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Approximation by Nörlund means of quadratical partial sums of double Walsh–Fourier series

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Dedicated to Professor Ferenc Móricz on the occasion of his seventieth birthday

Abstract. In this article we discuss the Nörlund means of cubical partial sums of Walsh–Fourier series of a function in L^p $(1 \le p \le \infty)$. We investigate the rate of the approximation by this means, in particular, in $\mathrm{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \le p \le \infty$. In case $p = \infty$ by L^p we mean C_W, the collection of the uniformly W-continuous functions. Our main theorems state that the approximation behavior of the two-dimensional Walsh-Nörlund means is so good as the approximation behavior of the one-dimensional Walsh-Nörlund means.

As special cases, we get the Nörlund logarithmic means of cubical partial sums of Walsh–Fourier series discussed recently by GÁT and GOGINAVA [5] in 2004 and the (C,β)-means of Marcinkiewicz type with respect to double Walsh–Fourier series discussed by GOGINAVA [10].

Earlier results on one-dimensional Nörlund means of the Walsh–Fourier series was given by MÓRICZ and SIDDIQI [14].

1. Introduction

Now, we give a brief introduction to the Walsh–Fourier analysis [15, 1].

Let denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbb{Z}_2$ is given in the way that the measure of a singleton

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is 1/2. Let

$$G := \mathop{\times}_{k=0}^{\infty} \mathbb{Z}_2,$$

G is called the Walsh group. The elements of G are sequences

$$x = (x_0, x_1, \ldots, x_k, \ldots) \quad \text{with} \quad x_k \in \{0,1\} \ (k \in \mathbb{N}).$$

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure (denoted by μ) and the topology are the product measure and topology. Dyadic intervals are defined by

$$I_0(x) := G, \quad I_n(x) := \Big\{ y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1} \ldots) \Big\}$$

for $x \in G$, $n \in \mathbb{P}$. They form a base for the neighborhoods of G. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of G and

$$I_n := I_n(0) \quad \text{for} \ \ n \in \mathbb{N}.$$

Set $e_i := (0, \ldots, 0, 1, 0, \ldots)$, where the ith coordinate is 1 and the rest are 0 $(i \in \mathbb{N})$.

Let L^p denote the usual Lebesgue spaces on G (with the corresponding norm $\|\cdot\|_p$). For the sake of brevity in notation, we agree to write L^∞ instead of C_W and set

$$\|f\|_\infty := \sup\{|f(x)| : x \in G\}.$$

Next, we define the modulus of continuity in L^p, $1 \le p \le \infty$, of a function $f \in L^p$ by

$$\omega_p(\delta, f) := \sup_{|t| < \delta} \|f(\cdot + t) - f(\cdot)\|_p, \quad \delta > 0.$$

The Lipschitz classes in L^p for each $\alpha > 0$ are defined by

$$\operatorname{Lip}(\alpha, p) := \Big\{ f \in L^p : \ \omega_p(\delta, f) = O(\delta^\alpha) \text{ as } \delta \to 0 \Big\}.$$

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, \ k \in \mathbb{N}).$$

Let the Walsh–Paley functions be the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0,1\} \ (i \in \mathbb{N}),$$

where only a finite number of n_i's different from zero. Let the order of $n > 0$ be denoted by

$$|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}.$$

Then the Walsh–Paley functions are $w_0 = 1$ and for $n \geq 1$,

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

The Dirichlet kernels are defined by

$$D_n^w := \sum_{k=0}^{n-1} w_k,$$

where $n \in \mathbb{P}$, $D_0^w := 0$. The 2^nth Dirichlet kernels have a closed form (see e.g. [15])

$$D_{2^n}^w = D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{otherwise } (n \in \mathbb{N}). \end{cases} \tag{1}$$

The nth Fejér mean and the Fejér kernel of the Fourier series of a function f (see e.g. [6]) is defined by

$$\sigma_n^w(f;x) := \frac{1}{n}\sum_{i=0}^{n} S_i^w(f;x), \quad K_n^w(x) := \frac{1}{n}\sum_{k=0}^{n} D_k^w(x) \qquad (x \in G),$$

and $K_0^w = 0$.

On G^2 we consider the two-dimensional system as

$$\left\{ w_{n^1}(x^1) \times w_{n^2}(x^2) : \ (n^1, n^2) \in \mathbb{N}^2 \right\}.$$

The two-dimenional Fourier coefficients, the rectangular partial sums of the Fourier series and Dirichlet kernels are defined in the usual way. Define the nth Marcinkiewicz kernel $\mathcal{K}_n^w$ by

$$\mathcal{K}_n^w(x^1, x^2) := \frac{1}{n}\sum_{k=0}^{n} D_k^w(x^1) D_k^w(x^2) \qquad (x = (x^1, x^2) \in G^2).$$

For $x \in G$ we define $|x|$ by

$$|x| := \sum_{j=0}^{\infty} x_j 2^{-j-1},$$

for $x = (x^1, x^2) \in G^2$ by

$$|x|^2 := (x^1)^2 + (x^2)^2.$$

Thus, for $f \in L^p(G^2)$ $(1 \leq p \leq \infty)$ the modulus of continuity $\omega_p(\delta, f)$ is well defined for $\delta > 0$. We define the mixed modulus of continuity as follows

$$\omega_{1,2}^p(\delta_1, \delta_2, f) := \sup \Big\{ \| f(\cdot + x^1, \cdot + x^2) - f(\cdot + x^1, \cdot) - f(\cdot, \cdot + x^2) + f(\cdot, \cdot)\|_p : \ |x^1| \leq \delta_1, |x^2| \leq \delta_2 \Big\},$$

where $\delta_1, \delta_2 > 0$.

2. Nörlund means

Let $\{q_k : k \geq 1\}$ be a sequence of nonnegative numbers. The Nörlund means t_n^w and kernels L_n^w for the Walsh–Fourier series are defined by

$$t_n^w(f,x) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k}S_k^w(f,x), \qquad L_n^w(x) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k}D_k^w(x),$$

where

$$Q_n := \sum_{k=1}^{n-1} q_k \qquad (n \geq 1).$$

We always assume that $q_1 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty. \tag{2}$$

In this case, the summability method generated by $\{q_k\}$ is regular (see [14]) if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0. \tag{3}$$

In particular, t_n^w are the Fejér means (for all k set $q_k = 1$) and t_n^w are the (C,β)-means when

$$q_k := A_k^\beta := \binom{\beta+k}{k} \quad \text{for } \ k \geq 1 \text{ and } \beta \neq -1, -2, \dots .$$

In [14] the rate of the approximation by Nörlund means for Walsh–Fourier series of a function in L^p (in particular, in $\mathrm{Lip}(\alpha,p)$, where $\alpha > 0$ and $1 \leq p \leq \infty$) was studied. In case $p = \infty$, by L^p we mean C_W, the collection of the uniform W-continuous functions. As special cases Móricz and Siddiqi [14] obtained the earlier results given by Yano [18], Jastrebova [11] and Skvortsov [16] on the rate of the approximation by Cesàro means. The approximation properties of the Cesàro means of negativ order was studied by Goginava [9] in 2002.

The case when $q_k = 1/k$ is not discussed in the paper of Móricz and Siddiqi [14], in this case t_n^w are called the Nörlund logarithmic means. The Nörlund logarithmic means for the Walsh–Fourier series was discussed by Gát, Goginava and Tkebuchava earlier [4, 8], for unbounded Vilenkin system by Blahota and Gát [2].

In 2004, Gát and Goginava [5] discussed the uniform and L-convergence of the Nörlund logarithmic means of cubical partial sums of the two-dimensional Walsh–Fourier series.

Motivated by the work of Gát and Goginava, we investigate the two-dimensional Nörlund means of cubical partial sums of the two-dimensional Walsh–Fourier series. Define the means and kernels by the usual way

$$\mathbf{t}_n^w(f,x^1,x^2) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k} S_{k,k}^w(f,x^1,x^2),$$

$$\mathcal{L}_n^w(x^1,x^2) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k} D_k^w(x^1) D_k^w(x^2).$$

The means $\mathbf{t}_n^w$ could be called Walsh-Nörlund means of Marcinkiewicz type. We mention that the case that $q_k := 1/k$ is not included in this paper. For Walsh system this case is discussed by Gát and Goginava [5]. If we choose

$$q_k := A_k^\beta = \binom{\beta+k}{k} \quad \text{for } k \geq 1 \text{ and } \beta \neq -1,-2,\dots,$$

then we get the (C,β)-means of Marcinkiewicz-type, which was discussed by Goginava [10] with respect to double Walsh–Fourier series.

3. The rate of the approximation

In the following lemma we give a decomposition of the Walsh–Nörlund kernels of Marcinkiewicz type. This lemma is the two-dimensional analogue of the Lemma proved by Móricz and Siddiqi in [14, Lemma 3].

Lemma 1. *Let $|n| = A \geq 1$ and $\{q_k\}$ be a sequence of nonnegative numbers. Then*

$$\begin{aligned}
Q_n\mathcal{L}_n^w(x^1,x^2) &= Q_{n-2^A+1} D_{2^A}(x^1) D_{2^A}(x^2) + \\
&+ D_{2^A}(x^1) r_A(x^2) Q_{n-2^A} L_{n-2^A}^w(x^2) + D_{2^A}(x^2) r_A(x^1) Q_{n-2^A} L_{n-2^A}^w(x^1) + \\
&+ r_A(x^1) r_A(x^2) Q_{n-2^A} \mathcal{L}_{n-2^A}^w(x^1,x^2) + \\
&+ \sum_{j=0}^{A-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1}) D_{2^j}(x^1) D_{2^j}(x^2) + \\
&+ \sum_{j=0}^{A-1} D_{2^j}(x^2) r_j(x^1) \sum_{i=1}^{2^j-2} (q_{n-2^j-i} - q_{n-2^j-i-1}) i K_i^w(x^1) + \\
&+ \sum_{j=0}^{A-1} D_{2^j}(x^2) r_j(x^1) q_{n-2^{j+1}+1} (2^j-1) K_{2^j-1}^w(x^1) +
\end{aligned}$$

$$+\sum_{j=0}^{A-1} D_{2^j}(x^1) r_j(x^2) \sum_{i=1}^{2^j-2} (q_{n-2^j-i} - q_{n-2^j-i-1}) i K_i^w(x^2) +$$

$$+\sum_{j=0}^{A-1} D_{2^j}(x^1) r_j(x^2) q_{n-2^{j+1}+1} (2^j-1) K_{2^j-1}^w(x^2) +$$

$$+\sum_{j=0}^{A-1} r_j(x^1) r_j(x^2) \sum_{i=1}^{2^j-2} (q_{n-2^j-i} - q_{n-2^j-i-1}) i \mathcal{K}_i^w(x^1,x^2) +$$

$$+\sum_{j=0}^{A-1} r_j(x^1) r_j(x^2) q_{n-2^{j+1}+1} (2^j-1) \mathcal{K}_{2^j-1}^w(x^1,x^2).$$

Proof. During the proof of Lemma 1 we use the following equations:

$$D_{2^A+j}^w(x) = D_{2^A}(x) + r_A(x) D_j^w(x), \qquad j = 0,1,\ldots,2^A-1. \tag{4}$$

Let $|n| = A$, then we may write

$$Q_n \mathcal{L}_n^w(x^1,x^2) = \sum_{k=1}^{2^A-1} q_{n-k} D_k^w(x^1) D_k^w(x^2) +$$

$$+\sum_{k=2^A}^{n-1} q_{n-k} D_k^w(x^1) D_k^w(x^2) =: I + II.$$

By means of (4), we decompose II.

$$II = \sum_{j=0}^{n-2^A-1} q_{n-2^A-j} D_{2^A+j}^w(x^1) D_{2^A+j}^w(x^2) =$$

$$= D_{2^A}(x^1) D_{2^A}(x^2) \sum_{j=0}^{n-2^A-1} q_{n-2^A-j} +$$

$$+ D_{2^A}(x^1) r_A(x^2) \sum_{j=0}^{n-2^A-1} q_{n-2^A-j} D_j^w(x^2) +$$

$$+ D_{2^A}(x^2) r_A(x^1) \sum_{j=0}^{n-2^A-1} q_{n-2^A-j} D_j^w(x^1) +$$

$$+ r_A(x^1) r_A(x^2) Q_{n-2^A} \mathcal{L}_{n-2^A}^w(x^1,x^2) =$$

$$= Q_{n-2^A+1} D_{2^A}(x^1) D_{2^A}(x^2) + D_{2^A}(x^1) r_A(x^2) Q_{n-2^A} L_{n-2^A}^w(x^2) +$$

$$+ D_{2^A}(x^2) r_A(x^1) Q_{n-2^A} L_{n-2^A}^w(x^1) + r_A(x^1) r_A(x^2) Q_{n-2^A} \mathcal{L}_{n-2^A}^w(x^1,x^2).$$

We use (4) for the discussion of I as follows:

$$I=\sum_{k=1}^{2^A-1} q_{n-k}D_k^w(x^1)D_k^w(x^2)=\sum_{j=0}^{A-1}\sum_{i=0}^{2^j-1} q_{n-2^j-i}D_{2^j+i}^w(x^1)D_{2^j+i}^w(x^2)=$$

$$=\sum_{j=0}^{A-1}(Q_{n-2^j+1}-Q_{n-2^{j+1}+1})D_{2^j}(x^1)D_{2^j}(x^2)+$$

$$+\sum_{j=0}^{A-1} D_{2^j}(x^2)r_j(x^1)\sum_{i=0}^{2^j-1} q_{n-2^j-i}D_i^w(x^1)+$$

$$+\sum_{j=0}^{A-1} D_{2^j}(x^1)r_j(x^2)\sum_{i=0}^{2^j-1} q_{n-2^j-i}D_i^w(x^2)+$$

$$+\sum_{j=0}^{A-1} r_j(x^1)r_j(x^2)\sum_{i=0}^{2^j-1} q_{n-2^j-i}D_i^w(x^1)D_i^w(x^2)=:I_1+I_2+I_3+I_4.$$

We use Abel's transformation to study I_2, I_3, I_4:

$$I_2=\sum_{j=0}^{A-1} D_{2^j}(x^2)r_j(x^1)\times$$

$$\times\Big(\sum_{i=0}^{2^j-2}(q_{n-2^j-i}-q_{n-2^j-i-1})iK_i^w(x^1)+q_{n-2^{j+1}+1}(2^j-1)K_{2^j-1}^w(x^1)\Big),$$

An analogous one is obtaied for I_3, and

$$I_4=\sum_{j=0}^{A-1} r_j(x^1)r_j(x^2)\times$$

$$\times\Big(\sum_{i=0}^{2^j-2}(q_{n-2^j-i}-q_{n-2^j-i-1})i\mathcal{K}_i^w(x^1,x^2)+q_{n-2^{j+1}+1}(2^j-1)\mathcal{K}_{2^j-1}^w(x^1,x^2)\Big).$$

For a sequence $q_k\downarrow$, we would like to reach the same result as for a sequence $q_k\uparrow$. To do this we have to decompose the expression I in another way into two parts. But, we did not write our result in the statement of Lemma 1.

Let $n\in\mathbb{N}$ be fixed and set $|n|=A$. We write for I that

$$I=\sum_{j=0}^{A-2}\sum_{i=0}^{2^j-1} q_{n-2^j-i}D_{2^j+i}^w(x^1)D_{2^j+i}^w(x^2)+$$

$$+\sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i}D_{2^{A-1}+i}^w(x^1)D_{2^{A-1}+i}^w(x^2)=:I^1+I^2.$$

I^1 was studied in Lemma 1, too. To decompose I^2 we will use the following formula in [14]:

$$D^w_{2^j+i} - D^w_{2^{j+1}} = -w_{2^{j+1}-1} D^w_{2^j-i} \quad (0 \le i < 2^j). \tag{5}$$

Now, we write for I^2 that

$$I^2 = \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} (D^w_{2^{A-1}+i}(x^1) - D_{2^A}(x^1)) D^w_{2^{A-1}+i}(x^2) + \\ + D_{2^A}(x^1) \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} D^w_{2^{A-1}+i}(x^2) = \\ = \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} (D^w_{2^{A-1}+i}(x^1) - D_{2^A}(x^1))(D^w_{2^{A-1}+i}(x^2) - D_{2^A}(x^2)) + \\ + D_{2^A}(x^2) \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} (D^w_{2^{A-1}+i}(x^1) - D_{2^A}(x^1)) + \\ + D_{2^A}(x^1) \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} (D^w_{2^{A-1}+i}(x^2) - D_{2^A}(x^2)) + \\ + D_{2^A}(x^1) D_{2^A}(x^2) \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} \,.$$

Substituting (5) into I^2 and using an Abel's transformation we get the decomposition

$$I^2 = (Q_{n-2^{A-1}+1} - Q_{n-2^A+1}) D_{2^A}(x^1) D_{2^A}(x^2) - \\ - D_{2^A}(x^2) w_{2^A-1}(x^1) \sum_{l=1}^{2^{A-1}-1} (q_{n-2^A+l} - q_{n-2^A+l+1}) l K^w_l(x^1) - \\ - D_{2^A}(x^1) w_{2^A-1}(x^2) \sum_{l=1}^{2^{A-1}-1} (q_{n-2^A+l} - q_{n-2^A+l+1}) l K^w_l(x^2) + \\ + D_{2^A}(x^2) w_{2^A-1}(x^1) q_{n-2^{A-1}} 2^{A-1} K^w_{2^{A-1}}(x^1) + \\ + D_{2^A}(x^1) w_{2^A-1}(x^2) q_{n-2^{A-1}} 2^{A-1} K^w_{2^{A-1}}(x^2) + \\ + w_{2^A-1}(x^1 + x^2) \sum_{l=1}^{2^{A-1}-1} (q_{n-2^A+l} - q_{n-2^A+l+1}) l \mathcal{K}^w_l(x^1, x^2) + \\ + w_{2^A-1}(x^1 + x^2) q_{n-2^{A-1}} 2^{A-1} \mathcal{K}^w_{2^{A-1}}(x^1, x^2).$$

This completes the proof of Lemma 1. □

By means of this lemma we have our main theorem which states that the approximation behavior of the two-dimensional Walsh-Nörlund means of Marcinkiewicz type is so good as the approximation behavior of the one-dimensional Walsh–Nörlund means. The last one was investigated by Móricz and Siddiqi [14]. Recently, Fridli, Manchanda and Siddiqi [3] generalized the result of Móricz and Siddiqi [14] for homogeneous Banach spaces and dyadic Hardy spaces.

Theorem 1. *Let* $f \in L^p$, $1 \le p \le \infty$, $|n| = A \ge 1$ *and* $\{q_k : k \ge 1\}$ *be a sequence of nonnegative numbers.*

If $\{q_k\}$ *is nondecreasing, in sign*: $\uparrow$, *then*

$$\|\mathbf{t}_n^w(f) - f\|_p \le \frac{c}{Q_n} \sum_{l=0}^{A-1} q_{n-2^l} 2^l \omega_p(2^{-l}, f) + O(\omega_p(2^{-A}, f)).$$

If $\{q_k\}$ *is nonincreasing, in sign*: $\downarrow$, *such that*

$$\frac{n}{Q_n^2} \sum_{k=1}^{n-1} q_k^2 = O(1), \tag{6}$$

then

$$\|\mathbf{t}_n^w(f) - f\|_p \le \frac{c}{Q_n} \sum_{l=0}^{A-1} q_{n-2^l} 2^l \omega_p(2^{-l}, f) + O(\omega_p(2^{-A}, f)).$$

To prove our theorem we need the following lemmas given by Móricz and Schipp [13], Yano [17], and Glukhov [7].

Lemma 2 (Móricz and Schipp [13]). *If the condition* (6) *is satisfied, then there exists a constant* C *such that*

$$\|L_n^w\|_1 \le C \quad (n \ge 1).$$

Lemma 3 (Yano [17]). *Let* $n \ge 1$, *then*

$$\|K_n^w\|_1 \le 2.$$

Lemma 4 (Glukhov [7]). *Let* $\alpha_1, \ldots, \alpha_n$ *be real numbers. Then*

$$\frac{1}{n}\Big\| \sum_{k=1}^{n} \alpha_k D_k^w D_k^w \Big\|_1 \le \frac{c}{\sqrt{n}} \Big(\sum_{k=1}^{n} \alpha_k^2 \Big)^{1/2},$$

where c *is an absolute constant.*

As a corollary of Lemma 4, we get that

$$\|\mathcal{K}_n^w\|_1 \le C \quad (n \ge 1), \tag{7}$$

where C is an absolute constant and that condition (6) implies

$$\|\mathcal{L}_n^w\|_1 \le C \quad (n \ge 1), \tag{8}$$

where C is an absolute constant.

Proof of Theorem 1. Clearly, condition (6) implies the regularity of the summability method. We present make the proof for $1 \le p < \infty$, since for $p = \infty$ the proof goes in a similar way (where $L^\infty = C_W$).

For a sequence $q_k \uparrow$, we use the decomposition given in Lemma 1, while for a sequence $q_k \downarrow$ we also use the decomposition in Lemma 1 and the decomposition of I^1 and I^2 in the proof of Lemma 1.

Let $n \in \mathbb{N}$ be fixed and set $|n| = A$. By Lemma 1 and the Minkowski inequality, we may write that

$$
\begin{aligned}
&Q_n \|\mathbf{t}_n^w(f) - f\|_p \le \\
&\le Q_{n-2^A+1} \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^A}(x^1) D_{2^A}(x^2)\, d\mu(x) \Big\|_p + \\
&+ Q_{n-2^A} \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^A}(x^1) r_A(x^2) L^w_{n-2^A}(x^2)\, d\mu(x) \Big\|_p + \\
&+ Q_{n-2^A} \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^A}(x^2) r_A(x^1) L^w_{n-2^A}(x^1)\, d\mu(x) \Big\|_p + \\
&+ Q_{n-2^A} \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) r_A(x^1) r_A(x^2) \mathcal{L}^w_{n-2^A}(x^1, x^2)\, d\mu(x) \Big\|_p + \\
&+ \sum_{j=0}^{A-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1}) \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^j}(x^1) D_{2^j}(x^2)\, d\mu(x) \Big\|_p + \\
&+ \sum_{j=0}^{A-1} \sum_{i=1}^{2^j-2} |q_{n-2^j-i} - q_{n-2^j-i-1}| i \times \\
&\times \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^j}(x^2) r_j(x^1) K^w_i(x^1)\, d\mu(x) \Big\|_p + \\
&+ \sum_{j=0}^{A-1} \sum_{i=1}^{2^j-2} |q_{n-2^j-i} - q_{n-2^j-i-1}| i \times \\
&\times \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^j}(x^1) r_j(x^2) K^w_i(x^2)\, d\mu(x) \Big\|_p + \\
&+ \sum_{j=0}^{A-1} q_{n-2^{j+1}+1} (2^j - 1) \times \\
&\times \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^j}(x^2) r_j(x^1) K^w_{2^j-1}(x^1)\, d\mu(x) \Big\|_p + \\
&+ \sum_{j=0}^{A-1} q_{n-2^{j+1}+1} (2^j - 1) \times \\
&\times \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^j}(x^1) r_j(x^2) K^w_{2^j-1}(x^2)\, d\mu(x) \Big\|_p +
\end{aligned}
$$

$$+\sum_{j=0}^{A-1}\sum_{i=1}^{2^j-2}|q_{n-2^j-i}-q_{n-2^j-i-1}|i\times$$

$$\times\Big\|\int_{G^2}(f(\cdot+x)-f(\cdot))r_j(x^1)r_j(x^2)\mathcal{K}_i^w(x^1,x^2)\,d\mu(x)\Big\|_p+$$

$$+\sum_{j=0}^{A-1}q_{n-2^{j+1}+1}(2^j-1)\times$$

$$\times\Big\|\int_{G^2}(f(\cdot+x)-f(\cdot))r_j(x^1)r_j(x^2)\mathcal{K}_{2^j-1}^w(x^1,x^2)\,d\mu(x)\Big\|_p=:\sum_{i=1}^{11}A_{n,i}.$$

By the above formula for a sequence $q_k\downarrow$, in the expressions $A_{n,5}$, ..., $A_{n,11}$ the sum goes upto $A-2$ and we have seven extra expressions $A_{n,12}$, ..., $A_{n,18}$ from the expression in the case of I^2, but we will discuss them in the second part of the proof of Theorem 1.

Now, we discuss $A_{n,1}$. By (1), we find that

$$\Big\|\int_{G^2}(f(\cdot+x)-f(\cdot))D_{2^A}(x^1)D_{2^A}(x^2)\,d\mu(x)\Big\|_p\le\int_{I_A^2}D_{2^A}(x^1)D_{2^A}(x^2)\times \tag{9}$$

$$\times\Big(\int_{G^2}|f(y+x)-f(y)|^p\,d\mu(y)\Big)^{1/p}d\mu(x)\le c\omega_p(2^{-A},f).$$

Thus, we immediately have

$$A_{n,1}\le cQ_{n-2^A+1}\omega_p(2^{-A},f)$$

and

$$A_{n,5}\le c\sum_{j=0}^{A-1}(Q_{n-2^j+1}-Q_{n-2^{j+1}+1})\omega_p(2^{-j},f).$$

If $q_k\uparrow$, then we get that

$$(Q_{n-2^j+1}-Q_{n-2^{j+1}+1})\le 2^jq_{n-2^j}$$

and

$$A_{n,5}\le c\sum_{j=0}^{A-1}2^jq_{n-2^j}\omega_p(2^{-j},f).$$

If $q_k\downarrow$, then we get that

$$(Q_{n-2^j+1}-Q_{n-2^{j+1}+1})\le 2^jq_{n-2^{j+1}}$$

and

$$A_{n,5}\le c\sum_{j=0}^{A-2}2^jq_{n-2^{j+1}}\omega_p(2^{-j},f)\le c\sum_{l=1}^{A-1}2^lq_{n-2^l}\omega_p(2^{-l+1},f).$$

To discuss $A_{n,2},\ A_{n,3},\ A_{n,6},\ A_{n,7},\ A_{n,8},\ A_{n,9}$ for any $\varepsilon \in G,\ y \in G^2$ and $A \in \mathbb{P}$ we write the following

$$
\begin{aligned}
(10)\qquad & \Big| \int_{I_A(\varepsilon)\times I_A} (f(y+x) - f(y)) r_A(x^1)\, d\mu(x) \Big| = \\
&= \Big| \int_{I_A(\varepsilon)\times I_A} f(y+x) r_A(x^1)\, d\mu(x) \Big| = \\
&= \Big| \int_{I_{A+1}(\varepsilon)\times I_A} f(y+x) r_A(x^1)\, d\mu(x) + \int_{I_{A+1}(\varepsilon+e_A)\times I_A} f(y+x) r_A(x^1)\, d\mu(x) \Big| = \\
&= \Big| \int_{I_{A+1}(\varepsilon)\times I_A} f(y+x) - f(y+x+e_A^1)\, d\mu(x) \Big| \le \\
&\le \int_{I_{A+1}(\varepsilon)\times I_A} |f(y+x) - f(y+x+e_A^1)|\, d\mu(x),
\end{aligned}
$$

where $e_A^1 := (e_A, 0)$ (and $e_A^2 := (0, e_A)$ we will use it later too).

To discuss $A_{n,6}$, for any $|j| \le k$ we write that

$$
B_j^k := \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^k}(x^2) r_k(x^1) K_j^w(x^1)\, d\mu(x) \Big\|_p =
$$

$$
= \Big\| \sum_{\substack{\varepsilon_i = 0 \\ i \in \{0,1,\dots,k-1\} \\ \varepsilon_l = 0, l \ge k}}^{1} \int_{I_k(\varepsilon)\times I_k} (f(\cdot + x) - f(\cdot)) D_{2^k}(x^2) r_k(x^1) K_j^w(x^1)\, d\mu(x) \Big\|_p .
$$

The function $K_j^w(x^1)$ is constant on the sets $I_k(\varepsilon)$ ($\varepsilon \in G,\ |j| \le k$). Thus, (10) and Lemma 3 imply that

$$
\begin{aligned}
B_j^k &= \Big\| \sum_{\substack{\varepsilon_i = 0 \\ i \in \{0,1,\dots,k-1\} \\ \varepsilon_l = 0,\ l \ge k}}^{1} 2^k K_j^w(\varepsilon) \int_{I_k(\varepsilon)\times I_k} (f(\cdot + x) - f(\cdot)) r_k(x^1)\, d\mu(x) \Big\|_p \le \\
&\le \sum_{\substack{\varepsilon_i = 0 \\ i \in \{0,1,\dots,k-1\} \\ \varepsilon_l = 0,\ l \ge k}}^{1} 2^k |K_j^w(\varepsilon)| \times \\
&\times \Big(\int_{G^2} \Big| \int_{I_k(\varepsilon)\times I_k} (f(y+x) - f(y)) r_k(x^1)\, d\mu(x) \Big|^p d\mu(y) \Big)^{1/p} \le \\
&\le \sum_{\substack{\varepsilon_i = 0 \\ i \in \{0,1,\dots,k-1\} \\ \varepsilon_l = 0,\ l \ge k}}^{1} 2^k |K_j^w(\varepsilon)| \times \\
&\times \Big(\int_{G^2} \Big(\int_{I_{k+1}(\varepsilon)\times I_k} |f(y+x) - f(y+x+e_k^1)|\, d\mu(x) \Big)^p d\mu(y) \Big)^{1/p} \le
\end{aligned}
$$

$$\leq \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,1,\ldots,k-1\} \\ \varepsilon_l=0,\ l\geq k}}^{1} 2^k|K_j^w(\varepsilon)| \times$$

$$\times \int_{I_{k+1}(\varepsilon)\times I_k} \Big(\int_{G^2} |f(y+x)-f(y+x+e_k^1)|^p\, d\mu(y)\Big)^{1/p} d\mu(x) \leq$$

$$\leq c \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,1,\ldots,k-1\} \\ \varepsilon_l=0,\ l\geq k}}^{1} 2^k|K_j^w(\varepsilon)|\omega_p(2^{-k},f)\int_{I_{k+1}(\varepsilon)\times I_k} d\mu(x) \leq$$

$$\leq c\omega_p(2^{-k},f)\|K_j^w\|_1 \leq c\omega_p(2^{-k},f).$$

That is, for any $|j|\leq k$ we have

(11)

$$B_j^k := \Big\| \int_{G^2} (f(\cdot+x)-f(\cdot))D_{2^k}(x^2)r_k(x^1)K_j^w(x^1)\, d\mu(x)\Big\|_p \leq c\omega_p(2^{-k},f).$$

This implies that

$$A_{n,6} = \sum_{j=0}^{A-1}\sum_{i=0}^{2^j-2} |q_{n-2^j-i}-q_{n-2^j-i-1}|iB_i^j \leq$$

$$\leq c\sum_{j=0}^{A-1}\sum_{i=0}^{2^j-2} |q_{n-2^j-i}-q_{n-2^j-i-1}|i\omega_p(2^{-j},f),$$

and the discussion of $A_{n,7}$ goes similarly to that of $A_{n,6}$.

Moreover, we have

$$A_{n,8} \leq c\sum_{j=0}^{A-1} q_{n-2^{j+1}+1}2^jB_{2^j-1}^j \leq c\sum_{j=0}^{A-1} q_{n-2^{j+1}+1}2^j\omega_p(2^{-j},f),$$

and the treatment of $A_{n,9}$ goes similarly to that of $A_{n,8}$.

If $q_k\uparrow$, then we get that

$$\sum_{i=0}^{2^j-2} |q_{n-2^j-i}-q_{n-2^j-i-1}|i \leq \tag{12}$$

$$\leq \sum_{i=1}^{2^j-2} q_{n-2^j-i} - (2^j-2)q_{n-2^{j+1}+1} \leq \sum_{i=1}^{2^j-2} q_{n-2^j-i} \leq 2^jq_{n-2^j}$$

and

$$A_{n,6} \leq c\sum_{j=0}^{A-1} 2^jq_{n-2^j}\omega_p(2^{-j},f).$$

Moreover, we have

$$A_{n,8} \leq c \sum_{j=0}^{A-1} q_{n-2^j} 2^j \omega_p(2^{-j}, f).$$

In the case when $q_k \downarrow$ we have

$$\sum_{i=0}^{2^j-2} |q_{n-2^j-i} - q_{n-2^j-i-1}| i \leq \tag{13}$$

$$\leq (2^j - 2) q_{n-2^{j+1}+1} - \sum_{i=1}^{2^j-2} q_{n-2^j-i} \leq 2^j q_{n-2^{j+1}}$$

and

$$A_{n,6} \leq c \sum_{j=0}^{A-2} 2^j q_{n-2^{j+1}} \omega_p(2^{-j}, f).$$

Moreover, we have

$$A_{n,8} \leq c \sum_{j=0}^{A-2} q_{n-2^{j+1}} 2^j \omega_p(2^{-j}, f).$$

Now, we introduce the notation $\widetilde{B}_j^A$ in order to discuss $A_{n,2}$, $A_{n,3}$. First, in the case when $q_k \downarrow$ set

$$\widetilde{B}_j^A := \Big\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^A}(x^2) r_A(x^1) L_j^w(x^1)\, d\mu(x) \Big\|_p.$$

The method above (see (11)), condition (6) and Lemma 2 imply that

$$\widetilde{B}_{n-2^A}^A \leq c\omega_p(2^{-A}, f) \|L_{n-2^A}^w\|_1 \leq c\omega_p(2^{-A}, f)$$

(we note that $|n - 2^A| \leq A - 1$) and

$$A_{n,3} \leq c Q_{n-2^A} \omega_p(2^{-A}, f).$$

Now, we consider the case when $q_k \uparrow$. We use Abel's transformation for $Q_{n-2^A} L_{n-2^A}^w$ to get

$$Q_{n-2^A} L_{n-2^A}^w = \sum_{j=1}^{n-2^A-2} (q_{n-2^A-j} - q_{n-2^A-j-1}) j K_j^w + q_1 (n - 2^A - 1) K_{n-2^A-1}^w,$$

the definition of B_j^A and (11) to conclude that

$$A_{n,3} \leq c\omega_p(2^{-A}, f) \Big(\sum_{j=1}^{n-2^A-2} |q_{n-2^A-j} - q_{n-2^A-j-1}| j + q_1 (n - 2^A - 1) \Big) \leq$$

$$\leq c\omega_p(2^{-A},f)\Big(\sum_{j=1}^{n-2^A-2} q_{n-2^A-j}+q_1(n-2^A-1)\Big)\leq$$
$$\leq c\omega_p(2^{-A},f)(Q_{n-2^A}+q_1(n-2^A-1)).$$

We note that if $q_n\uparrow$, then $Q_n\geq(n-1)q_1$ and the estimate of $A_{n,2}$ goes similarly.

At last, we discuss $A_{n,4}$, $A_{n,10}$, and $A_{n,11}$. First, we investigate $A_{n,4}$ and the estimate of the others go similarly, but we will write some words about it.

First, let $q_k\downarrow$. We note that $\mathcal{L}_j^w(x^1,x^2)$ is constant on the sets $I_A(\varepsilon)\times I_A(\rho)$ $(\varepsilon,\rho\in G)$. This and the generalized Minkowski inequality give

$$F_j^A:=\Big\|\int_{G^2}(f(\cdot+x)-f(\cdot))r_A(x^1)r_A(x^2)\mathcal{L}_j^w(x^1,x^2)\,d\mu(x)\Big\|_p=$$
$$=\Big\|\sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1}\ \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1}\int_{I_A(\varepsilon)\times I_A(\rho)}(f(\cdot+x)-f(\cdot))\times$$
$$\times r_A(x^1)r_A(x^2)\mathcal{L}_j^w(x^1,x^2)\,d\mu(x)\Big\|_p\leq$$
$$\leq\sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1}\ \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1}|\mathcal{L}_j^w(\varepsilon,\rho)|\times$$
$$\times\Big\|\int_{I_A(\varepsilon)\times I_A(\rho)}(f(\cdot+x)-f(\cdot))r_A(x^1)r_A(x^2)\,d\mu(x)\Big\|_p\leq$$
$$\leq\sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1}\ \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1}|\mathcal{L}_j^w(\varepsilon,\rho)|\times$$
$$\times\Big(\int_{G^2}\Big|\int_{I_A(\varepsilon)\times I_A(\rho)}(f(y+x)-f(y))r_A(x^1)r_A(x^2)\,d\mu(x)\Big|^p\,d\mu(y)\Big)^{1/p}$$

for $|j|\leq A$. Analogously to (10), we easily get that

$$\Big|\int_{I_A(\varepsilon)\times I_A(\rho)}(f(y+x)-f(y))r_A(x^1)r_A(x^2)\,d\mu(x)\Big|\leq \tag{14}$$
$$\leq\int_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)}\Delta_Af(x,y)\,d\mu(x),$$

where

$$\Delta_Af(x,y):=\Big|f(x+y)-f(x+y+e_A^2)-f(x+y+e_A^1)+f(x+y+e_A^1+e_A^2)\Big|.$$

Now, inequality (14), condition (6) and Lemma 4 (see equation (8)) imply that

$$F_j^A \leq \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0 \\ j\in\{0,\dots,A-1\}}}^{1} |\mathcal{L}_j^w(\varepsilon,\rho)| \times$$

$$\times \Big(\int_{G^2}\Big(\int_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)} \Delta_A f(x,y)\,d\mu(x)\Big)^p d\mu(y)\Big)^{1/p} \leq$$

$$\leq \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0 \\ j\in\{0,\dots,A-1\}}}^{1} |\mathcal{L}_j^w(\varepsilon,\rho)| \times$$

$$\times \int_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)} \Big(\int_{G^2}(\Delta_A f(x,y))^p\,d\mu(y)\Big)^{1/p} d\mu(x) \leq$$

$$\leq \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0 \\ j\in\{0,\dots,A-1\}}}^{1} \int_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)} |\mathcal{L}_j^w(\varepsilon,\rho)| \times$$

$$\times\ d\mu(x)\,\omega_{1,2}^p(2^{-A},2^{-A},f) \leq$$

$$\leq c\,\|\mathcal{L}_j^w\|_1\omega_{1,2}^p(2^{-A},2^{-A},f) \leq c\,\omega_{1,2}^p(2^{-A},2^{-A},f).$$

That is, for $|j|\leq A$, we have

$$F_j^A \leq c\|\mathcal{L}_j^w\|_1\omega_{1,2}^p(2^{-A},2^{-A},f) \leq c\omega_{1,2}^p(2^{-A},2^{-A},f) \tag{15}$$

and

$$A_{n,4} \leq Q_{n-2^A}F_{n-2^A}^A \leq cQ_{n-2^A}\omega_{1,2}^p(2^{-A},2^{-A},f) \leq cQ_{n-2^A}\omega_p(2^{-A},f).$$

Next, we discuss $A_{n,4}$ in the case when $q_k\uparrow$. By Abel's transformation, we write that

$$Q_{n-2^A}\mathcal{L}_{n-2^A}^w = \sum_{j=1}^{n-2^A-2}(q_{n-2^A-j}-q_{n-2^A-j-1})j\mathcal{K}_j^w + q_1(n-2^A-1)\mathcal{K}_{n-2^A-1}^w.$$

For $|j|\leq A$, we set

$$\widetilde{F}_j^A := \Big\|\int_{G^2}(f(\cdot+x)-f(\cdot))r_A(x^1)r_A(x^2)\mathcal{K}_j^w(x^1,x^2)\,d\mu(x)\Big\|_p.$$

The method of estimating F_j^A (see (15)) and Lemma 4 (see equation (7)) imply that

$$\widetilde{F}_j^A \leq c\omega_{1,2}^p(2^{-A},2^{-A},f)\|\mathcal{K}_j^w\|_1 \leq c\omega_p(2^{-A},f)$$

for $|j|\leq A$ and

$$A_{n,4} = c\omega_p(2^{-A},f)(Q_{n-2^A}+q_1(n-2^A-1)).$$

For more details, see $A_{n,2}$ and $A_{n,3}$, in the case when $q_k\uparrow$.

Finally, we discuss $A_{n,10}$ and $A_{n,11}$ as follows:

$$A_{n,10} = \sum_{j=0}^{A-1} \sum_{i=0}^{2^j-2} |q_{n-2^j-i} - q_{n-2^j-i-1}| i \widetilde{F}_i^j \leq$$

$$\leq c \sum_{j=0}^{A-1} \sum_{i=0}^{2^j-2} |q_{n-2^j-i} - q_{n-2^j-i-1}| i \omega_p(2^{-j}, f).$$

If $q_k \uparrow$, by (12) we get that

$$A_{n,10} \leq \sum_{j=0}^{A-1} 2^j q_{n-2^j} \omega_p(2^{-j}, f).$$

If $q_k \downarrow$, by (13) we get that

$$A_{n,10} \leq \sum_{j=0}^{A-2} 2^j q_{n-2^{j+1}} \omega_p(2^{-j}, f).$$

At last, we get

$$A_{n,11} \leq \sum_{j=0}^{A-1} q_{n-2^{j+1}+1} 2^j \widetilde{F}_{2^j-1}^j \leq c \sum_{j=0}^{A-1} q_{n-2^{j+1}+1} 2^j \omega_p(2^{-j}, f).$$

If $q_k \uparrow$, then we have

$$A_{n,11} \leq c \sum_{j=0}^{A-1} q_{n-2^j} 2^j \omega_p(2^{-j}, f),$$

while, if $q_k \downarrow$, then we have

$$A_{n,11} \leq c \sum_{j=0}^{A-2} q_{n-2^{j+1}} 2^j \omega_p(2^{-j}, f).$$

Combining our results on $A_{n,i}$ $(i = 1, \dots, 11)$ completes the proof of Theorem 1 for sequences $q_k \uparrow$.

The second part of the proof of Theorem 1. Let the sequence q_k be nonincreasing $(q_k \downarrow)$. We define $A_{n,i}$ $(i = 12, \dots, 18)$ analogously as we did in the case of $A_{n,1}, \dots, A_{n,11}$ (see the decomposition of I^2).

By (9), we have

$$A_{n,12} \leq c(Q_{n-2^{A-1}+1} - Q_{n-2^A+1}) \omega_p(2^{-A}, f).$$

To estimate $A_{n,13}$, $A_{n,14}$, $A_{n,15}$, $A_{n,16}$, we define $\widetilde{\widetilde{B}}_j^{A-1}$ as follows: for $|j| \leq A-1$ we set

$$\widetilde{\widetilde{B}}_j^{A-1} := \left\| (f(\cdot + x) - f(\cdot)) D_{2^A}(x^2) r_{A-1}(x^1) \omega_{2^{A-1}-1}(x^1) K_j^w(x^1) \, d\mu(x) \right\|_p.$$

The method of estimating B_j^A (see (11)) and Lemma 3 imply that

$$\tilde{\tilde{B}}_j^{A-1} \le c\omega_p(2^{-(A-1)}, f),$$

$$A_{n,15}, A_{n,16} \le cq_{n-2^{A-1}} 2^{A-1}\omega_p(2^{-(A-1)}, f)$$

and

$$A_{n,13}, A_{n,14} \le c \sum_{l=1}^{2^{A-1}-1} |q_{n-2^A+l} - q_{n-2^A+l+1}| l \tilde{\tilde{B}}_l^{A-1} \le$$

$$\le c \sum_{l=1}^{2^{A-1}-1} |q_{n-2^A+l} - q_{n-2^A+l+1}| l\omega_p(2^{-(A-1)}, f) \le$$

$$\le c(Q_{n-2^{A-1}} - Q_{n-2^A+1})\omega_p(2^{-A}, f).$$

To estimate $A_{n,17}$ and $A_{n,18}$, we define $\tilde{\tilde{F}}_j^{A-1}$ as follows: for $|j| \le A-1$ we set

$$\tilde{\tilde{F}}_j^{A-1} := \left\| (f(\cdot+x)-f(\cdot)) r_{A-1}(x^1+x^2)\omega_{2^{A-1}-1}(x^1+x^2)\mathcal{K}_j^w(x^1, x^2)\, d\mu(x) \right\|_p .$$

The method of estimating F_j^A (see (15)) and Lemma 4 imply that

$$\tilde{\tilde{F}}_j^{A-1} \le c\omega_{1,2}^p(2^{-(A-1)}, 2^{-(A-1)}, f) \le c\omega_p(2^{-(A-1)}, f).$$

As a result, we obtain

$$A_{n,18} \le q_{n-2^{A-1}} 2^{A-1} \tilde{\tilde{F}}_{2^{A-1}}^{A-1} \le cq_{n-2^{A-1}} 2^{A-1}\omega_p(2^{-(A-1)}, f)$$

and

$$A_{n,17} \le c \sum_{l=1}^{2^{A-1}-1} |q_{n-2^A+l} - q_{n-2^A+l+1}| l \tilde{\tilde{F}}_l^{A-1} \le$$

$$\le c \sum_{l=1}^{2^{A-1}-1} |q_{n-2^A+l} - q_{n-2^A+l+1}| l\omega_p(2^{-(A-1)}, f) \le$$

$$\le c(Q_{n-2^{A-1}} - Q_{n-2^A+1})\omega_p(2^{-A}, f).$$

These facts complete the proof of Theorem 1. □

We will discuss the following cases:

(a) if $q_k \uparrow$ and satisfies the condition

$$\frac{nq_{n-1}}{Q_n} = O(1). \tag{16}$$

In particular, this is the case when

$$q_k \asymp k^\beta \quad \text{or} \quad (\log k)^\beta \qquad \text{for some} \quad \beta > 0.$$

(b) if $q_k \downarrow$ and satisfies the condition

(bi) $q_k \asymp k^{-\beta}$ for some $0 < \beta < 1$, or

(bii) $q_k \asymp (\log k)^{-\beta}$ for some $0 < \beta$.

We note that condition (6) is satisfied in these cases. For more details, see [14].

The one-dimensional analogue of the following theorem was proved by Móricz and Siddiqi [14]. We mention that as special case (set $q_k := 1$ for all k) we get the so-called Marcinkiewicz means of Walsh–Fourier series. More generally, in the case when

$$q_k := A_k^\beta := \binom{\beta + k}{k} \quad \text{for } k \geq 1 \ (\beta \neq -1, -2, \ldots),$$

we have the (C, β) means of Marcinkiewicz type discussed by Goginava [10] with respect to double Walsh–Fourier series.

At last, we note that the following theorem states that the approximation behavior of the Nörlund means of square partial sums of double Walsh–Fourier series is so good as the approximation behavior of the one-dimensional Nörlund means of Walsh–Fourier series for Lipschitz functions showed by Móricz and Siddiqi [14].

Theorem 2. *Let $f \in \mathrm{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p \leq \infty$. If $\{q_k : k \geq 1\}$ is a sequence of nonnegative numbers such that in case $q_k \uparrow$ condition* (16) *is satisfied, while in case $q_k \downarrow$ either condition* (bi) *or* (bii) *is satisfied, then*

$$\|\mathbf{t}_n^w(f) - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if } 0 < \alpha < 1, \\ O(n^{-1} \log n), & \text{if } \alpha = 1, \\ O(n^{-1}), & \text{if } \alpha > 1. \end{cases}$$

Proof. First, let $q_k \uparrow$ satisfy condition (16). Theorem 1 and the method of Móricz and Siddiqi [14] immediately give our statement.

Second, let $q_k \downarrow$ satisfy condition (bi), that is,

$$q_k \asymp k^{-\beta} \text{ for some } 0 < \beta < 1, \quad \text{then} \quad Q_n \asymp n^{1-\beta}.$$

From Theorem 1 it follows that

$$\|\mathbf{t}_n^w(f) - f\|_p \leq \frac{c}{Q_n} \sum_{l=0}^{|n|-1} q_{n-2^l} 2^l 2^{-l\alpha} + O(2^{-|n|\alpha}).$$

For $0 \leq l \leq |n| - 1$, we have

$$2^{|n|-1} \leq n - 2^l \quad \text{and} \quad q_{n-2^l} \leq c 2^{-\beta(|n|-1)}.$$

Thus, we conclude that

$$\|\mathbf{t}_n^w(f)-f\|_p \le \frac{c}{n^{1-\beta}}\sum_{l=0}^{|n|-1} 2^{-\beta|n|}2^{l(1-\alpha)} + O(2^{-|n|\alpha}) \le$$

$$\le \frac{c}{n}\sum_{l=0}^{|n|-1} 2^{l(1-\alpha)} + O(2^{-|n|\alpha}) = \begin{cases} O(2^{|n|(1-\alpha)}/n), & \text{if } 0<\alpha<1,\\ O(|n|/n), & \text{if } \alpha=1,\\ O(\frac{1}{n}), & \text{if } \alpha>1.\end{cases}$$

Let condition (bii) be satisfied, that is,

$$q_k \asymp (\log k)^{-\beta} \text{ for some } 0<\beta, \quad \text{then} \quad Q_n \asymp n(\log n)^{-\beta}.$$

The proof goes along the same lines as that of case (bi). □

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Аппроксимация средними Нёрлунда сумм по квадратам двойного ряда Фурье–Уолша

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В работе рассматриваются средние Нёрлунда сумм Фурье–Уолша по квадратам для функций из L^p $(1 \leq p \leq \infty)$. Изучаются порядки приближений функций с помощью этих средних, в частности для функций из классов $\mathrm{Lip}(\alpha, p)$, где $\alpha > 0$ и $1 \leq p \leq \infty$. В случае $p = \infty$ мы считаем, что L^∞ это C_W, т.е. класс всех $W-$непрерывных функций. Наши основные теоремы утверждают, что для двумерных рядов Фурье–Уолша качество приближения средними Уолша–Нёрлунда не хуже, чем для одномерных рядов.

Как частные случаи наших результатов получаются оценки, недавно полученные в работе Гата и Гогинавы [5] для логарифмических средних сумм по кубам ряда Фурье–Уолша, а также $(C, \alpha)-$средних типа Марцинкевича для двойного ряда, которые изучал Гогинава [10].

Более ранние результаты для одномерных средних Нёрлунда ряда Фурье–Уолша были получены Морицем и Сиддики в работе [14].