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On the divergence of multiple Fourier–Haar series

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Abstract. It is proved that for any dimension $n \geq 2$, $L(\ln^+ L)^{n-1}$ is the widest integral class in which the almost everywhere convergence of spherical partial sums of multiple Fourier–Haar series is provided. Moreover,it is shown that the divergence effects of rectangular and spherical general terms of multiple Fourier–Haar series can be achieved simultaneously on a set of full measure by an appropriate rearrangement of values of arbitrary summable function f not belonging to $L(\ln^+ L)^{n-1}$.

1. Introduction

Let $n \geq 2$ and $\mathbb{I}^n = [0,1]^n$. For $f \in L(\mathbb{I}^n)$, $x \in \mathbb{I}^n$, $m \in \mathbb{N}^n$ and $r > 0$ by $S_m(f)(x)$, $S_r(f)(x)$, $G_m(f)(x)$ and $G_r(f)(x)$ denote a rectangular partial sum, a spherical partial sum, a rectangular general term and a spherical general term of the Fourier series of f at the point x with respect to the multiple Haar system $\{h_m\}_{m\in\mathbb{N}^n}$, respectively, i.e.,

$$S_m(f)(x) = \sum_{k_1=1}^{m_1} \cdots \sum_{k_n=1}^{m_n} c_k(f) h_k(x),$$

$$S_r(f)(x) = \sum_{k_1^2+\cdots+k_n^2 \leq r^2} c_k(f) h_k(x),$$

$$G_m(f)(x) = c_m(f) h_m(x),$$

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$$G_r(f)(x) = \sum_{k_1^2+\cdots+k_n^2=r^2} c_k(f)h_k(x),$$

where $m = (m_1, \ldots, m_n)$, $k = (k_1, \ldots, k_n)$, $c_k(f)$ is the k-th Fourier coefficient of f with respect to the multiple Haar system and a sum with respect to empty set of indexes is assumed to be equal to 0.

It is known that the convergence of rectangular partial sums of multiple Fourier–Haar series at every dyadic-irrational point $x \in \mathbb{I}^n$ is equivalent to the differentiability of $\int f$ with respect to the differentiation basis of n-dimensional dyadic intervals. Taking into account the well-known theorem of Jessen, Marcinkiewicz and Zygmund [6] on the strong differentiation of integrals, it follows that

$$\text{if } f \in L(\ln^+ L)^{n-1}(\mathbb{I}^n), \text{ then } \lim_{m\to\infty} S_m(f)(x) = f(x) \text{ a.e. on } \mathbb{I}^n.$$

On the other hand, Zerekidze [16] proved that if a function $f \in L(\mathbb{I}^n)$ is nonnegative, then for almost every point $x \in \mathbb{I}^n$ the convergence of $S_m(f)(x)$ is equivalent to the strong differentiability of $\int f$ at the point x. On the basis of this theorem and Saks' [13] negative result on the strong differentiability, it was shown in [16] that for every increasing function $\varphi : [0, \infty) \to [0, \infty)$ with

$$\lim_{t\to\infty} \frac{\varphi(t)}{t(\ln t)^{n-1}} = 0$$

there exists a non-negative function $f \in \varphi(L)(\mathbb{I}^n)$ such that

$$\limsup_{m\to\infty} S_m(f)(x) = \infty \text{ a.e. on } \mathbb{I}^n.$$

Moreover, taking into account Stokolos' [14] result instead of Saks', the following theorem holds true: For every non-negative $f \in L\backslash L(\ln^+ L)^{n-1}(\mathbb{I}^n)$ there exists a function g equimeasurable with f on $\mathbb{I}^n$ such that

$$\limsup_{m\to\infty} S_m(g)(x) = \infty \text{ a.e. on } \mathbb{I}^n.$$

As to the convergence of spherical partial sums of multiple Fourier–Haar series, the following theorems are known:

If $f \in L(\ln^+ L)^{n-1}(\mathbb{I}^n)$, *then* $\lim_{r\to\infty} S_r(f)(x) = f(x)$ *a.e. on* $\mathbb{I}^n$ (Kemkhadze [8]);

For every increasing function $\varphi : [0, \infty) \to [0, \infty)$ *with*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t \ln t} = 0$$

there exists a function $f \in \varphi(L)(\mathbb{I}^2)$ *such that*

$$\limsup_{r\to\infty} S_r(f)(x) = \infty \quad \text{a.e.} \quad \text{on } I^2$$

(Tkebuchava [15]);

For every nonnegative $f \in L \backslash L \ln^{+} L(\mathbb{I}^2)$ *there exists a function* g *equimeasurable with* f *on* $\mathbb{I}^2$ *such that*

$$\limsup_{m\to\infty} S_r(g)(x) = \infty \quad a.e. \quad on \ \ \mathbb{I}^2$$

(Oniani [10]).

Here we note that the first example of a function with double Fourier–Haar series spherically divergent on a set of positive measure was constructed by Kemkhadze [9].

Thus, fo any dimension $n \geq 2$, $L(\ln^{+} L)^{n-1}(\mathbb{I}^n)$ is the widest integral class in which the almost everywhere convergence of rectangular partial sums of multiple Fourier–Haar series is provided, while for spherical partial sums the same is known only in the two-dimensional case.

Getsadze [2] proved that in integral classes wider than $L \ln^{+} L(\mathbb{I}^2)$ divergence phenomena on sets of positive measure occur even for rectangular and spherical general terms of a double Fourier–Haar series. Namely, in [8] it is shown that: for every continuous, increasing, convex function $\varphi : [0,\infty) \to [0,\infty)$ with

$$\lim_{t\to\infty} \frac{\varphi(t)}{t \ln t} = 0$$

and for every set $E \subset \mathbb{I}^2$ with $|E| > 0$, there exist functions f and g from $\varphi(L)(\mathbb{I}^2)$ and a set $A \subset E$ with $|A| > 0$ such that

$$\limsup_{m\to\infty} |G_m(f)(x)| = \infty \quad \text{ and } \quad \limsup_{r\to\infty} |G_r(g)(x)| = \infty$$

for every $x \in A$. Getsadze in [3] also constructed functions with divergent spherical general terms with respect to double Walsh–Paley and trigonometric systems.

The following theorem shows that $L(\ln^{+} L)^{n-1}(\mathbb{I}^n)$ is the widest integral class for an dimension $n \geq 2$, in which the almost everywhere convergence of spherical partial sums of multiple Fourier–Haar series is provided; moreover, the divergence effects of rectangular and spherical general terms of multiple Fourier–Haar series can be achieved simultaneously on a set of full measure by an appropriate rearrangement of values of arbitrary summable function f not belonging to $L(\ln^{+} L)^{n-1}(\mathbb{I}^n)$.

Theorem 1. *For every* $n \geq 2$ *and* $f \in L \backslash L(\ln^{+} L)^{n-1}(\mathbb{I}^n)$ *there exists a function* g *equimeasurable with* f *on* $\mathbb{I}^n$ *such that*

$$\limsup_{m\to\infty} |G_m(g)(x)| = \infty \quad and \quad \limsup_{r\to\infty} |G_r(g)(x)| = \infty \quad a.e. \ on \ \ \mathbb{I}^n.$$

Theorem 1 was announced in [11].

This paper is organized as follows. In Section 2, we present some properties of the Haar system. In Section 3, we estimate the number of points and of nonisotropic points of a spectrum (i.e., the set of indices of Haar functions with nonzero values in a given point) lying on the sphere. Section 4 contains the reduction of the question to the divergence of the "general term" $G_I(f)$ of differentiation process of the integral with respect to the basis of dyadic intervals. Finally, in Section 5 we give the construction of appropriate functions with divergent general term $G_I(f)$ that we perform using modification of scheme known for the strong integral means that goes back to Saks [13].

2. Some properties of the Haar system

We shall use the following notation:

$\mathbb{Z}_0$ is the set of all nonnegative integers;

$\mathbb{S}_n(r)$ is the sphere with center at the origin and radius r, i.e.,

$$\mathbb{S}_n(r) = \{x \in \mathbb{R}^n : \|x\| = r\};$$

$\mathbb{R}^n_{\mathrm{d}}$ is the set of all dyadic-irrational points of the space $\mathbb{R}^n$;

$\mathbb{I}^n_{\mathrm{d}}$ is the set of all dyadic-irrational points of the unit cube $\mathbb{I}^n$;

$H(x)$ $(x \in \mathbb{I}^n)$ is the spectrum of the Haar system at x, i.e., the set

$$\{m \in \mathbb{N}^n : h_m(x) \neq 0\};$$

$\mathbf{1} = \mathbf{1}_n$ is the n-tuple $(1, \dots, 1)$;

2^k $(k \in \mathbb{Z}^n)$ is the n-tuple $(2^{k_1}, \dots, 2^{k_n})$;

Δ^i_k $(k \in \mathbb{Z},\ i \in \mathbb{Z})$ is the dyadic interval $[\frac{i-1}{2^k}, \frac{i}{2^k}]$;

$\widetilde{\Delta}^i_k (k \in \mathbb{Z},\ i \in \mathbb{Z})$ is the open dyadic interval $(\frac{i-1}{2^k}, \frac{i}{2^k})$;

Δ^i_k $(k \in \mathbb{Z}^n,\ i \in \mathbb{Z}^n)$ is n-the dimensional dyadic interval $\Delta^{i_1}_{k_1} \times \dots \times \Delta^{i_n}_{k_n}$;

$\mathbf{I} = \mathbf{I}_n$ is the family of all n-dimensional intervals;

$\mathbf{I}_{\mathrm{d}} = \mathbf{I}_{n,\mathrm{d}}$ is the family of all n-dimensional dyadic intervals;

$\mathbf{I}(x)(\mathbf{I}_{\mathrm{d}}(x))$ is the family of all $I \in \mathbf{I}(I \in \mathbf{I}_{\mathrm{d}})$ that contain a point x;

$k(I)$ is the rank of a dyadic interval I, i.e., $k \in \mathbb{Z}^n$ for which $I \in \{\Delta^i_k : i \in \mathbb{Z}^n\}$;

$\Delta_k(x)$ $(x \in \mathbb{I}^n_{\mathrm{d}}, k \in \mathbb{Z}^n)$ is the dyadic interval of rank k that contains x;

$M_I(f)$ $(f \in L(\mathbb{R}^n), I \in \mathbf{I})$ is the integral mean: $\frac{1}{|I|}\int_I f$.

Let us recall the definition of the Haar system $\{h_m\}_{m\in\mathbb{N}}$:

$$h_1(x) = 1 \quad (x \in \mathbb{I}^1),$$

and if

$$m = 2^k + i \quad (k \in \mathbb{Z}_0,\ i \in \overline{1, 2^k}),$$

then

$$h_m(x) = \begin{cases} 2^{k/2} & \text{if } x \in \widetilde{\Delta}_{k+1}^{2i-1}, \\ -2^{k/2} & \text{if } x \in \widetilde{\Delta}_{k+1}^{2i}, \\ 0 & \text{if } x \notin \Delta_k^i. \end{cases}$$

At inner points of discontinuity h_m is defined as the mean value of the limits from the right and from the left, and at the endpoints of $\mathbb{I}^1$ as the limits from inside of the interval.

The multiple Haar system $\{h_m\}_{m\in\mathbb{N}^n}$ is defined as follows:

$$h_m(x) = h_{m_1}(x_1) \times \cdots \times h_{m_n}(x_n) \quad (m \in \mathbb{N}^n,\ x \in \mathbb{I}^n).$$

The property below directly follows from the definition of the Haar system.

Proposition 1. *The following assertions hold*:

$$\mathrm{card}(H(x) \cap [2^k + 1, 2^{k+1}]) = 1 \quad (x \in \mathbb{I}_{\mathrm{d}}^1,\ k \in \mathbb{Z}_0)$$

and

$$H(x) = H(x_1) \times \cdots \times H(x_n) \quad (x \in \mathbb{I}^n).$$

For $x, y \in \mathbb{R}^n$, we write

$$x \le y\ (x < y) \quad \text{if} \quad x_i \le y_i\ (x_i < y_i) \quad \text{for every} \quad i \in \overline{1, n}.$$

For $x, y \in \mathbb{R}^n$ with $x \le y$, denote

$$[x, y] = [x_1, y_1] \times \cdots \times [x_n, y_n].$$

For $m \in \mathbb{N}^n$, we denote by $k(m)$ the n-tuple $k \in \mathbb{Z}_0^n$ for which $2^{k-\mathbf{1}} < m \le 2^k$.

The next property (see, e.g., [7, Ch.3, §1] or [1, Ch.1, §6]) connects rectangular convergence of Fourier–Haar series with the differentiation of integrals with respect to the basis of dyadic intervals.

Proposition 2. *If* $f \in L(\mathbb{I}^n)$, $x \in \mathbb{I}_{\mathrm{d}}^n$ *and* $k \in \mathbb{Z}_0^n$, *then*

$$S_{2^k}(f)(x) = M_{\Delta_k(x)}(f).$$

The next Proposition 3 follows from Propositions 1 and 2.

Proposition 3. *Let* $f \in L(\mathbb{I}^n)$, $x \in \mathbb{I}^n_{\rm d}$ *and* $m \in \mathbb{N}^n$. *Then the following assertions hold*:

a) *If* $H(x) \cap [2^{k(m)-\mathbf{1}} + \mathbf{1}, m] \neq \emptyset$, *then*

$$S_m(f)(x) = S_{2^{k(m)}}(f)(x) = M_{\Delta_{k(m)}(x)}(f).$$

b) *If* $H(x) \cap [2^{k(m)-\mathbf{1}} + \mathbf{1}, m] = \emptyset$, *then*

$$S_m(f)(x) = S_{2^{k(m)-\mathbf{1}}}(f)(x) = M_{\Delta_{k(m)-\mathbf{1}}(x)}(f).$$

Set

$$\Gamma_n = \{\gamma = (\gamma_1, \dots, \gamma_n) : \gamma_i \in \{0, 1\}, \quad i \in \overline{1, n}\}$$

and

$$|\gamma| = \gamma_1 + \dots + \gamma_n \quad \text{for } \gamma \in \Gamma_n.$$

The next lemma immediately follows from Proposition 3.

Lemma 1. *If* $f \in L(\mathbb{I}^n)$, $x \in \mathbb{I}^n_{\rm d}$, $m \in H(x)$ *and* $k(m) \geq \mathbf{1}$, *then*

$$G_m(f)(x) = \sum_{\gamma \in \Gamma_n} (-1)^{|\gamma|} S_{2^{k(m)-\gamma}}(f)(x).$$

For $I \in \mathbf{I}_{\rm d}$ and $\gamma \in \Gamma_n$, by $I(\gamma)$ denote the dyadic interval of rank $k(I) - \gamma$ that contains I.

For $f \in L(\mathbb{R}^n)$ and $I \in \mathbf{I}_{\rm d}$, denote

$$G_I(f) = \sum_{\gamma \in \Gamma_n} (-1)^{|\gamma|} M_{I(\gamma)}(f).$$

$G_I(f)$ is called a *general term* of the differentiation process of the integral of f with respect to the basis of dyadic intervals $\mathbf{I}_{\rm d}$.

Let $x \in \mathbb{I}^n_{\rm d}$, $I \in \mathbf{I}_{\rm d}(x)$ and $k(I) \geq \mathbf{1}$. By Proposition 1, there is a unique $m \in H(x)$ with $2^{k(I)-\mathbf{1}} < m \leq 2^{k(I)}$. Such m will be denoted by $m(I, x)$.

The next proposition is a direct consequence of Propositions 1, 3 and Lemma 1.

Lemma 2. *Let* $f \in L(\mathbb{I}^n)$ *and* $x \in \mathbb{I}^n_{\rm d}$. *Then the following assertions hold*:

a) *If* $m \in H(x)$ *and* $k(m) \geq \mathbf{1}$, *then* $G_m(f)(x) = G_{\Delta_{k(m)}(x)}(f)$;

b) *If* $I \in \mathbf{I}_{\rm d}(x)$ *and* $k(I) \geq \mathbf{1}$, *then* $G_I(f) = G_{m(I,x)}(f)(x)$.

3. Estimations for the spectrum

In the inequalities given in what follows, by $c(n), c_1(n), \dots$ we denote positive constants depending only on the dimension n.

Lemma 3. *For every* $x \in \mathbb{I}_{\mathrm{d}}^n$ *and* $r > 0$, *we have*

$$\operatorname{card}(H(x) \cap \mathbb{S}_n(r)) \le c(n).$$

The *parameter of nonisotropy* of $m \in \mathbb{N}^n$ is defined by

$$\pi(m) = \min\{m_{p(i+1)}/m_{p(i)} : \ i \in \overline{1, n-1}\},$$

where p is a permutation of $\overline{1,n}$ such that $m_{p(1)} \le \cdots \le m_{p(n)}$.

We also introduce the following notations:

$$\Pi(\alpha) = \Pi_n(\alpha) = \{m \in \mathbb{N}^n : \ \pi(m) \ge \alpha\} \qquad (\alpha \ge 1),$$

$$\Pi(\alpha, k) = \{m \in \Pi(\alpha) : \ m \ge k\mathbf{1}\} \qquad (\alpha \ge 1, k \in \mathbb{N}),$$

$$\widetilde{\mathbb{I}}_{\mathrm{d}}^n = \{x \in \mathbb{I}_{\mathrm{d}}^n : \ x_i \ne x_j \text{ if } i \ne j\}.$$

Lemma 4. *For every* $x \in \widetilde{\mathbb{I}}_{\mathrm{d}}^n$, *there exist* $k_x \in \mathbb{N}$ *and* $\alpha_x \ge 1$ *such that if*

$$H(x) \cap \mathbb{S}_n(r) \cap \Pi(\alpha_x, k_x) \ne \emptyset \qquad \textit{for some} \ \ r > 0$$

then

$$\operatorname{card}(H(x) \cap \mathbb{S}_n(r) \cap \Pi(\alpha_x)) = 1.$$

For the proofs of Lemmas 3 and 4 we shall need the following two "lacunarity" properties of spectrum $H(x)$ that was mentioned by Kemkhadze in [8] and [9], respectively.

Lemma A. *If* $x \in \mathbb{I}_{\mathrm{d}}^1$ *and* $H(x) = \{m_1, m_2, \ldots\}$, *where* $m_1 < m_2 < \cdots$, *then*

$$m_{k+1}/m_k = 2 \qquad \textit{or} \qquad m_{k+1}/m_k = 2 - 1/m_k.$$

Lemma B. *If* $x_1, x_2 \in \mathbb{I}_{\mathrm{d}}^1$ *and* $x_1 \ne x_2$, *then there are* $k(x_1, x_2) \in \mathbb{N}$ *and* $\lambda(x_1, x_2) \in (1, 2)$ *such that*

$$\max\{m_1/m_2, m_2/m_1\} \ge \lambda(x_1, x_2)$$

provided

$$m_1 \in H(x_1), \quad m_2 \in H(x_2) \quad \textit{and} \quad \max\{m_1, m_2\} \ge k(x_1, x_2).$$

Lemmas A and B easily follow from the definition of the Haar system.

Proof of Lemma 3. Taking into account Lemma A, it is easy to see the validity of the estimation for $n = 2$ with $c(2) = 2$. Let us assume that the assertion is true for $n - 1$.

Denote

$$E_k = \{m \in \mathbb{N}^n : m_k = \max_{i \in \overline{1,n}} m_i\} \qquad (k \in \overline{1,n}).$$

Let $k \in \overline{1,n}$,

$$P(t) = \{x \in \mathbb{R}^n : \ x_k = t\} \quad (t \in \mathbb{R})$$

and

$$\Omega = \{i \in \mathbb{N} : \ P(i) \cap E_k \cap H(x) \cap \mathbb{S}_n(r) \neq \emptyset\}.$$

Since $m \in E_k \cap H(x)$ we have that $r/\sqrt{n} \leq m_k \leq r$. Hence taking into account Lemma A gives

$$\operatorname{card}(\Omega) \leq \log_{3/2} \sqrt{n}.$$

Now, using the induction hypothesis for $r(i) = \sqrt{r^2 - i^2}$ and $x' = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \mathbb{I}_{\mathrm{d}}^{n-1}$, for each $i \in \Omega$ we write

$$\operatorname{card}(E_k \cap H(x) \cap \mathbb{S}_n(r)) = \sum_{i \in \Omega} \operatorname{card}(E_k \cap P(i) \cap H(x) \cap \mathbb{S}_n(r)) =$$

$$= \sum_{i \in \Omega} \operatorname{card}(H(x') \cap \mathbb{S}_{n-1}(r(i))) \leq c(n-1) \log_{3/2} \sqrt{n}.$$

Thus, we have

$$\operatorname{card}(H(x) \cap \mathbb{S}_n(r)) \leq \sum_{k=1}^{n} \operatorname{card}(E_k \cap H(x) \cap \mathbb{S}_n(r)) \leq nc(n-1) \log_{3/2} \sqrt{n}.$$

The lemma is proved. □

Proof of Lemma 4. Let

$$k_x^* = \max\{k(x_i, x_j) : \ i \neq j\},$$

$$\lambda_x^* = \min\{\lambda(x_i, x_j) : \ i \neq j\}, \qquad \alpha_x^* = \sqrt{\frac{n}{(\lambda_x^*)^2 - 1}}\,,$$

where $k(x_i, x_j)$ and $\lambda(x_i, x_j)$ are numbers from Lemma B corresponding to the couples of points x_i, x_j.

Suppose

$$m \in H(x) \cap \mathbb{S}_n(r) \cap \Pi(\alpha_x^*, k_x^*).$$

Let $i \in \overline{1,n}$ be an index such that $m_i = \max\{m_1, \dots, m_n\}$. Let us consider any $m' \in H(x) \cap \mathbb{S}_n(r)$ with $m'_i \neq m_i$. By choosing of α_x^* we have that $\lambda_x^* m_i > r$. Therefore, by Lemma B we find that

$$m'_j \leq m_i/\lambda_x^* \quad (j \in \overline{1,n}).$$

On the other hand, there is $j \in \overline{1,n}$ such that $m'_j \geq m_i/\sqrt{n}$. Hence it follows that $\pi(m') \leq \sqrt{n}/\lambda_x^* < \alpha_x^*$, and consequently, we have

$$m' \notin \Pi(\alpha_x^*). \tag{1}$$

Taking into account (1) we conclude the validity of the assertion of the Lemma for $n = 2$. Let us assume that the assertion is true for $n-1$, where $n \geq 3$.

Let us agree for $x \in \mathbb{R}^n$ and $i \in \overline{1,n}$ to denote

$$x(i) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Furthermore, assume that

$$k_x^{**} = \max\{k_{x(i)} : \ i \in \overline{1,n}\}, \qquad \alpha_x^{**} = \max\{\alpha_{x(i)} : \ i \in \overline{1,n}\},$$
$$k_x = \max\{k_x^*, k_x^{**}\} \quad \text{and} \quad \alpha_x = \max\{n, \alpha_x^*, \alpha_x^{**}\},$$

and suppose

$$m \in H(x) \cap \mathbb{S}_n(r) \cap \Pi(\alpha_x, k_x) \quad \text{and} \quad m' \in H(x) \cap \mathbb{S}_n(r).$$

Let $i \in \overline{1,n}$ be such that $m_i = \max\{m_1, \dots, m_n\}$. If $m_i' \neq m_i$, then by (1) and choosing of α_x we have: $m' \notin \Pi(\alpha_x)$. Now, let $m_i' = m_i$. Note that since $\pi(m) \geq n$, then we have

$$m_j' < m_i' \quad \text{for every} \quad j \neq i. \tag{2}$$

It is clear that

$$m(i) \geq k_x \mathbf{1}_{n-1} \geq k_{x(i)} \mathbf{1}_{n-1} \quad \text{and} \quad \pi(m(i)) \geq \alpha_x \geq \alpha_{x(i)}.$$

Therefore, by induction hypothesis we conclude:

$$m'(i) \notin \Pi_{n-1}(\alpha_{x(i)}).$$

Now, taking into account (2), we obtain that $m' \notin \Pi(\alpha_x)$. The lemma is proved. □

4. Reduction to the divergence of $G_I(f)$

The parameter $\pi(I)$ for an interval $I = I_1 \times \cdots \times I_n \in \mathbf{I}$ is defined by

$$\pi(I) = \min\left\{|I_{p(i+1)}|/|I_{p(i)}| : i \in \overline{1, n-1}\right\},$$

where p is a permutation of $\overline{1,n}$ such that $|I_{p(1)}| \leq \cdots \leq |I_{p(n)}|$.

For $\alpha \geq 1$, denote

$$\mathbf{I}(\alpha) = \{I \in \mathbf{I} : \pi(I) < \alpha\} \quad \text{and} \quad \mathbf{I}_{\mathrm{d}}(\alpha) = \mathbf{I}_{\mathrm{d}} \cap \mathbf{I}(\alpha).$$

A family B of sets with positive and finite measure and such that $\cup_{R \in B} R = \mathbb{R}^n$ is called a *basis*. For a point $x \in \mathbb{R}^n$, by $B(x)$ we denote the family of all sets from B containing x. The *maximal operator* M_B corresponding to a basis B is defined as follows:

$$M_B(f)(x) = \sup_{R \in B(x)} |M_R(f)| \quad (f \in L(\mathbb{R}^n), \ x \in \mathbb{R}^n),$$

where

$$M_R(f) = \frac{1}{|R|} \int_R f.$$

The *maximal operator* controlling the general terms $G_I(f)$ corresponding to a basis $B \subset \mathbf{I}_{\mathrm{d}}$ is denoted by G_B, i.e.,

$$G_B(f)(x) = \sup_{I \in B(x)} |G_I(f)| \quad (f \in L(\mathbb{R}^n),\ x \in \mathbb{R}^n_{\mathrm{d}}).$$

It is clear that

$$G_{\mathbf{I}_{\mathrm{d}}(\alpha)}(f)(x) \le 2^n M_{\mathbf{I}_{\mathrm{d}}(2\alpha)}(f)(x).$$

Lemma 5. *Let* $f \in L(\mathbb{I}^n)$ *and* $x \in \widetilde{\mathbb{I}}^n_{\mathrm{d}}$. *If*

$$\limsup_{I \in \mathbf{I}_{\mathrm{d}}(x), \operatorname{diam} I \to 0} |G_I(f)| = \infty$$

and

$$M_{\mathbf{I}_{\mathrm{d}}(\alpha)}(f)(x) < \infty \quad \textit{for every} \quad \alpha \ge 1,$$

then

$$\limsup_{m \to \infty} |G_m(f)(x)| = \infty \quad \textit{and} \quad \limsup_{r \to \infty} |G_r(f)(x)| = \infty.$$

Proof. For arbitrary $j, N \in \mathbb{N}$, let us choose $I \in \mathbf{I}_{\mathrm{d}}(x)$ with

$$\pi(I) > 2\alpha_x, \quad \operatorname{diam} I < 1/2^{k_x + j + 1}$$

and

$$|G_I(f)(x)| > N + c(n) 2^n M_{\mathbf{I}_{\mathrm{d}}(2\alpha_x)}(f)(x),$$

where $c(n)$ is the constant from Lemma 3. By Lemma 2, there exists an $m \in \mathbb{N}^n$ with

$$G_I(f) = G_m(f)(x) = c_m h_m(x) \quad \text{and} \quad 2^{k(I) - \mathbf{1}} < m \le 2^{k(I)}.$$

It is easy to see that

$$\pi(m) \ge \pi(2^{k(I)})/2 \ge \pi(I)/2 \ge \alpha_x, \quad m \ge 2^{k_x + j}\mathbf{1} \quad \text{and} \quad \|m\| > 2^{k_x + j}.$$

Now, using Lemmas 3 and 4 for $r = \|m\|$, we have

$$\operatorname{card}(H(x) \cap \mathbb{S}_n(r)) \le c(n), \quad H(x) \cap \mathbb{S}_n(r) \cap \Pi(\alpha_x) = \{m\}.$$

Let

$$H(x) \cap \mathbb{S}_n(r) = \{m, m_1, \dots, m_p\}.$$

Clearly, $p \le c(n)$. Since

$$\pi(m_i) < \alpha_x, \quad \text{for} \quad I_i = \Delta_{k(m_i)}(x) \quad (i \in \overline{1,p}),$$

we write

$$\pi(I_i) = \pi(2^{k(m_i)}) \le 2\pi(m_i) < 2\alpha_x.$$

Thus, $I_i \in \mathbf{I}_{\mathrm{d}}(2\alpha_x)(x)$ and therefore

$$|c_{m_i} h_{m_i}(x)| = |G_{m_i}(f)(x)| = |G_{I_i}(f)| \le G_{\mathbf{I}_{\mathrm{d}}(\alpha_x)}(f)(x) \le 2^n M_{\mathbf{I}_{\mathrm{d}}(2\alpha_x)}(f)(x).$$

Now, we can write that

$$|G_r(f)(x)| = \Big|c_m h_m(x) + \sum_{i=1}^{p} c_{m_i} h_{m_i}(x)\Big| \geq$$
$$\geq |G_I(f)| - c(n)2^n M_{\mathbf{I}_{\mathrm{d}}(2\alpha_x)}(f)(x) > N.$$

On the other hand, $G_m(f)(x) = G_I(f) > N$. Thus, we have

$$\limsup_{m\to\infty} |G_m(f)(x)| = \infty \quad \text{and} \quad \limsup_{r\to\infty} |G_r(f)(x)| = \infty.$$

The lemma is proved. □

5. Divergence of the general term $G_I(f)$

Theorem 1 is a consequence of Lemma 5 and the following theorem.

Theorem 2. *For every* $n \geq 2$ *and* $f \in L\backslash L(\ln^+ L)^{n-1}(\mathbb{I}^n)$ *there exists a function* g *equimeasurable with* f *on* $\mathbb{I}^n$ *such that*

$$\limsup_{I\in\mathbf{I}_{\mathrm{d}}(x),\ \mathrm{diam}\, I\to 0} |G_I(g)| = \infty \ \textit{a.e. on } \mathbb{I}^n_{\mathrm{d}}$$

and

$$M_{\mathbf{I}(\alpha)}(|g|)(x) < \infty \quad \textit{a.e. on } \mathbb{I}^n \quad \textit{for every } \alpha \geq 1.$$

Remark. In the proof of Theorem 2 functions defined on $\mathbb{I}^n$ and functions defined on $\mathbb{R}^n$ having support in $\mathbb{I}^n$ are identified.

Let $I \in \mathbf{I}_{\mathrm{d}}$ and $\alpha \geq 1$. By $\Omega(I,\alpha)$ we denote the family of all dyadic intervals J for which $I \subset J$ and $|J| < \alpha|I|$, and by $\Omega^*(I,\alpha)$ we denote the family of all intervals $I(\gamma)$, where $I \in \Omega(I,\alpha)$ and $\gamma \in \Gamma_n$.Obviously, $\Omega(I,\alpha) \subset \Omega^*(I,\alpha)$ and $\Omega^*(I,\alpha)$ is finite.

By $E(I,\alpha)$ and $E^*(I,\alpha)$ we denote dyadic *Bohr "staircase"* and its *"hull"* corresponding to I and α, respectively; exactly, let us denote

$$E(I,\alpha) = \bigcup_{J\in\Omega(I,\alpha)} J \quad \text{and} \quad E^*(I,\alpha) = \bigcup_{J\in\Omega^*(I,\alpha)} J.$$

It is easy to see that

$$E(I,\alpha) = \{M_{\mathbf{I}_{\mathrm{d}}}(\alpha\chi_I) > 1\}.$$

For dyadic interval $I = I_1 \times \cdots \times I_n$ and $\alpha > 1$, by αI we denote the dyadic interval $J_1 \times \cdots \times J_n$ such that $J_i (i \in \overline{1,n})$ is the maximal dyadic interval for which $I_i \subset J_i$ and $|J_i| < \alpha|I_i|$.

L e m m a 6. *If* $I \in \mathbf{I}_{\rm d}$ *and* $\alpha \geq 1$*, then*

$$I \subset E(I,\alpha) \subset E^*(I,\alpha) \subset 2\alpha I,$$

$$|E(I,\alpha)| \geq c_1(n)|E^*(I,\alpha)|, \qquad |E(I,\alpha)| \geq c_2(n)\alpha(\ln\alpha)^{n-1}|I|.$$

P r o o f. The first assertion of the lemma is obvious.

Let us recall the strong maximal inequality (see, e.g., [4, Ch. II, §3]) which is the quantitative version of the Jessen–Marcinkiewicz–Zygmund theorem: For every $f \in L(\mathbb{R}^n)$ and $\lambda > 0$, we have

$$|\{M_{\mathbf{I}}(|f|) > \lambda\}| \leq c(n)\int_{\mathbb{R}^n} \frac{|f|}{\lambda}\Big(1 + \ln^+ \frac{|f|}{\lambda}\Big)^{n-1}.$$

The second assertion of the lemma follows from the strong maximal inequality by taking into account the following evident inclusion:

$$E^*(I,\alpha) \subset \{M_{\mathbf{I}}(\chi_{E(I,\alpha)}) \geq 1/2^n\}.$$

Using standard technique, by induction with respect to dimension n (for details see, e.g., [17, p. 99] or [12, Ch. II, §1.3]) one can prove that

$$|\{M_{\mathbf{I}}(\alpha\chi_{_I}) > 1\}| \geq c(n)\alpha(\ln\alpha)^{n-1}|I|.$$

On the other hand, in [16, Lemma 2] it is proved that for every $f \in L(\mathbb{R}^n)$ and $\lambda > 0$, we have

$$|\{M_{\mathbf{I}_{\rm d}}(f) > \lambda/3^n\}| \geq |\{M_{\mathbf{I}}(f) > \lambda\}|/5^n.$$

Now, taking into account that

$$E(I,\alpha) = \{M_{\mathbf{I}_{\rm d}}(\alpha\chi_{_I}) > 1\},$$

we conclude the validity of the third assertion of the lemma. The lemma is proved. □

Denote by $l(I)$ the minimal length of the edges of an interval I. For $f \in L(\mathbb{R}^n)$ and positive numbers λ, η, δ, where $\eta < \delta$, by $\Lambda^{\eta,\delta}(f,\lambda)$, we denote the family of all dyadic intervals I such that

$$G_I(f) > \lambda, \quad l(I) \geq \eta \quad \text{and} \quad \operatorname{diam} I < \delta.$$

L e m m a 7. *Let* $I \in \mathbf{I}_{\rm d}$, $\lambda > 0$ *and* $\alpha > 2^n\lambda$. *If* $f \in L(\mathbb{R}^n)$ *is a function such that*

$$f \geq \alpha\chi_I \quad \textit{and} \quad f(x) = 0 \quad \textit{when } x \in E^*(I,\alpha/2^n\lambda) \setminus I,$$

then

$$\Omega(I,\alpha/2^n\lambda) \subset \Lambda^{\eta,\delta}(\alpha\chi_I,\lambda) \subset \Lambda^{\eta,\delta}(f,\lambda),$$

where $\delta = (\alpha/2^n\lambda)\operatorname{diam} I$ *and* $\eta = l(I)$.

P r o o f. Let $J \in \Omega(I, \alpha/2^n\lambda)$. Then it is easy to see that

$$E^*(I, \alpha/2^n\lambda) \supset J(\gamma) \supset J \supset I \quad (\gamma \in \Gamma_n),$$

$$|J(\gamma)| = 2^{|\gamma|}|J| \;\; (\gamma \in \Gamma_n), \qquad \operatorname{diam} J < \delta, \quad l(J) \geq l(I) = \eta.$$

Taking into account the conditions of the lemma, hence it follows that

$$G_J(f) \geq G_J(\alpha\chi_I) = \sum_{\gamma \in \Gamma_n} (-1)^{|\gamma|} \frac{M_I(f)}{2^{|\gamma|}|J|} = \frac{1}{2^n}\frac{M_I(f)}{|J|} > \lambda.$$

The lemma is proved. □

R e m a r k. Lemmas 6 and 7 provide a "partial resonance" of order $t(\ln t)^{n-1}$ for the general term $G_I(f)$ similar to the one known for the strong integral means. On the other hand, $G_I(f)$ are not nonnegative even for nonnegative functions (as distinct from integral means) and therefore, it is needed certain mechanism avoiding cancelation of them, while summarizing partial resonances. The lemmas below provide also such mechanism.

It is easy to check the validity of the following

L e m m a 8. *Let $I \in \mathbf{I}_\mathrm{d}$, $\alpha > 1$, H be a homothety and T be a translation. If*

$$\{TH(J) : \; J \in \Omega^*(I, \alpha)\} \subset \mathbf{I}_\mathrm{d},$$

then

$$\{TH(J) : J \in \Omega(I, \alpha)\} = \Omega(TH(I), \alpha)$$

and

$$\{TH(J) : J \in \Omega^*(I, \alpha)\} = \Omega^*(TH(I), \alpha).$$

The next statement is a dyadic version of the lemma given in [4, Ch. III, §1].

L e m m a 9. *Let G be a nonempty bounded open set, $\delta > 0$ and Ω be a nonempty finite subfamily of $\mathbf{I}_\mathrm{d}$ with $K = \bigcup_{J \in \Omega} J$ being contained in some dyadic cube. Then there are homotheties $\{H_m\}_{m \in \mathbb{N}}$ and translations $\{T_m\}_{m \in \mathbb{N}}$ such that*

$$(T_m H_m)(K) \cap (T_{m'} H_{m'})(K) = \emptyset \;\; (m \neq m'); \qquad \operatorname{diam}(T_m H_m)(K) < \delta,$$

$$\{(T_m H_m)(J) : J \in \Omega\} \subset \mathbf{I}_\mathrm{d} \;\; (m \in \mathbb{N}) \quad \text{and} \quad \Big|G \setminus \bigcup_m (T_m H_m)(K)\Big| = 0.$$

The proof of Lemma 9 is almost the same as the one given in [4], but it must be used dyadic cubes instead of arbitrary ones throughout the proof.

For a function $f \in L(\mathbb{I}^n)$, by F_f we denote the *distribution function* of f, i.e.,

$$F_f(\lambda) = |\{x \in \mathbb{I}^n : f(x) > \lambda\}| \quad (\lambda \in \mathbb{R}).$$

Let us recall that f and g are called *equimeasurable* if

$$F_f(\lambda) = F_g(\lambda) \quad \text{for every} \quad \lambda \in \mathbb{R}.$$

Lemma 10. *Let $f \in L(\mathbb{I}^n)$. Then for every sets $A, E \subset \mathbb{I}^n$ with $|A| \le |E|$, there exists a function $g \in L(\mathbb{I}^n)$ such that g is equimeasurable with $f\chi_A$,*

$$\operatorname{supp} g \subset E \quad \textit{and} \quad M_{\mathbf{I}}(|g|)(x) < \infty \quad \textit{for a.e. } x \in \mathbb{I}^n.$$

Proof. First, let us assume that $A \subset \{f > 0\}$. Let $f_* : [0,1] \to [0,\infty)$ be an increasing function equimeasurable with $f\chi_A$.

Let y be a density point of E. Set

$$E(x) = B[y, \|x - y\|] \cap E \quad \text{for} \quad x \in E,$$

where $B[y,\varepsilon]$ denote the closed ball with the center at y and radius ε. Define a function g as follows

$$g(x) = \begin{cases} f_*(1 - |E(x)|) & \text{when} \quad x \in E, \ |E(x)| > 0; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\operatorname{supp} g \subset A$, g is equimeasurable with f_* and therefore with $f\chi_A$, and g is bounded outside of $B[y,\varepsilon]$ for every $\varepsilon > 0$.

Let x be such that $x_k \ne y_k$ for every $k \in \overline{1,n}$. Let us choose $\varepsilon > 0$ and $C > 0$ so that

$$|x_k - y_k| > 2\varepsilon \ \text{ for every } k \in \overline{1,n} \quad \text{and} \quad |g(x)| \le C \ \text{ if } x \notin B[y,\varepsilon].$$

Then for $I \in \mathbf{I}(x)$, we have

$$M_I(|g|) \le C \ \text{ if } I \cap B[y,\varepsilon] = \emptyset \quad \text{and} \quad M_I(|g|) \le \|g\|_1/\varepsilon^n \ \text{ if } I \cap B[y,\varepsilon] \ne \emptyset.$$

Consequently, we have $M_{\mathbf{I}}(|g|)(x) < \infty$.

Now, let us consider the general case. Let the sets A_1, A_2, E_1, E_2 be such that

$$A_1 = A \cap \{f > 0\}, \quad A_2 = A \cap \{f < 0\},$$

$$E_1, E_2 \subset E, \quad |A_1| = |E_1| \quad \text{and} \quad |A_2| = |E_2|.$$

Suppose g_1 and g_2 are the functions corresponding to f, A_1, E_1 and to $(-f)$, A_2, E_2,respectively, according to the considered case. Then as it is easy to see $g_1 - g_2$ is the needed function. The lemma is proved. □

For $0 < \eta < \delta$, let us introduce the following truncated maximal operators:

$$G^{(\eta,\delta)}_{\mathbf{I}_{\mathrm{d}}}(f)(x) = \sup\{G_I(f) :\ I \in \mathbf{I}_{\mathrm{d}}(x),\ \operatorname{diam} I < \delta,\ l(I) \geq \eta\},$$

$$M^{(\eta)}_{\mathbf{I}_{\mathrm{d}}}(f)(x) = \sup\{M_I(f) :\ I \in \mathbf{I}_{\mathrm{d}}(x),\ l(I) \geq \eta\}.$$

Lemma 11. *For every function $f \in L \setminus L(\ln^+ L)^{n-1}(\mathbb{I}^n)$, taking natural values and every $m \in \mathbb{N}$, there exist a measurable set $A \subset \mathbb{I}^n$, a number $\eta > 0$ and a measurable function $g : \mathbb{R}^n \to [0,\infty)$ such that*

1) *$A \subset \{f \geq 2^m\}$ and $f\chi_A$ is a bounded function*;

2) *g is equimeasurable with $f\chi_A$ and* $\operatorname{supp} g \subset \mathbb{I}^n$;

3) *there exists a set $E \subset \mathbb{I}^n$ with the following properties: for every dyadic cube $Q \subset \mathbb{I}^n$ with the length of edges equal to $1/2^m$,*

$$|E \cap Q| \geq c(n)|Q|,$$

and for every function $v \in L(\mathbb{I}^n)$ with $\operatorname{supp} v = \operatorname{supp} g$ *and $v \geq g$,*

$$\{G^{(\eta,1/2^m)}_{\mathbf{I}_{\mathrm{d}}}(v) > m\} \supset E;$$

4) $M^{(1/2^m)}_{\mathbf{I}_{\mathrm{d}}}(g)(x) \leq 1/2^m$ *for every $x \in \mathbb{I}^n$.*

Proof. Since $f \in L \setminus L(\ln^+ L)^{n-1}(\mathbb{I}^n)$, it is easy to find $p \in \mathbb{N}$, sets $\{A_i\}_{i=1}^p$ and natural numbers $\{\alpha_i\}_{i=1}^p$ such that

$$|A_i| > 0 \quad (i \in \overline{1,p}), \qquad A_i \cap A_j = \emptyset \quad (i \neq j),$$

$$\alpha_i \geq 2^m \quad \text{and} \quad \frac{c_2(n)}{2^n m} \ln^{n-1} \frac{\alpha_i}{2^n m} > 2^m \quad (i \in \overline{1,p}),$$

$$f(x) = \alpha_i \quad (x \in A_i,\ i \in \overline{1,p}),$$

$$\sum_{i=1}^{p} c_2(n) \frac{\alpha_i}{2^n m} \ln^{n-1} \frac{\alpha_i}{2^n m} |A_i| \geq 1.$$

Let us choose dyadic cubes I_i $(i \in \overline{1,p})$ so that $|I_i| = |A_i|$. Taking into account Lemma 6, we can assume that the sets $E^*(I_i, \alpha_i/2^n m)$ are disjoint and their union is contained in some fixed dyadic cube.

Further let us divide $\mathbb{I}^n$ into 2^{nm} equal dyadic cubes Q_s $(s \in \overline{1, 2^{nm}})$. Let T_j and H_j be a finite number of translations and homotheties corresponding to

$$G = Q_1, \qquad \Omega = \bigcup_{i \in \overline{1,p}} \Omega^*(I_i, \alpha_i/2^n m)$$

and

$$\delta > 0 \quad \text{with} \quad (\max_i \alpha_i)(1/2^n m)\delta < 1/2^m,$$

according to Lemma 9, such that

$$\Big|Q_1 \setminus \bigcup_j (T_j H_j)(K)\Big| < \frac{|Q_1|}{2},$$

where

$$K = \bigcup_{J\in\Omega} J = \bigcup_{i\in\overline{1,p}} E^*(I_i, \alpha_i/2^n m).$$

Let $\widetilde{T}_s$ $(s \in \overline{1, 2^{nm}})$ be the translation mapping Q_1 into Q_s.

Denote by $I^s_{i,j}$, $E^s_{i,j}$, $E^{*,s}_{i,j}$ the images under the mapping $\widetilde{T}_s T_j H_j$ of the sets I_i, $E(I_i, \alpha_i/2^n m)$, $E^*(I_i, \alpha_i/2^n m)$, respectively. Denote also

$$E^s = \bigcup_{i,j} E^s_{i,j}, \qquad E^{*,s} = \bigcup_{i,j} E^{*,s}_{i,j}, \qquad \widetilde{I}_i = \bigcup_{j,s} I^s_{i,j}.$$

By virtue of Lemmas 6–9 and taking into account the construction, we have:

$$\Omega(I^s_{i,j}, \alpha_i/2^n m) = \{\widetilde{T}_s T_j H_j(J) : J \in \Omega(I_i, \alpha_i/2^n m)\} \quad \text{and}$$
$$E^s_{i,j} = E(I^s_{i,j}, \alpha_i/2^n m);$$
$$\Omega^*(I^s_{i,j}, \alpha_i/2^n m) = \{\widetilde{T}_s T_j H_j(J) : J \in \Omega^*(I_i, \alpha_i/2^n m)\} \quad \text{and}$$
$$E^{*,s}_{i,j} = E^*(I^s_{i,j}, \alpha_i/2^n m);$$
$$E^{*,s}_{i,j} \cap E^{*,s'}_{i',j'} = \emptyset \quad \text{if} \ (i,j,s) \neq (i',j',s');$$
$$\operatorname{diam} J < 1/2^m \quad \text{and}$$
$$l(J) \geq \min\{l(I^s_{i,j}) : i,j,s\} > 0 \quad \text{for every} \ J \in \Omega(I^s_{i,j}, \alpha_i/2^n m);$$
$$|E^s| \geq c_1(n)|E^{*,s}| > c_1(n)\frac{|Q_s|}{2} \quad \text{and} \quad |\widetilde{I}_i| \leq |A_i|.$$

Set

$$g = \sum_{i,j,s} \alpha_i \chi_{I^s_{i,j}}.$$

Let A'_i be a subset of A_i with $|A'_i| = |\widetilde{I}_i|$ and let $A = \bigcup_i A'_i$. Then it is clear that A and g satisfy the first and second conditions of the lemma.

Let $\eta = \min\{l(I^s_{i,j}) : i,j,s\}$. From Lemma 7 it follows that

$$\Omega(I^s_{i,j}, \alpha_i/2^n m) \subset \Lambda^{(\eta, 1/2^m)}(\alpha_i \chi_{I^s_{i,j}}, m).$$

Applying Lemma 7 again and disjointness of the sets $E^{*,s}_{i,j}$ we find that

$$\{G^{(\eta,1/2^m)}_{\mathbf{I}_\mathrm{d}}(v) > m\} \supset E^s_{i,j} \cap \mathbb{R}^n_\mathrm{d},$$

for every function $v \in L(\mathbb{I}^n)$ with $\operatorname{supp} v = \operatorname{supp} g$ and $v \geq g$, and for every i, j, s. Thus, we have

$$\{G^{(\eta,1/2^m)}_{\mathbf{I}_\mathrm{d}}(v) > m\} \supset \bigcup_{i,j,s} E^s_{i,j} \cap \mathbb{R}^n_\mathrm{d}.$$

Now, defining E as the union in the above inclusion and taking into account the estimate

$$|E^s| \geq \frac{c_1(n)}{2}|Q_s|,$$

we conclude the validity of the third condition of the lemma.

Using the estimate (see Lemma 6)

$$|E^s_{i,j}| \geq c_2(n)\frac{\alpha_i}{2^n m}\ \ln^{n-1}\frac{\alpha_i}{2^n m}|I^s_{i,j}|$$

and choosing α_i as in Lemma 11, we obtain

$$|E^s_{i,j}| \geq 2^m \alpha_i |I^s_{i,j}|.$$

Hence it follows that

$$\int_{Q_s} g = \sum_{i,j} \alpha_i |I^s_{i,j}| \leq \sum_{i,j} \frac{1}{2^m}|E^s_{i,j}| \leq \frac{1}{2^m}|Q_s|.$$

Therefore, taking into account that every $I \in \mathbf{I}_\mathrm{d}$ with $I \subset \mathbb{I}^n$ and $l(I) \geq 1/2^m$ is an union of certain cubes Q_s, we have

$$M^{(1/2^m)}_{\mathbf{I}_\mathrm{d}}(g)(x) \leq 1/2^m \quad \text{for every} \ \ x \in \mathbb{I}^n.$$

The lemma is proved. □

Proof of Theorem 2. *Step* 1. f takes natural values and

$$f \in L(\ln^+ L)^{n-2} \setminus L(\ln^+ L)^{n-1}(\mathbb{I}^n).$$

For every $k \in \mathbb{N}$, let us choose $m_k \in \mathbb{N}$, $\eta_k > 0$ and g_k according to Lemma 11 so that

$$m_k < m_{k+1}, \qquad m_k > 2^{n+1}\sum_{i=1}^{k-1}\|g_i\|_{L^\infty}, \qquad \eta_k > \frac{1}{2^{m_{k+1}}}.$$

Suppose $v = \sup_{k\in\mathbb{N}} g_k$ and denote

$$S_k = \operatorname{supp} g_k, \qquad F_k = S_k \setminus \bigcup_{i=k+1}^{\infty} S_i, \qquad v_k = v\chi_{F_k}.$$

It is clear that $v = \sum_{k=1}^{\infty} v_k$ and v is equimeasurable with $f\chi_A$ for some $A \subset \mathbb{I}^n$.

Let $k > 1$. Denote

$$v_{k,k} = v\chi_{S_k} \quad \text{and} \quad v_{k,i} = v\chi_{F_i\setminus S_k} \quad \text{for} \ \ i \neq k.$$

It is easy to check that for any $k \in \mathbb{N}$,

$$v = \sum_{i=1}^{\infty} v_{k,i}, \ \ v_{k,k} \geq g_k, \ \ \operatorname{supp} v_{k,k} = \operatorname{supp} g_k = F_k \ \text{ and } \ v_{k,i} \leq g_i \text{ if } i \neq k.$$

From Lemma 11 it follows that

$$\{G^{(\eta_k,1/2^{m_k})}_{\mathbf{I}_\mathrm{d}}(v_{k,k}) > m_k\} \supset E_k,$$

where E_k is the set from Lemma 11 corresponding to m_k, η_k and g_k.

Let $x \in E_k$. Let us consider $I \in \mathbf{I}_{\mathrm{d}}(x)$ such that

$$G_I(v_{k,k}) > m_k, \quad l(I) \ge \eta_k \quad \text{and} \quad \operatorname{diam} I < 1/2^{m_k}.$$

Then by construction and Lemma 11, we write

$$|G_I(v)| = \Big|\sum_{i=1}^{\infty} G_I(v_{k,i})\Big| \ge |G_I(v_{k,k})| - \Big|\sum_{i=1}^{k-1} G_I(v_{k,i})\Big| -$$

$$- \Big|\sum_{i=k+1}^{\infty} G_I(v_{k,i})\Big| = |G_I(v_{k,k})| - a_1 - a_2;$$

$$|G_I(v_{k,k})| > m_k;$$

$$|a_1| \le 2^n \sum_{i=1}^{k-1} \|v_{k,i}\|_{L^\infty} \le 2^n \sum_{i=1}^{k-1} \|g_i\|_{L^\infty} < \frac{m_k}{2}\,;$$

$$|a_2| \le \sum_{i=k+1}^{\infty} 2^n M_{\mathbf{I}_{\mathrm{d}}}^{(\eta_k)}(g_i)(x) \le \sum_{i=k+1}^{\infty} 2^n M_{\mathbf{I}_{\mathrm{d}}}^{(1/2^{m_i})}(g_i)(x) \le \sum_{i=k+1}^{\infty} \frac{2^n}{2^{m_i}} \le 2^n.$$

Thus, we have

$$|G_I(v)| \ge \frac{m_k}{2} - 2^n,$$

and therefore we also have

$$\{G_{\mathbf{I}_{\mathrm{d}}}^{(\eta_k, 1/2^{m_k})}(v) > m_k/2 - 2^n\} \supset E_k.$$

From the condition 3) of Lemma 11 it follows easily that

$$|\limsup_{k\to\infty} E_k| = 1.$$

Now, from the last two conclusions we obtain

$$\limsup_{I\in\mathbf{I}_{\mathrm{d}}(x),\ \operatorname{diam} I\to 0} |G_I(v)| = \infty \quad \text{for a.e.} \quad x \in \mathbb{I}_{\mathrm{d}}^n.$$

Let v' be the function corresponding to the function f and sets $\mathbb{I}^n \setminus A$, $\mathbb{I}^n \setminus \operatorname{supp} v$, according to Lemma 10. Then it is easy to see that $g = v + v'$ is equimeasurable with f and g satisfies the first condition of the theorem.

On the other hand, for any $\alpha \ge 1$ the estimate

$$M_{\mathbf{I}(\alpha)}(|f|) \le \alpha M_B(|f|),$$

holds true, where B is the basis consisting of all n-dimensional intervals such that 2 of the n side lenghts are equal and the other $n-2$ are arbitrary. For M_B an analogue of strong maximal inequality with $n-2$ in power of logarithm instead of $n-1$ is valid (see [5] or [4, Ch. II, §3]). Taking into account that $g \in L(\ln^+ L)^{n-2}(\mathbb{I}^n)$, we conclude the validity of the second condition of Theorem 2 for g.

Step 2. Let f be a function with arbitrary nonnegative values and belong to $L(\ln^+ L)^{n-2}(\mathbb{I}^n)$. Suppose

$$f^*(x) = \begin{cases} 1 & \text{if } 0 \le f(x) < 1, \\ [f(x)] & \text{otherwise,} \end{cases}$$

where $[\cdot]$ denotes the integer part of a number. Let g^* be an equimeasurable with f^* function satisfying the conditions of Theorem 2.

Let

$$E_k = \{f^* = k\} \quad \text{and} \quad E_k^* = \{g^* = k\} \quad (k \in \mathbb{N}),$$

then

$$E_k = \{k \le f < k+1\} \quad \text{and} \quad |E_k| = |E_k^*|.$$

Let g_k $(k \in \mathbb{N})$ be the function corresponding to the function f and the sets E_k, E_k^* according to Lemma 10. Then we have

$$g = \sum_{k=1}^{\infty} g_k \ \text{ is equimeasurable with } f \text{ and } \|g - g^*\|_{L^\infty} \le 1.$$

Hence we easily conclude that g is the needed function.

Step 3. Let f be an arbitrary function from $L \setminus L(\ln^+ L)^{n-1}(\mathbb{I}^n)$. Let us choose a set A_1 so that

$$f\chi_{A_1} \in L(\ln^+ L)^{n-2} \setminus L(\ln^+ L)^{n-1}(\mathbb{I}^n)$$

and

$$\text{either } \ A_1 \subset \{f > 0\} \ \text{ or } \ A_1 \subset \{f < 0\}.$$

Without loss of generality, we may assume that $A_1 \subset \{f > 0\}$.

Let g_1 be the function equimiesurable with $f\chi_{A_1}$ that satisfies the conditions of the theorem.

Denote

$$A_2 = \{f > 0\} \setminus A_1, \quad A_3 = \{f < 0\}, \quad f_2 = f\chi_{A_2} \ \text{ and } \ f_3 = |f|\chi_{A_3}.$$

Let us choose the sets E_2 and E_3 so that

$$E_2, E_3 \subset \mathbb{I}^n \setminus A_1, \quad E_2 \cap E_3 = \emptyset, \quad |E_2| = |A_2| \ \text{ and } \ |E_3| = |A_3|.$$

Now, if we consider the functions g_i $(i = 2, 3)$ corresponding to the function f_i, and the sets A_i, E_i according to Lemma 10, then it is easy to see that $g = g_1 + g_2 - g_3$ is the needed function. The theorem is proved. □

Remark 1. For any $\varepsilon > 0$ it is possible to construct a function g in Theorems 1 and 2 such that $|\{g \ne f\}| < \varepsilon$.

Remark 2. As it is seen from the proofs of Theorems 1 and 2, the upper limit of the general terms of g is equal to ∞ a.e. if the function f is nonnegative. Moreover, it is possible to sharpen Theorems 1 and 2 by achieving the fullfilment of

$$\limsup = \infty \quad \text{and} \quad \liminf = -\infty \quad \text{simultaneously a.e.}$$

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О расходимости кратных рядов Хаара

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Доказано, что для любой размерности $n \geq 2$, $L(\ln^+ L)^{n-1}$ является наиболее широким интегральным классом, в котором обеспечено почти всюду сходимость кратных рядов Фурье–Хаара. Более того, показано, что можно одновременно добиться эффектов расходимости как прямоугольных, так и сферических общих членов кратных рядов Фурье–Хаара на множестве полной меры путем подходящей перестановки значений произвольной суммируемой функции f, не принадлежащей классу $L(\ln^+ L)^{n-1}$.