}{2}\Big(\frac{x}{\tanh x} - 1\Big)\Big) < \frac{\sinh x}{x} < \exp\Big(\Big(\frac{x}{\tanh x} - 1\Big)\Big).$$

The second inequalities in Theorems 1.5 and 1.6 are known [19]. For these, however, here a new method of proof is offered.

2. Preliminaries

In this section we give few lemmas, they will be used in the proofs of our theorems.

For $|x| < \pi$, the following power series expansions can be found in [8, 1.3.1.4(2)–(3)]:

$$x\cot x = 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n}, \tag{2.1}$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}, \tag{2.2}$$

and

$$\coth x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}, \tag{2.3}$$

where B_{2n} are the even-indexed Bernoulli numbers, see [7, p. 231]. The following expansions can be obtained directly from (2.2) and (2.3):

$$\frac{1}{(\sin x)^2} = -(\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|(2n-1)x^{2n-2}, \tag{2.4}$$

$$\frac{1}{(\sinh x)^2} = -(\coth x)' = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}(2n-1)|B_{2n}|x^{2n-2}. \tag{2.5}$$

The following result is due to BIERNACKI and KRZYŻ [4], which will be very useful in studying the monotonicity of certain power series. This result was also simply proved in [6] by HEIKKALA et al.

Lemma 2.6. *For* $0 < R \le \infty$, *let*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad B(x) = \sum_{n=0}^{\infty} b_n x^n$$

be two real power series converging on the interval $(-R, R)$. *If the sequence* $\{a_n/b_n\}$ *is increasing (decreasing) and* $b_n > 0$ *for all* n, *then the function* $A(x)/B(x)$ *is also increasing (decreasing) on* $(0, R)$.

The following result, which is sometime called Monotone l'Hôpital rule, was proved in [2] by Anderson et al.

Lemma 2.7. *For* $-\infty < a < b < \infty$, *let* $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$, *and be differentiable on* (a, b). *Let* $g'(x) \neq 0$ *on* (a, b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a, b), *then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.8. (1) *The function*

$$f(x) = x + \sin x - \frac{x^2}{\tan x/2}$$

is strictly increasing from $(0, \pi/2)$ *onto* $(0, c)$, $c = (4+2\pi-\pi^2)/4 \approx 0.10336$. *In particular, for* $x \in (0, \pi/2)$ *we have*

$$1 + \frac{\sin x}{x} - \frac{c}{x} < \frac{x}{\tan x/2} < 1 + \frac{\sin x}{x}.$$

(2) *The function*

$$g(x) = \frac{1}{x}\Big(x\frac{\cos x}{\sin x} - 1\Big)\Big(\frac{\cos x}{\sin x} + \frac{1}{\sin x}\Big)$$

is strictly increasing from $(0, \pi/2)$ *onto* $(-4/3, -4/\pi)$.

Proof. For the proof of (1), we get

$$f'(x) = \frac{1}{2}\Big(\frac{1}{\sin x/2}\Big)^2 (x - \sin x)^2 > 0,$$

the limiting values are clear. For (2), derivation gives

$$g'(x) = \frac{1}{x^2(1-\cos x)}\Big(x + \sin x - \frac{x^2}{\tan x/2}\Big),$$

which is positive by part (1), and the limiting values follow from l'Hôpital rule. This completes the proof. □

Lemma 2.9. (1) *The function*

$$f(x) = \cos x - \Big(1 - \frac{x^2}{3}\Big)^{3/2}$$

is strictly decreasing from $(0, \pi/2)$ *onto* $(0, c_1)$*, where*

$$c_1 = -(12-\pi^2)^{3/2}/(24\sqrt{3}) \approx -0.07480\,.$$

In particular, for $x \in (0, \pi/2)$ *we have*

$$\Big(1 - \frac{x^2}{3}\Big)^{3/2} - c_1 < \cos x < \Big(1 - \frac{x^2}{3}\Big)^{3/2}, \tag{2.10}$$

(2) *For* $x \in (0, \pi/2)$ *we have*

$$\frac{x}{4}\Big(\frac{5}{2} - \frac{x}{2\tan x}\Big) < \tan\frac{x}{2}\,. \tag{2.11}$$

(3) *For* $x \in (0, \infty)$ *we have*

$$\frac{x}{4}\Big(\frac{5}{2} - \frac{x}{2\tanh x}\Big) < \tanh\frac{x}{2}\,. \tag{2.12}$$

Proof. By utilizing the inequality

$$1 - \frac{x^2}{6} < \frac{\sin x}{x}$$

from [9, Theorem 3.1], we get

$$f'(x) = x(1 - x^2/3)^{1/2} - \sin x < x((1 - x^2/3)^{1/2} - (1 - x^2/6)) = x\,g(x),$$

which is negative, since

$$g'(x) = \frac{x}{3}\left(1 - \frac{1}{(1 - x^2/3)^{1/2}}\right) < 0 \quad \text{and} \quad g(x) = 0.$$

This implies that f is decreasing, and the limiting values are clear.

For the proof of Part (2), we use the indentity $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$ and write the inequality (2.11) as

$$\frac{\sin x}{x} < \frac{8(1-\cos x) + x^2\cos x}{5x^2}, \qquad x \in (0, \pi/2). \tag{2.13}$$

Let

$$h(x) = 5x \sin x - 8(1-\cos x) + x^2 \cos x, \qquad x \in (0, \pi/2).$$

Applying the inequality

$$(\cos x)^{1/3} < \frac{\sin x}{x},$$

we get

$$h'(x) = 3x\cos x - (3-x^2)\sin x < x(3\cos x - (3-x^2)(\cos x)^{1/3}) =$$
$$= 3x(\cos x)^{1/3}\Big((\cos x)^{2/3} - \Big(1 - \frac{x^2}{3}\Big)\Big),$$

which is negative by Part (1), and $h(0) = 0$. The proof of Part (3) is similar to the proof of Part (2), if we use the indentity

$$\tanh\frac{x}{2} = \frac{\cosh x - 1}{\sinh x}. \qquad \square$$

3. Proofs of the theorems

3.1. Proof of Theorem 1.3. Let $f(x) = f_1(x)/f_2(x)$, where

$$f_1(x) = \log\frac{x}{\sin x}, \qquad f_2(x) = \log\frac{1}{\sqrt{(1+\cos x)/2}},$$

and clearly $f_1(0) = f_2(0) = 0$. We get

$$\frac{f_1'(x)}{f_2'(x)} = -\frac{1}{x}\Big(x\frac{\cos x}{\sin x} - 1\Big)\Big(\frac{\cos x}{\sin x} + \frac{1}{\sin x}\Big),$$

which is strictly decreasing by Lemma 2.8 Part (2). Hence the function f is strictly decreasing by Lemma 2.7, and by l'Hôpital rule we have

$$\lim_{x\to 0} f(x) = 4/3 \quad \text{and} \quad \lim_{x\to\pi/2} f(x) = 2\log(\pi/2)/\log(2) = 1.30299\ldots = \alpha.$$

This implies the following inequalities:

$$\Big(\sqrt{\frac{1+\cos x}{2}}\Big)^{4/3} < \frac{\sin x}{x} < \Big(\sqrt{\frac{1+\cos x}{2}}\Big)^{\alpha}. \tag{3.2}$$

By utilizing the indentity $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$ we get (1.4). $\square$

In [18], it is proved that

$$\frac{\sin x}{x} < \sqrt{\frac{1+\cos x}{2}}\,\frac{\sqrt{(1+\cos x)/2}+2}{3} = \frac{t^{1/2}(t^{1/2}+2)}{3} = \frac{t+2t^{1/2}}{3}, \tag{3.3}$$

if one denotes $t = (\cos x + 1)/2$. The right-hand side of (3.2) cannot be compared with inequality (3.3), and this inequality improves the Cusa–Huygens inequality $(\sin x)/x < (\cos x + 2)/3 = (2t + 1)/3$.

The following theorem is known (see in [21]). Here another proof appears, but both proofs use Lemma 2.7.

Theorem 3.4. *The function*

$$f(x) = \frac{\log((\sinh x)/x)}{\log(\cosh(x/2))}$$

is strictly decreasing from $(0, \infty)$ *onto* $(4/3, 2)$. *In particular, we have*

$$\Big(\frac{1 + \cosh x}{2}\Big)^{2/3} < \frac{\sinh x}{x} < \Big(\frac{1 + \cosh x}{2}\Big).$$

Proof. Let

$$f_1(x) = \log\Big(\frac{\sinh x}{x}\Big), \quad f_2(x) = \log\Big(\frac{\cosh x}{x}\Big), \quad \text{and} \quad f_1(0) = f_2(0) = 0.$$

We get

$$\frac{f_1'(x)}{f_2'(x)} = 2g_1(x),$$

where

$$g_1(x) = x\Big(1 + \frac{\cosh x}{\sinh x}\Big)\frac{1}{(\sinh x)^2}\Big(\frac{\cosh x}{x} + \frac{\sinh x}{x}\Big).$$

Differentiation gives

$$g_1'(x) = \frac{1}{2x^2} g_2(x)\,,$$

where

$$g_2(x) = 2\frac{\cosh(x/2)}{\sinh(x/2)} + \frac{x}{(\sinh(x/2))^2}\Big(1 - x\frac{\cosh(x/2)}{\sinh(x/2)}\Big),$$

which is positive since

$$g_2'(x) = \frac{x}{2(\sinh(x/2))^4}(x(2 + \cosh x) - 3\sinh x) > 0,$$

due to the inequality

$$\frac{\sinh x}{x} < \frac{2 + \cosh x}{3}\,,$$

and $g_2(x)$ tends to 0 as x tends to 0. This implies that f_1'/f_2' is increasing, Hence by Lemma 2.7, f is increasing, and the limiting values follow from l'Hôpital rule. This completes the proof. $\square$

Corollary 3.5. *For* $x \in (0, \infty)$, *we have*

$$(\cosh x)^{1/3} < \Big(\frac{1 + \cosh x}{2}\Big)^{2/3} < \frac{\sinh x}{x} < \frac{1 + \cosh x}{2} < \Big(\frac{2 + \cosh(x/2)}{3}\Big)^4.$$

P r o o f. The proof of the first inequality is trivial, since it can be simplified as $0 < (1 - \cosh x)^2$. The second and third inequalities follow from Theorem 3.4. For the proof of the fourth inequality, it is enough to prove that the function

$$h(y) = 3\Big(\frac{y+2}{3}\Big)^{1/4} - \Big(\frac{y+1}{2}\Big)^{1/2} - 2$$

is negative for $y \in (0, \infty)$. We get

$$h'(x) = \frac{3^{3/4}}{4(1+y)^{1/2}}\Big(h_1(y) - \frac{\sqrt{2}}{3^{3/4}}\Big),$$

where

$$h_1(x) = \frac{(1+y)^{1/2}}{(2+y)^{3/4}}.$$

Now, we prove that $\log h_1$ is negative, since

$$(\log(h_1(y)))' = -\frac{y-1}{4(2+3y+y^2)} < 0,$$

and $h_1(1) = \sqrt{2}/3^{3/4}$. This implies that h is decreasing, and h tends to 0 when y tends to 1, hence the proof of the fourth inequality is completed. □

3.6. P r o o f o f T h e o r e m 1.5. It is well known that

$$L(a, b) < (a + b)/2, \qquad \text{where } L(a, b) = \frac{b-a}{\log b - \log a}$$

denotes the logarithmic mean of two distint real numbers a and b. In particular, we have

$$L(t, 1) < (t + 1)/2, \qquad \text{for } \ t > 1.$$

This inequality can be written as

$$\log t > 2(t - 1)/(t + 1).$$

Letting

$$t = \frac{\tan(x/2)}{x - \tan(x/2)} > 1$$

and using (2.11), we get

$$\log\Big(\frac{\tan(x/2)}{x - \tan(x/2)}\Big) > \frac{2(\tan(x/2) - x)}{x} > \frac{\tan x - x}{2\tan x},$$

this implies the proof of the first inequality.

For the proof of the second and the third inequality, we define $f(x) = f_1(x)/f_2(x)$, $x \in (0, \pi/2)$, where

$$f_1(x) = \log\Big(\frac{x}{\sin x}\Big), \qquad f_2(x) = 1 - \frac{x}{\tan x},$$

and clearly $f_1(0)=f_2(0)=0$. Differentiation with respect x gives

$$\frac{f_1'(x)}{f_2'(x)}=\frac{A(x)}{B(x)}\,,$$

where

$$A(x)=1-x\frac{\cos x}{\sin x}\qquad\text{and}\qquad B(x)=\frac{x^2}{\sin x^2}-x\frac{\cos x}{\sin x}\,.$$

By (2.1) and (2.4), we get

$$A(x)=\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n}=\sum_{n=1}^{\infty}a_nx^{2n},$$

and

$$B(x)=\sum_{n=1}^{\infty}\frac{2^{2n}2n}{(2n)!}|B_{2n}|x^{2n}=\sum_{n=1}^{\infty}b_nx^{2n}.$$

The sequence $c_n=a_n/b_n=1/(2n)$ is decreasing in $n=1,2,\ldots$. This implies from Lemma 2.6 that f_1'/f_2' is decreasing, and the function f is decreasing in $x\in(0,\pi/2)$, due to Lemma 2.7. By l'Hôpital rule, we get

$$\lim_{x\to 0}f(x)=1/2\qquad\text{and}\qquad\lim_{x\to\pi/2}f(x)=\log(\pi/2),$$

this completes the proof of the second and the third inequalities. □

3.7. Proof of Theorem 1.6. The proof of the first inequality is similar to the proof of the first inequality in Theorem 1.5 if we use (2.12). For the proof of the second and the third inequalities, we define the function $g(x)=g_1(x)/g_2(x)$, $x\in(0,\infty)$, where

$$g_1(x)=\log\Big(\frac{\sinh(x)}{x}\Big)\qquad\text{and}\qquad g_2(x)=\frac{x}{\tanh(x)}-1,$$

and $g_1(0)=g_2(0)=0$. Differentiating g with respect x gives $g'(x)=A(x)/B(x)$, where

$$A(x)=1-x\coth(x)\quad\text{and}\quad B(x)=\Big(\frac{x}{\sinh x}\Big)^2-x\coth x.$$

Now, the monotonicity of g follows from (2.2), (2.5), Lemma 2.6 and 2.7. Applying the l'Hôpital's rule, we get

$$\lim_{x\to 0}g(x)=1/2\qquad\text{and}\qquad\lim_{x\to\infty}=1,$$

this completes the proof. □

The second inequality in Theorem 1.5 can be written as

$$\log\Big(\frac{x}{\sin x}\Big) < \frac{\sin x - x\cos x}{2\sin x}\,,$$

and this appears in [19]. A similar observation applies for the second inequality of Theorem 1.6, which appears in [19] as

$$\log\Big(\frac{\sinh x}{x}\Big) < \frac{x\cosh x - \sinh x}{2\sinh x}\,.$$

Corollary 3.8. *The following relations hold true:*

(1) $$x\tan\Big(\frac{x}{2}\Big) < \log\Big(\frac{1}{\cos x}\Big) < \frac{x}{2}\tan x, \qquad x\in(0,\pi/2),$$

(2) $$\frac{x}{2}\tanh x < \log(\cosh x) < \frac{x^2}{2}\,, \qquad x\in(0,\infty),$$

(3) $$\frac{x}{2}\tanh x < \log(\cosh x) < \frac{\sinh x\tanh x}{2}\,, \qquad x\in(0,\infty).$$

Proof. Applying the Hadamard inequalities (see in [17]) to the convex function $\tan t$, we get:

$$x\tan\Big(\frac{x+0}{2}\Big) = x\tan(x/2) < \int_0^x \tan t\,dt < \frac{x(\tan 0 + \tan x)}{2} = (x\tan x)/2,$$

so (1) follows. For Part (2), it is sufficient to prove that the functions

$$m(x) = \log(\cosh x) - \frac{x}{2}\tanh x \quad\text{and}\quad n(x) = \frac{x^2}{2} - \log(\cosh x)$$

are strictly increasing for $x > 0$. This follows from

$$m'(x) = \frac{\sinh x\cosh x - x}{(\cosh x)^2} > 0$$

as $\sinh x > x$ and $\cosh x > 1$ for $x > 0$. On the other hand, one has

$$n'(x) = \frac{x\cosh x - \sinh x}{\cosh x} > 0$$

by $\tanh x < x$. It is interesting to note that the right-hand side of (2) cannot be compared to the right-hand side of (3). □

Lemma 3.9. *The following inequalities hold:*

(1) $$\log\Big(\frac{1}{\cos x}\Big) < \frac{1}{2}\sin x\tan x\,, \qquad x\in(0,\pi/2),$$

(2) $$\log\Big(\frac{x}{\sin x}\Big) > \frac{\sin x - x\cos x}{2x}\,, \qquad x\in(0,\pi/2),$$

(3) $$\log\Big(\frac{\sinh x}{x}\Big) < \frac{x\cosh x - \sinh x}{2x}\,, \qquad x > 0,$$

(4) $$\log\Big(\frac{\sinh x}{x}\Big) > \frac{x\cosh x - \sinh x}{2\sinh x}\,, \qquad x > 0.$$

Proof. Letting

$$t(x)=\frac{\sin x\tan x}{2}-\log\Big(\frac{1}{\cos x}\Big),$$

we get

$$t'(x)=\frac{\sin x(\cos x-1)^2}{(\cos x)^2}>0.$$

Hence it follows that t is strictly increasing, this implies (1). As the logarithmic mean of t and 1 is $L(t,1)=(t-1)/\log t$, from the inequality

$$\log t>1-1/t \quad \text{for} \quad t>1,$$

we get $L(t,1)<t$. Putting $t=x/\sin x$, we obtain

$$\log\frac{x}{\sin x}>1-\frac{\sin x}{x}.$$

Thus, it is sufficient to prove that

$$1-\frac{\sin x}{x}>\frac{\sin x}{2x}-\frac{\cos x}{2},$$

which is equivalent to write

$$\frac{\sin x}{x}<\frac{\cos x+2}{3}.$$

This is a known inequality, namely the so-called Cusa–Huygens inequality (1.1), thus (2) follows.

For the proof of (3) we use similar method as in the circular case. By inequality $\log t<t-1$ for $t>1$ applied to $t=(\sinh x)/x$, to deduce (3) it is sufficient to prove that

$$\frac{\sinh x}{x}-1<\frac{\cosh x}{2}-\frac{\sinh x}{2x}.$$

But this is equivalent to the following:

$$\frac{\sinh x}{x}<\frac{\cosh x+2}{3},$$

which is known as the hyperbolic Cusa–Huygens inequality (1.2). For the proof of (4), see [21]. □

The last inequality in Theorem 1.6 can be written as

$$\log\Big(\frac{\sinh x}{x}\Big)<\frac{x\cosh x-\sinh x}{\sinh x}. \tag{3.10}$$

We observe that inequality (3) in Lemma 3.9 cannot be compared with (3.10).

Theorem 3.11. *The following functions*

$$F_1(x) = \frac{\log(x/\sin x)}{\log(\cosh x)}\,, \quad F_2(x) = \frac{\log(\cosh x)}{\log(\cos x)}\,, \quad F_3(x) = \frac{\log(x/\sin x)}{\log(\cos x)}$$

are strictly increasing in $x \in (0, \pi/2)$.

The proof in the case of the function F_1 can be found in [16]. The proof in the case of F_2 is a consequence of Corollary 3.8, by considering the derivative

$$F_2'(x) = \tanh x \log(\cos x) + \cos x \log(\cosh x) > 0.$$

The proof in the case for the F_3 can be found in [12], here we give a simpler proof. Let $F_3(x) = u(x)/v(x)$, the proof follows if $(u(x)/v(x))'$ is negative or one has to show that $u'(x)v(x) < u(x)v'(x)$. This is equivalent to the following

$$\log\Big(\frac{1}{\cos x}\Big)\Big(\frac{\sin x - x\cos x}{x \sin x}\Big) < \log\Big(\frac{x}{\sin x}\Big)\tan x. \tag{3.12}$$

The inequalitiy (3.12) follows immediately from Lemma (3.9) (1) and (2). This completes the proof. □

Corollary 3.13. *For* $x \in (0, \pi/2)$ *we have*

$$\frac{3\sin x}{2+\cos x} < \frac{8\sin(x/2) - \sin x}{3} < x < \frac{\sin x}{((\cos x + 1)/2)^{2/3}}\,.$$

Proof. By applying the Mitrinovic–Adamovic inequality

$$\frac{\sin t}{t} > (\cos t)^{1/3} \quad \text{for} \quad t = x/2,$$

we get

$$\Big(\frac{2\sin(x/2)\cos(x/2)}{x}\Big)^3 > (\cos x/2)^4,$$

by multiplying both sides with $\cos(x/2)$. Since

$$\cos(x/2) = (1+\cos x)^{1/2},$$

we get the inequality

$$\frac{\sin x}{x} > \Big(\frac{\cos x + 1}{2}\Big)^{2/3}.$$

By putting $x/2$ in the Cusa–Huygens inequality $(\sin x)/x < (\cos x + 2)/3$, we get

$$\frac{\sin(x/2)}{x/2} < \frac{\cos(x/2) + 2}{3}\,. \tag{3.14}$$

Clearly,

$$(\cos x - 1)^2 > 0, \quad t = \cos(x/2),$$

this implies

$$\frac{\cos t + 2}{3} < \frac{3}{4 - \cos t}. \tag{3.15}$$

By (3.14) and (3.15), we have

$$\sin(x/2) < \frac{3x}{2(4 - \cos(x/2))},$$

this is stronger than the Cusa–Huygens inequality $(\sin x)/x < (2 + \cos x)/3$. Indeed, we will prove that

$$\frac{3 \sin x}{2 + \cos x} < \frac{1}{3}\Big(8 \sin \frac{x}{2} - \sin x\Big). \tag{3.16}$$

By letting $t = \cos(x/2)$ in (3.16), after some simple transformations, we have to prove that

$$2t^3 - 8t^2 + 10t - 4 < 0,$$

this is equivalent to

$$(t-1)^2(t-2) < 0,$$

which is obvious. □

Remark 3.17. The first two inequalities in Corollary 3.13 offer an improvement of the Cusa–Huygens inequality, while the last inequality is in fact the first relation of (3.2), with an elementary proof. The authors have found in a book on history of number π, that the second inequality of Corollary 3.13 was discovered by geometric and heuristic arguments (see [5]) by C. Huygens.

References

[1] M. Abramowitz and I. Stegun, *Handbook of mathematical functions with formulas, graphs and mathematical tables*, National Bureau of Standards (Dover–New York, 1965).

[2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, Monotonicity Rules in Calculus, *Amer. Math. Monthly*, **113**(2006), 805–816.

[3] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Monotonicity of some functions in calculus*, www.math.auckland.ac.nz/Research/Reports/Series/538.pdf.

[4] M. Biernacki and J. Krzyż, On the monotonicity of certain functionals in the theory of analytic functions, *Ann. Univ. Mariae. Curie-Sklodowska*, **2**(1955) 134–145.

[5] F. T. Campan, *The history of number pi*, Albatros (Romania, 1977) (in Romanian).

[6] V. Heikkala, M. K. Vamanamurthy, and M. Vuorinen, Generalized elliptic integrals, http://arxiv.org/pdf/math/0701436v2.pdf.

[7] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer (New York–Berlin–Heidelberg, 1990).
[8] A. Jeffrey, *Handbook of Mathematical Formulas and Integrals*, Elsevier Academic Press (2004).
[9] R. Klén, M. Visuri, and M. Vuorinen, On Jordan type inequalities for hyperbolic functions, *J. Ineqal. Appl.*, vol. 2010, pp. 14.
[10] D.S. Mitrinović, *Analytic Inequalities*, Springer (Berlin, 1970).
[11] E. Neuman, Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions, *Adv. Inequal. Appl.*, **1**(2012), 1–11.
[12] E. Neuman, On some means derived from the Schwab-Borchardt mean II, *J. Math. Inequal.*, **8**(2014), 359–368.
[13] E. Neuman, Inequalities involving hyperbolic functions and trigonometric functions, *Bull. Int. Math. Vir. Int.*, **2**(2012), 87–92.
[14] E. Neuman, Inequalities involving jacobian elliptic trigonometric and hyperbolic functions, *J. Ineq. Spec. Funct.*, **3**(2012), 16–21.
[15] E. Neuman and J. Sándor, On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa–Huygens, Wilker, and Huygens inequalities, *Math. Inequal. Appl.*, **13**(2010), 715–723.
[16] E. Neuman and J. Sándor, Optimal inequalities for hyperbolic and trigonometric functions, *Bull. Math. Anal. Appl.*, **3**(2011), 177–181.
[17] J. Sándor, On new refinements of Kobers and Jordans trigonometric inequalities, *Notes Number Theory Discrete Math.*, **19**(2013), 73–83.
[18] J. Sándor, On certain inequalities for means. III, *Arch. Math.* (*Basel*), **76**(2001), 34–40.
[19] J. Sándor, Two sharp inequalities for trigonometric and hyperbolic functions, *Math. Inequal. Appl.*, **15**(2012), 409–413.
[20] J. Sándor, Sharp Cusa–Huygens' and related inequalities, *Notes Number Theory Discrete Math.*, **19**(2013), 50–54.
[21] J. Sándor and R. Oláh-Gál, On Cusa–Huygens type trigonometric and hyperbolic inequalities, *Acta Univ. Sapientiae Math.*, **4**(2012), 145–153.
[22] J. Sándor and M. Bencze, On Huygens trigonometric inequality, *RGMIA Res. Rep. Collection*, **8**(2005), No. 3, Article 14.

О некоторых старых и новых тригонометрических и гиперболических неравенствах

БАРКАТ АЛИ БАЙО и ЙОЖЕФ ШАНДОР

В этой статье мы изучаем двусторонние неравенства для тригонометрических и гиперболических функций.