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Turán's extremal problem on locally compact abelian groups

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Abstract. Let G be a locally compact abelian group (LCA group) and Ω be an open, 0-symmetric set. Let $\mathcal{F} := \mathcal{F}(\Omega)$ be the set of all continuous functions $f : G \to \mathbb{R}$ which are supported in Ω and are positive definite. The Turán constant of Ω is then defined as

$$\mathcal{T}(\Omega) := \sup\left\{ \int_\Omega f : \ f \in \mathcal{F}(\Omega), \ f(0) = 1 \right\}.$$

Mihalis Kolountzakis and the author has shown that structural properties — like spectrality, tiling or packing with a certain set Λ — of subsets Ω in finite, compact or Euclidean (i.e., $\mathbb{R}^d$) groups and in $\mathbb{Z}^d$ yield estimates of $\mathcal{T}(\Omega)$. However, in these estimates some notion of the size, i.e., density of Λ played a natural role, and thus in groups where we had no grasp of the notion, we could not accomplish such estimates.

In the present work a recent generalized notion of asymptotic uniform upper density is invoked, allowing a more general investigation of the Turán constant in relation to the above structural properties. Our main result extends a result of Arestov and Berdysheva, (also obtained independently and along different lines by Kolountzakis and the author), stating that convex tiles of a Euclidean space necessarily have

$$\mathcal{T}_{\mathbb{R}^d}(\Omega) = |\Omega|/2^d.$$

In our extension $\mathbb{R}^d$ could be replaced by any LCA group, convexity is considerably relaxed to Ω being a difference set, and the condition of tiling is also relaxed to a certain packing type condition and positive asymptotic uniform upper density of the set Λ.

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Also our goal is to give a more complete account of all the related developments and history, because until now an exhaustive overview of the full background of the so-called Turán problem was not delivered.

1. Introduction

1.1. The Turán problem

We study the following problem, generally investigated under the name of *"Turán's Problem"*, following Stechkin [61], who recalls a question posed to him in personal discussion.

Problem 1. *Given an open set* Ω*, symmetric about* 0*, and a continuous, positive definite, integrable function* f*, with* $\operatorname{supp} f \subseteq \Omega$ *and with* $f(0) = 1$*, how large can* $\int f$ *be?*

Although this name for the problem is quite widespread, one has to note that all the important versions of the problem were investigated well before the beginning of the seventies, when the discussion of Turán and Stechkin took place.

About the same time when Turán discussed the question with Stechkin, American researchers already investigated in detail the *square integral version of the problem*, see [24, 54, 17]. Their reason for searching the extremal function and value came from radar engineering problems at the Jet Propulsion Laboratory.

More importantly, Problem 1 appears as early as in the thirties [60], when Siegel considered the question for Ω being a ball, or an ellipsoid in Euclidean space $\mathbb{R}^d$, and established the right extremal value $|\Omega|/2^d$. The question occurred to Siegel as a theoretical possibility to sharpen the Minkowski Latice Point Theorem. Although Siegel concluded that, due to the extremal value being just as large as the Minkowski Lattice Point Theorem would require, this geometric statement can not be further sharpened through improvement on the extremal problem, nevertheless he works out the extremal problem fully and exhibits some nice applications in the theory of entire functions.

Furthermore, the same Problem 1 appeared in a paper of Boas and Kac [12] in the forties, even if the main direction of the study there was a different version, what is nowadays generally called the *pointwise Turán problem*. However, as is realized partially in [12] and fully only later in [46], the pointwise Turán problem — formulated in the classical setting of Fourier series, but nevertheless equivalent to the Euclidean space settings of [12] — goes back already to CARATHEODORY [13] and FEJÉR [21, 22].

The Turán problem was considered on an interval in the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by Stechkin [61] and in $\mathbb{R}$ by BOAS and KAC [12], but extensions were to follow in several directions.

Such a question is interesting in the study of sphere packings [26, 14, 15], in additive number theory [58, 39, 53, 30] and in the theory of Dirichlet characters and exponential sums [48], among other things. In their short survey of results on Problem 1, EHM, GNEITING and RICHARDS [18] also mentions applications of several variants in optics, antenna design and statistics.

1.2. One dimensional case of the Turán problem

Already the symmetric interval case in one dimension presents non-trivial complications, which were resolved satisfactorily only recently. We discuss the development of the problem from the outset to date.

Actually, Turán's interest might have come from another area in number theory, namely Diophantine approximation. (Let us point out that [2] starts with the sentence: "With regard to applications in number theory, P. Turán stated the following problem:", while at the end of the paper there is special expression of gratitude to Professor Stechkin for his interest in this work. Also, GORBACHEV writes in [25, p. 314]: "Studying applications in number theory, P. Turán posed the problem ...")

One can hypothesise that Turán thought of the elegant proof of the well-known Dirichlet approximation theorem, stating that for any given $\alpha \in \mathbb{R}$ at least one multiple $n\alpha$ in the range $n = 1, \dots, N$ have to approach some integer as close as $1/(N+1)$. The proof, which uses Fourier analysis and Fejér kernels in particular, is presented in [53, p. 99], and in a generalized framework it is explained in [11], but it is remarked in [53, p. 105] that the idea comes from SIEGEL [60], so Turán could have been well aware of it. Let us briefly present the argument right here.

If we wish to detect multiples $n\alpha$ of $\alpha \in \mathbb{R}$ which fall in the δ-neighborhood of an integer, that is which have $\{\{n\alpha\}\} < \delta$ (where, as usual in this field, $\{\{x\}\} := \operatorname{dist}(x, \mathbb{Z})$), then we can use that for the triangle function

$$F(x) := F_\delta(x) := (1 - \{\{x\}\}/\delta)_+ := \max(1 - \{\{x\}\}/\delta, 0),$$

we have

$$F(n\alpha) > 0 \quad \text{iff} \quad \{\{n\alpha\}\} < \delta.$$

So if with an arbitrary $\delta > 1/(N+1)$ we can work through a proof of $F(n\alpha) > 0$ for some $n \in [1, N]$, then the proof yields the sharp form of the Dirichlet approximation theorem. (It is indeed sharp, because for no $N \in \mathbb{N}$ can any better statement hold true, as the easy example of $\alpha := 1/(N+1)$ shows.)

So we take now

$$S := \sum_{n=1}^{N} \Big(1 - \frac{n}{N+1}\Big) F(n\alpha),$$

or, since F is even and $F(0) = 1$, consider the more symmetric sum

$$2S + 1 = \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big) F(n\alpha).$$

Note that

$$\widehat{F_\delta}(t) = \delta\Big(\frac{\sin(\pi\delta t)}{\pi\delta t}\Big)^2,$$

so in particular with the nonnegative coefficients $\widehat{F}(k) = c_k$ we can write (with $e(t) := e^{2\pi i t}$)

(1)

$$F_\delta(x) = \sum_{k=-\infty}^{\infty} c_k e(kx), \qquad c_0 = \delta, \quad c_k = \delta\Big(\frac{\sin(\pi k\delta)}{\pi k\delta}\Big)^2 \quad (k = \pm 1, \pm 2, \ldots).$$

It suffices to show $S > 0$. With the Fejér kernels

$$\sigma_N(x) := \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big) e(nx) = \frac{1}{N+1}\Big(\frac{\sin(\pi(N+1)x)}{\pi x}\Big)^2 \geq 0,$$

after a change of the order of summation we are led to

$$2S + 1 = \sum_{k=-\infty}^{\infty} c_k \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big) e(nk\alpha) =$$

$$= c_0\sigma_N(0) + 2\sum_{k=1}^{\infty} c_k \sigma_N(k\alpha) \geq c_0\sigma_N(0) = \delta(N+1) > 1,$$

which concludes the argument.

Now if in place of the triangle function with $\delta = 1/(N+1)$ another positive definite (i.e., $\widehat{f} \geq 0$) function f could be put with $\operatorname{supp} f \subset [-\delta, \delta]$ and $f(0) = 1$ but with $\widehat{f}(0) > \delta$, then the above argument with f in place of F would give $S > 0$ even for $\delta = 1/(N+1)$, clearly a contradiction since the Dirichlet approximation theorem cannot be further sharpened. That round-about argument already gives that for h a reciprocal of an integer, the triangle function F_h is extremal in the Turán problem for $[-h, h]$. In other words, we obtain STECHKIN's result [61], (see also below) already from considerations of Diophantine approximation.

So Turán asked Stechkin if for any $h > 0$ the triangle function provides the largest possible integral among all positive definite functions vanishing outside $[-h, h]$ and normalized by attaining the value 1 at 0. (Note that this formulation is slightly different of the general formulation in Problem 1, (which form became standard only later) in the extent that setting $\Omega = [-h, h]$ means use of a *closed* set for Ω. In the interval setting an easy limiting process easily shows the equivalence to the general formulation, in view of availability of approximations of any interval, and functions supported on that interval, by dilates. That is, however, an argument not available for non-convex sets, or in general topological groups without proper interpretations of dilation.)

Stechkin derived that this extremality of the triangle function is the case for h being the reciprocal of a natural number: by monotonicity in h for other values he could conclude an estimate. Anticipating and slightly abusing the general notations below, denote the extremal value by $T(h)$: then Stechkin obtained

$$T(h) = h + O(h^2).$$

This was sharpened later by GORBACHEV [25] and POPOV [55] (cited in [27, p. 77]) to $h + O(h^3)$.

The corresponding Turán extremal value $T_{\mathbb{R}}(h)$ on the real line is, by simple dilation, depends linearly on the interval length and is just $hT_{\mathbb{R}}(1)$ for any interval $I = [-h, h]$. On the other hand, it follows already from

$$\lim_{h \to 0+} T(h)/h = 1$$

that e.g. for the unit interval $[-1, 1]$ the extremal function must be the triangle function and $T_{\mathbb{R}}(1) = 1$, hence $T_{\mathbb{R}}(h) = h$. In fact, this case was already settled earlier by BOAS and KATZ in [12] as a byproduct of their investigation of the pointwise question.

But there is another observation, seemingly well-known although no written source can be found. Namely, it is also known for some time that for *h not being a reciprocal of an integer number*, the triangle function *can indeed be improved upon* a little. Indeed, the triangle function F_h has Fourier transform which vanishes precisely at integer multiples of $1/h$, and in case $1/h \notin \mathbb{N}$, some multiples fall outside $\mathbb{Z}$. And then the otherwise double zeroes of $\widehat{F_h}$ can even be substituted by a product of two close-by zero factors, allowing a small interval in between, where the Fourier transform can be negative. This negativity spoils positive definiteness regarding the function on $\mathbb{R}$: but on $\mathbb{T}$ it does not, for only the values at integer increments must be nonnegative in order that a function be positive definite on $\mathbb{T}$. With a detailed calculus (using also the symmetric pair of zeroes) such an improvement upon the triangle function is indeed possible. (Note that here $\widehat{F}$, so also $\int \widehat{F} = F(0)$ is perturbed while $\widehat{F}(0) = \int F$ is unchanged.) I have heard this construction explained in lectures during my university studies [33]; in Russia, a similar observation was communicated by Popov [55] and later recorded in writing in [29, 31, 27].

As said above, the computation of exact values of $T(h)$ started with Stechkin for $h = 1/q$, $q \in \mathbb{N}$: these are the only cases when $T(h) = h$. Further values, already deviating from this simple formula, were much more difficult to compute exactly. At the turn of the century, Gorbachev and Manoshina [29, 31, 50] reduced the Turán problem for $h \in \mathbb{Q}$ to a discrete Fejér type optimization problem. Also they showed that the extremal function in the Turán problem is a piecewise linear function connecting discrete values of the discrete extremal polynomial solution of the Fejér problem. In 2000 Ivanov in his seminar lectures in Tula State University formulated the right conjecture about the form of the solution of the Fejér type problem and the extremal polynomial in the corresponding direct and dual linear programming problem. Then the goal became to prove positiveness of coefficients of the hypothetically extremal discrete trigonometric polynomials. Via this approach, Gorbachev and Manoshina [29, 31] solved the Turán problem for *some* rational values. The full conjecture on the solution of the discrete Fejér problem was finally proved by Ivanov and Rudomazina [37, 27, 28], which implied also the solution of the Turán extremal problem *for all rational* h and furnished the solution of the so-called Montgomery problem, too.

Finally, in 2006 Ivanov [36] solved *even the case of irrational* h, and thus *completed the solution of Turán's problem on the torus*. Ivanov's paper also established that for $[-h, h] \subset \mathbb{T}$ the Turán extremal problem and the Delsarte extremal problem (see below in §1.4.) have the same extremal value (and extremal functions). Note that this coincidence does not hold true in general.

However, it seems that almost nothing is known about Turán extremal values of other, one would say "dispersed" sets not being intervals. A natural conjecture is that e.g. on $\mathbb{R}$ (or perhaps even on $\mathbb{T}$?) a set $\Omega \subset \mathbb{R}$ of fixed measure $|\Omega| = m$ can have maximal Turán constant value if only it is a zero-symmetric interval $[-m/2, m/2]$. What we know from [47, Theorem 6] is that we certainly have $T(\Omega) \leq m/2$, that is, in $\mathbb{R}$ no "better sets", than zero-symmetric intervals, can exist. However, uniqueness is not known, not even for $\mathbb{R}$. In [47] there is a more general estimate in terms of the prescribed measure m, but for higher dimensions it is far less precise. Also, regarding the discrete group $\mathbb{Z}$ one must observe that zero-symmetric intervals $[-N, N] \subset \mathbb{Z}$ have the same Turán extremal values as their homothetic copies $k[-N, N]$ $(k \in \mathbb{N})$ which already destroys the hope for "uniqueness only for intervals". In higher dimensions not even the right class of the corresponding "condensed sets", like intervals in dimension one, has been identified.

1.3. Turán's problem in the multivariate setting

Already as early as in the 1930's, Siegel [60] proved that for an ellipsoid $\Omega \subset \mathbb{R}^d$ the extremal value in Problem 1 is $|\Omega|/2^d$.

In the 1940's, Boas and Katz [12] mentioned that Poisson summation may be used to treat similar questions in higher dimensions. Besides mentioning the group settings, Garcia et al. [24] and Domar [17] also touches upon the question without going into further details. The packing problem by balls in Euclidean space has already been treated by many authors via multivariate extremal problems of the type, but there the optimal approach is to pose a closely related, still different variant, named Delsarte- (and also as Logan- and Levenshtein-) problem. See, e.g., [26, 14] and the references therein.

As a direct generalization of Stechkin's work, Andreev [1] calculated the Turán constants of cubes Q_h^d in $\mathbb{T}^d$ obtaining $h^d + O(h^{d+1})$. Moreover, he estimated the Turán constant of the cross-polytope (ℓ_1-ball) O_h^d in $\mathbb{T}^d$: his estimates are asymptotically sharp when $d = 2$. Gorbachev [25] simultaneously sharpened and extended these results proving that for any centrally symmetric body $D \subset [-1, 1]^d$ and for all $0 < h < 1/2$ we always have

$$\mathcal{T}_{\mathbb{T}^d}(hD) = \mathcal{T}_{\mathbb{R}^d}(D) \cdot h^d + O(h^{d+2})$$

(where the notation $\mathcal{T}_G(\Omega)$ for the Turán extremal value can be anticipated already here although it is introduced formally only below in Definition 1).

Arestov and Berdysheva [5] offer a systematic investigation of the multivariate Turán problem collecting several natural properties. They also prove that the hexagon has Turán constant exactly one fourth of the area of itself. Gorbachov [25] proved that the unit ball $B_d \subset \mathbb{R}^d$ has Turán constant $2^{-d}|B_d|$, where $|B_d|$ is the volume (d-dimensional Lebesgue measure) of the ball. Another proof of this fact can be found in [45], but we have already noted that the result goes back to Siegel [60].

There is a special interest in the case which concerns Ω being a (centrally symmetric) convex subset of $\mathbb{R}^d$ [5, 6, 25, 45], since in this case the natural analog of the triangle function, the self-convolution (convolution square) of the characteristic function $\chi_{\frac{1}{2}\Omega}$ of the half-body $\frac{1}{2}\Omega$ is available showing that

$$\mathcal{T}_{\mathbb{R}^d}(\Omega) \geq |\Omega|/2^d.$$

The natural conjecture is that for a symmetric convex body this convolution square is extremal, and

$$\mathcal{T}_{\mathbb{R}^d}(\Omega) = |\Omega|/2^d.$$

(Note that this fails in $\mathbb{T}^d$, already for $d = 1$, for some sets Ω.) Convex bodies with this property may be called Turán type, or Stechkin-regular, or, perhaps, *Stechkin–Turán domains*, while symmetric convex bodies in $\mathbb{R}^d$ with

$$\mathcal{T}_{\mathbb{R}^d}(\Omega) > |\Omega|/2^d$$

as anti-Turán or *non-Stechkin–Turán* domains. Thus the above mentioned result about the ball can be reworded saying that the ball is of Stechkin–Turán type.

To date, no non-Stechkin–Turán domains are known, although the family of known Stechkin–Turán domains is also quite meager (apart from $d = 1$ when everything is clear for the intervals).

In [5, 6] Arestov and Berdysheva prove that if $\Omega \subseteq \mathbb{R}^d$ is a convex polytope which can tile space when translated by the lattice $\Lambda \subseteq \mathbb{R}^d$ (this means that the copies $\Omega + \lambda$, $\lambda \in \Lambda$, are non-overlapping and almost every point in space is covered) then

$$\mathcal{T}_{\mathbb{R}^d}(\Omega) = |\Omega|/2^d.$$

Whence the class of Stechkin–Turán domains includes, by the result of Arestov and Berdysheva, convex lattice tiles.

Kolountzakis and Révész [45] showed the same formula for all convex domains in $\mathbb{R}^d$ which are *spectral*. For the definition and some context see §2.2, where it will be explained that all convex tiles are spectral, and so the result of Arestov and Berdysheva is also a consequence of the result in [45].

For not necessarily convex sets, further results are contained in [47] for $\mathbb{R}^d$, $\mathbb{T}^d$ and $\mathbb{Z}^d$.

1.4. Variants and relatives of the Turán problem

Let $\mathcal{F}$ be a a class of functions. There are several related quantities which we may want to maximize, which induce several Turán-type problems. The two most natural versions concern the *square-integral* of $f \in \mathcal{F}$, henceforth called the *square-integral Turán problem*, and the *function value* at some arbitrarily prescribed point $z \in \Omega$, called the *pointwise Turán problem.*

The square-integral Turán problem occurred for applied scientists in connection with radar design (radar ambiguity and overall signal strength maximizing), see [54, 24]. Further interesting results were obtained in [17]. Nevertheless, already on the torus $\mathbb{T}$ the exact answer is not known, even if [54] provides convincing computational evidence for certain conjectures in case $h = \pi/n$, and the existence of *some* extremal function is known.

The natural pointwise analogue of Problem 1 is the maximization of the function value $f(z)$, for given, fixed $z \in \Omega$, in place of the integral, over functions from the same class than in Problem 1. (Actually, the question can as well be posed in any LCA group.) For intervals in $\mathbb{T}$ or $\mathbb{R}$ this was studied in [7] under the name of "the pointwise Turán problem", although the same problem was already settled in the relatively easy case of an interval $(-h, h) \subset \mathbb{R}$ by Boas and Kac in [12]. For general domains in arbitrary dimension the problem was further studied in [46]. Further ramifications are obtained by considering different variations of the above definitions. E.g. Belov and Konyagin [8, 9] consider functions with integer coefficients, and periodic even functions

$$f \sim \sum_k a_k \cos(kx) \quad \text{ with } \quad \sum_k |a_k| = 1$$

but with not necessarily $a_k \geq 0$, i.e., not necessarily positive definite.

Berdysheva and Berens [10] consider the multivariate question restricted to the class of ℓ_1-radial functions.

A very natural version of the same problem is the Delsarte problem [15] (also known under the names of Logan and Levenshtein): here the only change in the conditioning of the extremal problem is that we assume, instead of vanishing of f outside a given set Ω, only the less restrictive condition that f be nonpositive outside the given set. Both extremal problems are suitable in deriving estimates of packing densities through Poisson summation: this is exploited in particular for balls in Euclidean space, see, e.g., [16, 38, 49, 3, 15, 4, 26, 14].

There are several other rather similar, yet different extremal problems around. E.g. one related intriguing question [59], dealt with by several authors, is the maximization of $\int f$ for real functions f supported in $[-1, 1]$, admitting $\|f\|_\infty = 1$, but instead of being positive definite, (which in $\mathbb{R}$ is equivalent to being represented as $g * \widetilde{g}$, with

$$\widetilde{g}(x) := \overline{g(-x)}, \qquad x \in \mathbb{R}),$$

having a representation $f = g * g$ with some $g \geq 0$ supported in the half-interval $[-1/2, 1/2]$.

Here we do not consider these relatives of the Turán problem.

1.5. Extension of the problem to LCA groups

Some authors have already extended the investigations, although not that systematically as in case of the multivariate setting, to locally compact abelian groups (LCA groups henceforth). This is the natural setting for a general investigation, since the basic notions used in the formulation of the question — positive definiteness, neighborhood of zero, support in and integral over a 0-symmetric set Ω — can be considered whenever we have the algebraic and topological structure of an LCA group. Note that we always have the Haar measure, which makes the consideration of the integral over a compact set (hence over the support of a compactly supported positive definite function) well defined. Also recall that on a LCA group G a function f is called positive definite if the inequality

$$\sum_{n,m=1}^{N} c_n\overline{c_m}f(x_n-x_m)\geq 0 \quad (\forall\, x_1,\dots,x_N\in G,\ \forall\, c_1,\dots,c_N\in\mathbb{C}) \tag{2}$$

holds true. Note that positive definite functions are not assumed to be continuous. Still, all such functions f are necessarily bounded by $f(0)$ (see [57, p. 19, Eqn (3)]). Moreover,

$$f(x)=\widetilde{f}(x):=\overline{f(-x)} \quad \text{for all} \quad x\in G$$

(see [57, p. 19, Eqn (2)]), hence the support of f is necessarily symmetric, and the condition $\operatorname{supp} f\subset\Omega$ implies also $\operatorname{supp} f\subset\Omega\cap(-\Omega)$. The latter set being symmetric, without loss of generality we can assume at the outset that Ω is symmetric itself. So in this paper the set Ω will always be taken to be a 0-symmetric, open set in G.

We find the first mention of the group case in [24], and a more systematic use of the settings (for the square-integral Turán problem) in [17]. Utilizing also the work in [5] on extensions to the several dimensional case, the framework below was set up in [47]. There we obtained some fairly general results for compact LCA groups as well as for the most classical non-compact groups: $\mathbb{R}^d$ and $\mathbb{Z}^d$.

In this paper we study the problem in the generality of LCA groups. This simplifies and unifies many of the existing results and gives several new estimates and examples. If G is a LCA group a continuous function $f\in L^1(G)$ is positive definite if its Fourier transform $\widehat{f}:\widehat{G}\to\mathbb{C}$ is everywhere nonnegative on the dual group $\widehat{G}$. For the relevant definitions of the Fourier transform we refer to [40, Chapter VII] or [57].

We say that f belongs to the class $\mathcal{F}(\Omega)$ of functions if $f\in L^1(G)$ is continuous, positive definite and is supported on a closed subset of Ω. For any positive definite function f it follows that $f(0) \geq f(x)$ for any $x \in G$. This leads to the estimate

$$\int_G f \leq |\Omega| f(0) \quad \text{for all } \; f \in \mathcal{F},$$

which is called (following ANDREEV [1]) the *trivial estimate* from now on.

Definition 1. The *Turán constant* $\mathcal{T}_G(\Omega)$ of a 0-symmetric, open subset Ω of a LCA group G is the supremum of the quantity

$$\int_G f/f(0), \qquad \text{where} \;\; f \in \mathcal{F}(\Omega),$$

i.e., $f \in L^1(G)$ is continuous and positive definite, and

$$\operatorname{supp} f := \overline{\{x \; : \; f(x) \neq 0\}}$$

is a closed set contained in Ω.

In fact, depending on the precise requirements on the functions considered, here we have certain variants of the problem: an account of these is presented below in §1.6.

Remark 1. The quantity $\mathcal{T}_G(\Omega)$ depends on which normalization we use for the Haar measure on G. If G is discrete we use the counting measure and if G is compact and non-discrete we normalize the measure of G to be 1. (Note that normalizing $\mathcal{T}_G(\Omega)$ by the measure of Ω would be inconvenient for several reasons, in particular when it is infinite.)

The *trivial upper estimate* or *trivial bound* for the Turán constant is thus $\mathcal{T}_G(\Omega) \leq |\Omega|$.

1.6. Various equivalent forms of the Turán problem

It is worth noting that Turán type problems can be, and have been considered with various settings, although their relation has not been fully clarified yet. Thus in extending the investigation to LCA groups or to domains in Euclidean groups which are not convex, the issue of equivalence has to be dealt with. One may consider the following function classes (with $\Subset$ denoting compact subsets).

$$\mathcal{F}_1(\Omega) := \Big\{ f \in L^1(G) : \; \operatorname{supp} f \subset \Omega, \; f \text{ positive definite} \Big\}, \tag{3}$$

$$\mathcal{F}_{\&}(\Omega) := \Big\{ f \in L^1(G) \cap C(G) : \; \operatorname{supp} f \subset \Omega, \; f \text{ positive definite} \Big\}, \tag{4}$$

$$\mathcal{F}_c(\Omega) := \Big\{ f \in L^1(G) : \ \operatorname{supp} f \Subset \Omega, \ f \ \text{ positive definite} \Big\}, \tag{5}$$

$$\mathcal{F}(\Omega) := \Big\{ f \in C(G) : \ \operatorname{supp} f \Subset \Omega, \ f \ \text{ positive definite} \Big\}. \tag{6}$$

In $\mathcal{F}_1$, $\mathcal{F}_\&$ $\operatorname{supp} f$ is assumed to be merely closed and not necessarily compact, and in $\mathcal{F}_1, \mathcal{F}_c$ the function f may be discontinuous.

The respective Turán constants are

$$\begin{aligned} & \mathcal{T}_G^{(1)}(\Omega) \ \text{ or } \ \mathcal{T}_G^{\&}(\Omega) \ \text{ or } \ \mathcal{T}_G^{c}(\Omega) \ \text{ or } \ \mathcal{T}_G(\Omega) := \\ & := \sup \left\{ \frac{\int_G f}{f(0)} \ : \ f \in \mathcal{F}_1(\Omega) \ \text{ or } \ \mathcal{F}_\&(\Omega) \ \text{ or } \ \mathcal{F}_c(\Omega) \ \text{ or } \ \mathcal{F}(\Omega), \ \text{ resp.} \right\}. \end{aligned} \tag{7}$$

In general we should consider functions $f : G \to \mathbb{C}$. However, it is easy to see from (2) that together with f, also $\overline{f}$ is positive definite. Whence even $\varphi := \operatorname{Re} f$ is positive definite, while belonging to the same function class. As we also have

$$f(0) = \varphi(0) \quad \text{ and } \quad \int f = \int \varphi,$$

restriction to real valued functions does not change the values of the Turán constants.

For a detailed introduction to positive definite functions, and for a proof of the following theorem, we refer to [47].

Theorem 1 (Kolountzakis–Révész). *In any LCA group the above defined versions of the Turán constants coincide:*

$$\mathcal{T}_G^{(1)}(\Omega) = \mathcal{T}_G^{\&}(\Omega) = \mathcal{T}_G^{c}(\Omega) = \mathcal{T}_G(\Omega)\,. \tag{8}$$

Note that the original formulation, presented also above in Definition 1, corresponds to $\mathcal{T}_G^{\&}(\Omega)$. Also note that with this setup, e.g. the interval case $\Omega = [-h, h] \subset \mathbb{T}$ or $\mathbb{R}$ admits no extremal function, because the support of Δ_h is the full $\overline{\Omega}$, not a closed subset of the open set $(-h, h)$. In this case an obvious limiting process is neglected in the formulation of the results above.

Remark 2. It is not fully clarified what happens for functions vanishing only outside of Ω, but having nonzero values up to the boundary $\partial\Omega$.

Our main result in this paper appears in Theorem 7. This is an essential extension of the above mentioned result of Arestov and Berdysheva about convex lattice tiles in Euclidean spaces being of the Stechkin–Turán type. To arrive at the result we need some preparations. So in the next section we describe the structural context, including without proofs a different extension of the result of Arestov and Berdysheva — in the direction of spectrality — already given in [45]. Also we explain the relevant new notion of asymptotic uniform upper density and its computation or estimation in relation with packing, covering and tiling. The main result then appears in §3.

2. Structural properties of sets — tiling, packing, spectrality, and asymptotic uniform upper density

2.1. Tiling and packing

Suppose G is a LCA group.

Definition 2. We say that a nonnegative function $f \in L^1(G)$ *tiles* G *by translation with a set* $\Lambda \subseteq G$ *at level* $c \in \mathbb{R}_+$ if

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = c$$

for a.e. $x \in G$, with the sum converging absolutely. We then write "$f + \Lambda = cG$".

We say that f *packs* G *with the translation set* Λ *at level* $c \in \mathbb{R}_+$, and write $f + \Lambda \le cG$, if

$$\sum_{\lambda\in\Lambda} f(x-\lambda) \le c$$

for a.e. $x \in G$. When the same properties hold with constant $c = 1$ for a characteristic function χ_Ω of some Borel measurable set Ω with compact closure, then we simply say that Ω *tiles* or *packs* G, and write $\Omega + \Lambda = G$, $\Omega + \Lambda \le G$, respectively.

Neglecting some measure zero sets, packing occurs when for any point $x \in G$ $x - \lambda \in \Omega$ for at most one point λ of Λ, which in turn is equivalent to $\lambda + \Omega$ being disjoint for different $\lambda \in \Lambda$. This explains the term "packing". On the other hand this latter statement is equivalent to saying that

$$\lambda + x = \lambda' + x' \quad \text{with } \lambda, \lambda' \in \Lambda \text{ and } x, x' \in \Omega$$

can occur only if $\lambda = \lambda'$ and hence also $x = x'$. Writing this in the form of differences, $\lambda - \lambda' = x' - x$ only for both sides being 0, that is,

$$(\Lambda - \Lambda) \cap (\Omega - \Omega) = \{0\}.$$

This is an equivalent condition to Ω packing with Λ. More generally, we will say that the set S satisfies a "packing type condition" with L, if

$$(L - L) \cap S \subset \{0\},$$

irrespectively of the situation whether S can be represented as a difference set of some other Ω or not.

So in an Euclidean space about a nonnegative $f \in L^1(\mathbb{R}^d)$ we say that f tiles with Λ at level ℓ if

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = \ell, \qquad \text{a.e. } x.$$

We denote this latter condition by $f + \Lambda = \ell\mathbb{R}^d$.

In particular, a measurable set $\Omega \subseteq \mathbb{R}^d$ is a *translational tile* if there exists a set $\Lambda \subseteq \mathbb{R}^d$ such that almost all (Lebesgue) points in $\mathbb{R}^d$ belong to exactly one of the translates

$$\Omega + \lambda, \qquad \lambda \in \Lambda.$$

We denote this condition by $\Omega + \Lambda = \mathbb{R}^d$.

In any tiling the translation set has some properties of density, which hold uniformly in space. A set $\Lambda \subseteq \mathbb{R}^d$ has (uniform) density ρ if (with

$$\lim_{R\to\infty} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|} \to \rho \quad \text{ with } \; B_R(x) := \{y \in \mathbb{R}^d \; : \; |y-x| \le R\})$$

uniformly in $x \in \mathbb{R}^d$. We write $\rho = \operatorname{dens}\Lambda$. We say that Λ has density bounded uniformly by ρ, if the fraction above is bounded by the constant ρ uniformly for $x \in \mathbb{R}$ and $R > 1$.

Remark 3. It is not hard to prove (see for example [42, Lemma 2.3], where it is proved in dimension one — the proof extends verbatim to higher dimension) that in any tiling $f + \Lambda = \ell\mathbb{R}^d$ the set Λ has density $\ell/\int f$.

When the group is finite (and we do not, therefore, have to worry about the set Λ being finite or not) the tiling condition $f + \Lambda = cG$ means precisely $f * \chi_\Lambda = c$. Taking Fourier transform, this is the same as $\widehat{f}\widehat{\chi_\Lambda} = c|G|\chi_{\{0\}}$, which is in turn equivalent to the condition

$$\operatorname{supp}\widehat{\chi_\Lambda} \subseteq \{0\} \cup \{\widehat{f} = 0\} \quad \text{ and } \quad c = \frac{|\Lambda|}{|G|}\sum_{x\in G} f(x). \tag{9}$$

The packing type condition $\Omega \cap (\Lambda - \Lambda) = \{0\}$ will be used in Theorem 7 below. This result will be an essential extension of the earlier result of Arestov and Berdysheva, stating that in $\mathbb{R}^d$ a convex lattice tile is necessarily of the Stechkin–Turán type. Another generalization of this result appears in the next section, through another structural property of sets, namely spectrality.

2.2. Spectral sets

Definition 3. Let G be a LCA group and $\widehat{G}$ be its dual group, that is the group of all continuous group homomorphisms (characters)

$$G \to \mathbb{C}_1 := \{z \in \mathbb{C} \ : \ |z| = 1\}.$$

We say that the set $T \subseteq \widehat{G}$ is a *spectrum* of $H \subseteq G$ if and only if the restrictions of the characters from T form an orthogonal basis for $L^2(H)$.

In particular, let Ω be a measurable subset of $\mathbb{R}^d$ and Λ be a discrete subset of $\mathbb{R}^d$. We write

$$e_\lambda(x) = \exp(2\pi i \langle \lambda, x \rangle), \qquad (x \in \mathbb{R}^d),$$

and

$$E_\Lambda = \{e_\lambda : \ \lambda \in \Lambda\} \subset L^2(\Omega).$$

The inner product and norm on $L^2(\Omega)$ are

$$\langle f, g \rangle_\Omega = \int_\Omega f\overline{g}, \qquad \text{and} \qquad \|f\|_\Omega^2 = \int_\Omega |f|^2.$$

The pair (Ω, Λ) is called a *spectral pair* if E_Λ is an orthogonal basis for $L^2(\Omega)$. A set Ω will be called *spectral* if there is $\Lambda \subset \mathbb{R}^d$ such that (Ω, Λ) is a spectral pair. The set Λ is then called a *spectrum* of Ω.

Example 1. If $Q_d = (-1/2, 1/2)^d$ is the cube of unit volume in $\mathbb{R}^d$ then $(Q_d, \mathbb{Z}^d)$ is a spectral pair, as is well known by the ordinary L^2 theory of multiple Fourier series.

Fuglede [23] formulated the following famous conjecture in 1974.

Conjecture 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Then Ω is spectral if and only if there exists $L \subset \mathbb{R}^d$ such that $\Omega + L = \mathbb{R}^d$ is a tiling.*

One basis for the conjecture was that the lattice case of this conjecture is easy to show, (see for example [23, 41]). In the following result the dual lattice Λ^* of a lattice Λ is defined as usual by

$$\Lambda^* = \big\{x \in \mathbb{R}^d : \ \forall\, \lambda \in \Lambda \ \langle x, \lambda \rangle \in \mathbb{Z}\big\}.$$

Theorem 2 (Fuglede [23]). *The bounded, open domain Ω admits translational tilings by a lattice Λ if and only if E_{Λ^*} is an orthogonal basis for $L^2(\Omega)$.*

Note that in Fuglede's Conjecture no relation is claimed between the translation set L and the spectrum Λ.

Conjecture 1 in its full generality was recently disproved. First, Tao [62] showed that in $\mathbb{R}^5$ there exists a spectral set, which however fails to tile space. The method, roughly speaking, is to construct counterexamples in finite groups, and then "lift them up" first to $\mathbb{Z}^d$ and finally to $\mathbb{R}^d$. Soon after that breakthrough, Tao's construction was further sharpened to provide non-tiling spectral sets in $\mathbb{R}^4$ (see [51]) and finally even in dimension 3 (see [44]).

Furthermore, the converse implication was also disproved, first in dimension 5 by Kolountzakis and Matolcsi [43]. Subsequently, examples of tiling, but non-spectral sets were constructed in $\mathbb{R}^4$ by Farkas and Révész [20], and then even in $\mathbb{R}^3$ by Farkas, Matolcsi and Móra [19].

Positive results are far more meager, and basically restrict to special sets on the real line. However, for *planar convex domains*, Conjecture 1also holds true (see [35]).

As for application of spectrality for estimating the Turán constant, essentially the following was proved in [45].

Theorem 3 (Kolountzakis–Révész [45]). *If H is a bounded open set in $\mathbb{R}^d$ which is spectral, then for the difference set $\Omega = H - H$ we have $\mathcal{T}_{\mathbb{R}^d}(\Omega) = |H|$. So in particular in such cases $|H|$ is uniquely determined by $\Omega = H - H$ even if H may not be unique.*

Originally, we formulated in [45] only the following special case of the above result. The possibility of getting Theorem 3 from essentially the same proof, was noted only in [47].

Corollary 1. (Kolountzakis–Révész [47]). *Let $\Omega \subseteq \mathbb{R}^d$ be a convex domain. If Ω is spectral, then it has to be a Stechkin–Turán type domain as well.*

Proof. First let us note that convex spectral domains are necessarily symmetric according to the result in [41]. Let now Ω be a symmetric convex domain. Then taking $H := \frac{1}{2}\Omega$, we have $H - H = \Omega$. Moreover, if Ω is spectral, say with spectrum Λ, then also H is clearly spectral with the dilated spectrum 2Λ. So Theorem 3 applies and we are done, in view of

$$|H| = \left|\frac{1}{2}\Omega\right| = |\Omega|/2^d. \qquad \square$$

Corollary 2 (Arestov–Berdysheva [6]). *Suppose the symmetric convex domain $\Omega \subseteq \mathbb{R}^d$ is a translational tile. Then it is a Stechkin–Turán domain.*

Proof of Corollary 2. We start with the following result which claims that every convex tile is also a lattice tile.

Theorem 4 (Venkov [63] and McMullen [52]). *Suppose that a convex body K tiles space by translation. Then it is necessarily a symmetric polytope and there is a lattice L such that*

$$K + L = \mathbb{R}^d.$$

A complete characterization of the tiling polytopes is also among the conclusions of the Venkov–McMullen Theorem but we do not need it here and choose not to give the full statement as it would require some more definitions.

So, if a convex domain is a tile, it is also a lattice tile, hence spectral by Theorem 2, and as such it is Stechkin–Turán type, by Corollary 1. $\square$

Remark 4. If one wants to avoid using the Venkov–McMullen Theorem in the proof of Corollary 2 one should enhance the assumption of Corollary 2 to state that Ω is a lattice tile. Arestov and Berdysheva [6] prove Corollary 2 without going through spectral domains.

The result of [5] about the hexagon being a Stechkin–Turán type domain is thus a special case of our Corollary 2, but not the result in [60] and [25] about the ball being Stechkin–Turán type. The ball, and essentially every smooth convex body, is known not to be spectral, in accordance with the Fuglede Conjecture (see [34]).

2.3. The notion of asymptotic uniform upper density on LCA groups

First let us recall the frequently used definition of asymptotic uniform upper density in $\mathbb{R}^d$. Let $K \subset \mathbb{R}^d$ be a *fat body*, i.e. a set with

$$0 \in \operatorname{int} K, \qquad K = \overline{\operatorname{int} K} \qquad \text{and} \qquad K \text{ compact}.$$

Then *asymptotic uniform upper density* of a measurable set $A \subset \mathbb{R}^d$ with respect to K is defined as

$$\overline{D}_K(A) := \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d} |A \cap (rK + x)|}{|rK|}. \tag{10}$$

It is obvious that the notion is translation invariant. It is also well known, that $\overline{D}_K(A)$ gives the same value for all nice — e.g., for all convex — bodies $K \subset \mathbb{R}^d$, although this fact does not seem immediate from the formulation.

Note also the following ambiguity in the use of densities in literature. Sometimes even in continuous groups a discrete set Λ is considered in place of A, and then the definition of the asymptotic uniform upper density of the sequence (discrete set) $\Lambda \subset \mathbb{R}^d$ is

$$\overline{D}^{\#}_K(\Lambda) := \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d} \#(\Lambda \cap (rK + x))}{|rK|}. \tag{11}$$

This motivates the general definition of asymptotic uniform upper densities of *measures*, say measure ν with respect to measure μ, whether equal or not. E.g., $\nu := \#$ is the cardinality or counting measure in (11), while $\mu := |\cdot|$ is just the volume. The general formulation of *a.u.u.d.* (this shorthand version standing for the expression *asymptotic uniform upper density*) in $\mathbb{R}^d$ is thus

$$\overline{D}_K(\nu) := \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d} \nu(rK+x)}{|rK|} . \tag{12}$$

Two notions of asymptotic uniform upper densities of measures ν with respect to a translation invariant, nonnegative, locally finite (outer) measure μ were defined in general LCA groups in [56]. Considering such groups are natural for they have an essentially unique translation invariant Haar measure μ_G (see e.g. [57]), what we fix to be our μ. By construction, μ is a Borel measure, and the sigma algebra of μ-measurable sets is just the sigma algebra of Borel mesurable sets, denoted by $\mathcal{B}$ throughout. To avoid questions of infinite measure, we consider the subset $\mathcal{B}_0$ of Borel measurable sets having compact closure.

Note if we consider the discrete topological structure on any abelian group G, it makes G a LCA group with Haar measure $\mu_G = \#$, the counting measure. This is the natural structure for $\mathbb{Z}^d$, e.g. On the other hand all σ-finite groups admit the same structure as well, i.e. are LCA groups with the discrete topology and the counting measure being the natural Haar measure. This unifies considerations. (Note that e.g. $\mathbb{Z}^d$ is not a σ-finite group since it is *torsion-free*, i.e. has no finite subgroups.)

The other measure ν can be defined, e.g. , as the *trace* of μ on the given set A, that is,

$$\nu(H) := \nu_A(H) := \mu_G(H\cap A),$$

or can be taken as the counting measure of the points included in some set Λ derived from the cardinality measure similarly:

$$\gamma(H) := \gamma_\Lambda(H) := \#(H\cap\Lambda).$$

Definition 4. Let G be a LCA group and $\mu := \mu_G$ be its Haar measure. If ν is another measure on G with the sigma algebra of measurable sets being $\mathcal{S}$, then we define

$$\overline{D}(\nu;\mu) := \inf_{C\Subset G} \sup_{V\in\mathcal{S}\cap\mathcal{B}_0} \frac{\nu(V)}{\mu(C+V)} . \tag{13}$$

In particular, if $A\subset G$ is Borel measurable and $\nu=\mu_A$ is the trace of the Haar measure on the set A, then we get

$$\overline{D}(A) := \overline{D}(\nu_A;\mu) := \inf_{C\Subset G} \sup_{V\in\mathcal{B}_0} \frac{\mu(A\cap V)}{\mu(C+V)} . \tag{14}$$

If $\Lambda \subset G$ is any (e.g. discrete) set and

$$\gamma := \gamma_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$$

is the counting measure of Λ, then we get

$$\overline{D}^{\#}(\Lambda) := \overline{D}(\gamma_\Lambda; \mu) := \inf_{C \Subset G} \sup_{V \in \mathcal{B}_0} \frac{\#(\Lambda \cap V)}{\mu(C + V)}\,. \tag{15}$$

Proposition 1. *Let K be any convex body in $\mathbb{R}^d$ and normalize the Haar measure of $\mathbb{R}^d$ to be equal to the volume $|\cdot|$. Let ν be any measure with sigma algebra of measurable sets $\mathcal{S}$. Then we have*

$$\overline{D}(\nu; |\cdot|) = \overline{D}_K(\nu)\,. \tag{16}$$

The same statement applies also to $\mathbb{Z}^d$. For heuristical considerations and comparisons to existing notions and approaches, as well as for the proofs and for some examples we refer to [56].

2.4. Packing, covering, tiling and asymptotic uniform upper density

Proposition 2. *Assume that $H \in \mathcal{B}$ and that $H + \Lambda \leq G$ (H packs G with $\Lambda \subset G$), i.e.,*

$$(H - H) \cap (\Lambda - \Lambda) \subseteq \{0\}.$$

Then Λ must satisfy

$$\overline{D}^{\#}(\Lambda) \leq 1/\mu(H).$$

Proof. Let $B \Subset H$ and $V \in \mathcal{B}_0$ be arbitrary. Denote $L := \Lambda \cap V$. Then

$$B + V \supset B + L = \bigcup_{\lambda \in L} (B + \lambda),$$

and this union being disjoint (as

$$(B + \lambda) \cap (B + \lambda') \subset (H + \lambda) \cap (H + \lambda') = \emptyset$$

unless $\lambda = \lambda'$), from additivity and translation invariance of the Haar measure we obtain

$$\mu(B + V) \geq \mu(B + L) = \#L\mu(B).$$

This yields

$$\#L/\mu(B + V) \leq 1/\mu(B),$$

therefore

$$\sup_{V \in \mathcal{B}_0} \#(\Lambda \cap V)/\mu(B + V) \leq 1/\mu(B).$$

Approximating $\mu(H)$ by $\mu(B)$ of $B \Subset H$ arbitrarily closely, we thus obtain

$$\inf_{B \Subset H} \sup_{V \in \mathcal{B}_0} \#(\Lambda \cap V)/\mu(B+V) \leq 1/\mu(H).$$

However, $\overline{D}^{\#}(\Lambda)$ is a similar infimum extended to a larger family of compact sets, so it can not be larger, and the assertion follows. □

Proposition 3. *Assume that $H \in \mathcal{B}_0$ and that it covers G with $\Lambda \subset G$ ("$H + \Lambda \geq G$"), i.e., $H + \Lambda$ contains μ-almost all points of G. Then we necessarily have*

$$\overline{D}^{\#}(\Lambda) \geq 1/\mu(H).$$

Proof. Let $C \Subset G$ be arbitrary, and take $W := \overline{H} - C$, which is again a compact set of G by assumption on H and in view of the continuity of the group operation on G. So the Theorem in §2.6.7. on p. 52 of [57] applies to the compact set W and to any given $\varepsilon > 0$, and we find some Borel measurable set $U = U_{\varepsilon,C} \in \mathcal{B}_0$ satisfying

$$\mu(U - W) < (1+\varepsilon)\mu(U).$$

Consider now

$$V := V_{\varepsilon,C} := U - H \in \mathcal{B}_0.$$

Then

$$\mu(C+V) = \mu(C+U-H) \leq \mu(U - (\overline{H} - C)) = \mu(U-W) < (1+\varepsilon)\mu(U).$$

Denote $L := \Lambda \cap V$. Then

$$L = \{\lambda \in \Lambda : \ \exists h \in H, \ \lambda + h \in U\} = \{\lambda \in \Lambda : \ (\lambda + H) \cap U \neq \emptyset\},$$

and so clearly

$$U \cap (\Lambda + H) \subset \bigcup_{\lambda \in L} (\lambda + H),$$

while

$$U_0 := U \setminus (U \cap (\Lambda + H))$$

is of measure zero by assumption on the covering property of H with Λ. So in all

$$\mu(U) \leq \mu(U_0) + \sum_{\lambda \in L} \mu(\lambda + H) = 0 + \#L\mu(H) \text{ and } \mu(C+V) < (1+\varepsilon)\#L\mu(H).$$

It follows that with the arbitrarily chosen $C \Subset G$ and for all $\varepsilon > 0$ we have

$$\frac{\#(\Lambda \cap V_{\varepsilon,C})}{\mu(C + V_{\varepsilon,C})} \geq \frac{1}{(1+\varepsilon)^2 \mu(H)}$$

with a certain $V_{\varepsilon,C} \in \mathcal{B}_0$, so taking supremum over all $V \in \mathcal{B}_0$ we even get

$$\sup_{V\in\mathcal{B}_0} \#(\Lambda \cap V)/\mu(C+V) \geq 1/\mu(H).$$

This holding for all $C \Subset G$, taking infimum over C does not change the lower estimation, so finally we arrive at

$$\overline{D}^{\#}(\Lambda) \geq 1/\mu(H),$$

whence the proposition follows. □

Tiling means simultaneously packing and covering. Therefore, from the above two propositions the following corollary obtains immediately.

Corollary 3. *Assume that $H \in \mathcal{B}_0$ tiles with the set of translations $\Lambda \subset G: \ H + \Lambda = G$. Then we also have $\overline{D}^{\#}(\Lambda) = 1/\mu(H)$.*

3. Upper bound from packing

3.1. Bounds from packing in some special cases

In the type of results we now present, some kind of "packing" condition is assumed on Ω which leads to an upper bound for $\mathcal{T}_G(\Omega)$. The first result we present here is taken from [47]: we repeat it here for sake of a simpler situation which nevertheless may shed light on the general case.

Theorem 5 (Kolountzakis–Révész [47]). *Suppose that G is a compact abelian group, $\Lambda \subseteq G$, $\Omega \subseteq G$ is a 0-symmetric open set and $(\Lambda - \Lambda) \cap \Omega \subseteq \{0\}$. Suppose also that $f \in L^1(G)$ is a continuous positive definite function supported on Ω. Then*

$$\int_G f(x)\,dx \leq \frac{\mu(G)}{\#\Lambda} f(0). \tag{17}$$

In other words,

$$\mathcal{T}_G(\Omega) \leq \mu(G)/\#\Lambda.$$

Observe that the conditions imply that Λ is finite.

Proof. Define $F : G \to \mathbb{C}$ by

$$F(x) = \sum_{\lambda,\mu\in\Lambda} f(x + \lambda - \mu).$$

In other words, $F = f * \delta_\Lambda * \delta_{-\Lambda}$, where δ_A denotes the finite measure on G that assigns a unit mass to each point of the finite set A. It follows that

$$\widehat{F} = \widehat{f}\left|\widehat{\delta_\Lambda}\right|^2 \geq 0$$

so that F is continuous and positive definite. Moreover, we also have

$$\operatorname{supp} F \subseteq \operatorname{supp} f + (\Lambda - \Lambda) \subseteq \Omega + (\Lambda - \Lambda) \tag{18}$$

and

$$F(0) = \#\Lambda f(0), \tag{19}$$

since $\Omega \cap (\Lambda - \Lambda) \subseteq \{0\}$. Finally

$$\int_G F = \#\Lambda^2 \int_G f. \tag{20}$$

Applying the trivial upper bound

$$\int_G F \le F(0)\mu(\Omega + (\Lambda - \Lambda))$$

to the positive definite function F and using (19) and (20) we get

$$\int_G f \le \frac{\mu(\Omega + (\Lambda - \Lambda))}{\#\Lambda} f(0). \tag{21}$$

Estimating trivially $\mu(\Omega + (\Lambda - \Lambda))$ from above by $\mu(G)$ we obtain the required

$$\mathcal{T}_G(\Omega) \le \mu(G)/\#\Lambda. \qquad \square$$

Corollary 4. *Let G be a compact abelian group and suppose $\Omega, H, \Lambda \subseteq G$, $H + \Lambda \le G$ is a packing at level 1, that $\Omega \subseteq H - H$ and that $f \in \mathcal{F}(\Omega)$. Then (17) holds.*

In particular, if $H + \Lambda = G$ is a tiling, we have

$$\mathcal{T}_G(\Omega) \le \mu(H). \tag{22}$$

Proof. Since $H + \Lambda \le G$ it follows that

$$(H - H) \cap (\Lambda - \Lambda) = \{0\}.$$

Since $\Omega \subseteq H - H$ by assumption it follows that Ω and $\Lambda - \Lambda$ have at most 0 in common. Theorem 5 therefore applies and gives the result. If $H + \Lambda = G$ then $\mu(G)/\#\Lambda = \mu(H)$ and this proves (22). $\square$

A partial extension of the result to the non-compact case was also worked out in [47]. However, it used the notion of a.u.u.d. which then restricted considerations to classical groups only.

Theorem 6 (Kolountzakis–Révész [47]). *Suppose that G is one of the groups $\mathbb{R}^d$ or $\mathbb{Z}^d$, that $\Lambda \subseteq G$ is a set of asymptotic uniform upper density $\rho > 0$, and $\Omega \subseteq G$ is a 0-symmetric open set such that*

$$\Omega \cap (\Lambda - \Lambda) \subseteq \{0\}.$$

Let also $f \in L^1(G)$ be a continuous positive definite function on G whose support is a compact set contained in Ω. Then

$$\int_G f(x)\,dx \le \frac{1}{\rho} f(0). \tag{23}$$

In other words, $\mathcal{T}_G(\Omega) \le 1/\rho$.

For sharpness and examples we refer to [47]. Note that some parts of the proof in [47] for this theorem will be used in the proof for our more general result, see the end of Lemma 1.

3.2. Bounds from packing in general LCA groups

Now we have ready a notion of a.u.u.d. as defined in §2.3. With this notion, we have the following general version of the above particular results.

Theorem 7. *Let $\Omega \subset G$ be a 0-symmetric open neighborhood of 0 and $\Lambda \subset G$ be a subset satisfying the "packing-type condition" $\Omega \cap (\Lambda - \Lambda) = \{0\}$. If $\rho := \overline{D}^{\#}(\Lambda) > 0$, then we have $\mathcal{T}_G(\Omega) \le 1/\rho$.*

Proof. Let $\varepsilon > 0$ be fixed small, but arbitrary. By Theorem 1, there exists $f \in \mathcal{F}(\Omega)$, normalized to satisfy

$$f(0) = 1, \quad \text{with} \quad \int_G f > \mathcal{T}_G(\Omega) - \varepsilon.$$

Denote $S := \operatorname{supp} f$, which is a compact subset of Ω in view of $f \in \mathcal{F}(\Omega)$.

In the following we consider a compact, 0-symmetric neighborhood of 0 which we denote by W. We require W to be the closure of a 0-symmetric open subset O containing $S - S$ in it.

Let us consider the subgroup G_0 of G, generated by W. Here we repeat the construction on [57, p. 52]. First, by [57, Lemma 2.4.2], $\langle W \rangle = G_0$ implies that there exists a closed subgroup $K \le G_0$ which is isomorphic to $\mathbb{Z}^k$ with some natural number k and satisfies $W \cap K = \{0\}$, so that $H := G_0/K$ is then compact. Let ϕ be the natural homomorphism (projection) of G_0 onto H.

Since $S - S \subset \operatorname{int} W$, there exists an open neighborhood X_1 of S such that $X_1 - X_1 \subset W$, whence

$$\phi(x) - \phi(y) = 0 \in H \quad \text{with} \quad x, y \in X_1$$

would imply

$$x - y \in \ker \phi = K, \quad \text{i.e.} \quad x - y \in K \cap W = \{0\}$$

and thus $x = y$. In other words, ϕ is a homeomorphism on X_1, and $Y_1 := \phi(X_1) \subset H$ is open. By compactness of H, finitely many translates of Y_1, say Y_1, Y_2, ..., Y_r will cover H, and there are open subsets X_i of G_0 with compact closure such that ϕ maps X_i onto Y_i homeomorphically for each $i = 1, \dots, r$. If

$$Y'_1 := Y_1, \quad Y'_i := Y_i \setminus \Big(\bigcup_{j=1}^{i-1} Y_j \Big) \quad (i = 2, \dots, r)$$

and

$$X'_i := X_i \cap \phi^{-1}(Y'_i) \quad (i = 1, \dots, r),$$

then

$$E := \bigcup_{i=1}^{r} X'_i$$

is a Borel set in G_0 with compact closure, ϕ is one-to-one on E, and $\phi(E) = H$, i.e., each $x \in G_0$ can be uniquely represented as $x = e + n$, with $e \in E$ and $n \in K$. We will call this *the standard decomposition* of the element $x \in G_0$.

In the following, we put

$$\|n\| := \max_{1 \le j \le k} |n_j|, \quad \text{where } (n_1, \dots, n_k) \in \mathbb{Z}^k$$

is the element corresponding to $n \in K$ under the fixed isomorphism from K to $\mathbb{Z}^k$. Note also that

$$S \subset X_1 = X'_1 \subset E$$

and that $\overline{E}$ is compact. Hence also $E + E - E$ has compact closure, and the discrete set K can intersect it only in finitely many points. So we put

$$s := \max\{\|n\| : \ n \in (E + E - E) \cap K\},$$

which is finite. Next we define

$$V_N := \bigcup \Big\{ E + n : \ n \in K, \ \|n\| \le N \Big\} \quad (N \in \mathbb{N}). \tag{24}$$

Note that

$$\mu(V_N) = (2N+1)^k \mu(E) \quad \text{for all } \ N \in \mathbb{N},$$

and the V_N are Borel sets with compact closure. Let $N, M \in \mathbb{N}$, and

$$x = e + n, \quad y = f + m$$

be the standard decomposition of two elements $x \in V_N$ and $y \in V_M$ in terms of $E + K$, that is, $e, f \in E$ and $n, m \in K$. Then

$$x + y = e + f + n + m = g + p + n + m,$$

where $e + f$ has the standard decomposition $g + p$, and so

$$p = e + f - g \in (E + E - E),$$

therefore in $(E+E-E)\cap K$, and we find $\|p\|\le s$. In all, we find $x+y\in E+q$, where $q:=p+n+m$ satisfies $\|q\|\le N+M+s$, and so $x+y\in V_{N+M+s}$. It follows that

$$V_N+V_M\subset V_{N+M+s}.$$

For the sake of the next lemma we introduce a notation extending the notion of the Turán constant from open sets even to Borel (i.e. Haar) measurable sets V. For this we pick up the function class $\mathcal{F}(V)$ with continuous positive definite functions compactly supported in V and write

$$\mathcal{T}_G(V):=\sup\Big\{\int f/f(0):\ f\in\mathcal{F}(V)\Big\}.$$

L e m m a 1. *With the above notations we have*

$$\mathcal{T}_{G_0}(V_N)\le(N+s+1)^k\mu(E)$$

for arbitrary $N\in\mathbb{N}$.

P r o o f. Our proof will run analogously to [47, Proposition 3], but, since we consider here measurable sets, we give a full proof.

Recall that the natural homeomorphism (projection) $\phi:G_0\to G_0/K=:H$ maps surjectively onto H with H a compact subgroup and $K\cong\mathbb{Z}^k$ a closed discrete subgroup, hence a LCA group itself. By definition of the topology of G_0/K, ϕ is an open and continuous mapping. (Compare §§B.2 and B.6 in [57, Appendix B].)

For the determination of the Turán constants, the choice of the Haar measure is relevant. Haar measures are unique up to a constant factor: we can always choose the Haar measures μ_K and $\mu_{G_0/K}$ so that $d\mu_{G_0}=d\mu_K\,d\mu_{G_0/K}$, in the sense of (2) in [57, §2.7.3]. Considering G_0/K as a factor group, this is the natural choice: for distinguishing from the normalized Haar measure of $G_0/K=H$, we denote $\nu:=\mu_{G_0/K}$. On the other hand fixing a particular Haar measure μ_H of H always leaves open the question of compatibility with the fixed measure $\nu=\mu_{G_0/K}$ (and the mapping ϕ). Recall that under our convention, the discrete group K admits $\mu_K=\#$, while for the compact group $H\le G_0$ the natural normalized Haar measure μ_H has $\mu_H(H)=1$. Let us denote $C:=d\mu_H/\,d\nu$.

Obviously

$$V_N^{(g)}:=V_N\cap(K+g)$$

is a discrete, hence closed subset for any $g\in G$, together with the full coset $K+g$. Let us choose arbitrarily a representative $g(h)\in G_0$ of each coset $\phi^{-1}(h)$ of K to all $h\in H$. Now for any uniformly continuous function (so in particular to any compactly supported continuous function) $f : G_0 \to \mathbb{C}$ we can define with $\mu_K = \#_K$

$$F(h) := \int_K f(g(h) + k)\, d\mu_K(k) = \tag{25}$$

$$= \int_{\varphi^{-1}(h)} f(x)\, d\mu_K(x - g(h)) \ \Big(= \sum_{k \in K + g(h)} f(k) \Big).$$

From now on let $\operatorname{supp} f \Subset V_N$. Since f is compactly supported, the sum is always finite, and the function $F : H \to \mathbb{C}$ is continuous,

$$F(0) = \int_K f\, d\mu_K,$$

and by Fubini's Theorem (denoting $[g(h)] := g(h) + K = \phi^{-1}(h)$ the coset of K in G_0, i.e. the element of G_0/K, corresponding to h)

$$\int_H F(h)\, d\mu_H(h) = \int_H \int_K f(g(h) + k)\, d\mu_K(k) C\, d\nu(h) = \tag{26}$$

$$= C \int_{H \times K} f(g(h) + k)\, d\mu_K(k)\, d\mu_{G_0/K}([g(h)]) = C \int_{G_0} f\, d\mu_{G_0}\,,$$

taking into account the choice of normalization of the Haar measures for K and G_0/K.

Next we prove that F is positive definite on H in case f is positive definite on G_0. Indeed, for any character χ on H there is a character $\gamma := \chi \circ \phi$ on G_0, and applying (26) to $f\gamma$ yields

$$\int_H F(h)\chi(h)\, d\mu_H(h) = C \int_G f(g)\gamma(g)\, d\mu_{G_0}(g) \geq 0\,.$$

Note that

$$\int_H F\, d\mu_H \leq F(0)\mu_H(H) = F(0)$$

in view of the trivial estimate and the normalization of the Haar measure μ_H. Furthermore, also $f|_K$ is positive definite on K, hence we also have

$$F(0) = \int_{K \cap V_N} f \ \ d\mu_K \leq \mathcal{T}_K(K \cap V_N) f(0).$$

Comparing these inequalities with (26) yields

$$C \int_{G_0} f \ \ d\mu_{G_0} \leq \mathcal{T}_K(K \cap V_N) f(0),$$

and taking supremum of $\int_{G_0} f\, d\mu_{G_0}/f(0)$ yields

$$\mathcal{T}_{G_0}(V_N) \leq \frac{1}{C} \mathcal{T}_K(V_N \cap K), \qquad C := \frac{d\mu_H}{d\nu}\,. \tag{27}$$

Next, let us compute C. It suffices to consider one test function, which we chose to be χ_E, the characteristic function of E. We obtain

$$\mu(E) = \mu_{G_0}(E) = \int_{G_0} \chi_E \, d\mu_{G_0} = \tag{28}$$
$$= \int_{G_0/K} \int_K \chi_E(x+y)\, d\mu_K(y)\, d\mu_{G_0/K}([x]) =$$
$$= \int_{G_0/K} 1 \, d\mu_{G_0/K}([x]) = \mu_{G_0/K}(G_0/K),$$

in view of

$$\#\{y \in K \ : \ x+y \in E\} = 1$$

and by the above unique representation (the standard decomposition) of G_0 as $E+K$. It follows that

$$C = \frac{\mu_H(H)}{\mu_{G_0/K}(G_0/K)} = \frac{1}{\mu(E)}\,, \tag{29}$$

and we are led to

$$\mathcal{T}_{G_0}(V_N) \le \mu(E)\mathcal{T}_K(V_N \cap K). \tag{30}$$

Let us write from now on

$$Q_M := Q_{2M}(0) := \{m : \ m \in K, \ \|m\| \le M\}.$$

We know that $V_N \cap K \subset Q_{N+s}$, because for any $e \in E \cap K$ we necessarily have $\|e\| \le s$. These observations yield

$$\mathcal{T}_{G_0}(V_N) \le \mu(E)\mathcal{T}_K\Big(\{m \in K : \ \|m\| \le N+s\}\Big) = \mu(E)\mathcal{T}_{\mathbb{Z}^k}(Q_{N+s}),$$

by the isomorphism of K and $\mathbb{Z}^k$. It remains to recall that for $Q_L = Q_{2L}(0)$ we have

$$\mathcal{T}_{\mathbb{Z}^k}(Q_L) \le (L+1)^k,$$

in view of [47, formula (26)] from the proof of Theorem 6 in [47]. □

Lemma 2. *Let V be any Borel measurable subset of G with compact closure and let ν be a Borel measure on G with $\overline{D}_G(\nu;\mu) = \rho > 0$. If $\varepsilon > 0$ is given, then there exists $z \in G$ such that*

$$\nu(V+z) \ge (\rho-\varepsilon)\mu(V). \tag{31}$$

Proof. Let $D := -V$. D is a Borel set with compact closure $\overline{D} \Subset G$. So by Definition 4 we can find, according to the assumption on $\overline{D}_G(\nu;\mu) = \rho$, some $Z \in \mathcal{B}_0$ (i.e. $Z \in \mathcal{B}$ with $\mu(Z) < \infty$) which satisfies

$$\nu(Z) \ge (\rho-\varepsilon)\mu(Z+\overline{D}) \ge (\rho-\varepsilon)\mu(Z+D). \tag{32}$$

We can then write

$$\int \chi_Z(t)\, d\nu(t) \geq (\rho-\varepsilon)\mu(Z+D). \tag{33}$$

For $t \in Z$ and $u \in D(=-V)$ also $t+u \in Z+D$ holds. Hence $\chi_{Z+D}(t+u) = 1$, and we get

$$\chi_Z(t) \leq \frac{1}{\mu(D)} \int \chi_{Z+D}(t+u)\chi_D(u)\, d\mu(u) \tag{34}$$

for all $t \in Z$. But for $t \notin Z$ $\chi_Z(t) = 0$ and the right hand side being nonnegative, inequality (34) holds for all $t \in G$, hence (33) implies

$$(\rho-\varepsilon)\mu(Z+D) \leq \frac{1}{\mu(D)} \int\int \chi_{Z+D}(t+u)\chi_D(u)\, d\mu(u)\, d\nu(t) = \tag{35}$$
$$= \int \chi_{Z+D}(y)\Big(\frac{1}{\mu(D)}\int \chi_D(y-t)\, d\nu(t)\Big)\, d\mu(y) =$$
$$= \int \chi_{Z+D}(y) f(y)\, d\mu(y) = \int_{Z+D} f\, d\mu \quad \Big(\text{with } f(y) := \frac{\nu(y-D)}{\mu(D)}\Big).$$

It follows that there exists $z \in Z+D \subset G$ satisfying $f(z) \geq (\rho-\varepsilon)$. That is, we find

$$\nu(z+V) = \nu(z-D) \geq (\rho-\varepsilon)\mu(D) = (\rho-\varepsilon)\mu(V). \qquad \square$$

Lemma 3. *If $\overline{D}_G(\nu;\mu) = \rho > 0$ with $\mu = \mu_G$ and ν any given Borel measure on the LCA group G, then for any open subgroup G' of G, compact $D \Subset G'$ and $\varepsilon > 0$ there exist $x \in G$ and $Z \subset G'$, $Z \in \mathcal{B}_0$ so that*

$$\nu(Z+x) \geq (\rho-\varepsilon)\mu(Z+D).$$

Remark 5. One would be tempted to assert that on some coset $G'+x$ of G' the relative density of ν must be at least $\rho-\varepsilon$, i.e.

$$\overline{D}_{G'}(\nu_x;\mu|_{G'}) = \rho-\varepsilon \quad \text{with } \nu_x(Z) := \nu(Z+x) \text{ for } Z \subset G' \text{ and } x \in G.$$

However, this stronger statement does not hold true. Consider, e.g.,

$$G = \mathbb{Z}^2, \quad G' := \mathbb{Z}\times\{0\}, \quad A := \{(k,l) :\ k \in \mathbb{N},\ l \geq k\},$$

and $\nu := \mu_A$ the trace of the counting measure μ of $\mathbb{Z}^2$ on A. Since A contains arbitrarily large squares, $\overline{D}(\nu;\mu) = 1$. (In fact, ν has a positive asymptotic density $\delta(\nu;\mu) = 1/8$, too, where

$$\delta(\nu,\mu) := \lim_{r\to\infty} \nu(B(0,r)/\mu(B(0,r)$$

whenever the limit exists.) However, for each coset $G'+x = \mathbb{Z}\times\{m\}$ of G' the intersection $A \cap G'$ is only finite and $\overline{D}_{G'}(\nu_x;\mu|_{G'}) = 0$.

P r o o f o f L e m m a 3. By condition, for $D \Subset G' \leq G$ there exists $V \Subset G$ such that

$$\nu(V) \geq (\rho - \varepsilon)\mu(V + D). \tag{36}$$

Let now U be an open set containing $V+D$ and with compact closure $\overline{U} \Subset G$. Because the cosets of G' cover G, we have

$$V + D = \bigcup_{x \in G} \Big((V + D) \cap (G' + x) \Big) \subset \bigcup_{x \in G} \Big(U \cap (G' + x) \Big).$$

Since both U and G' are open, and $V + D$ is compact, the covering on the right hand side has a finite subcovering; moreover, we can select all covering cosets only once, hence arrive at a disjoint covering

$$V + D \subset \bigcup_{j=1}^{m} U_j \qquad (U_j := U \cap (G' + x_j), \quad j = 1, \ldots, m).$$

Take now

$$V_j := U_j \cap (V + D).$$

As the U_j are disjoint, so are the V_j; and as the U_j together cover $V + D$, so do the V_j. So we have the disjoint covering

$$V + D = \bigcup_{j=1}^{m} V_j.$$

Furthermore, we can write

$$V_j = (V + D) \cap (G' + x_j)$$

in place of the above definition of V_j, that is, we can drop the set U from the intersection defining V_j. Indeed, it is clear that

$$V_j \subset (V + D) \cap (G' + x_j) \qquad (j = 1, \ldots, m),$$

and as already the cosets $G' + x_j$ were chosen to be disjoint, we have

$$V_i \cap (G' + x_j) = \emptyset \qquad \text{unless} \quad i = j,$$

hence

$$V_j \subseteq (V + D) \cap (G' + x_j) = \bigcup_{i=1}^{m} V_i \cap (G' + x_j) = V_j \cap (G' + x_j) = V_j$$

and we have equality throughout.

By this we can see that the sets V_j are necessarily compact sets for all $j = 1, \ldots, m$. Indeed, $V + D$ is compact and $G' + x_j$ is closed, as G'

is closed, the latter being a general property of open subgroups in a locally compact group because

$$G' = G \setminus \bigcup_{(G'+y)\cap G'=\emptyset} (G'+y)$$

expresses the open subgroup as a complement of an open set.

Next we define $W_j := V \cap V_j$. Plainly, $W_j \Subset G$ and disjoint, and

$$V = \bigcup_{j=1}^{m} W_j.$$

Moreover, $W_j + D = V_j$; indeed,

$$W_j + D = (V \cap (G' + x_j)) + D = (V + D) \cap (G' + x_j)$$

since $D \subset G'$ and $G' \leq G$. So we find

$$\nu(V) = \sum_{j=1}^{m} \nu(W_j) \tag{37}$$

and also

$$\mu(V + D) = \sum_{j=1}^{m} \mu(V_j) = \sum_{j=1}^{m} \mu(W_j + D) = \sum_{j=1}^{m} \mu(W_j - x_j + D). \tag{38}$$

Collecting (37), (36) and (38) we conclude

$$\sum_{j=1}^{m} \nu(W_j) \geq (\rho - \varepsilon) \sum_{j=1}^{m} \mu(W_j - x_j + D), \tag{39}$$

hence for some appropriate $j \in [1, m]$ we also have

$$\nu(W_j) \geq (\rho - \varepsilon)\mu(W_j - x_j + D).$$

Taking $Z := W_j - x_j$ and $x = x_j$ concludes the proof. □

E n d o f t h e p r o o f o f T h e o r e m 7. Let now $\nu := \delta_\Lambda$ be the counting measure of the (discrete) set $\Lambda \subset G$. Then

$$\overline{D}_G(\nu; \mu) = \overline{D}_G^{\#}(\Lambda) = \rho > 0$$

and Lemma 2 applies providing some $z := z_N \in G$ with

$$M := \#(\Lambda \cap (V_N + z)) \geq (\rho - \varepsilon)\mu(V_N). \tag{40}$$

Take now

$$\Lambda' := \Lambda \cap (V_N + z) = \{\lambda_m : \ m = 1, \dots, M\},$$

and put $F := f \star \delta_{\Lambda'} \star \delta_{-\Lambda'}$, i.e.,

$$F(x) := \sum_{m=1}^{M} \sum_{n=1}^{M} f(x + \lambda_m - \lambda_n),$$

which is a positive definite continuous function, compactly supported in

$$S + (V_N + z) - (V_N + z) = S + V_N - V_N =$$
$$= S + E - E + Q_{2N} \subset E + E - E + Q_{2N} \subset V_{2N+s}.$$

Furthermore, as $S \subset G_0$,

$$\int_{G_0} F = M^2 \int_{G_0} f \geq M^2(\mathcal{T}_G(\Omega) - \varepsilon) \tag{41}$$

and

$$F(0) = \sum_{m=1}^{M} \sum_{n=1}^{M} f(\lambda_m - \lambda_n) = Mf(0) = M, \tag{42}$$

because if $\lambda_m - \lambda_n \in S$, then

$$\lambda_m - \lambda_n \in S \cap (\Lambda - \Lambda) \subset \Omega \cap (\Lambda - \Lambda) = \{0\} \quad \text{and} \quad \lambda_m = \lambda_n,$$

i.e., $n = m$. By this construction we derive that

$$\mathcal{T}_{G_0}(V_{2N+s}) \geq \frac{1}{F(0)} \int_{G_0} F \geq M(\mathcal{T}_G(\Omega) - \varepsilon) \geq \tag{43}$$
$$\geq (\rho - \varepsilon)(\mathcal{T}_G(\Omega) - \varepsilon)\mu(V_N) = (\rho - \varepsilon)(\mathcal{T}_G(\Omega) - \varepsilon)(2N+1)^k \mu(E).$$

On the other hand, Lemma 1 provides us

$$\mathcal{T}_{G_0}(V_{2N+s}) \leq (2N + 2s + 1)^k \mu(E). \tag{44}$$

On comparing (43) and (44) we conclude

$$(\rho - \varepsilon)(\mathcal{T}_G(\Omega) - \varepsilon)(2N+1)^k \mu(E) \leq (2N + 2s + 1)^k \mu(E),$$

that is,

$$\mathcal{T}_G(\Omega) - \varepsilon \leq \frac{1}{\rho - \varepsilon}\Big(\frac{2N + 2s + 1}{2N+1}\Big)^k.$$

Letting $N \to \infty$ and $\varepsilon \to 0$ gives the assertion. □

Corollary 5. *Suppose that $\Omega \subset G$ is an open and symmetric set and $\Omega = H - H$, where H tiles space with $\Lambda \subset G$. Moreover, assume that H has compact closure $\overline{H} \Subset G$ and is measurable, i.e. $H \in \mathcal{B}_0$. Then $\mathcal{T}_G(\Omega) = \mu(H)$.*

Proof. First, observe that for any $A \Subset H$ we have

$$f := \chi_A * \chi_{-A} \in \mathcal{F}_{\&}(\Omega).$$

Indeed, $\widetilde{\chi_A} = \chi_{-A}$ because χ_A is real valued, also $\chi_A \in L^2(G)$, and such a convolution representation guarantees that $f \in C(G) \cap L^1(G)$ is positive definite; furthermore, if $f(x) \neq 0$, then necessarily $x = a - a'$ with some $a, a' \in A \subset H$, hence $\operatorname{supp} f \subset \Omega$.

Therefore, calculating with the admissible function f, we find

$$\mathcal{T}_G(\Omega) \geq \int_G f/f(0) = \mu(A)^2/\mu(A) = \mu(A).$$

Since H is Borel measurable, its measure can be approximated arbitrarily closely by measures of inscribed compact sets A: therefore, taking supremum over compact sets $A \Subset H$, we obtain the lower estimate $\mathcal{T}_G(\Omega) \geq \mu(H)$.

On the other hand, $H + \Lambda = G$ entails that H packs with Λ, and so an application of Theorem 7 gives

$$\mathcal{T}_G(\Omega) \leq 1/\overline{D}^{\#}(\Lambda),$$

while by Corollary 3, we have $\overline{D}^{\#}(\Lambda) = \mu(H)$, whence the assertion follows. □

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Экстремальная задача Турана на локально компактных абелевых группах

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Пусть G локально компактная абелева группа (ЛКА группа) и Ω — открытое множество, симметрическое относительно 0. Пусть $\mathcal{F} := \mathcal{F}(\Omega)$ обозначает множество всех непрерывных положительно определённых функций $f : G \to \mathbb{R}$ с носителем G. Тогда константа Турана множества Ω определяется следующим соотношением:

$$\mathcal{T}(\Omega) := \sup \left\{ \int_\Omega f : \ f \in \mathcal{F}(\Omega), \ f(0) = 1 \right\}.$$

М. Колунзакис и автор показали, что структурные свойства подмножеств Ω — такие как спектральность, разбиения или упаковки с помощью некоторого множества Λ в конечных, компактных или Евклидовых (т.е. $\mathbb{R}^d$) группах и в $\mathbb{Z}^d$ влекут выполнение оценок $\mathcal{T}(\Omega)$. Однако в упомянутых оценках естественную роль играло некоторое понятие размера, т.е. плотности Λ, и поэтому для групп, в которых такое понятие неясно, неясными оставались и оценки. В настоящей работе применяется недавно возникшее обобщенное понятие асимптотической равномерной верхней плотности, и это позволяет более общее исследование константы Турана в связи с вышеуказанными структурными свойствами. Наш основной результат обобщает некоторый результат Арестова и Бердышевой (независимо доказанный также и автором совместно с Колунзакисом) о том, что для выпуклых разбиений Евклидова пространства выполняется

$$\mathcal{T}_{\mathbb{R}^d}(\Omega) = |\Omega|/2^d.$$

Наш подход позволяет заменить $\mathbb{R}^d$ на любую ЛКА группу, избавиться от условия выпуклости, а также ослабить условие разбиваемости до некоторого условия типа упаковки и положительности асимптотической равномерной верхней плотности множества Λ.